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Global Boundedness of Discrete-time Adaptive Control Just Using Estimator Projection*

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A global boundedness result is found for a simple indirect certainty equivalence adaptive controller without modifications like data normalization, deadzones or injection of persistence of excitation.

Key Words—Adaptive control; parameter estimation; stability; control theory.

Abstract—In this paper, we study a discrete-time indirect adaptive control algorithm which contains a constrained gradient parameter estimator and a pole assignment control law synthesis module. This adaptive control algorithm does not involve modifications like data normalization, use of deadzones, or injection of persistently excited signals. Also it requires no *a priori* knowledge of system modelling errors. It is shown that global robustness properties still hold when this simple algorithm is applied to systems with bounded disturbances and arbitrarily small fast parasitic dynamics. The problem of indirect decentralized adaptive control of interconnected systems is also considered. We use the above adaptive algorithm to design completely decentralized local adaptive controllers for each isolated subsystem by ignoring interactions between subsystems. We show that the local controllers designed in this way are robust in the sense that all signals in the closed loop adaptive system are bounded for bounded initial conditions, reference inputs, disturbances and an arbitrarily small amount of interaction between subsystems and unmodelled dynamics of each subsystem.

1. INTRODUCTION

AT THE END OF the 1970s and in the early 1980s, correct proofs were obtained for stability of adaptive control systems under ideal conditions (Egardt, 1979; Goodwin *et al.*, 1980; Narendra *et al.*, 1980; Morse, 1980). A summary of these basic adaptive algorithms and their stability analysis which suits our needs here can be found in Goodwin and Sin (1984). Simulation results and some analysis showed that violation of the ideal conditions can cause adaptive control algorithms to go unstable (Rohrs *et al.*, 1982; Ioannou and Kokotovic, 1984; Egardt, 1979). To counteract instability and improve robustness with respect to bounded disturbances and unmodelled dynamics, several groups of researchers came up with modifications to these basic algorithms. Some overview of the progress can be found in Åström (1987), Anderson *et al.* (1986), Middleton *et al.* (1988) and Ortega and Yu (1987). The major modifications include normalizations with parameter projection (Praly, 1983, 1984), σ-modifications (Ioannou and Tsakalis, 1986; Ioannou and Kokotovic, 1983), relative deadzones with parameter projection (Kreisselmeier and Anderson, 1986) and other combinations (Ioannou and Sun, 1988). To implement these algorithms, one requires some *a priori* knowledge of modelling errors.

Consider the algorithm modification based on use of relative deadzones (Kreisselmeier and Anderson, 1986; Middleton *et al.*, 1988; Middleton and Goodwin, 1988). We need to know an upper bound on the system unmodelled dynamics for the implementation of the parameter estimator. This bound should be sufficiently small to ensure stability of the adaptive system. Clearly, this is complicated when insufficient *a priori* information is available. This robust adaptive algorithm has been extended to use on inter-connected systems (Hill *et al.*, 1988). We need information from other subsystems and some bounds on interactions to build relative deadzone functions in local adaptive controllers. In this way, the local controllers are actually partially decentralized. Allowing for the case where the *a priori* information is not available, we are led to look for a robust algorithm which does not involve the requirement on the knowledge of system modelling errors and can be used to design totally decentralized controllers. Another way to enhance robustness of an adaptive control algorithm is to inject perturbation signals such

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that the regression vector is persistently exciting (Åström, 1987; Anderson *et al.*, 1986). However this method has only been proved to handle bounded disturbances effectively, and global consequences in the presence of unmodelled dynamics remain unsolved. Again, successful implementation depends on some *a priori* knowledge to achieve 'dominate richness'. Further, sometimes signal injection may not be desirable during the operation of the closed loop adaptive system. Thus it makes sense to consider what robustness remains if it is not used.

In a previous paper (Wen and Hill, 1989), an indirect adaptive control algorithm was studied which involves a basic parameter estimator subject to parameter projection as the only modification and pole assignment control law synthesis module. It was shown there that some useful local robustness properties hold. In this paper, we will re-examine the robustness properties of this adaptive algorithm. It is shown that this simple algorithm can actually counteract the instability caused by system modelling errors including fast parastics and bounded disturbances such that global stability can be ensured. We can also achieve ϵ small tracking error in the mean as defined in Ioannou and Tao (1987) and Middleton and Wang (1988). Unlike some other robust algorithms (Kreisselmeier and Anderson, 1986; Egardt, 1979), it can preserve results established in earlier global convergence proofs (Goodwin and Sin, 1984) when modelling errors are removed. Robustness of constrained estimation was also considered in Kreisselmeier and Narendra (1982). There a direct adaptive scheme was used and an upper bound for the norm of the desired controller parameter vector is assumed known such that the search procedure is confined to a known set. However, only bounded disturbances were considered as modelling error and unmodelled dynamics which can no longer be assumed bounded were not addressed. A recent report (Ydstie, 1989) which became available during preparation of the current paper, considers the unmodelled dynamics in a direct scheme and shows global stability in a quite different analysis to that presented here.

Decentralized adaptive control of interconnected systems is a subject that many researchers have worked on. Results relevant to the current discussion can be found in Hill *et al.* (1988), Ioannou and Kokotovic (1985), Ioannou (1986), Praly and Trulsson (1986), Gavel and Siljak (1989), Wen and Hill (1990), Yang and Papavassilopoulos (1985) and Reed and Ioannou (1988). The σ-modification is introduced to design local adaptive controllers for a restricted class of systems in Ioannou and Kokotovic (1985) and Ioannou (1986). In Praly and Trulsson (1986) and Wen and Hill (1990), some types of normalization signals together with parameter projection are used. In Gavel and Siljak (1989), only systems satisfying certain structural restrictions placed on interactions between subsystems are considered.

Here we will consider the class of interconnected systems presented in Praly and Trulsson (1986) and Wen and Hill (1990) and apply the above mentioned algorithm to design local adaptive controllers for isolated subsystems by neglecting interactions. It is shown that all states in the closed loop of the adaptive system so designed can be guaranteed to be bounded for bounded initial conditions, reference inputs, disturbances and a certain amount of interactions and unmodelled dynamics for each subsystem. The result considerably improves that in Wen and Hill (1990) where a special estimator normalization and order bounds on unmodelled dynamics were required. Further, by using a uniform bound device for all loops, the problem of handling loop interactions is seen to be essentially the same as that of studying single loop robustness.

The paper is organized as follows. We examine the robustness of the adaptive control algorithm applied to single input–single output systems in the following three sections. The class of scalar systems are given in Section 2. Section 3 presents the adaptive control algorithm and establishes some useful properties of the simple parameter estimator. We analyze stability of the adaptive system in Section 4. Then the indirect decentralized adaptive control problem of interconnected systems is studied in Section 5. Section 6 concludes the paper.

2. MODELS OF SCALAR SYSTEMS

Let $A(q^{-1})$ and $B(q^{-1})$ be polynomials of degree n in the inverse shift operator q^{-1}, i.e.

$$A(q^{-1}) = 1 + a_1 q^{-1} + \cdots + a_n q^{-n},$$
$$B(q^{-1}) = b_1 q^{-1} + \cdots + b_n q^{-n}.$$

The class of plants we consider first are single input–single output systems and can be mathematically modelled as in the following equation

$$A(q^{-1})y(t) = B(q^{-1})u(t) + m(t), \quad (1)$$

where u and y represent the input and output, respectively. This can be expressed as

$$y(t) = \phi^T(t-1)\theta_* + m(t), \quad (2)$$

where $\phi(t-1)$ is a regression vector and θ_* denotes a vector containing unknown parameters of the nominal system model, i.e.

$$\phi^T(t-1) = [y(t-1), \ldots, y(t-n), u(t-1), \ldots, u(t-n)],$$

$$\theta_*^T = [-a_1, \ldots, a_n, b_1, \ldots, b_n].$$

The modelling error $m(t)$ consists of terms representing bounded extraneous disturbances $\omega(t)$ and unmodelled dynamics $\eta(t)$

$$m(t) = \eta(t) + \omega(t). \quad (3)$$

Assumptions 2.1.

