A Robust Adaptive Controller with Minimal Modifications for Discrete Time-Varying Systems

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***Abstract*—The goal of this paper is to show that an indirect adaptive controller with parameter projection as the only modification on the basis of conventional adaptive control algorithms can globally stabilize systems having fast parasitics, bounded external disturbances, and time-varying parameters without any restriction on signals in the closed-loop system such as persistence of excitation. Further, the controller can still retain the properties of earlier unmodified conventional adaptive controllers when the controlled plant satisfies so-called "ideal assumptions" or the rates at which the plant parameters' change belong to the l_1 (or l_2) space.**

I. Introduction

Earlier robust adaptive control algorithms involve various modifications such as parameter projection, together with normalization [1], σ-modification with normalization [2], and dead zones [3], [9]. A summary of the progress can be found in [4]–[6]. Those modified algorithms contain some critical parameters to be chosen to ensure global stability when unmodeled dynamics appear. Also, assumptions on unmodeled dynamics are made for implementations. Taking the use of relative dead zone [3], [6], [9], [11] as an example, we need to know an upper bound of the gain of the unmodeled dynamics. The stability condition, on the other hand, requires this bound to be sufficiently small. Clearly, the choice of such parameters makes it complicated to implement the algorithm.

To avoid the choice of such parameters related to unknown modeling errors, we studied the robustness of an *indirect* conventional adaptive algorithm which involves a basic parameter estimator subject to parameter projection as the only modification and a *pole assignment* control synthesis module [7]. In this paper, we will reexamine the robustness properties of this adaptive algorithm by applying it to a plant which is allowed to have a *time-varying* reduced-order model. It is shown that the adaptive controller can still globally stabilize this type of time-varying system in the presence of modeling errors, including unmodeled dynamics and external disturbances.

The continuous-time version of the above adaptive controller was studied in [11] for time-varying systems without modeling errors. A *relative dead zone* is built in when the system studied has a modeling error such as unmodeled dynamics and external disturbance. The robustness of a *direct model reference adaptive control* scheme with parameter projection was also studied for systems with a *time-invariant* reduced-order model in [12]–[14].

II. System Models

The class of controlled time-varying plants we consider can be modeled as in the equation

$$A(q^{-1}, t)y(t) = B(q^{-1}, t)u(t) + m(t) \tag{1}$$

where u and y represent the input and output, respectively, $A(q^{-1}, t)$ and $B(q^{-1}, t)$ are time-varying polynomials of degree n in the backward shift operator q^{-1}, i.e.,

$$A(q^{-1}) = 1 + a_1(t)q^{-1} + \cdots + a_n(t)q^{-n}$$

$$B(q^{-1}) = b_1(t)q^{-1} + \cdots + b_n(t)q^{-n}$$

and $m(t)$ denotes the modeling error consisting of bounded disturbances $d(t)$ and unmodeled dynamics $\eta(t)$, i.e.,

$$m(t) = \eta(t) + d(t). \tag{2}$$

Now, rewrite (1) in a regression form as

$$y(t) = \phi^T(t-1)\theta(t) + m(t) \tag{3}$$

where $\phi(t-1)$ is a regression vector and $\theta(t)$ denotes a vector containing unknown time-varying parameters of the nominal system model (reduced-order model), i.e.,

$$\phi^T(t-1) = [y(t-1), \cdots, y(t-n),\ u(t-1), \cdots, u(t-n)]$$

$$\theta^T = [-a_1(t), \cdots, -a_n(t),\ b_1(t), \cdots, b_n(t)].$$

For the plant in (1), we have the following.

Assumption 2.1: 1) Unmodeled dynamics term $\eta(t)$ satisfies

$$|\eta(t)| \le \bar{d} + \epsilon r_0(t) \tag{4}$$

where $\bar{d}$ is a constant which bounds the initial value η_0, ϵ is a small nonnegative constant, and $r_0(t)$ is defined as

$$r_0(t) = \mu_0 r_0(t-1) + \|\phi(t-1)\|, \qquad r_0(0) = 0 \tag{5}$$

where μ_0 is a constant satisfying $|\mu_0| < 1$.

2) For $d(t)$, there exists a constant d' such that

$$|d(t)| \le d'. \qquad \square \tag{6}$$

Since $|\mu_0| < 1$ in (5), we can have

$$r_0(t) \le c_\eta \max_{0,\cdots,t-1} \|\phi(\tau)\| \tag{7}$$

where c_η is a constant.

