(5.2) is R^2-approximately boundary controllable, i.e., without any constraints posed on the controls, iff $\beta_1\beta_2 \neq 0$ [6].

VI. Conclusions

In this paper, constrained approximate controllability problems for linear abstract dynamical systems with linear unbounded control operator and piecewise polynomial controls have been investigated. Using some very general results taken from the paper [20], necessary and sufficient conditions for constrained approximate controllability in finite time for linear continuous dynamical systems or equivalently in finite number of steps for linear discrete dynamical systems have been formulated and proven. Moreover, the relationships between approximate and exact controllability have been explained and discussed. Finally, two simple illustrative examples have been studied in detail. These examples represent linear distributed parameters dynamical system described by partial differential equation of the parabolic type with different boundary conditions and boundary piecewise polynomial controls.

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Discrete-Time Robust Backstepping Adaptive Control for Nonlinear Time-Varying Systems

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***Abstract*—This paper studies the problem of adaptive control for a class of nonlinear time-varying discrete-time systems with nonparametric uncertainties. The plant parameters considered here are not necessarily slowly time-varying in a uniform way. They are allowed to have finite number of big jumps. By using the backstepping procedures with parameter projection update laws, a robust adaptive controller can be designed to achieve adaptive tracking of a reference signal for this class of systems. It is shown that the proposed controller can guarantee the global boundedness of the states of the whole adaptive system in the presence of parametric and nonparametric uncertainties. It can also ensure that the tracking error falls within a compact set whose size is proportional to the size of the uncertainties and disturbances. In the ideal case when there is no nonparametric uncertainties and time-varying parameters, perfect tracking will be achieved.**

***Index Terms*—Adaptive control, backstepping, discrete-time system, nonlinear controller, parameter projection, time-varying system.**

I. Introduction

Adaptive control of nonlinear systems is an increasingly active area of research. Much progress has been achieved for continuous-time systems [1]–[7]. In contrast, the effort devoted to the adaptive control of nonlinear discrete-time systems is less. This is mainly because it is usually difficult to find a discrete-time Lyapunov function such that its increment is a linear function with respect to the increments of its variables. Thus, some developed control techniques such as the backstepping design scheme, which are based on Lyapunov theory and have been shown very effective to control of a large class of continuous-time systems, cannot be parallelly extended to treat nonlinear discrete-time systems.

Recently, this problem was considered in [8]. By employing the basic parameter estimators in [9] as update laws and ultilizing the properties of these estimators, the global boundedness and convergence can be achieved without employing Lyapunov functions in the backstepping procedures. But the results of [8] were obtained only in the ideal case neglecting uncertainties such as time-varying parameters, unmodeled dynamics, and external disturbances which usually inevitable in practical situations. Under the same condtions for the nominal system in [8], a robust design scheme was proposed in [10]. However, in order to obtain the stability of the adaptive system subject to the proposed controller, a constant which depends on the system initial states is used in the design of the parameter estimation adaptive laws in [10]. Therefore, only local stability can be guaranteed in [10].

In this paper, a robust backstepping adaptive controller is designed without using such a constant in the adaptive laws as in [10]. It is shown, though the procedures are more complex than those in [10], that the proposed controller can achieve global stability results for a class of nonlinear discrete-time systems with time-varying parameters and nonparametric uncertainties. In our design, the plant parameters are not necessarily slowly time-varying in a conventional uniform way as in [13]. They are allowed to have a finite number of large jumps. It is also shown that the proposed adaptive controller can also ensure

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a small-in-the-mean tracking error in the presence of parametric and nonparametric uncertainties. When the uncertainties disappear, perfect tracking is ensured.

II. Problem Formulation

It has been shown in [8] that under certain geometric conditions a large class of nonlinear discrete-time systems can be transformed into a *parametric-strict-feedback* form. If some uncertainties are also considered, the class of nonlinear systems to which the adaptive control can be applied will be enlarged. This motivates us to consider a class of uncertain nonlinear time-varying discrete-time systems described by

$$\begin{aligned} x_1^{t+1} &= x_2^t + \theta^T(t)\alpha_1\left(x_1^t\right) + \eta_1(t) \\ x_2^{t+1} &= x_3^t + \theta^T(t)\alpha_2\left(x_1^t, x_2^t\right) + \eta_2(t) \\ &\vdots \\ x_{n-1}^{t+1} &= x_n^t + \theta^T(t)\alpha_{n-1}\left(x_1^t, x_2^t, \cdots, x_{n-1}^t\right) + \eta_{n-1}(t) \\ x_n^{t+1} &= \theta^T(t)\alpha_n\left(x_1^t, x_2^t, \cdots, x_n^t\right) + \eta_n(t) + u(t) \\ y(t) &= x_1(t) \end{aligned} \tag{1}$$

where $u(t)$ and $y(t)$ represent the system input and output respectively, and $\theta(t)$ is the unknown time-varying parameter vector in R^p. For each $1 \le i \le n$, $\alpha_i(x_1^t, \cdots, x_i^t)$ are known nonlinear functions which are continuous and satisfies $\alpha_i(0) = 0$. For simplicity of illustration, $\alpha_i(x_1^t, x_2^t, \cdots, x_i^t)$ are denoted by $\alpha_i(t)$ for each $i = 1, 2, \cdots, n$ in the remaining parts of the paper.

Two types of uncertainties are considered in the discrete-time system described by (1). One is parametric uncertainty denoted by the unknown time-varying parameter vector $\theta(t)$. Usually we have some *a priori* knowledge about the range of $\theta(t)$, which is characterized by the following assumption.

Assumption A.1: $\theta(t)$ lies in a known convex compact set Θ, i.e.,

$$\theta(t) \in \Theta = \{\theta(t): \|\theta(t)\| \le k_\theta; \|\theta(t) - \theta'(t)\| \le k_\theta, \forall\, \theta'(t) \in \Theta\} \tag{2}$$

where k_θ is a positive constant.

In addition, the time variation of the parameters satisfies the following.

Assumption A.2:

$$\sum_{t=t_0+1}^{t_0+N} \|\theta(t) - \theta(t-1)\| \le k_\epsilon + \epsilon_\theta N, \quad \forall\, t_0 \ge 0,\ N \ge 1 \tag{3}$$

where k_ϵ and ϵ_θ are constants and ϵ_θ can be reduced to sufficiently small. As no smallness restriction is imposed on k_ϵ, this assumption not only allows for slowly time-varying parameters in a uniform way as in [13], but also takes into account time-varying parameters with big jumps.

Another kind of uncertainty appearing in (1) is the nonparametric uncertainty denoted by the unknown functions $\eta_i(t)$, which may often be due to modeling errors and external disturbances. As shown in [12], they satisfy the following assumption.