(1) Unmodelled dynamics term $\eta(t)$ satisfies

$$|\eta(t)| \le \bar{d} + \epsilon r_0(t), \quad (4)$$

where $\bar{d}$ is a constant which bounds the initial value $\eta(0)$, ϵ is a small non-negative constant and $r_0(t)$ is defined as

$$r_0(t) = \mu_0 r_0(t-1) + \|\phi(t-1)\|, \quad r_0(0) = 0, \quad (5)$$

where μ_0 is a constant less than 1.

(2) For $\omega(t)$, there exists a constant d' such that

$$|\omega(t)| \le d'. \quad (6)$$

Since $0 < \mu_0 < 1$ in (5), we can have

$$r_0(t) \le c_\eta \max_{0,\ldots,t-1} \|\phi(\tau)\|, \quad (7)$$

where c_η is a constant. Thus if ϕ is bounded according to $\|\phi(\tau)\| \le M$ for $\tau = 0, \ldots, t-1$, we can have

$$|\eta(t)| \le \bar{d} + c_\eta \epsilon M. \quad (8)$$

and

$$|m(t)| \le c_\eta \epsilon M + d, \quad (9)$$

where

$$d = \bar{d} + d'.$$

Comments 2.1.

(1) The system model given above was studied in detail in Praly (1990). It has been shown that a broad class of discrete time systems can be represented by this model. Indeed, it includes the systems considered in Kreisselmeier and Anderson (1986), Wen and Hill (1989) and is equivalent to the one in Praly (1983). If the true system is purely linear, then $\eta(t)$ is the output of a strictly proper system with fast modes and/or nearly cancellable pole–zeros and $u(t)$ or/and $y(t)$ as the inputs.

(2) A properly chosen stable filter may be introduced to system (1) so that the deterministic disturbance can be eliminated and also the effect of unmodelled dynamics be mitigated (Middleton, 1988).

(3) Knowledge of ϵ, μ_0 and d is not required to implement the adaptive control algorithm studied later. This contrasts with the approaches given in Middleton *et al.* (1988), Ioannou and Kokotovic (1983) Praly (1983) (also see Åström (1987)).

Usually, we have some knowledge of the range of unknown parameter vector θ_* of the nominal system model. This is given in the following assumption.

Assumption 2.2.

(1) θ_* lies in a known convex compact region $\mathscr{C}$.

(2) The polynomials $\hat{A}(q^{-1})$, $\hat{B}(q^{-1})$ induced by an arbitrary (nonzero) vector $\hat{\theta}$ in $\mathscr{C}$ are uniformly coprime.

Assumption 2.2 gives that

$$\|\theta_1 - \theta_2\| \le k_\theta \quad \forall \theta_1, \theta_2 \in \mathscr{C}, \quad (10)$$

$$\|\theta_3\| \le k_c \quad \forall \theta_3 \in \mathscr{C}, \quad (11)$$

where k_θ, k_c are constants. k_0 reflects the size of $\mathscr{C}$ and k_c the maximum distance from $\mathscr{C}$ to the origin.

Suppose y^* is a given reference set-point for output y. The control problem is to design a controller such that the resulting system is bounded input bounded state (BIBS) stable and the tracking error is small. Moreover, these properties are to be robust to the modelling error $m(t)$.

Comment 2.2.

For the implementation of the adaptive control algorithm, the only knowledge of the plant needed is the nominal system order n and a convex compact region that the nominal parameters θ_* lie in.

3. ADAPTIVE CONTROL ALGORITHM

In this section, an indirect adaptive control algorithm is presented. The parameter estimator is a basic one used to establish earlier global convergence results (Goodwin and Sin, 1984) subject to parameter projection.

3.1. *Parameter estimation algorithm*

For simplicity of analysis, we use the gradient estimation algorithm

$$\hat{\theta}(t) = \mathscr{P}\left\{\hat{\theta}(t-1) + \frac{\phi(t-1)e(t)}{1 + \phi^T(t-1)\phi(t-1)}\right\}, \quad (12)$$

where $\hat{\theta}(t)$ denotes the estimate of θ_* at t and $\mathscr{P}$ represents the projection operator necessary to ensure $\hat{\theta}(t) \in \mathscr{C} \forall t$. $e(t)$ is the prediction error defined as

$$e(t) = y(t) - \phi^T(t-1)\hat{\theta}(t-1). \quad (13)$$

Now suppose M_0 is a constant such that

$\frac{d}{M_0} \le \delta$, where $0 < \delta < 1$. Also let M be a constant such that $M^2 = k_1 M_0^2 + k_2$ where k_1 and k_2 are positive constants specified in later sections.

The properties of estimator (12), (13) given below will be used in our stability analysis.

Lemma 3.1. Consider the estimator (12), (13), applied to system (1). Assuming

$$\|\phi(t_0 - 1)\| \le M_0, \quad \|\phi(\tau)\| > M_0, \quad \tau = t_0, \ldots, t-1,$$

and $\|\phi(\tau_1)\| \le M$, $\tau_1 = 0, \ldots, t-1$, where $t \ge t_0 + 1$, then we have:

(1)

$$|e(t_0)| \le (k_\theta + a_1)M_0 + a_1, \tag{14}$$

and

$$|\bar{e}(t)| := \frac{e(t)}{(1 + \|\phi(t-1)\|^2)^{1/2}}, \le k_\theta + a_1, \quad t \ge t_0 + 1, \tag{15}$$

where

$$a_1 = (k_1^{1/2} + k_2^{1/2})c_\eta \epsilon + \delta,$$

(2)

$$\sum_{\tau=t_0+1}^{t} |\bar{e}(\tau)|^2 \le k_\theta^2 + (a_2 + a_3)(t - t_0), \tag{16}$$

where

$$a_2 = 2(k_\theta(k_1^{1/2} + k_2^{1/2}) + 2c_\eta(k_1 + k_2)\epsilon)c_\eta \epsilon, \tag{17}$$

$$a_3 = 2(2\delta + k_\theta)\delta, \tag{18}$$

(3)

$$\|\hat{\theta}(t) - \hat{\theta}(t-1)\| \le |\bar{e}(t)| \; \forall t. \tag{19}$$

Proof.

(1) From (2) and (13), we have

$$e(t) = -\phi^T(t-1)\tilde{\theta}^T(t-1) + m(t), \tag{20}$$

where

$$\tilde{\theta}(t) = \hat{\theta}(t) - \theta_*.$$

Then (20), (10) and (9) give

$$|e(t)| \le k_\theta \|\phi(t-1)\| + c_\eta \epsilon (k_1 M_0^2 + k_2)^{1/2} + d, \tag{21}$$

where $M^2 = k_1 M_0^2 + k_2$ has been used. For $t = t_0$, we have

$$|e(t_0)| \le k_\theta M_0 + a_1 M_0 + a_1,$$

using $\|\phi(t_0 - 1)\| \le M_0$.

Under the assumption of the lemma, it follows that in general

$$|\bar{e}(t)| \le \frac{k_\theta \|\phi(t-1)\| + c_\eta \epsilon (k_1 M_0^2 + k_2)^{1/2} + d}{(1 + \|\phi(t-1)\|^2)^{1/2}},$$
$$\le k_\theta + \frac{k_1^{1/2} M_0 + k_2^{1/2}}{(1 + M_0^2)^{1/2}} c_\eta \epsilon + \delta \quad \text{for } t - 1 \ge t_0,$$
$$\le k_\theta + \alpha_1.$$

(2) Let $\hat{\theta}_{np}(t)$ denote the parameter estimate before projection, i.e. $\hat{\theta}(t) = \mathcal{P}\{\hat{\theta}_{np}(t)\}$. Thus

$$\hat{\theta}_{np}(t) - \hat{\theta}(t-1) = \frac{\phi(t-1)e(t)}{1 + \phi^T(t-1)\phi(t-1)}. \tag{22}$$

We consider the function $v(t) = \tilde{\theta}^T(t)\tilde{\theta}(t)$. Then

$$\begin{aligned} v(t) - v(t-1) &\le \tilde{\theta}_{np}^T(t)\tilde{\theta}_{np}(t) - \tilde{\theta}^T(t-1)\tilde{\theta}(t-1) \le [\tilde{\theta}_{np}^T(t) \\ &\quad - \tilde{\theta}^T(t-1)][\tilde{\theta}_{np} - \tilde{\theta}(t-1) + 2\tilde{\theta}(t-1)] \\ &\le \frac{e^2(t)}{1 + \|\phi(t-1)\|^2} + \frac{2\phi^T(t-1)\tilde{\theta}(t-1)e(t)}{1 + \|\phi(t-1)\|^2}. \end{aligned} \tag{23}$$