Thus, if $\phi(\tau)$ is bounded according to $\|\phi(\tau)\| \le M$ for $\tau = 0, \cdots, t-1$, we can have

$$|\eta(t)| \le \bar{d} + c_\eta \epsilon M \tag{8}$$

and

$$|m(t)| \le c_\eta \epsilon M + d \tag{9}$$

where

$$d = \bar{d} + d'.$$

Comment 2.1: Assumption 2.1 is also required for nonadaptive controllers, designed based on a known reduced-order system, to give a stable closed-loop system. Therefore, Assumption 2.1 is a natural extension from nonadaptive control to adaptive control for robustness against fast parasitics and bounded disturbances.

Usually, we have some knowledge on the range of unknown time-varying parameter vector $\theta(t)$ of the nominal system model. This is given in the following assumption.

Assumption 2.2: $\theta(t)$ lies in a known (large) convex compact region C for all t, and C has the property that the polynomials $\hat{A}(q^{-1})$, $\hat{B}(q^{-1})$ induced by an arbitrary (nonzero) vector $\hat{\theta}$ in C are uniformly coprime. $\square$

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Comment 2.2: The coprimeness of $\hat{A}(q^{-1})$, $\hat{B}(q^{-1})$ is only required when pole assignment control law synthesis is used. When some other adaptive control strategies [10] are employed, this requirement can be relaxed.

Assumption 2.2 gives

$$\|\theta_1 - \theta_2\| \le k_\theta \qquad \forall \theta_1, \theta_2 \in \mathcal{C} \tag{10}$$

$$\|\theta_3\| \le k_c \qquad \forall \theta_3 \in \mathcal{C} \tag{11}$$

where k_θ, k_c are constants. k_θ reflects the size of $\mathcal{C}$ and k_c the maximum distance from $\mathcal{C}$ to the origin.

Regarding the time variation of plant parameters, we have the following.

Assumption 2.3:

$$\sum_{t=t_0+1}^{t_0+N} \|\theta(t) - \theta(t-1)\| \le k_\epsilon + \epsilon_\theta N \qquad \forall t_0 \ge 0, \quad N \ge 1 \tag{12}$$

where k_ϵ and ϵ_θ are nonnegative constants and ϵ_θ can be sufficiently small. Note that the rates at which the system parameters change are ϵ_θ-small in the mean. This implies that the parameters are not necessarily slowly time varying in a uniform way as in [11].

Suppose y^* is a given reference set point for output y. The control problem is to design a controller such that the resulting system is bounded input bounded state (BIBS) stable and the tracking error is small. Moreover, these properties are to be robust to the modeling error $m(t)$.

III. Adaptive Control Algorithm

In this section, an indirect adaptive control algorithm is presented. The parameter estimator is a basic one used to establish earlier global convergence results (see [10]) subject to parameter projection.

A. Parameter Estimation Algorithm

For simplicity of analysis, we use the gradient estimation algorithm

$$\hat{\theta}(t) = \mathcal{P}\left\{\hat{\theta}(t-1) + \frac{\phi(t-1)e(t)}{1 + \phi^T(t-1)\phi(t-1)}\right\} \tag{13}$$

where $\hat{\theta}(t)$ denotes the estimates of $\theta(t)$ at t and $\mathcal{P}$ represents the projection operator necessary to ensure $\hat{\theta} \in \mathcal{C}\ \forall t$. $e(t)$ is the prediction error defined as

$$e(t) = y(t) - \phi^T(t-1)\hat{\theta}(t-1). \tag{14}$$

Now, suppose M_0 is a constant such that $d/M_0 \le \delta$, where $0 < \delta < 1$. Also, let M be a constant such that $M^2 = k_1 M_0^2 + k_2$ where k_1 and k_2 are nonnegative constants with $k_1 > 1$ specified in later sections. Then, some properties of estimator (13), (14) can be established as below and will be used in our stability analysis.