Assumption A.3: There exist constants ϵ and d such that

$$\eta_i(t) \le c_\eta \epsilon \max_{0 \le \tau \le t-1} \left\| [x_1^\tau, x_2^\tau, \cdots, x_n^\tau]^T \right\| + d \tag{4}$$

where c_η is a known constant. It will be shown later that knowledge of ϵ and d is not required to implement the adaptive controller.

Remark 2.1: From (4), it is noted that the modeling error $\eta_i(t)$ can have infinite memory as the function $\max_{0\le\tau\le t-1} \|\cdot\|$ is included. However, this makes the stability analysis more difficult especially when the knowledge of ϵ and d is not available.

The adaptive control problem is to obtain a control law for plant (1) such that all the signals in the resulting closed-loop system are bounded for arbitrary bounded reference set-point $y_m(t)$ and initial conditions, and the tracking error $|y(t) - y_m(t)|$ is small in some sense. To solve the problem, an additional assumption on the nonlinear functions $\alpha_i(t)$ is required.

Assumption A.4: All the known nonlinear functions $\alpha_i(t)$ satisfy the following two conditions:

$$k'_\alpha \left\| \left[x_1^t, x_2^t, \cdots, x_n^t\right]^T \right\| \le \|\alpha_i(t)\| \le k_\alpha \left\| \left[x_1^t, x_2^t, \cdots, x_n^t\right]^T \right\| \tag{5}$$

$$\|\alpha_i(\xi(t)) - \alpha_i(\xi'(t))\| \le k_\alpha \|\xi(t) - \xi'(t)\|, \quad \forall \xi(t), \xi'(t) \in R^i \tag{6}$$

where k'_α and k_α are constants. All the norms in this paper are vector norms.

III. Adaptive Control Design Using Backstepping Technique

The desired controller can be obtained by performing the following backstepping procedures.

Step 1: Let

$$z_1^t = x_1^t \tag{7}$$

$$z_2^t = x_2^t + \hat{\theta}_1(t)^T \alpha_1(t). \tag{8}$$

Then

$$z_1^{t+1} = z_2^t + (\theta(t) - \hat{\theta}_1(t))^T \alpha_1(t) + \eta_1(t). \tag{9}$$

The update law for $\hat{\theta}_1(t)$ is obtained by the following projection algorithm:

$$\hat{\theta}_1(t+1) = \wp\left\{\hat{\theta}_1(t) + \frac{\alpha_1(t) e_1(t+1)}{1 + \|\alpha_1(t)\|^2}\right\} \tag{10}$$

where

$$e_1(t+1) \triangleq z_1^{t+1} - z_2^t \tag{11}$$

and $\wp\{\cdot\}$ denotes a projection operator.

Step j $(2 \le j \le n-1)$: To proceed, the following functions are needed:

$$\overline{\alpha}_{1,1}(t) \triangleq \alpha_1\left(z_1^t\right), \quad \overline{\alpha}_{1,2}(t) \triangleq \alpha_1\left(z_2^t\right) \tag{12}$$

$$\overline{\alpha}_{i,j}(t) \triangleq \alpha_i \Bigg(z_{j-i+1}^t, z_{j-i+2}^t - \hat{\theta}_1(t)^T \overline{\alpha}_{1,j-i+1}(t), \cdots, z_{j-i+l}^t - \sum_{k=1}^{l-1} \hat{\theta}_k(t)^T \overline{\alpha}_{k,j-i+l-1}(t), \cdots, z_j^t - \sum_{k=1}^{i-1} \hat{\theta}_k(t)^T \overline{\alpha}_{k,j-1}(t) \Bigg) \tag{13}$$

where $1 \le i \le j-1$.

Let

$$z_{j+1}^t = x_{j+1}^t + \hat{\theta}_j(t)^T \alpha_j(t) + \sum_{k=1}^{j-1} \hat{\theta}_k(t)^T \overline{\alpha}_{k,j}(t). \tag{14}$$

Then

$$z_j^{t+1} = z_{j+1}^t + (\theta(t) - \hat{\theta}_j(t))^T \alpha_j(t) + \chi_j(t+1) + \eta_j(t) \tag{15}$$

where

$$\chi_k(t+1) \triangleq \sum_{k=1}^{j-1} \hat{\theta}_k(t+1)^T \overline{\alpha}_{k,j-1}(t+1) - \sum_{k=1}^{j-1} \hat{\theta}_k(t)^T \overline{\alpha}_{k,j}(t). \quad (16)$$

The update law for $\hat{\theta}_j(t)$ is obtained by

$$\hat{\theta}_j(t+1) = \wp \left\{ \hat{\theta}_j(t) + \frac{\alpha_j(t) e_j(t+1)}{1 + \|\alpha_j(t)\|^2} \right\} \quad (17)$$

where

$$e_j(t+1) \triangleq z_j^{t+1} - z_{j+1}^t - \chi_j(t+1). \quad (18)$$

Step n: The control law is taken as

$$u(t) = y_m(t+n) - \sum_{i=1}^{n} f_i z_i^t - \hat{\theta}_n(t)^T \alpha_n(t) - \sum_{k=1}^{n-1} \hat{\theta}_k(t)^T \overline{\alpha}_{k,n}(t). \quad (19)$$

where

$$\overline{\alpha}_{i,n}(t) \triangleq \alpha_i \left(z_{n-i+1}^t, z_{n-i+2}^t - \hat{\theta}_1 \overline{\alpha}_{1,n-i+1}(t), \cdots, z_j^t - \sum_{k=1}^{i-1} \hat{\theta}_k(t)^T \overline{\alpha}_{k,n-1}(t) \right) \quad (20)$$

and f_i $(i = 1, 2, \cdots, n)$ are the coefficients of a strictly stable polynomial $F(q^{-1})$, i.e., $F(q^{-1}) = 1 + f_n q^{-1} + \cdots + f_1 q^{-n}$.

Then

$$z_n^{t+1} = y_m(t+n) - \sum_{i=1}^{n} f_i z_i^t + (\theta(t) - \hat{\theta}_n(t))^T \alpha_n(t) + \chi_n(t+1) + \eta_n(t) \quad (21)$$

where

$$\chi_n(t+1) = \sum_{k=1}^{n-1} \hat{\theta}_k(t+1)^T \overline{\alpha}_{k,n-1}(t+1) - \sum_{k=1}^{n-1} \hat{\theta}_k(t)^T \overline{\alpha}_{k,n}(t+1). \quad (22)$$

The update law for $\hat{\theta}_n(t)$ is obtained by

$$\hat{\theta}_n(t+1) = \wp \left\{ \hat{\theta}_n(t) + \frac{\alpha_n(t) e_n(t+1)}{1 + \|\alpha_n(t)\|^2} \right\} \quad (23)$$

where

$$e_n(t+1) \triangleq z_n^{t+1} + \sum_{i=1}^{n} f_i z_i^t - y_m(t+n) - \chi_n(t+1). \quad (24)$$

The resulting closed-loop system is expressed by (25)–(27) shown at the bottom of the page, where

$$F = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -f_1 & -f_2 & -f_3 & \cdots & -f_n \end{bmatrix}, \quad z(t) = \begin{bmatrix} z_1^t \\ z_2^t \\ \vdots \\ z_n^t \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in R^n, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in R^n \quad (28)$$

$$\mathbf{e}(t+1) \triangleq [e_1(t+1), e_2(t+1), \cdots, e_n(t+1)]^T \quad (29)$$

$$\Psi(t+1) \triangleq [\chi_1(t+1), \chi_2(t+1), \cdots, \chi_n(t+1)]^T \quad (30)$$

$$(31)$$

with $\chi_1(t+1) \triangleq 0$.