Now using (20) and (10) gives

$$\begin{aligned} 2\phi^T(t-1)\tilde{\theta}(t-1)e(t) &= 2(-e(t) + m(t))e(t) \\ &\le -2e^2 + 2|m|(k_\theta \|\phi(t-1)\| + |m|) \\ &\le -2e^2 + 2k_\theta \|\phi(t-1)\| \, |m| + 2|m|^2. \end{aligned} \tag{24}$$

From (9), (20), (23) and (24), we get

$$\begin{aligned} v(\tau) - v(\tau-1) \le &- \frac{e^2}{1 + \|\phi(\tau-1)\|^2} + \frac{2k_\theta c_\eta \epsilon (k_1 M_0^2 + k_2)^{1/2} \|\phi(\tau-1)\|}{1 + \|\phi(\tau-1)\|^2} \\ &+ \frac{4c_\eta^2 \epsilon^2 (k_1 M_0^2 + k_2)}{1 + \|\phi(\tau-1)\|^2} + \frac{4d^2 + 2k_\theta d \|\phi(\tau-1)\|}{1 + \|\phi(\tau-1)\|^2}. \end{aligned}$$

Thus we have

$$\bar{e}^2(\tau) \le v(\tau-1) - v(\tau) + a_2 + a_3, \tag{25}$$

for $\tau = t_0 + 1, \ldots, t$ and $t \ge t_0 + 1$. Summing in (25) gives

$$\sum_{\tau=t_0+1}^{t} |\bar{e}(\tau)|^2 \le \tilde{\theta}^T(t_0)\tilde{\theta}(t_0) - \tilde{\theta}^T(t)\tilde{\theta}(t) + (a_2 + a_3)(t - t_0).$$

Using (10) again, the results follows.

(3)

$$\|\hat{\theta}(t)-\hat{\theta}(t-1)\| \le \|\hat{\theta}_{np}(t)-\hat{\theta}(t-1)\| \le \frac{\|\phi(t-1)\|\,|e(t)|}{1+\|\phi(t-1)\|^2} \le |\bar{e}(t)|.$$

Comments 3.1.

(1) The least squares algorithm is more commonly used in practical algorithms (Goodwin and Sin, 1984). Similar properties for this estimator can be derived by defining a different Lyapunov type function $v(t)$, but the analysis is more tedious.

(2) Note that a_1 and a_2 are functions of ϵ and can be made arbitrarily small by reducing ϵ. Also a_3 depends on δ.

(3) Suppose the system modelling errors are removed, i.e. ϵ and d in (9) are identically zeros. In this case, $m(t)=0$ and the above estimator has those properties established in earlier global convergence results (Goodwin and Sin (1984)).

3.2. *Adaptive controller design*

Following the Certainty Equivalence Principle, we will use the parameter estimates obtained from the estimator (12) and (13) to compute the parameters of the controller. Here the pole assignment strategy is utilized. This is just one choice of many control schemes which could be used (Goodwin and Sin, 1984). The control $u(t)$ is given by the equation

$$\hat{L}(t-1)u(t) = -\hat{P}(t-1)(y(t)-y^*(t)), \quad (26)$$

where y^* is the given set-point and

$$\hat{L}(t) = 1+\hat{l}_1(t)q^{-1}+\cdots+\hat{l}_n(t)q^{-n}, \quad (27)$$

$$\hat{P}(t) = \hat{p}_1(t)q^{-1}+\cdots+\hat{p}_n(t)q^{-n}, \quad (28)$$

$\hat{L}$ and $\hat{P}$ are obtained by solving the following Diophantine equation

$$\hat{A}(t)\hat{L}(t)+\hat{B}(t)\hat{P}(t) = A^*, \quad (29)$$

where A^* is a given monic strictly (discrete-time) Hurwitz constant polynomial in shift operator q^{-1} of degree $2n$. From Assumption 2.2, we see that the coefficients in $\hat{L}(t)$ and $\hat{P}(t)$ obtained from equation (29) are bounded (Goodwin and Sin, 1984).

4. ROBUSTNESS ANALYSIS

In this section, we will examine the robustness of adaptive control algorithm (12), (13), (26)–(29) applied to system (1). It will be shown that there exists a class of unmodelled dynamics, i.e. a ϵ^* such that for each ϵ given in (4) satisfying $\epsilon\in[0,\epsilon^*]$, all states in the closed adaptive system are bounded for any bounded initial conditions, bounded set-point and extraneous disturbances.

Before going to present the details of our result and its proof, we first establish and make some analysis of the closed loop system equation. By combining (13) and (26), we can get

$$\phi(t+1) = \bar{A}(t)\phi(t)+B_1e(t+1)+B_2r(t+1), \quad (30)$$

where

$$\bar{A}(t)=\begin{pmatrix} -\hat{a}_1(t) & -\hat{a}_2(t) & \cdots & -\hat{a}_n(t) & \hat{b}_1(t) & \cdots & & \hat{b}_n(t) \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ & & & 0 & 0 & \cdots & 0 & 0 \\ & & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\hat{p}_1(t) & & \cdots & -\hat{p}_n(t) & -\hat{l}_1(t) & \cdots & & -\hat{l}_1(t) \\ 0 & & \cdots & 0 & 1 & \cdots & 0 & 0 \\ 0 & & \cdots & 0 & & & & 0 \\ 0 & & \cdots & 0 & & & 1 & 0 \end{pmatrix}, \quad (31)$$

$$B_1^T = [1, 0, \ldots, 0], \quad (32)$$

$$B_2^T = [0, \ldots, 0, 1, \ldots, 0], \quad (33)$$

$$r(t+1) = \hat{P}(t)y^*(t+1). \quad (34)$$

Since $\hat{P}(t)$ is bounded, then $|r(t+1)|\le c_p\,|y^*(t+1)|$ where c_p is a constant. From Lemma 3.1, we can obtain

Lemma 4.1. The matrix $\bar{A}(t)$ defined in (31) satisfies

(1) $\bar{A}(t)$ is bounded $\forall t$.

(2) $\sum_{\tau=t_0+1}^{t}\|\bar{A}(\tau)-\bar{A}(\tau-1)\|$

$$\le k(k_\theta^2+a_2(t-t_0)+a_3(t-t_0)) \quad (35)$$

for

$$t>t_0 \quad \text{if } \|\phi(\tau)\|\le M,\ \tau=0,\ldots,t-1$$

and

$$\|\phi(\tau)\|>M_0, \quad \tau=t_0,\ldots,t-1$$

where k is a constant.

(3) $\lambda_i(\bar{A}(t)) = \lambda_i(A^*)\ \forall t$

Proof.

(1) This follows from Assumption 2.2 and $\hat{\theta}(t) \in C \forall t$.

(2) From assumption 2.2, we have

$$\sum_{\tau=t_0+1}^{t} \|\bar{A}(\tau) - \bar{A}(\tau-1)\|^2$$
$$\leq k \sum_{\tau=t_0+1}^{t} \|\hat{\theta}(\tau) - \hat{\theta}(\tau-1)\|^2$$
for some constant k
$$\leq k \sum_{\tau=t_0+1}^{t} |\bar{e}(\tau)|^2$$
using (19)
$$\leq k(k_\theta^2 + a_2(t-t_0) + a_3(t-t_0))$$
using (16).

(3) This is easy to verify from (31).

The following lemma is similar to one given in Kreisselmeier (1986) and a discrete-time version to one in Middleton *et al.* (1988).

Lemma 4.2. Consider a linear time-varying system of the form

$$x(t+1) = A(t)x(t). \tag{36}$$

Suppose

(i) $A(t)$ is bounded

(ii) $\sum_{\tau=t_0+1}^{t} \|A(\tau) - A(\tau-1)\|^2$
$$\leq \beta_0 + \beta_1(t-t_0) \quad \text{for } t > t_0,$$
where β_0, β_1 are positive constants with β_1 sufficiently small

(iii) $|\lambda_i(A(t)| < 1$ for all t and $i = 1, \ldots, n$.

Then the transition matrix of (36), denoted $\psi_1(t, \tau)$, satisfies

$$\|\psi_1(t, \tau)\| \leq C'\mu^{t-\tau} \quad \text{for } t \geq \tau, \tag{37}$$

where $\mu \in (0, 1)$ and C' is a constant.