Lemma 3.1: Consider the estimator (13) and (14), applied to system (1). Assuming $\|\phi(t_0 - 1)\| \le M_0$, $\|\phi(\tau)\| > M_0$, $\tau = t_0, \cdots, t-1$ and $\|\phi(\tau_1)\| \le M$, $\tau_1 = 0, \cdots, t-1$, where $t \ge t_0 + 1$, then we have

1)

$$|\tilde{e}(t)| = \frac{e(t)}{(1 + \|\phi(t-1)\|^2)^{1/2}} \le \begin{cases} (k_\theta + \alpha_1)M_0 + \alpha_1, & t = t_0 \\ k_\theta + \alpha_1 + \delta, & t \ge t_0 + 1 \end{cases} \tag{15}$$

where

$$\alpha_1 = (k_1^{1/2} + k_2^{1/2})c_\eta \epsilon + \delta. \tag{16}$$

2)

$$\sum_{\tau=t_0+1}^{t} |\tilde{e}(\tau)|^2 \le \overline{k}_\theta^2 + \alpha_2(t - t_0) + \alpha_3(t - t_0) \tag{17}$$

where

$$\overline{k}_\theta^2 = k_\theta^2 + k_\epsilon(2.5k_\theta + \alpha_1 + \delta)$$

$$\alpha_2 = 2(k_\theta(k_1^{1/2} + k_2^{1/2}) + 2c_\eta(k_1 + k_2)\epsilon)c_\eta\epsilon + 2\epsilon_\theta\left(2.5k_\theta + \alpha_1 + \delta + \frac{1}{2}\epsilon_\theta\right) \tag{18}$$

$$\alpha_3 = 2(2\delta + k_\theta)\delta \tag{19}$$

3)

$$\|\hat{\theta}(t) - \hat{\theta}(t-1)\| \le |\tilde{e}(t)| \qquad \forall t. \tag{20}$$

Proof: From (3) and (14), we have

$$e(t) = -\phi^T(t-1)\tilde{\theta}(t-1) + m(t) \tag{21}$$

where

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta(t).$$

1) It follows by similar analysis to that in [7].

2) Let $\hat{\theta}_{np}(t)$ denote the parameter estimate before projection, i.e., $\hat{\theta}(t) = \mathcal{P}\{\hat{\theta}_{np}(t)\}$. Thus,

$$\hat{\theta}_{np} - \hat{\theta}(t-1) = \frac{\phi(t-1)e(t)}{1 + \phi^T(t-1)\phi(t-1)}. \tag{22}$$

We consider the function $v(t) = \tilde{\theta}^T(t)\tilde{\theta}(t)$.

Then

$$\begin{aligned} v(t) - v(t-1) &\le \left\|\tilde{\theta}_{np}(t)\right\|^2 - \left\|\tilde{\theta}(t-1)\right\|^2 \qquad (23)\\ &\le [\tilde{\theta}_{np}^T(t) - \tilde{\theta}^T(t-1)][\tilde{\theta}_{np}(t) - \tilde{\theta}(t-1) + 2\tilde{\theta}(t-1)] \\ &= \frac{|e(t)|^2}{1 + \phi^T(t-1)\phi(t-1)} + \frac{2\phi^T(t-1)\tilde{\theta}(t-1)e(t)}{1 + \phi^T(t-1)\phi(t-1)} \\ &\quad - 2[\theta(t) - \theta(t-1)]^T \\ &\quad \cdot \left[\tilde{\theta}(t-1) + \frac{\phi(t-1)e(t)}{1 + \phi^T(t-1)\phi(t-1)} - \frac{1}{2}(\theta(t) - \theta(t-1))\right] \qquad (24) \end{aligned}$$

where (23) follows from the convexity of region $\mathcal{C}$. Now, using (21) and (10) gives

$$\begin{aligned} 2\phi^T(t-1)\tilde{\theta}(t-1)e(t) &= 2(-e(t) + m(t))e(t) \\ &\le -2e^2(t) + 2|m(t)|(k_\theta\|\phi(t-1)\| + |m(t)|) \\ &\le -2e^2(t) + 2k_\theta\|\phi(t-1)\||m(t)| + 2|m(t)|^2. \qquad (25) \end{aligned}$$

From (9), (15), (21), (24), and (25), we get

$$\begin{aligned} v(\tau) - v(\tau-1) &\le -\frac{e^2(t)}{1 + \|\phi(\tau-1)\|^2} + \frac{2k_\theta c_\eta\epsilon(k_1M_0^2 + k_2)^{1/2}\|\phi(\tau-1)\|}{1 + \|\phi(\tau-1)\|^2} \\ &\quad + \frac{4k_\theta c_\eta^2\epsilon^2(k_1M_0^2 + k_2)}{1 + \|\phi(\tau-1)\|^2} + \frac{4d^2 + 2k_\theta d\|\phi(\tau-1)\|}{1 + \|\phi(\tau-1)\|^2} \\ &\quad + 2\|\theta(t) - \theta(t-1)\|(2.5k_\theta + \alpha_1 + \delta). \end{aligned}$$