Some useful properties of the estimator, which will aid our robust stability analysis, are established in Appendix A.

IV. Stability Analysis

In this section we show that there exist small constants ϵ^* and ϵ_θ^* such that for each $\epsilon \in [0, \epsilon^*]$ and $\epsilon_\theta \in [0, \epsilon_\theta^*]$, all the signals in the closed-loop system (25) are bounded for any bounded initial conditions, bounded tracking reference signal, and external disturbances. Similar stability analysis methods as in [11] and [12], where the inductive strategy is adopt, are used to come up with the conclusion.

The stability together with a tracking property of the closed-loop system are stated in the following theorem.

Theorem 1: Consider the adaptive system consisting of plant (1), update laws (26), (27), and controller (19). Under Assumptions A.1–A.4, there exist constants ϵ^* and ϵ_θ^* such that for each $\epsilon \in [0, \epsilon^*]$ and $\epsilon_\theta \in [0, \epsilon_\theta^*]$, $\|z(t)\|$ is bounded for all bounded initial conditions and setpoints. In addition the tracking error satisfies

$$\sum_{\tau=t_0}^{t-1} \left| y(\tau) - \frac{1}{K} y_m(\tau) \right| \le \beta_1 + \beta_2 0(\epsilon, \epsilon_\theta)(t - t_0) \quad (32)$$

where $K = 1 + \sum_{i=1}^{n} |f_i|$, β_1 and β_2 are constants, and $0(\epsilon, \epsilon_\theta)$ is a function such that $\lim_{\epsilon \to 0, \epsilon_\theta \to 0} 0(\epsilon, \epsilon_\theta) = 0$.

Proof: We introduce an intermediate positive constant M_0 which satisfies that $\|x(0)\| \le M_0$, $\|y_m(t)\|_\infty < M_0$, and $d/(k'_\alpha M_0) < \delta$ for $0 < \delta < 1$, where $x(0)$ denotes the initial conditions of the system. Using this constant, the time interval is divided into two subsequences

$$N_1 \triangleq \{t \in Z_+ \mid \|x(t)\| > M_0\} \quad (33)$$

$$\begin{cases} z(t+1) = Fz(t) + by_m(t+n) + \Psi(t+1) + \mathbf{e}(t+1) \\ y(t) = c^T z(t) \end{cases} \quad (25)$$

$$\hat{\theta}_i(t+1) = \wp \left\{ \hat{\theta}_i(t) + \frac{(z_i^{t+1} - z_i^t - \chi_i(t)) \alpha_i(t)}{1 + \|\alpha_i(t)\|^2} \right\}, \quad 1 \le i \le n-1 \quad (26)$$

$$\hat{\theta}_n(t+1) = \wp \left\{ \hat{\theta}_n(t) + \frac{\left(z_n^{t+1} + \sum_{i=1}^{n} f_i z_i^t - y_m(t+n) - \chi_n(t) \right) \alpha_n(t)}{1 + \|\alpha_n(t)\|^2} \right\} \quad (27)$$

$$N_2 \triangleq \{t \in Z_+ \mid \|x(t)\| \le M_0\} \tag{34}$$

where Z_+ denotes all positive integers.

Clearly, from Lemma A.1, it is sufficient to show that $\|z(t)\|$ is bounded for $t \in N_1$ to obtain the boundedness of $z(t)$ in the whole time interval $[0, \infty)$. To this end, we choose time instant t_0 such that $t_0 - 1 \in N_2$ and $[t_0, t-1] \in N_1$. The inductive strategy is adopted to prove the result. Firstly, note that $\|x(0)\| \le M_0$. Thus it follows from Lemma A.1 that there exists a constant M such that $\|z(0)\| \le b_l M_0 \le M$. Next we assume that $\|z(\tau)\| \le M$ for $\tau = 0, 1, \cdots, t-1$, then we show that $\|z(t)\| < M$.

The solution of system (25) is

$$\begin{aligned} z(t) &= \Phi(t, t_0) z(t_0) + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1) \\ &\quad \times [b y_m(\tau+n) + \Psi(\tau+1) + \mathrm{e}(\tau+1)] \\ &= \Phi(t, t_0)[F z(t_0 - 1) + b y_m(t_0 + n - 1) + \Psi(t_0) + \mathrm{e}(t_0)] \\ &\quad + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)[b y_m(\tau+n) + \Psi(\tau+1) + \mathrm{e}(\tau+1)] \end{aligned} \tag{35}$$

where $\Phi(t, \tau)$ is the transition function of the system $z(t+1) = F z(t)$. Since F is strictly stable, there exist contants C and $\sigma < 1$ such that $\|\Phi(t, \tau)\| \le C\sigma^{t-\tau}$.

From the definition of $\tilde{e}_i(t+1)$ and Lemma A.1, we have

$$\begin{aligned} |e_i(t+1)| &\le \left(1 + \|\alpha_i(t)\|^2\right)^{1/2} |\tilde{e}_i(t+1)| \\ &\le (1 + \|\alpha_i(t)\|) |\tilde{e}_i(t+1)| \\ &\le \left(1 + \frac{k_\alpha}{b_l} \|z(t)\|\right) |\tilde{e}_i(t+1)| \end{aligned} \tag{36}$$

where $(a^2 + b^2)^{1/2} \le (|a| + |b|)$ is used in the second inequality.

Noting that $\|z(t_0 - 1)\| \le M_0$ and using (A-2), (A-8), (A-9), (35), and (36), it follows from the inductive assumptions and $t \in N_1$ that

$$\begin{aligned} \|z(t)\| &\le C\sigma^{t-t_0}[(C_1 + C_2 a_1) M_0 + C_3 a_1] \\ &\quad + \sum_{\tau=t_0}^{t-1} C\sigma^{t-\tau-1}[C_4 \|z(\tau)\| \, \|\tilde{\mathrm{e}}(\tau+1)\| \\ &\quad + C_5 \|\tilde{\mathrm{e}}(\tau+1)\| + C_6 M_0], \qquad \forall t \in N_1 \end{aligned} \tag{37}$$

where $\tilde{\mathrm{e}}(t+1) \triangleq [\tilde{e}_1(t+1), \tilde{e}_2(t+1), \cdots, \tilde{e}_n(t+1)]^T$, and $C_i, (i = 1, 2, \cdots, 6)$ are constants.