From Lemmas 4.1 and 4.2, we can study the stability of the linear system

$$\phi(t+1) = \bar{A}(t)\phi(t). \tag{38}$$

Let $\Phi(t, \tau)$ denote the state transition matrix of system (38). Then we have the property

$$\|\Phi(t, \tau)\| \leq C\sigma^{t-\tau} \quad \text{for } t \geq \tau \geq t_0, \tag{39}$$

if $\|\phi(\tau)\| \leq M$, $\tau = 0, \ldots, t-1$; $\|\phi(\tau)\| > M_0$, $\tau = t_0, \ldots, t-1$ and a_2, a_3 obey bounds $a_2 \leq \bar{a}_2^*$, $a_3 \leq \bar{a}_3^*$. These bounds are given by $\bar{a}_2^* = (\beta_1 - ka_3)/k$ with $a_3 \leq \bar{a}_3^*$ ensuring $\beta_1 - ka_3 > 0$. C is a constant and $\sigma \in (0, 1)$. From (18) the bound $\bar{a}_3^*$ is equivalent to a bound $\bar{\delta}^*$ such that $\delta \leq \bar{\delta}^*$ where $\bar{\delta}^*$ is just a number. Now we are in the position to present our stability result.

Theorem 4.1. Consider the adaptive system consisting of plant (1), estimator (12) to (13) and controller (26) to (29). Under Assumptions 2.1 and 2.2, there exists ϵ^* such that $\epsilon \leq \epsilon^*$ ensures $\|\phi(t)\|$ bounded $\forall t$ for all bounded initial conditions and setpoints.

Proof. In order to clarify the development of the proof, we begin with some preliminary motivating steps.

(1) From the stability assumption on modelling error $m(t)$ given in Assumption 2.1, we see that $m(t)$ can be bounded by a function of $\|\phi(0)\|, \|\phi(1)\|, \ldots, \|\phi(t-1)\|$, i.e. past values of $\|\phi(t)\|$.

(2) Also note that for any bounded initial conditions $\phi(0)$, set points y^* and disturbances $\omega(t)$, there always exists a number M_0 such that $\|\phi(0)\| \leq M_0$, $\|r(t)\|_\infty \leq M_0$ and $\frac{d}{m_0} \leq \delta$ for a sufficiently small δ, where $r(t)$ is given by (34). From the observation in point 1 above and the fact that $\|\phi(0)\| \leq M_0 < M$ defined as a function of M_0 given earlier, we can ensure that $\|\phi(1)\| \leq M$ and then $\|\phi(2)\| \leq M$ under certain conditions independent of M_0.

(3) The above two points motivate us to use an inductive proof by assuming that $\|\phi(\tau)\| \leq M$, $\tau = 0, \ldots, t-1$ for $t \geq 1$ and proving $\|\phi(t)\| \leq M$.

(4) Under this inductive assumption, we can bound the modelling error $m(t)$ by a function of M, i.e. M_0, as in (9). Then using (9) and the normalizing term $1 + \|\phi(t-1)\|$ in the estimator (12), we can invoke the estimator properties given in Lemma 3.1. Also note that a_1 and a_2 are independent of M_0 and can be made arbitrarily small by restricting the gain ϵ of unmodelled dynamics.

(5) In order to apply Lemma 3.1 and the exponential stability property of $\bar{A}(t)$ in the closed loop equation in (30), we divide the time interval Z_+ into two subsequences (Praly and Trulsson, 1986; Wen and Hill, 1989)

$$Z_1 := \{t \in Z_+ \mid \|\phi(t)\| > M_0\},$$
$$Z_2 := \{t \in Z_+ \mid \|\phi(t)\| \leq M_0\}.$$

Clearly, the result is proved if we can show that $\|\phi(t)\| \leq M$ for $t \in Z_1$ since $M > M_0$. To do this, we choose t_0 so that $t_0 \geq 1$, $t_0 - 1 \in Z_2$ and $t_0, \ldots, t-1 \in Z_1$.

(6) Following somewhat standard steps in proving stability of adaptive systems given in the literature (Middleton *et al.*, 1988; Wen and Hill, 1989), we outline the proof of $\|\phi(t)\| \le M$ as follows.

The general solution of (30) is

$$\phi(t) = \Phi(t, t_0)\phi(t_0) + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)[B_1 e(\tau+1) + B_2 r(\tau+1)],$$

i.e.

$$\phi(t) = \Phi(t, t_0)[\bar{A}(t_0-1)\phi(t_0-1) + B_1 e(t_0) + B_2 r(t_0)] + \sum_{\tau=t_0}^{t-1} \Phi(t, \tau+1)[B_1 e(\tau+1) + B_2 r(\tau+1)].$$

Using (14), (39), boundedness of $\|\bar{A}(t_0-1)\|$ and inequality

$$|e(\tau+1)| \le (1 + \|\phi(\tau)\|)\,|\bar{e}(\tau+1)|,$$

gives

$$\|\phi(t)\| \le C\sigma^{t-t_0}[(C_1 + a_1)M_0 + a_1] + \sum_{\tau=t_0}^{t-1} C\sigma^{t-\tau-1}[|\bar{e}(\tau+1)|\,\|\phi(\tau)\| + |\bar{e}(\tau+1)| + M_0], \quad (40)$$

where C and C_1 are constants depending on k_θ, σ and $\bar{\delta}^*$.

Squaring both sides of (40) and applying the Schwarz inequality, we get

$$\|\phi(t)\|^2 \le C_3\sigma^{2(t-t_0)}[(C_1 + a_1)^2 M_0^2 + a_1^2] + C_4 \sum_{\tau=t_0}^{t-1} \sigma^{t-\tau}[\|\phi(\tau)\|^2\,|\bar{e}(\tau+1)|^2 + |\bar{e}(\tau+1)|^2 + M_0^2],$$

where C_3 and C_4 are constants. Multiplying by σ^{-t} gives

$$\sigma^{-t}\|\phi(t)\|^2 \le s^2(t) + C_4 \sum_{\tau=t_0}^{t-1} \sigma^{-\tau}\|\phi(\tau)\|^2\,|\bar{e}(\tau+1)|^2, \quad (41)$$

where

$$s^2(t) = C_3\sigma^{-t_0}[(C_1 + a_1)^2 M_0^2 + a_1^2] + C_4 \sum_{\tau=t_0}^{t-1} \sigma^{-\tau}(|\bar{e}(\tau+1)|^2 + M_0^2). \quad (42)$$

Then the discrete Grownwall lemma (Desoer and Vidyasagar, 1975) can be applied to (41) to yield

$$\|\phi(t)\|^2 \le \sigma^t s^2(t) + \sum_{\tau=t_0}^{t-1} \Gamma(t, \tau)\sigma^\tau s^2(\tau), \quad (43)$$

where

$$\Gamma(t, \tau) = (\sigma C_4 |\bar{e}(\tau+1)|^2) \prod_{\tau_1=\tau+2}^{t} \sigma(1 + C_4|\bar{e}(\tau_1)|^2), \quad (44)$$

and the product term is 1 for $\tau = t-1$. The "Theorem of the Arithmetic and Geometric Means" (Hardy *et al.*, 1952) gives

$$\prod_{i=1}^{n} a_i \le \left\{\frac{1}{n}\sum_{i=1}^{n} a_i\right\}^n, \quad (45)$$

for a sequence of non-negative numbers. Applying (45) to (44) gives that

$$|\Gamma(t, \tau)| \le \left\{\frac{1}{t-\tau}\left[\sum_{\tau_1=\tau+1}^{t} \sigma(1 + C_4|\bar{e}(\tau_1)|^2)\right]\right\}^{t-\tau} \le \left\{\frac{\sigma C_4 k_\theta^2}{t-\tau} + \sigma(C_4 a_2 + C_4 a_3 + 1)\right\}^{t-\tau}, \quad (46)$$

using (16). Choose $\sigma < \sigma_c^* < 1$. Then from (17) and (18), it is clear that there exists $\bar{\bar{a}}_2^*$, $\bar{\bar{\delta}}^*$ which are small enough to guarantee $\sigma_c^* - \sigma(1 + C_4 a_3) > 0$ for $\delta \le \bar{\bar{\delta}}^*$ and

$$\sigma(C_4 a_2 + C_4 a_3 + 1) < \sigma_c^*, \quad (47)$$

for $a_2 \le \bar{\bar{a}}_2^*$. (This will be set by choice of ϵ.) From (46)

$$|\Gamma(t, \tau)| \le (\sigma_c^*)^{t-\tau}\left\{1 + \frac{\sigma C_4 k_\theta^2}{\sigma_c^*(t-\tau)}\right\}^{t-\tau} \le (\sigma_c^*)^{t-\tau} \exp\left\{\frac{\sigma C_4 k_\theta^2}{\sigma_c^*}\right\},$$

using the inequality $\left(1 + \frac{1}{x}\right)^x \le e$.