Thus, we have

$$\hat{e}^2(\tau) \le v(\tau-1) - v(\tau) + \frac{2k_\theta c_\eta \epsilon (k_1 M_0^2 + k_2)^{1/2} \|\phi(\tau-1)\|}{1+\|\phi(\tau-1)\|^2} + \frac{4k_\theta c_\eta^2 \epsilon^2 (k_1 M_0^2 + k_2)}{1+\|\phi(\tau-1)\|^2} + \frac{4d^2 + 2k_\theta d\|\phi(\tau-1)\|}{1+\|\phi(\tau-1)\|^2} + 2\|\theta(t) - \theta(t-1)\|(2.5k_\theta + \alpha_1 + \delta) \quad (26)$$

for $\tau = t_0 + 1, \cdots, t$ and $t \ge t_0 + 1$. Summing (26) and using (10) and (12), the result follows.

3)

$$\left\|\hat{\theta}(t) - \hat{\theta}(t-1)\right\| \le \left\|\hat{\theta}_{np}(t) - \hat{\theta}(t-1)\right\| \le \frac{\|\phi(t-1)\||e(t)|}{1+\|\phi(t-1)\|^2} \le |\hat{e}(t)|. \quad \square$$

Comments 3.1: 1) The least squares algorithm is more commonly used in practical algorithms [10]. Similar properties for this estimator can be derived by defining a different Lyapunov-type function $v(t)$, but the analysis is more tedious.

2) Note that α_1 and α_2 are functions of ϵ, ϵ_θ and can be made arbitrarily small by reducing ϵ and ϵ_θ. Also, α_3 depends on δ.

3) Comparing the derivation of the estimator properties in Lemma 3.1 with those in [7], the establishment of (17) is much more involved due to the time-varying nature of the system parameters. But it can be noted that the properties in both cases are similar in their final forms.

4) Suppose the system modeling errors are removed and the system is time invariant or the rates of the plant parameters variation belong to the l_1 (or l_2) space, i.e., ϵ, d in (9) and ϵ_θ in (12) are identically zeros. In this case, $m(t) = 0$ and the above estimator has those properties established in earlier global convergence results [10].

B. Adaptive Controller Design

Following the *Certainty Equivalence Principle*, we will use the parameter estimates obtained from the estimator (13) and (14) to adjust the parameters of the controller. Here, the pole assignment strategy is utilized. This is just one choice of many control schemes which could be used [10]. The control $u(t)$ is given by the equation

$$\hat{L}(t-1)u(t) = -\hat{P}(t-1)(y(t) - y^*(t)) \quad (27)$$

where y^* is the given set point and

$$\hat{L}(t) = 1 + \hat{l}_1(t)q^{-1} + \cdots + \hat{l}_n(t)q^{-n} \quad (28)$$

$$\hat{P}(t) = \hat{p}_1(t)q^{-1} + \cdots + \hat{p}_n(t)q^{-n}. \quad (29)$$

$\hat{L}$ and $\hat{P}$ are obtained by solving the following Diophantine equation:

$$\hat{A}(t)\hat{L}(t) + \hat{B}(t)\hat{P}(t) = A^* \quad (30)$$

where A^* is a given monic strictly (discrete-time) Hurwitz constant polynomial in backward shift operator q^{-1} of degree $2n$. From Assumption 2.2, we see that the coefficients in $\hat{L}(t)$ and $\hat{P}(t)$ obtained from (30) are bounded [10].

IV. Stability of the Adaptive Control System

In this section, we will study the robustness and stability of the adaptive control algorithm (13), (14), (27)–(30) applied to system (1). It will be shown that if the parameters of the reduced-order model are sufficiently slowly varying, i.e., ϵ_θ in (12) satisfying $\epsilon_\theta \in [0, \epsilon_\theta^*]$ where ϵ_θ^* is a sufficiently small constant, then there exists a class of unmodeled dynamics, i.e., a ϵ^* such that for each ϵ given in (4) satisfying $\epsilon \in [0, \epsilon^*]$, all states in the closed adaptive system are bounded for any bounded initial conditions, bounded set points, and extraneous disturbances.