Performing similar procedures as in [11], which includes squaring both sides of (37), applying the Schwarz inequality and the discrete Grownwall lemma, and using that fact that the arithmetic mean of a sequence of nonnegative numbers is greater than the geometric mean of the same sequence, we can show that there exit constants $\overline{\epsilon}^*$, ϵ_θ^*, and $\overline{\sigma}^*$ such that

$$\begin{aligned} \|z(t)\|^2 &\le \left[C_7 + C_8 c_\eta^2 \epsilon^2 (k_1 (k'_\alpha)^2 + k_2)\right] M_0^2 + C_9, \\ &\text{for } \epsilon < \overline{\epsilon}^*, \ \epsilon_\theta < \epsilon_\theta^*, \ \sigma < \overline{\sigma}^* \end{aligned} \tag{38}$$

where C_9 is a constant combining k_α, k'_α, and k_θ.

Let

$$k_2 = C_9 / b_u^2 \tag{39}$$

$$k_1 = \frac{1}{b_u^2 (k'_\alpha)^2} \max\left\{1, \frac{C_7 + C_8 c_\eta^2 \overline{\overline{\epsilon}}^* C_9}{1 - C_8 c_\eta^2 \overline{\overline{\epsilon}}^*}\right\} \tag{40}$$

where $\overline{\overline{\epsilon}}^*$ is a constant satisfying $C_8 c_\eta^2 \overline{\overline{\epsilon}}^* \le 1$.

Then it follows from (38) that

$$\|z(t)\|^2 \le b_u^2 \left(k_1 (k'_\alpha M_0)^2 + k_2\right). \tag{41}$$

Therefore taking $\epsilon^* = \max\{\overline{\epsilon}^*, \overline{\overline{\epsilon}}^*\}$ and $M^2 = \max\{b_u^2 (k_1 (k'_\alpha M_0)^2 + k_2), b_l^2 M_0^2\}$ confirms the first part of theorem.

Since the boundedness of all the states in the closed-loop system has been established, it follows immediately from the definitions of $\tilde{e}_i(t+1)$ that

$$\sum_{\tau=t_0}^{t-1} \|\mathrm{e}(t)\| \le \beta_3 + \beta_4 0(\epsilon, \epsilon_\theta)(t - t_0) \tag{42}$$

where β_3, β_4 are constants.

Using (A-8), we have

$$\sum_{\tau=t_0}^{t-1} \|\Psi(\tau+1)\| \le \beta_5 + \beta_6 0(\epsilon, \epsilon_\theta)(t - t_0) \tag{43}$$

where β_5 and β_6 are constants. Applying (42) and (43) to (25), (32) follows. $\square$

Remark 4.1: It is noted that if there is no nonparametric uncertianty and the system parameters are constants, i.e., $\epsilon = 0$, $\epsilon_\theta = 0$ and $\delta = 0$, $\|\hat{\theta}_i(t+1) - \hat{\theta}_i(t)\| \to 0$, $|e_i(t+1)| \to 0$, and $|y(t) - (1/K) y_m(t)| \to 0$, which implies perfect tracking is achieved.

Remark 4.2: It is noted that the adaptive controller in Section III was obtained by employing an update law in each backstepping step. This results in overparameterization. This problem can be avoided by postponing the determination of the update law until in the last step; see [10] for details.

V. CONCLUSION

In this paper, a scheme of designing adaptive controller for a class of nonlinear time-varying discrete-time systems is developed by using the backstepping tool and parameter projection. With this scheme, the global boundedness of the adaptive closed-loop system is guaranteed for any bounded initial conditions, set-point signals, and external disturbances, and small-in-the-mean tracking error can be achieved. It is also clear that those stability and convergence results obtained in the ideal case are still preserved if there are no modeling errors, external disturbances, and time-varying parameters. Particularly in the ideal case, perfect tracking of a reference trajectory is achieved. Since both the parametric and nonparametric uncertainties are considered, the class of the nonlinear discrete-time systems for which the adaptive control can be ultilized has been enlarged.

APPENDIX A
PARAMETER ESTIMATOR PROPERTIES

It is shown in the proof of Theorem 1 that the properties of the adaptive laws are crucial to fulfill the stability analysis. All these necessary properties are given in this section. For convenience of illustration, we denote

$$\begin{aligned} x(t) &\triangleq \left[x_1^t, x_2^t, \cdots, x_n^t\right]^T \in R^n \\ \mathrm{z}_{[i,j]}(t) &\triangleq \left[z_i^t, z_{i+1}^t, \cdots, z_j^t\right]^T \in R^{(j-i+1)} \\ \mathrm{e}_{[i,j]}(t) &\triangleq [e_i(t), e_{i+1}(t), \cdots, e_j(t)]^T \in R^{(j-i+1)} \\ \tilde{\mathrm{e}}_{[i,j]}(t) &\triangleq [\tilde{e}_i(t), \tilde{e}_{i+1}(t), \cdots, \tilde{e}_j(t)]^T \in R^{(j-i+1)} \end{aligned}$$

and

$$\chi_{[i,j]}(t) \triangleq [\chi_i(t), \chi_{i+1}(t), \cdots, \chi_j(t)]^T \in R^{(j-i+1)}.$$

From the definitions of z_j^t, it is trivial to show that the relationship between the new state variable $z(t)$ and the original state $x(t)$ can be specified by the following lemma.

Lemma A.1: For $z(t)$ obtained by (7), (8), and (14), we have

$$b_l\|x(t)\| \le \|z(t)\| \le b_u\|x(t)\| \tag{A-1}$$

where b_l and b_u are constants which depend on k_α and k_θ. □

The properties of the estimator given (26) and (27) are summarized in the following lemma, which is used in the robust stability analysis.