Thus there exists $K > 1$ such that

$$|\Gamma(t, \tau)| \le K(\sigma_c^*)^{t-\tau}. \quad (48)$$

Now consider the term $\sigma^\tau s^2(\tau)$ appearing in (43).

$$\sigma^\tau s^2(\tau) \le C_3\sigma^{\tau-t_0}[(C_1 + a_1)^2 M_0^2 + a_1^2] + C_4 \sum_{\tau_1=t_0}^{\tau-1} \sigma^{\tau-\tau_1}(|\bar{e}(\tau_1+1)|^2 + M_0^2) \le C_3\sigma^{\tau-t_0}[(C_1 + a_1)^2 M_0^2 + a_1^2] + C_5 + C_6 a_1^2 + C_7 M_0^2, \quad (49)$$

using (15) where C_5, C_6, C_7 are constants depending on k_θ^2, σ and $\bar{\delta}^*$. Substituting (48) and (49) into (43), we get

$$\|\phi(t)\|^2 \le C_3\sigma^{t-t_0}[(C_1 + a_1)^2 M_0^2 + a_1^2] + C_5 + C_6 a_1^2 + C_7 M_0^2 + \sum_{\tau=t_0}^{t-1} K(\sigma_c^*)^{t-\tau} \times \{C_3\sigma^{\tau-t_0}[(C_1 + a_1)^2 M_0^2 + a_1^2] + C_5 + C_6 a_1^2 + C_7 M_0^2\} \le C_8 + C_9 M_0^2 + C_{10} a_1^2 M_0^2 + C_{11} a_1^2, \quad (50)$$

using $\sigma < \sigma_c^* < 1$, where C_8, C_9, C_{10} and C_{11} are constants depending on k_0, σ, σ_c^* and $\bar{\delta}^*$. Choose a number a_1^* such that $a_1 \le a_1^*$ for all ϵ satisfying $\epsilon \le \bar{\epsilon}_1^*$.

From (15), we have

$$\|\phi(t)\|^2 \le [C_{13} + C_{14}c_\eta^2\epsilon^2(k_1 + k_2)]M_0^2 + C_{12}, \tag{51}$$

where C_{12} is a constant combining the previous C_8, C_{11} terms. Let $k_2 = C_{12}$ and $\bar{\epsilon}^*$ be a constant satisfying

$$C_{14}c_\eta^2(\bar{\epsilon}^*)^2 \le 1. \tag{52}$$

By taking k_1 as

$$k_1 = \max\left\{1, \frac{C_{13} + C_{14}c_\eta^2(\bar{\epsilon}^*)^2C_{12}}{1 - C_{14}c_\eta^2(\bar{\epsilon}^*)^2}\right\}, \tag{53}$$

we can have

$$\|\phi(t)\|^2 \le k_1M_0^2 + k_2,$$

i.e.

$$\|\phi(t)\|^2 \le M^2. \tag{54}$$

It remains to clarify the role of ϵ at (4) in establishing (54). From the above argument, we can see that k_1 and k_2 are constants depending only on system parameters k_θ, σ, c_η and numbers $\sigma_c^*, \bar{\delta}^*$ and $\bar{\epsilon}^*$. Now let $a_2^* = \min\{\bar{a}_2^*, \bar{\bar{a}}_2^*\}$ where $\bar{a}_2^*$ and $\bar{\bar{a}}_2^*$ were defined to ensure (39) and (47) are satisfied. From (17), we see that there exists an $\bar{\bar{\epsilon}}^*$ such that $\epsilon \le \bar{\bar{\epsilon}}^*$ gives $a_2 \le a_2^*$. Finally, taking $\epsilon^* = \min\{\bar{\epsilon}^*, \bar{\bar{\epsilon}}^*, \bar{\epsilon}_1^*\}$ where $\bar{\epsilon}^*$ was defined in (52), we have proved the results.

Comments 4.1.

(1) For a given system, there always exists a M_0 such that $\|\phi(0)\| \le M_0$, $\|r(t)\|_\infty \le M_0$ and $d/M_0 \le \delta^*$ for any bounded initial condition, set point and disturbances $\omega(t)$, where $\delta^* = \min\{\bar{\delta}^*, \bar{\bar{\delta}}^*\}$, $\bar{\delta}^*$ and $\bar{\bar{\delta}}^*$ are sufficiently small numbers to ensure (39) and (47) satisfied. Note that $\bar{\delta}^*$ and $\bar{\bar{\delta}}^*$ were defined in such a way that they are independent of M_0 in establishing (39) and (47). Since the stability condition does not depend on M_0, we do not need to know it's value in proving our result.

(2) Once a system is given and A^* is chosen, then k_θ, c_η and σ are fixed. There exist $\bar{\delta}^*$ to satisfy $\beta_1 - k\bar{a}_3^* > 0$ in establishing (39), $\bar{\epsilon}^*$ to satisfy (52) and we can choose a number $\sigma_c^* < 1$. Thus constants k_1 and k_2 are set in (53). Then a ϵ^* can be found and we can ensure that $\|\phi(t)\|$ is bounded for all t and $\epsilon \le \epsilon^*$.

(3) In Wen and Hill (1989), a weaker local stability result was obtained. The improvement here comes from more careful use of the normalizing term $1 + \|\phi(t-1)\|$ in allowing for the modelling errors in the estimator properties. The effect of unmodelled dynamics is bounded by a function of past inputs and outputs of the system—see (4). Through induction, this effect is bounded by the constant M which depends on initial states of the system. The devices used here enable the dependence of the stability condition on initial conditions to be cancelled out to obtain a (global) bounded-input–bounded-state type stability statement.

(4) An interesting observation on the use of a relative deadzone (Middleton *et al.*, 1988; Kreisselmeier and Anderson, 1986) is given here. We see that if an incorrect upper bound of ϵ at (4) is used to build a relative deadzone function in the adaptive controller, the closed loop system is still stable for sufficiently small ϵ from our analysis given above.

(5) If there is no modelling error appearing in the system, we obtain the results that basic adaptive control algorithms achieve for ideal plants (Goodwin and Sin, 1984). One of them is that perfect tracking can be achieved since the prediction error tends to zero.

(6) Suppose disturbance $\omega(t)$ is identically zero or satisfies

$$S(q^{-1})\omega(t) = 0,$$

where $S(q^{-1})$ is a known polynomial of q^{-1} with all roots on the unit circle. Also note that the term $\bar{d}$ in (4) can be an exponentially decaying function. Since the prediction error $e(t)$ is a continuous function of ϵ given at (4) (which reflects the effect of unmodelled dynamics of the plant), we can expect that $e(t)$ should be small if ϵ is. Thus having established boundedness of all states in the closed loop system, it can be shown that the prediction error $e(t)$ for a given system with given initial condition is ϵ small in the mean, i.e. $e(t)$ satisfies

$$\sum_{\tau=t_0+1}^{t} e^2(t) \le \beta_{11} + \beta_{12}0(\epsilon)(t - t_0),$$

where β_{11}, β_{21} are constants and $0(\epsilon)$ satisfies $\lim_{\epsilon\to 0} 0(\epsilon) = 0$. Clearly, if $\epsilon = 0$, then $e(t) \in l_2$. If control synthesis strategies including use of the internal model principle (Goodwin and Sin, 1984; Middleton *et al.*, 1988; Middleton and Wang, 1988) are used, we can prove that the tracking error $|y - y^*|$ is ϵ small in the mean, by using the similar methods of analysis in Goodwin and Sin (1984), Ioannou and Tao (1987) and Middleton and Wang (1988).