First, we derive an equation to describe the closed-loop system by combining (14) and (27).

$$\phi(t+1) = \overline{A}(t)\phi(t) + B_1 e(t+1) + B_2 r(t+1) \quad (31)$$

where (see (32) at the bottom of the page)

$$B_1^T = [1, 0, \cdots, 0], \qquad B_2^T = [0, \cdots, 0, 1, \cdots, 0] \quad (33)$$

$$r(t+1) = \hat{P}(t)y^*(t+1). \quad (34)$$

Since $\hat{P}(t)$ is bounded, then $\|r(t+1)\|_\infty \le c_p\|y^*(t+1)\|_\infty$ where c_p is a constant.

From Lemma 3.1, Assumption 2.2, and (30), we can show that the transition matrix $\Phi(t, \tau)$ of the homogeneous part in system (31) satisfies

$$\|\Phi(t, \tau)\| \le c\sigma^{t-\tau} \qquad \text{for } t \ge \tau \ge t_0 \quad (35)$$

where $|\sigma| \in (0, 1)$ and c is a constant, if $\|\phi(\tau)\| \le M$, $\tau = 0, \cdots, t-1$, $\|\phi(\tau_1)\| > M_0$, $\tau_1 = t_0, \cdots, t-1$, and $\alpha_2 \le \overline{\alpha}_2^*$, $\delta \le \overline{\delta}^*$ for some sufficiently small $\overline{\alpha}_2^*$, $\overline{\delta}^*$. Now, we are in a position to present our stability result.

Theorem 4.1: Consider the adaptive system consisting of plant (1), estimator (13)–(14), and controller (27)–(30). Under Assumptions 2.1–2.3, $\exists\epsilon^*$ and ϵ_θ^* such that $\epsilon \le \epsilon_*$ and $\epsilon_\theta \le \epsilon_\theta^*$ ensure $\|\phi(t)\|$ bounded $\forall t$ for all bounded initial conditions, set points, and external disturbances.

Proof: The methods of analysis used here are similar to that in [7], and thus only the major steps are outlined. For more details including some preliminary motivations, see [7].

Note that for any bounded initial conditions $\phi(0)$, set points y^*, and disturbances $d(t)$, there always exists a number M_0 such that $\|\phi(0)\| \le M_0$, $\|r(t)\|_\infty \le M_0$ and $d/M_0 \le \delta$ for a sufficiently small δ, where $r(t)$ is given by (34). We will use an inductive proof by assuming that $\|\phi(\tau)\| \le M$, $\tau = 0, \cdots, t-1$ for $t \ge 1$, and prove that $\|\phi(t)\| \le M$, where $M > M_0$ and is defined in Section III. To apply Lemma 3.1 and the exponential stability property of $\overline{A}(t)$ in the closed-loop equation (31), we divide the time interval Z_+ into two subsequences

$$Z_1 := \{t \in Z_+ | \|\phi(t)\| > M_0\}$$

$$Z_2 := \{t \in Z_+ | \|\phi(t)\| \le M_0\}.$$

$$\overline{A}(t) = \begin{bmatrix} -\hat{a}_1(t) & -\hat{a}_2(t) & \cdots & -\hat{a}_n(t) & \hat{b}_1(t) & \cdots & \hat{b}_{n-1}(t) & \hat{b}_n(t) \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\hat{p}_1(t) & & \cdots & -\hat{p}_n(t) & -\hat{l}_1(t) & \cdots & -\hat{l}_{n-1}(t) & -\hat{l}_n(t) \\ 0 & & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & & \cdots & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad (32)$$

Clearly, the result is proved if we can show that $\|\phi(t)\| \leq M$ and $t \in Z_1$ since $M > M_0$. To do this, we choose t_0 so that $t_0 \geq 1$, $t_0 - 1 \in Z_2$, and $t_0, \cdots, t-1 \in Z_1$.