Lemma A.2: Assume that there exist constants M_1, M_0, k_1, and k_2 such that

$$\begin{aligned}\|x(t_0-1)\| &\le M_0\\ \|x(\tau)\| &> M_0, \quad \tau = t_0, t_0+1, \cdots, t-1\\ \|x(\tau_1)\| &< M_1, \quad \tau_1 = 0, 1, \cdots, t-1\\ M_1^2 &= k_1(k'_\alpha M_0)^2 + k_2 > M_0.\end{aligned}$$

Then

1)

$$|e_i(t_0)| \le (k_\alpha k_\theta + a_1)M_0 + a_1 \tag{A-2}$$

$$|\tilde{e}_i(t+1)| \le \frac{k_\alpha}{k'_\alpha}\, k_\theta + a'_1, \qquad \forall\, t \ge t_0 \tag{A-3}$$

where

$$\tilde{e}_i(t+1) \triangleq \frac{e_i(t+1)}{(1+\|\alpha_i(t)\|^2)^{1/2}} \tag{A-4}$$

$$a_1 = c_\eta \epsilon \left(k'_\alpha k_1^{1/2} + k_2^{1/2}\right) + k'_\alpha \delta$$

$$a'_1 = c_\eta \epsilon \left(k_1^{1/2} + k_2^{1/2}\right) + \delta. \tag{A-5}$$

2)

$$\|\hat{\theta}_i(t+1) - \hat{\theta}_i(t)\| \le |\tilde{e}_i(t+1)|. \tag{A-6}$$

3)

$$\sum_{\tau=t_0}^{t-1} |\tilde{e}_i(\tau)|^2 \le \overline{k}_\theta^2 + (a_2+a_3)(t-t_0) \tag{A-7}$$

where

$$\begin{aligned}a_2 &= 2\left(k_\theta\left(k_1^{1/2}+k_2^{1/2}\right) + 2c_\eta\epsilon(k_1+k_2)\right)c_\eta\epsilon\\ &\quad + 2\epsilon_\theta\left(\left(\frac{3}{2}+\frac{k_\alpha}{k'_\alpha}\right)k_\theta + a'_1 + \frac{1}{2}\epsilon_\theta\right)\\ a_3 &= 2\delta\,(2\delta + k_\theta)\\ \overline{k}_\theta^2 &= k_\theta^2 + 2k_\epsilon\left(\left(\frac{3}{2}+\frac{k_\alpha}{k'_\alpha}\right)k_\theta + a'_1\right).\end{aligned}$$

4)

$$\begin{aligned}\|\chi_i(t+1)\| &\le c_1\left\|\mathrm{e}_{[1,i-1]}(t+1)\right\|\\ &\quad + c_2\left\|\mathrm{z}_{[1,i]}(t)\right\|\left\|\tilde{\mathrm{e}}_{[1,i-1]}(t+1)\right\|\end{aligned} \tag{A-8}$$

$$\|\chi_i(t_0)\| \le (c_3 + c_4 a_1)M_0 + c_5 a_1 \tag{A-9}$$

where $c_j, (j = 1, 2, \cdots, 5)$ are constants depending on k'_α, k_α, and k_θ.

Proof:

1) From the definitions of $e_i(t+1)$, we have

$$\begin{aligned}e_i(t+1) &\triangleq z_i^{t+1} - z_{i+1}^t - \chi_i(t+1)\\ &= (\theta - \hat{\theta}_i(t))^T\alpha_i(t) + \eta_i(t)\\ &= -\tilde{\theta}_i(t)^T\alpha_i(t) + \eta_i(t).\end{aligned} \tag{A-10}$$

Applying the Assumptions A.1, A.3, and A.4 gives

$$|e_i(t+1)| \le k_\theta\|\alpha_i(t)\| + c_\eta\epsilon \max_{0<\tau\le t-1}\|x(\tau)\| + d \tag{A-11}$$

$$\begin{aligned}&\le k_\theta k_\alpha\left\|\left[x_1^t, x_2^t, \cdots, x_n^t\right]^T\right\|\\ &\quad + c_\eta\epsilon(k_1(k'_\alpha M_0)^2 + k_2)^{1/2} + d\end{aligned} \tag{A-12}$$

where $M_1^2 = k_1(k'_\alpha M_0)^2 + k_2$ is used.

Since $\|x(t_0-1)\| \le M_0$, it follows immediately that

$$\begin{aligned}|e_i(t_0)| &\le k_\alpha k_\theta M_0 + c_\eta\epsilon\left(k_1(k'_\alpha M_0)^2 + k_2\right)^{1/2} + d\\ &\le k_\alpha k_\theta M_0 + c_\eta\epsilon\left(k'_\alpha k_1^{1/2} + k_2^{1/2}\right)(M_0+1) + d\\ &\le k_\alpha k_\theta M_0 + c_\eta\epsilon\left(k'_\alpha k_1^{1/2} + k_2^{1/2}\right)M_0\\ &\quad + \delta k'_\alpha M_0 + c_\eta\epsilon\left(k'_\alpha k_1^{1/2} + k_2^{1/2}\right) + \delta k'_\alpha\\ &\le (k_\alpha k_\theta + a_1)M_0 + a_1.\end{aligned} \tag{A-13}$$

From (A-4) and (A-12), we have

$$\begin{aligned}|\tilde{e}_i(\tau+1)| &= \frac{|e_1(\tau+1)|}{(1+\|\alpha_i(\tau)\|^2)^{1/2}}\\ &\le \frac{k_\alpha}{k'_\alpha}\,k_\theta + \frac{c_\eta\epsilon\left(k_1(k'_\alpha M_0)^2+k_2\right)^{1/2} + d}{(1+\|\alpha_i(\tau)\|^2)^{1/2}}\\ &\le \frac{k_\alpha}{k'_\alpha}\,k_\theta + \frac{c_\eta\epsilon\left(k_1(k'_\alpha M_0)^2+k_2\right)^{1/2} + d}{(1+k'^2_\alpha M_0^2)^{1/2}}\\ &\qquad \text{for } \tau \ge t_0\\ &\le \frac{k_\alpha}{k'_\alpha}\,k_\theta + \delta + c_\eta\epsilon\left(k_1^{1/2}+k_2^{1/2}\right)\\ &\le \frac{k_\alpha}{k'_\alpha}\,k_\theta + a'_1.\end{aligned} \tag{A-14}$$

2) Let $\hat{\theta}_{ip}(\tau)$ denote a parameter estimate before applying a projector $\wp$, i.e.,

$$\hat{\theta}_{ip}(\tau+1) - \hat{\theta}_i(\tau) = \frac{\alpha_i(\tau)e_i(\tau+1)}{1+\|\alpha_i(\tau)\|^2}.$$

Then

$$\begin{aligned}\|\hat{\theta}_i(\tau+1) - \hat{\theta}_i(\tau)\| &\le \|\theta_{ip}(\tau+1) - \hat{\theta}_i(\tau)\|\\ &= \frac{\|\alpha_i(\tau)\|\,|e_i(\tau+1)|}{(1+\|\alpha_i(\tau)\|^2)^{1/2}}\\ &\le |\tilde{e}_i(\tau+1)|, \qquad \forall\,\tau.\end{aligned} \tag{A-15}$$