Example 4.1. The system to be controlled in a z-domain description is

$$\frac{Y(z)}{U(z)} = \frac{0.4}{z + 1.06}\left(1 + \frac{d}{z + c}\right).$$

The nominal model used for adaptive controller design is a first order model given by

$$\frac{Y(z)}{U(z)} = \frac{b}{z + a}.$$

Clearly, there is a multiplicative plant perturbation and the plant has a unstable mode.

In the design of an adaptive controller, we choose A^* to be $z^2 - 1.4z + 0.5368$, which gives a damping of 0.707 and a natural frequency of 0.88 rad sec^{-1} when the sampling period is $\Delta = 0.5$ sec (Åström and Wittenmark, 1984). This choice also satisfies the constraint imposed by the unstable mode according to a design rule given in Middleton and Goodwin (1989).

Suppose that we know $-1.5 \leq a \leq 1.5$ and $0.1 \leq b \leq 1.2$. Thus we choose $\hat{a}(0) = -0.8$ and $\hat{b}(0) = 1.0$. The command signal y^* is a square wave with period 40 sec. The tracking problem is not considered in this design example.

When $c = 0.4$ and $d = 0.2$ the system is stable and the responses are shown in Figs 1–4. When c is increased to 0.82 or d to 0.6, the system becomes unstable. Figures 5 and 6 show the responses of the system when $c = 0.82$ and $d = 0.4$. Figures 7 and 8 give the responses when $c = 0.4$ and $d = 0.6$.

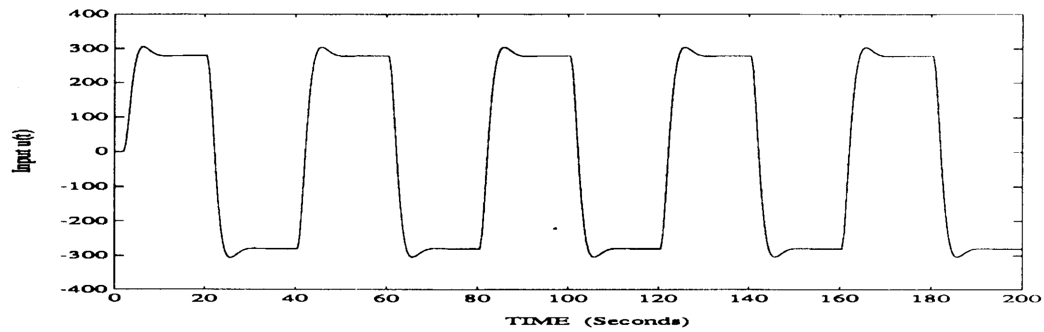

FIG. 1. Control signal $u(t)$.

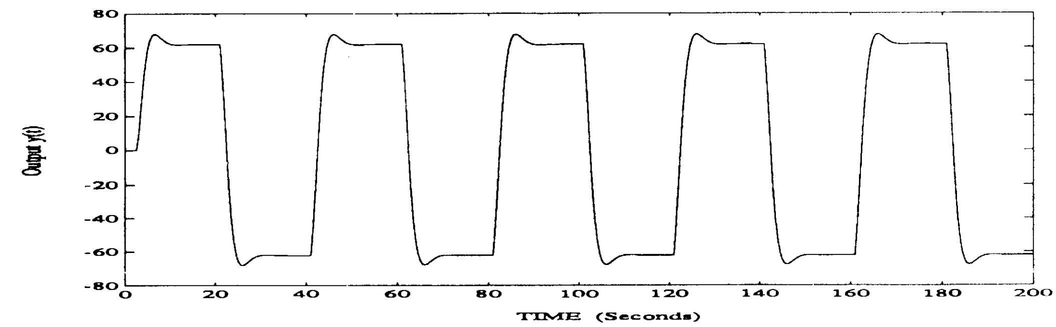

FIG. 2. Plant output $y(t)$.

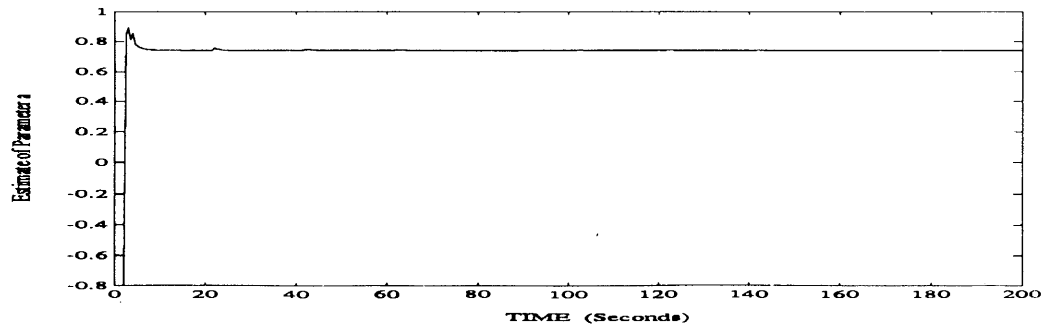

FIG. 3. Parameter estimate $\hat{a}(t)$.

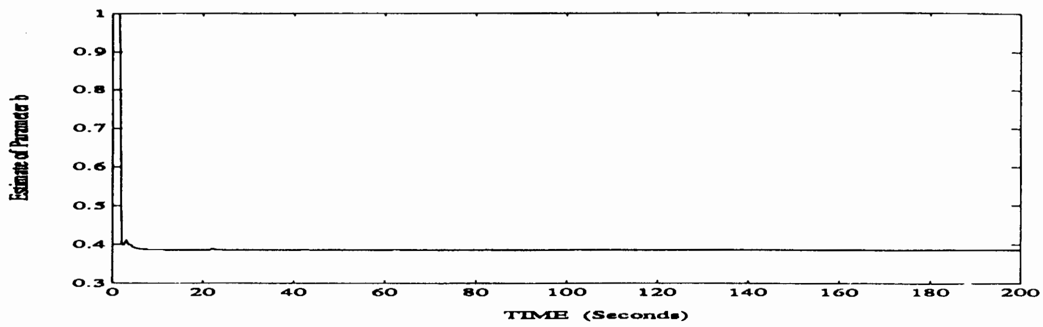

FIG. 4. Parameter estimate $\hat{b}(t)$.

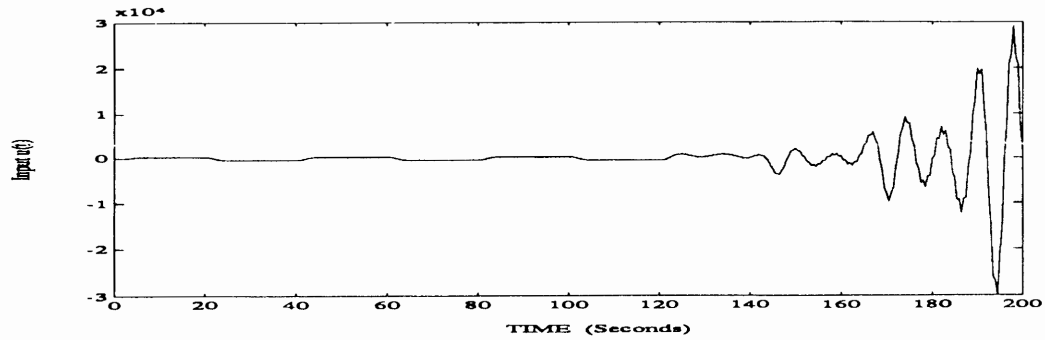

FIG. 5. Control signal $u(t)$.

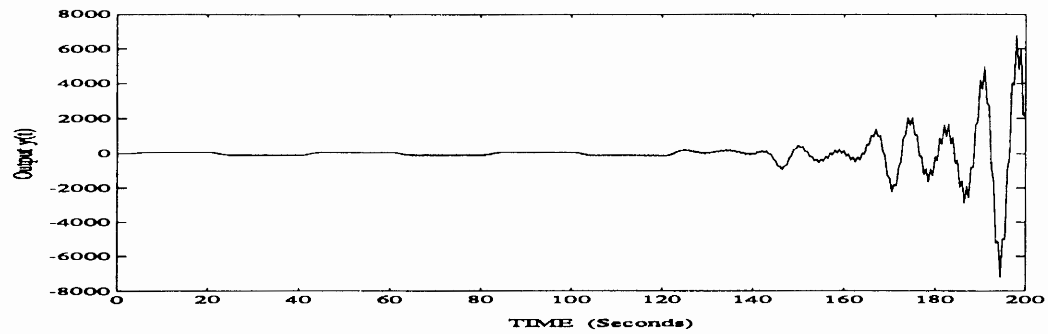

FIG. 6. Plant output $y(t)$.