The general solution of (31) is

$$\phi(t) = \Phi(t, t_0)\phi(t_0) + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau)[B_1 e(\tau+1) + B_2 r(\tau+1)],$$

i.e.,

$$\phi(t) = \Phi(t, t_0)[\overline{A}(t_0-1)\phi(t_0-1) + B_1 e(t_0) + B_2 r(t_0)] + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau)[B_1 e(\tau+1) + B_2 r(\tau+1)].$$

Using (15), (35), the boundedness of $\|\overline{A}(t_0-1)\|$, and inequality

$$|e(\tau+1)| \leq (1 + \|\phi(\tau)\|)|\tilde{e}(\tau+1)|$$

gives

$$\|\phi(t)\| \leq c\sigma^{t-t_0}[(c_1+\alpha_1)M_0 + c_2 + \alpha_1] + \sum_{\tau=t_0}^{t-1} c\sigma^{t-\tau}[|\tilde{e}(\tau+1)|\|\phi(\tau)\| + |\tilde{e}(\tau+1)| + M_0] \quad (36)$$

where c_1 and c_2 are constants depending on k_θ, σ, and $\overline{\delta}^*$. Following similar steps to those in [7] involving the use of the Schwarz inequality, Lemma 3.1, the Grownwall lemma, and the theorem of the arithmetic and geometric means, we obtain

$$\|\phi(t)\|^2 \leq c_3 + c_4 M_0^2 + c_5 \alpha_1^2 M_0^2 + c_6 \alpha_1^2 \quad (37)$$

if $\alpha_2 \leq \bar{\bar{\alpha}}_2^*$, $\delta \leq \bar{\bar{\delta}}^*$ for some sufficiently small $\bar{\bar{\alpha}}_2^*$, $\bar{\bar{\delta}}^*$. From (16),

$$\|\phi(t)\|^2 \leq [c_7 + c_8 c_\eta^2 \epsilon^2 (k_1+k_2)]M_0^2 + c_9 + c_{10} c_\eta^2 \epsilon^2 (k_1+k_2). \quad (38)$$

Then, constants k_1 and k_2 can be chosen as the solution of the equations

$$k_1 = c_7 + c_8 c_\eta^2 \epsilon^2 (k_1 + k_2)$$

$$k_2 = c_9 + c_{10} c_\eta^2 \epsilon^2 (k_1 + k_2),$$

i.e.,

$$[k_1, k_2]^T = [c_7, c_9]^T [I - N]^{-T} \quad (39)$$

where N is a matrix given by

$$N = \begin{bmatrix} c_8 c_\eta^2 \epsilon^2, & c_8 c_\eta^2 \epsilon^2 \\ c_{10} c_\eta^2 \epsilon^2, & c_{10} c_\eta^2 \epsilon^2 \end{bmatrix}.$$

Clearly, there exists a constant $\overline{\epsilon}^*$ such that $I - N$ is an M-matrix [15], and therefore k_1 and k_2 have positive solutions for all $\epsilon \leq \overline{\epsilon}^*$. Thus, we can have

$$\|\phi_i(t)\|^2 \leq k_1 M_0^2 + k_2 = M^2. \quad (40)$$

It now remains to clarify the roles of ϵ at (4) and ϵ_θ at (12) in establishing (40). From the above argument, we can see that k_1 and k_2 are constants depending only on system parameters k_θ, σ, c_η and numbers $\overline{\delta}^*$, $\overline{\epsilon}^*$. Now, let $\alpha_2^* = \min\{\overline{\alpha}_2^*, \bar{\bar{\alpha}}_2^*\}$ where $\overline{\alpha}_2^*$ and $\bar{\bar{\alpha}}_2^*$ were defined to ensure that (35) and (37) are satisfied. From (18), we see that there exist an $\bar{\bar{\epsilon}}^*$ and an ϵ_θ^* such that $\epsilon \leq \bar{\bar{\epsilon}}^*$ and $\epsilon_\theta \leq \epsilon_\theta^*$ give that $\alpha_2 \leq \alpha_2^*$. Finally, taking $\epsilon^* = \min\{\overline{\epsilon}^*, \bar{\bar{\epsilon}}^*\}$, we have proved the result.

Comments 4.1:

1) A remark on the use of a relative dead zone in [11] is given here. We see that if an incorrect upper bound of ϵ at (4) is used to build a relative dead zone function in the adaptive controller, the closed-loop system is still BIBS stable for sufficiently small ϵ and ϵ_θ from our analysis given above.

2) If there is no modeling error appearing in the system and the system parameters are time invariant, we can still obtain the results that basic adaptive control algorithms can achieve for ideal plants [10] (in fact, from the proof of Lemma 3.1, this is also true if $\|\theta(t) - \theta(t-1)\| \in \ell_1$, i.e., $\epsilon_\theta = 0$ in (12)). One of them is that perfect tracking can be achieved or the prediction error tends to zero.