3) Introducing $v_i(t+1) = \tilde{\theta}_i^T(t+1)\tilde{\theta}_i(t+1)$, we get

$$\begin{aligned}&v_i(\tau+1) - v_i(\tau)\\ &\quad \le \tilde{\theta}_{ip}(\tau+1)^T\tilde{\theta}_{ip}(\tau+1) - \tilde{\theta}_i(\tau)^T\tilde{\theta}_i(\tau)\\ &\quad \le [\tilde{\theta}_{ip}(\tau+1) - \tilde{\theta}_i(\tau)]^T\\ &\qquad \times [\tilde{\theta}_{ip}(\tau+1) - \tilde{\theta}_i(\tau) + 2\tilde{\theta}_i(\tau)]\end{aligned} \tag{A-16}$$

$$\begin{aligned}&\quad = \frac{\|\alpha_i(\tau)\|^2 e_i^2(\tau+1)}{(1+\|\alpha_i(\tau)\|^2)^2} + \frac{2\alpha_i(\tau)^T\tilde{\theta}_i(\tau)e_i(\tau+1)}{1+\|\alpha_i(\tau)\|^2}\\ &\qquad - 2[\theta_i(\tau) - \theta_i(\tau-1)]^T\left[\tilde{\theta}_i(\tau-1)\right.\\ &\qquad \left. + \frac{\alpha_i(\tau)e_i(\tau+1)}{1+\|\alpha_i(\tau)\|^2} - \frac{1}{2}\left(\theta_i(\tau) - \theta_i(\tau-1)\right)\right]\\ &\quad \le \frac{e_i^2(\tau+1)}{1+\|\alpha_i(\tau)\|^2} + \frac{2\alpha_i(\tau)^T\tilde{\theta}_i(\tau)e_i(\tau+1)}{1+\|\alpha_i(\tau)\|^2}\\ &\qquad - 2[\theta_i(\tau) - \theta_i(\tau-1)]^T\left[\tilde{\theta}_i(\tau-1)\right.\\ &\qquad \left. + \frac{\alpha_i(\tau)e_i(\tau+1)}{1+\|\alpha_i(\tau)\|^2} - \frac{1}{2}\left(\theta_i(\tau) - \theta_i(\tau-1)\right)\right].\end{aligned} \tag{A-17}$$

From (A-10), we have

$$\begin{aligned}
&2\alpha_i(\tau)^T\tilde{\theta}_i(\tau)e_i(\tau+1)\\
&\quad= 2(\eta_i(\tau)-e_i(\tau+1))e_i(\tau+1)\\
&\quad\le -2e_i^2(\tau+1)+2k_\theta\|\alpha_i(\tau)\|\,|\eta_i(\tau)|+2|\eta_i(\tau)|^2. \qquad \text{(A-18)}
\end{aligned}$$

Combining (4), (A-3), (A-4), (A-10), (A-17), and (A-18), we have

$$\begin{aligned}
&v(\tau+1)-v(\tau)\\
&\le \frac{-e_i^2(\tau+1)}{1+\|\alpha_i(\tau)\|^2}+2\,\frac{k_\theta\|\alpha_i(\tau)\|\,|\eta_i(\tau)|}{1+\|\alpha_i(\tau)\|^2}+2\,\frac{|\eta_i(\tau)|^2}{1+\|\alpha_i(\tau)\|^2}\\
&\quad -2[\theta_i(\tau)-\theta_i(\tau-1)]^T\left[\tilde{\theta}_i(\tau-1)\right.\\
&\quad \left.+\frac{\alpha_i(\tau)e_i(\tau+1)}{1+\|\alpha_i(\tau)\|^2}-\frac{1}{2}\left(\theta_i(\tau)-\theta_i(\tau-1)\right)\right]\\
&\quad +2\|\theta_i(\tau)-\theta_i(\tau-1)\|\left(\left(\frac{3}{2}+\frac{k_\alpha}{k'_\alpha}\right)k_\theta+a'_1\right)\\
&\le -\tilde{e}_i(\tau+1)^2+2k_\theta\,\frac{c_\eta\epsilon\left(k_1(k'_\alpha M_0)^2+k_2\right)^{1/2}}{1+\|\alpha_i(\tau)\|^2}\,\|\alpha_i(\tau)\|\\
&\quad +\frac{4c_\eta^2\epsilon^2\left(k_1(k'_\alpha M_0)^2+k_2\right)}{1+\|\alpha_i(\tau)\|^2}+\frac{4d^2+2dk_\theta\|\alpha_i(\tau)\|}{1+\|\alpha_i(\tau)\|^2}\\
&\quad +2\|\theta_i(\tau)-\theta_i(\tau-1)\|\left(\left(\frac{3}{2}+\frac{k_\alpha}{k'_\alpha}\right)k_\theta+a'_1\right)\\
&\le -\tilde{e}_i(\tau+1)^2+2k_\theta c_\eta\epsilon\left(k_1^{1/2}+k_2^{1/2}\right)+4c_\eta^2\epsilon^2(k_1+k_2)\\
&\quad +4\delta^2+2k_\theta\delta+2\|\theta(\tau)-\theta(\tau-1)\|\\
&\quad \times\left(\left(\frac{3}{2}+\frac{k_\alpha}{k'_\alpha}\right)k_\theta+a'_1\right), \qquad \forall\,\tau\ge t_0. \qquad \text{(A-19)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\tilde{e}_i(\tau+1)^2 \le\; & v(\tau)-v(\tau+1)+2k_\theta c_\eta\epsilon\left(k_1^{1/2}+k_2^{1/2}\right)\\
&+4c_\eta^2\epsilon^2(k_1+k_2)+4\delta^2+2k_\theta\delta\\
&+2\|\theta(\tau)-\theta(\tau-1)\|\\
&\times\left(\left(\frac{3}{2}+\frac{k_\alpha}{k'_\alpha}\right)k_\theta+a'_1\right). \qquad \text{(A-20)}
\end{aligned}$$

Summing both sides of (A-20) gives

$$\begin{aligned}
\sum_{\tau=t_0}^{t-1}|\tilde{e}_i(\tau)|^2 \le\; & \|\tilde{\theta}_i(t_0)\|^2-\|\tilde{\theta}_i(t)\|^2\\
&+2k_\epsilon\left(\left(\frac{3}{2}+\frac{k_\alpha}{k'_\alpha}\right)k_\theta+a'_1\right)\\
&+(a_2+a_3)(t-t_0) \qquad \text{(A-21)}
\end{aligned}$$

which confirms (A-7) by applying Assumption A.2.