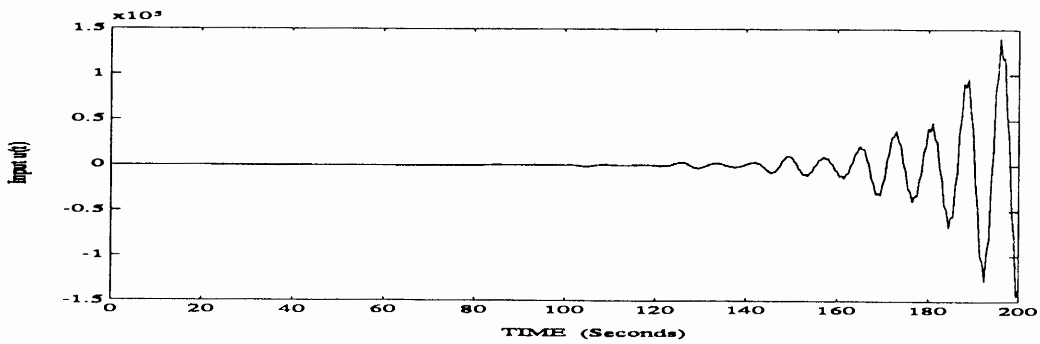

FIG. 7. Control signal $u(t)$.

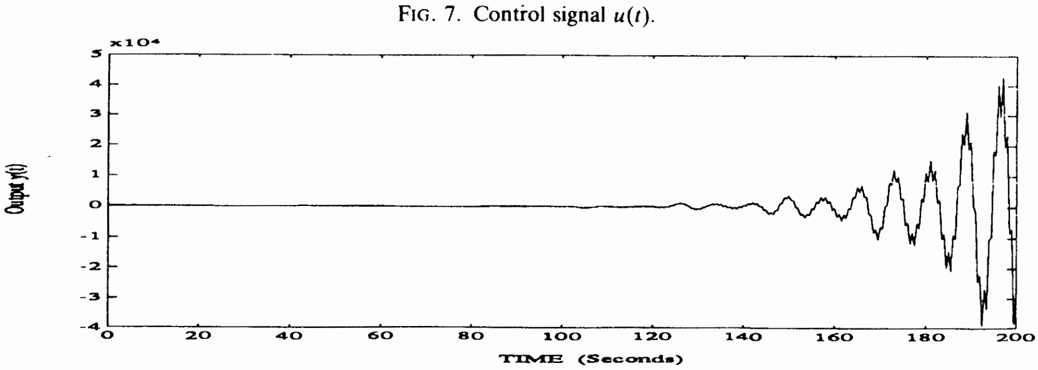

FIG. 8. Plant output $y(t)$.

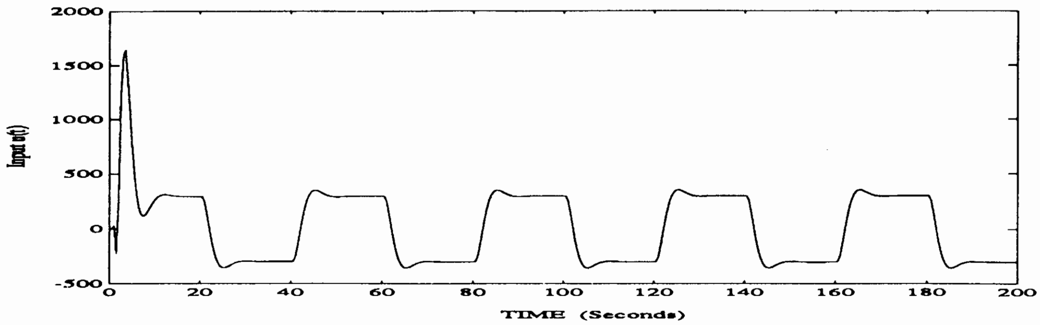

FIG. 9. Control signal $u(t)$.

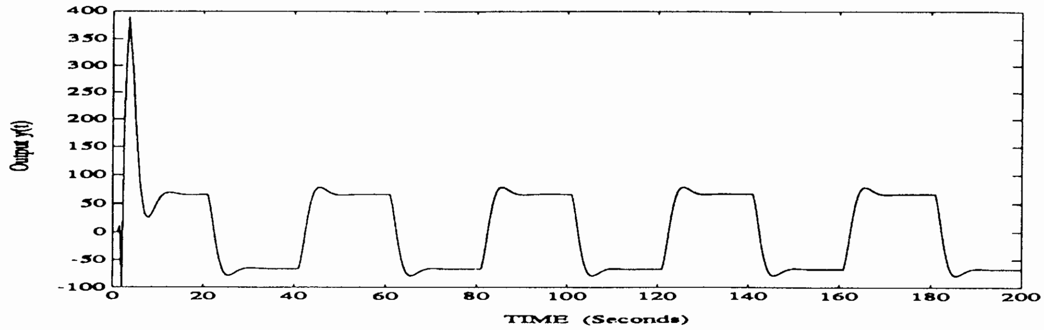

FIG. 10. Plant output $y(t)$.

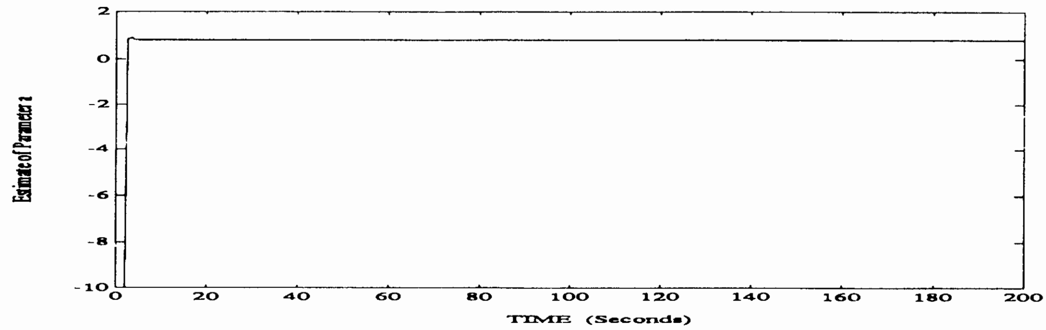

FIG. 11. Parameter estimate $\hat{a}(t)$.

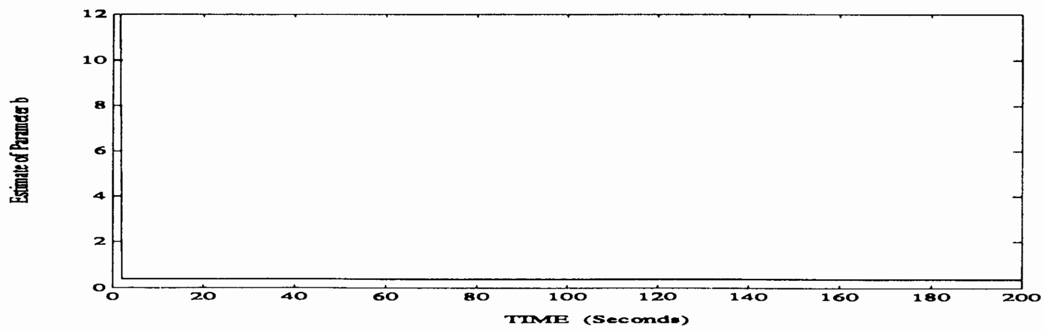

FIG. 12. Parameter estimate $\hat{b}(t)$.

If the allowable region for a and b is enlarged, then the performance of the system responses during the early stage is degraded. Figures 9–12 present the responses when initial values of $\hat{a}(0)$ and $\hat{b}(0)$ are chosen to be -10 and 12, respectively.