3) Suppose disturbance $d(t)$ is identically zero or satisfies

$$S(q^{-1})d(t) = 0$$

where $S(q^{-1})$ is a known polynomial of q^{-1} with all roots on the unit circle. Also note that $\overline{d}$ at (4) can be an exponentially decaying function. Thus, having established boundedness of all states in the closed-loop system, from (17) in Lemma 3.1 we can notice that the prediction error $e(t)$ for a given system with given initial condition is ϵ and ϵ_θ small in the mean, i.e., $e(t)$ satisfies

$$\sum_{\tau=t_0+1}^{t} e^2(t) \leq \beta_{11} + \beta_{12} o(\epsilon, \epsilon_\theta)(t - t_0)$$

where β_{11}, β_{12} are constants and $o(\epsilon, \epsilon_\theta)$ satisfies $\lim_{\epsilon\to 0,\ \epsilon_\theta \to 0} = 0$. If the internal model principle [10], [6], [8] is used, we can show that the tracking error $|y - y^*|$ is ϵ and ϵ_θ small in the mean, or tends to zero if $\epsilon = 0$ and $\epsilon_\theta = 0$ by employing the similar methods of [6] and [8].

V. Conclusion

In this paper, we reexamined the basic adaptive control algorithm studied in [7], which consists of a gradient estimator, subject to parameter projection as the only modification plus a pole assignment controller. The only *a priori* information required for the implementation of the algorithm is the range that each unknown time-varying parameter of the reduced-order plant lies in, which is quite reasonable.

It has been shown that the above adaptive controller can also globally stabilize a slowly time-varying system with modeling error including bounded disturbances and unmodeled dynamics. Small in the mean tracking error is possible if appropriate adaptive control schemes with the internal model principle are used. It is also clear that those results established in earlier global convergence analysis of ideal situations are preserved if plants to be controlled satisfy the "ideal assumptions" (see [10]) or the rates at which the plant parameters' change belong to the l_1 (or l_2) space. In particular, we can guarantee perfect tracking in this case.

References

[1] L. Praly, "Robustness of indirect adaptive control based on pole-placement design," in *Proc. IFAC Workshop Adaptive Contr.*, San Francisco, CA, 1983 (preliminary version).

[2] P. A. Ioannou and K. S. Tsakalis, "A robust direct adaptive controller," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 1033–1043, 1986.

[3] G. Kreisselmeier and B. D. O. Anderson, "Robust model reference adaptive control," *IEEE Trans. Automat. Contr.*, vol. AC-31, pp. 127–133, 1986.

[4] K. J. Astrom, "Adaptive feedback control," *Proc. IEEE*, vol. 75, pp. 185–217, 1987.

[5] R. Ortega and T. Yu, "Robustness of adaptive controllers—A survey," *Automatica*, vol. 5, no. 5, pp. 651–677, 1989.

[6] R. H. Middleton, G. C. Goodwin, D. J. Hill, and D. Q. Mayne, "Design issues in adaptive control," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 50–58, 1988.

[7] C. Wen and D. J. Hill, "Global boundedness of discrete-time adaptive control just using estimator projection," *Automatica*, pp. 1143–1157, Nov. 1992.
[8] R. H. Middleton and Y. Wang, "The internal model principle in the adaptive control of time varying linear systems," Tech. Rep. EE8830, Univ. Newcastle, Australia, 1988.
[9] B. Egardt, *Stability of Adaptive Controllers.* New York: Springer-Verlag, 1979.
[10] G. C. Goodwin and K. S. Sin, *Adaptive Filtering Prediction and Control.* Englewood Cliffs, NJ: Prentice-Hall, 1984.
[11] R. H. Middleton and G. C. Goodwin, "Adaptive control of time varying linear systems," *IEEE Trans. Automat. Contr.*, vol. 33, pp. 150–155, 1988.
[12] B. E. Ydstie, "Stability of discrete model reference control revisited," *Syst. Contr. Lett.*, vol. 13, pp. 429–438, 1989.
[13] ——, "Stability of the direct self-tuning regulator," in *Foundations of Adaptive Control*, P. V. Kokotovic, Ed., 1991, pp. 201–238.
[14] S. M. Naik, P. R. Kumar, and B. E. Ydstie, "Robust continuous-time adaptive control by parameter projection," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 182–197, 1992.
[15] H. Minc, *Nonnegative Matrices.* New York: Wiley, 1988.