4) To show (A-8) and (A-9), the following inequality is required:

$$\begin{aligned}
&|\tilde{\theta}_i(t+1)^T\overline{\alpha}_{i,k}(t+1)-\tilde{\theta}_i(t)^T\overline{\alpha}_{i,k+1}(t)|\\
&\quad\le c_{1,i}\left\|\mathbf{e}_{[k-i+1,k]}(t+1)\right\|+c_{2,i}\left\|\chi_{[k-i+1,k]}(t+1)\right\|\\
&\quad\quad +c_{3,i}\left\|\mathbf{z}_{[k-i+2,k+1]}(t)\right\|\left\|\tilde{\mathbf{e}}_{[1,i]}(t+1)\right\| \qquad \text{(A-22)}
\end{aligned}$$

Here an inductive strategy is adopted to verify (A-22). First, consider $i=1$. From the definitions of $\overline{\alpha}_{1,k}(t)$ and $e_k(t+1)$, (2), (6), and (A-6), we have

$$\begin{aligned}
&|\hat{\theta}_1(t+1)^T\overline{\alpha}_{1,k}(t+1)-\hat{\theta}_1(t)^T\overline{\alpha}_{1,k+1}(t)|\\
&\quad\le |\hat{\theta}_1(t+1)^T\overline{\alpha}_{1,k}(t+1)-\hat{\theta}_1(t+1)^T\overline{\alpha}_{1,k+1}(t)|\\
&\quad\quad +|\hat{\theta}_1(t+1)^T\overline{\alpha}_{1,k+1}(t+1)-\hat{\theta}_1(t)^T\overline{\alpha}_{1,k+1}(t)|\\
&\quad\le k_\theta k_\alpha\left|z_k^t-z_{k+1}^t\right|+k_\alpha\left|z_{k+1}^t\right|\,\|\hat{\theta}_1(t+1)-\hat{\theta}_1(t)\|\\
&\quad\le k_\theta k_\alpha(|e_k(t+1)|+|\chi_k(t+1)|)\\
&\quad\quad +k_\alpha\left|z_{k+1}^t\right|\,|\tilde{e}_1(t+1)| \qquad \text{(A-23)}
\end{aligned}$$

which obviously supports inequality (A-22). Particularly, if $k=1$,

$$\chi_2(t+1)\le k_\theta k_\alpha|e_1(t+1)|+k_\alpha|z_2^t|\,|\tilde{e}_1(t+1)| \qquad \text{(A-24)}$$

where $\chi_1(t+1)=0$ is used. This actually verifies (A-8) for $i=1$.

Then consider $i=2$, we have

$$\begin{aligned}
&|\hat{\theta}_2(t+1)^T\overline{\alpha}_{2,k}(t+1)-\hat{\theta}_2(t)^T\overline{\alpha}_{2,k+1}(t)|\\
&\quad\le |\hat{\theta}_2(t+1)^T\overline{\alpha}_{2,k}(t+1)-\hat{\theta}_2(t+1)^T\overline{\alpha}_{2,k+1}(t)|\\
&\quad\quad +|\hat{\theta}_2(t+1)^T\overline{\alpha}_{2,k+1}(t+1)-\hat{\theta}_2(t)^T\overline{\alpha}_{2,k+1}(t)|\\
&\quad\le k_\theta k_\alpha\left(\left\|\begin{bmatrix} z_{k-1}^t-z_k^t\\ z_k^t-z_{k+1}^t\end{bmatrix}\right\|+(|\hat{\theta}_1(t+1)^T\overline{\alpha}_{1,k}(t+1)\right.\\
&\quad\quad\quad \left.-\hat{\theta}_1(t)^T\overline{\alpha}_{1,k+1}(t)\,|)\right)\\
&\quad\quad +k_\alpha\left\|\begin{bmatrix} z_k^t\\ z_{k+1}^t\end{bmatrix}\right\|\,\|\hat{\theta}_2(t+1)-\hat{\theta}_2(t)\|. \qquad \text{(A-25)}
\end{aligned}$$

Subsituting (A-23) into (A-25) and using the definition of $e_k(t+1)$ gives

$$\begin{aligned}
&|\hat{\theta}_2(t+1)^T\overline{\alpha}_{2,k}(t+1)-\hat{\theta}_2(t)^T\overline{\alpha}_{2,k+1}(t)|\\
&\quad\le c_{1,2}\left\|\begin{bmatrix} e_{k-1}(t+1)\\ e_k(t+1)\end{bmatrix}\right\|+c_{2,2}\left\|\begin{bmatrix} \chi_{k-1}(t+1)\\ \chi_k(t+1)\end{bmatrix}\right\|\\
&\quad\quad +c_{3,2}\left\|\begin{bmatrix} z_k^t\\ z_{k+1}^t\end{bmatrix}\right\|\left\|\begin{bmatrix} \tilde{e}_1(t+1)\\ \tilde{e}_2(t+1)\end{bmatrix}\right\| \qquad \text{(A-26)}
\end{aligned}$$

where $c_{1,2}, c_{2,2}$, and $c_{3,2}$ are constants combining k_θ and k_α. Thus (A-22) holds for $i=2$.

Finally assume (A-22) holds for all $1\le p\le i-1$, i.e.,

$$\begin{aligned}
&|\tilde{\theta}_p(t+1)^T\overline{\alpha}_{p,k}(t+1)-\tilde{\theta}_p(t)^T\overline{\alpha}_{p,k+1}(t)|\\
&\quad\le c_{1,p}\left\|\mathbf{e}_{[k-p+1,k]}(t+1)\right\|\\
&\quad\quad +c_{2,p}\left\|\chi_{[k-p+1,k]}(t+1)\right\|\\
&\quad\quad +c_{3,p}\left\|\mathrm{z}_{[k-p+2,k+1]}(t)\right\|\left\|\tilde{\mathrm{e}}_{[1,p]}(t+1)\right\| \qquad \text{(A-27)}
\end{aligned}$$

where $c_{1,p}$, $c_{2,p}$, and $c_{3,p}$ are constants depending upon k_α and k_θ. Then we show that (A-22) also holds for $p=i$.

From the definitions of $\overline{\alpha}_{i,k}(t)$, it follows that

$$\begin{aligned}
&|\hat{\theta}_i(t+1)^T\overline{\alpha}_{i,k}(t+1)-\hat{\theta}_i(t)^T\overline{\alpha}_{i,k+1}(t)|\\
&\quad\le |\hat{\theta}_i(t+1)^T\overline{\alpha}_{i,k}(t+1)-\hat{\theta}_i(t+1)^T\overline{\alpha}_{i,k+1}(t)|\\
&\quad\quad +|\hat{\theta}_i(t+1)^T\overline{\alpha}_{i,k+1}(t)-\hat{\theta}_i(t)^T\overline{\alpha}_{i,k+1}(t)|\\
&\quad\le k_\theta\left|\alpha_i\left(z_{k-i+1}^{t+1}, z_{k-i+2}^{t+1}-\hat{\theta}_1^T\alpha_{1,k-i+1}(t+1),\right.\right.\\
&\quad\quad\quad \left.\cdots, z_k^{t+1}-\sum_{l=1}^{i-1}\hat{\theta}_l(t+1)^T\overline{\alpha}_{l,k-1}(t+1)\right)\\
&\quad\quad\quad -\alpha_i\left(z_{k-i+2}^t, z_{k-i+3}^t-\hat{\theta}_1^T\alpha_{1,k-i+2}(t),\cdots,\right.\\
&\quad\quad\quad \left.\left. z_{k+1}^t-\sum_{l=1}^{i-1}\hat{\theta}_l(t+1)^T\overline{\alpha}_{l,k}(t+1)\right)\right|\\
&\quad\quad +\|\overline{\alpha}_{i,k+1}(t)\|\,\|\hat{\theta}_i(t+1)-\hat{\theta}_i(t)\|. \qquad \text{(A-28)}
\end{aligned}$$

Using (6), (A-6) and noting that $\overline{\alpha}_{i,k+1}(t)$ is a function of $z_{k+1}^t, z_k^t, \cdots, z_{k-i+2}^t$, we have

$$\begin{aligned}
&|\tilde{\theta}_i(t+1)^T\overline{\alpha}_{i,k}(t+1)-\tilde{\theta}_i(t)^T\overline{\alpha}_{i,k+1}(t)|\\
&\quad\le k_\theta k_\alpha\left\|\mathbf{z}_{[k-i+1,k]}(t+1)-\mathbf{z}_{[k-i+2,k+1]}(t)\right\|\\
&\quad\quad +K'\sum_{l=1}^{i-1}|\hat{\theta}_l(t+1)^T\overline{\alpha}_{l,k-l}(t+1)-\hat{\theta}_l(t)^T\overline{\alpha}_{l,k-l+1}(t)|\\
&\quad\quad +k_\alpha\left\|\mathbf{z}_{[k-i+2,k+1]}(t)\right\|\,|\tilde{e}_i(t+1)| \qquad \text{(A-29)}
\end{aligned}$$

where K' is a constant depending on k_θ and k_α.

Substituting (18) and (A-27) into (A-29) gives

$$\begin{aligned}|\tilde{\theta}_i(t+1)^T\overline{\alpha}_{i,k}(t+1) - \tilde{\theta}_i(t)^T\overline{\alpha}_{i,k+1}(t)| \\ \leq c_{1,i}\left\|\mathrm{e}_{[k-i+1,k]}(t+1)\right\| + c_{2,i}\left\|\chi_{[k-i+1,k]}(t+1)\right\| \\ + c_{3,i}\left\|\mathrm{z}_{[k-i+2,k+1]}(t)\right\|\left\|\tilde{\mathrm{e}}_{[1,i]}(t+1)\right\| \end{aligned} \quad \text{(A-30)}$$

where $c_{m,i}, (m = 1,2,3)$ are constants combining $c_{m,p}, (m = 1,2,3; 1 \leq p \leq i-1)$, k_α and k_θ. Thus $c_{m,i}(m = 1,2,3)$ are dependent of k_θ and k_α only. So far we have proved the inequality (A-22).

Using (A-22), it follows immediately from the definition of $\chi_i(t+1)$ that

$$\begin{aligned}|\chi_i(t+1)| \leq c'_{1,i}\left\|\mathrm{e}_{[1,i-1]}(t+1)\right\| + c'_{2,i}\left\|\chi_{[1,i-1]}(t+1)\right\| \\ + c'_{3,i}\left\|\mathrm{z}_{[1,i]}(t)\right\|\left\|\tilde{\mathrm{e}}_{[1,i-1]}(t+1)\right\|\end{aligned} \quad \text{(A-31)}$$

where $c'_{m,i}, (m = 1,2,3)$ are constants.

Since $\chi_1(t+1) = 0$ and $\chi_2(t+1) \leq k_\theta k_\alpha |e_1(t+1)| + k_\alpha |z_2^t| \, |\tilde{e}_1(t+1)|$, it can be shown from (A-31) that

$$\begin{aligned}|\chi_i(t+1)| \leq c''_{1,i}\left\|\mathrm{e}_{[1,i-1]}(t+1)\right\| \\ + c''_{2,i}\left\|\mathrm{z}_{[1,i]}(t)\right\|\left\|\tilde{\mathrm{e}}_{[1,i-1]}(t+1)\right\|\end{aligned} \quad \text{(A-32)}$$

where $c''_{1,i}$ and $c''_{2,i}$ are constants combining k_θ and k_α.

Taking $c_1 = \max_{1\leq i\leq n}\{c''_{1,i}\}$ and $c_2 = \max_{1\leq i\leq n}\{c''_{2,i}\}$, (A-8) follows.

Using (A-2) and inequality

$$\begin{aligned}\|z(t_0-1)\|\,|\tilde{e}_i(t_0)| &\leq \frac{\|z(t_0-1)\|\,|e_i(t_0)|}{(1+\|\alpha_i(t_0-1)\|^2)^{1/2}} \\ &\leq \frac{b_u\|x(t_0-1)\|\,|e_i(t_0)|}{(1+k'_\alpha\|x(t_0-1)\|^2)^{1/2}} \\ &\leq \frac{b_u}{k'_\alpha}\left((k_\theta k_\alpha + a_1)M_0 + a_1\right) \end{aligned} \quad \text{(A-33)}$$

(A-9) follows. □

Remark A.1: Note that M_0 is not a design parameter. For any bounded $x(0)$ and $y_m(t)$, such a contant M_0 always exists.

Remark A.2: In Lemma A.2, it is noted that the update law has the same properties as those given in [8] if the nonparametric uncertainties are removed and all the system parameters are considered to be constants. Moreover, the constants a_1, a'_1, and a_2 are functions of ϵ and ϵ_θ. They can be made sufficiently small by specifying sufficiently small ϵ and ϵ_θ.

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When Is (D, G)-Scaling Both Necessary and Sufficient

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***Abstract*—It is shown that the well-known (D, G)-scaling upper bound of the structured singular value is a nonconservative test for robust stability with respect to certain linear time-varying uncertainties.**

***Index Terms*—Duality, IQC, linear matrix inequalities, mixed structured singular values, robustness, time-varying systems.**

I. INTRODUCTION

Is the closed-loop stable in Fig. 1 for all Δ's in a given set of stable operators $\mathcal{B}$? That, roughly, is the fundamental robust stability problem.

There is an intriguing result by Megretski and Treil [4] and Shamma [8] which says, loosely speaking, that if M is a stable LTI operator and the set of Δ's is the set of contractive linear time-varying operators of some fixed block diagonal structure

$$\Delta = \operatorname{diag}(\Delta_1, \Delta_2, \ldots, \Delta_{m_F}) \quad (1)$$

that then the closed loop is robustly stable—that is, stable for all such Δ's—if and only if the $\mathcal{H}_\infty$-norm of DMD^{-1} is less than one for some constant diagonal matrix D that commutes with the Δ's. The problem can be decided in polynomial time, and it is a problem that has since long been associated with an *upper bound* of the structured singular value. The intriguing part is that the result holds for any number of LTV blocks Δ_i, which is in stark contrast with the case that the Δ_i's are assumed time-invariant.

Paganini [6] extended this result by allowing for the more general block diagonal structure

$$\Delta = \operatorname{diag}\left(\delta_1 I_{n_1}, \ldots, \delta_{m_c} I_{n_{m_c}}, \Delta_1, \ldots, \Delta_{m_F}\right). \quad (2)$$

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