Robust Stability of Discrete-Time Systems Under Parametric Perturbations

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Abstract—**Stability robustness analysis of a system under parametric perturbations is concerned with characterizing a region in the parameter space in which the system remains stable. In this paper, two methods are presented to estimate the stability robustness region of a linear, time-invariant, discrete-time system under multiparameter additive perturbations. An inherent difficulty, which originates from the nonlinear appearance of the perturbation parameters in the inequalities defining the robustness region, is resolved by transforming the problem to stability of a higher order continuous-time system. This allows for application of the available results on stability robustness of continuous-time systems to discrete-time systems. The results are also applied to stability analysis of discrete-time interconnected systems, where the interconnections are treated as perturbations on decoupled stable subsystems.**

I. Introduction

An essential feature of complex dynamic systems is the uncertainty in the system parameters, which may arise due to modeling errors or change of operating conditions. The analysis of stability in the presence of uncertainty is the subject of the robust stability problem. A common approach to stability robustness analysis is to model the uncertainty as perturbations on a nominal stable model. A measure of degree of stability of the nominal system can then be used to obtain bounds on the perturbations which the system can tolerate without going unstable.

Lyapunov's direct method provides a convenient way to estimate the degree of stability. It also directly yields bounds on tolerable perturbations [1]. This feature of the Lyapunov approach has been used among many others in [2]–[5] to obtain explicit robustness bounds for state-space models of continuous-time systems under additive perturbations. Some of these results have also been reproduced for discrete-time systems (see, for example [6]–[8]).

The main objective of this paper is to link the stability robustness problem of discrete-time systems to that of continuous-time systems. We show, using two different approaches, that stability robustness of a discrete-time system can be reformulated as that of an auxiliary continuous-time system. One of these approaches makes use of Lyapunov theory and yields a sufficient condition. The second approach, which is based on the properties of Kronecker products, provides a necessary and sufficient condition at the expense of an increase in the dimensionality. This is a pleasing development, since it allows for a direct application of the known results on stability robustness bounds for continuous-time systems to discrete-time systems. The results are applied to stability analysis of interconnected systems, where the interconnections are treated as perturbations on a collection of stable subsystems. This demonstrates how a knowledge of the structure of perturbations can be exploited to obtain simple robustness bounds.

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II. Problem Statement

Consider a discrete-time system under additive multiparameter perturbations, which is described as

$$\mathcal{D}\colon x(k+1) = A(p)x(k) \tag{2.1}$$

where $x(k) \in \mathcal{R}^n$ is the state of $\mathcal{D}$ at the discrete time instant $k \in \mathcal{Z}_+$, $p = [p_1\ p_2\ \cdots\ p_m]^T \in \mathcal{R}^m$ is a vector of real perturbation parameters, and

$$A(p) = A + \sum_{r=1}^{m} p_r E_r \tag{2.2}$$

with A and E_r, $r = 1, 2, \cdots, m$, being constant $n \times n$ real matrices. We assume that the matrix $A(0) = A$ is Schur-stable, that is, has all the eigenvalues in the open unit disk in the complex plane.

We would like to describe an open neighborhood of the origin in the parameter space in which $\mathcal{D}$ remains stable. More precisely, we are interested in a region

$$\Omega = \{p \mid A(\alpha p)\ \text{is Schur-stable for all}\ \alpha \in [0,\ 1]\} \tag{2.3}$$

in the parameter space $\mathcal{R}^m$. Since, in general, it is difficult to characterize Ω explicitly in terms of the perturbation parameters, we aim at obtaining estimates of Ω as regular volumes embedded in Ω which can be characterized explicitly.

III. Estimation of Robustness Region Via Lyapunov Theory

Our first approach to estimating Ω is through Lyapunov theory. Let $V(x) = x^T Px$ be a Lyapunov function for the nominal system corresponding to $p = 0$, where $P \in \mathcal{R}^{n\times n}$ is the unique, symmetric, positive-definite solution of the discrete-time Lyapunov equation

$$A^T PA - P = -Q \tag{3.1}$$

for some symmetric, positive-definite matrix $Q \in \mathcal{R}^{n\times n}$. The difference of $V(x)$ along the solutions of the perturbed system $\mathcal{D}$ in (2.1) is computed as

$$\Delta V(x(k))\,|_{\mathcal{D}_p} = x^T(k)Q^{1/2}[Q(p) - I]Q^{1/2}x(k) \tag{3.2}$$