5. DECENTRALIZED INDIRECT ADAPTIVE CONTROL

Having shown the robustness of adaptive control algorithm (12), (13), (26)–(29) against modelling errors in scalar systems, we now turn to study the problem of decentralized adaptive control of multi-input–multi-output systems. The class of systems we consider consists of l single-input–single-output subsystems and the ith subsystem can be modelled as follows

$$A^i(q^{-1})y_i(t) = B^i(q^{-1})u_i(t) + m_i(t), \quad (55)$$

where $i = 1, \ldots, l$ and

$$A^i(q^{-1}) = 1 + a_1^i q^{-1} + \cdots + a_{n_i}^i q^{-n_i},$$
$$B^i(q^{-1}) = b_1^i q^{-1} + \cdots + b_{n_i}^i q^{-n_i},$$

$m_i(t)$ denotes the modelling uncertainty including interactions from other subsystems, unmodelled dynamics of the ith subsystem and bounded disturbances.

Equation (55) can be equivalently rewritten as

$$y_i(t) = \phi_i^T(t-1)\theta_*^i + m_i(t), \quad (56)$$

where $i = 1, \ldots, l$ and

$$\phi_i^T(t-1) = [y_i(t-1), \ldots, y_i(t-n_i), u_i(t-1), \ldots, u_i(t-n_i)],$$
$$\theta_*^i = [-a_1^i, \ldots, -a_{n_i}^i, b_1^i, \ldots, b_{n_i}^i]^T.$$

For the above system, the following assumptions similar to that of Section 2 are made.

Assumption 5.1.

The modelling uncertainty $m_i(t)$ satisfies

$$|m_i(t)| \le d_i + \epsilon_i \sum_{j=1}^{l} r_0^i(t), \quad (57)$$

where d_i and ϵ_i are nonnegative constants, and $r_0^i(t)$ is defined as

$$r_0^i(t) = \mu_0^i r_0^i(t-1) + \|\phi_i(t-1)\|, \quad r_0^i(0) = 0, \quad (58)$$

where μ_0^i is a positive constant less than 1.

As in Section 2, we can obtain

$$r_0^i(t) \le c_\eta^i \max_{0,\ldots,t-1} \|\phi_i(\tau)\|, \quad (59)$$

where c_η^i is a constant. Thus for each subsystem i, $i = 1, \ldots, l$, if $\|\phi_i(\tau)\| \le M$ for $\tau = 0, \ldots, t-1$, we can have

$$|m_i(t)| \le c_\eta \epsilon M + d, \quad (60)$$

where c_η is a constant and

$$\epsilon = \max_{1,\ldots,l} \{\epsilon_i\},$$
$$d = \max_{1,\ldots,l} \{d_i\}.$$

Assumption 5.2.

(1) θ_*^i lies in a known convex compact region $\mathscr{C}_i$.

(2) The polynomials $\hat{A}^i(q^{-1})$, $\hat{B}^i(q^{-1})$ induced by an arbitrary (nonzero) vector $\hat{\theta}^i$ in $\mathscr{C}_i$ are uniformly coprime.

Note that the class of systems given above is exactly equivalent to that in Praly and Trulsson (1986) for decentralized systems where a bounded sequence and an operator with a finite $(2, \mu)$-exponential gain are used to characterize the modelling uncertainty $m_i(t)$.

For each subsystem described in (55) or (56), we use the adaptive algorithm given earlier, for scalar systems to design a local adaptive controller, which is composed of a parameter estimator and a control law synthesis module, obtained by ignoring term $m_i(t)$.

5.1. *Parameter estimator*

$$\hat{\theta}^i(t) = \mathscr{P}\left\{\hat{\theta}^i(t-1) + \frac{\phi_i(t-1)e_i(t)}{1 + \phi_i^T(t-1)\phi_i(t-1)}\right\}, \quad (61)$$

where $\hat{\theta}^i(t)$ represents the estimate of θ_*^i at t and $\mathscr{P}$ represents the projection operator necessary to ensure $\hat{\theta}^i(t) \in \mathscr{C}_i\ \forall t$. $e_i(t)$ is the prediction error defined as

$$e_i(t) = y_i(t) - \phi_i^T(t-1)\hat{\theta}^i(t-1). \quad (62)$$

From above, we see that estimator (61) and (62) only requires the input and output data of the ith subsystem only. This contrasts with some existing partially decentralized indirect adaptive schemes (Hill *et al.*, 1988; Yang and Papavassilopoulos, 1985; Reed and Ioannou, 1988).

Now suppose M_0 is a constant such that $\frac{d}{M_0} \le \delta$, where $0 < \delta < 1$ and will be further restricted. Also let M be a constant such that $M^2 = \bar{k}_1 M_0^2 + \bar{k}_2$ where $\bar{k}_1$ and $\bar{k}_2$ are constants like k_1 and k_2 given in an earlier section.

Lemma 5.1. The estimator (61), (62) applied to system (55), has the following properties. For each i, $(i = 1, \ldots, l)$, assuming $\|\phi_i(t_{0_i} - 1)\| \le M_0$, $\|\phi_i(\tau)\| > M_0$, $\tau = t_{0_i}, \ldots, t-1$ and $\|\phi_i(\tau_1)\| \le M$, $\tau_1 = 0, \ldots, t-1$, where $t \ge t_{0_i} + 1$, then we have

(1)

$$|e_i(t_{0_i})| \le (k_{\theta_i} + a_1)M_0 + a_1, \quad (63)$$

and

$$|\bar{e}_i(t)| = \frac{e_i(t)}{(1 + \|\phi_i(t-1)\|^2)^{1/2}},$$
$$\le k_{\theta i} + a_1, \quad t \ge t_{0_i} + 1, \quad (64)$$

where $k_{\theta i}$ is a constant reflecting the size of $\mathscr{C}_i$ and

$$a_1 = (\bar{k}_1^{1/2} + \bar{k}_2^{1/2})c_\eta \epsilon + \delta,$$

(2)

$$\sum_{\tau = t_{0_i}+1}^{t} |\bar{e}_i(\tau)|^2 \le k_\theta^2 + (a_2 + a_3)(t - t_{0_i}), \quad (65)$$

where

$$k_\theta = \max_{i=1,\ldots,l} \{k_{\theta^i}\}$$

$$a_2 = 2(k_\theta(\bar{k}_1^{1/2} + \bar{k}_2^{1/2}) + 2c_\eta(\bar{k}_1 + \bar{k}_2)\epsilon)c_\eta\epsilon, \tag{66}$$

$$a_3 = 2(2\delta + k_\theta)\delta, \tag{67}$$

(3)

$$\|\hat{\theta}^i(t) - \hat{\theta}^i(t-1)\| \leq |\bar{e}_i(t)| \; \forall t. \tag{68}$$

Proof. The results follow by using similar arguments as in the proof of Lemma 3.1.

5.2. *Adaptive controller design*

The estimates from estimator (61) and (62) are used for the tuning of local controller parameters. The control $u_i(t)$ of the ith subsystem is generated from the equation

$$\hat{L}_i(t-1)u_i(t) = -\hat{P}_i(t-1)(y_i(t) - y_i^*(t)), \tag{69}$$

for $i = 1, \ldots, l$, where y_i^* is the given set-point and

$$\hat{L}_i(t) = 1 + \hat{l}_1^i(t)q^{-1} + \cdots + \hat{l}_{n_i}^i(t)q^{-n_i}, \tag{70}$$

$$\hat{P}_i(t) = \hat{p}_1^i q^{-1} + \cdots + \hat{p}_{n_i}^i(t)q^{-n_i}, \tag{71}$$

$\hat{L}_i$ and $\hat{P}_i$ are obtained by solving the following Diophantine equation

$$\hat{A}_i(t)\hat{L}_i(t) + \hat{B}_i(t)\hat{P}_i(t) = A_i^*, \tag{72}$$

where A_i^* is a given monic strictly (discrete-time) Hurwitz constant polynomial in shift operator q^{-1} of degree $2n_i$. From Assumption 5.2, we see that the coefficients in $\hat{L}_i(t)$ and $\hat{P}_i(t)$ obtained from equation (72) are bounded (Goodwin and Sin, 1984).

Now we examine the robustness of adaptive control algorithm (61), (62), (69)–(72) applied to system (55). The question we need to answer is as follows. Does there exist a class of modelling uncertainty, i.e. a ϵ^* (or ϵ_i^*) such that for each ϵ given in (60) (or ϵ_i in (57)) satisfying $\epsilon \in [0, \epsilon_i^*]$ (or $\epsilon_i \in [0, \epsilon_i^*]$ for $i = 1, \ldots, l$), such that all states in the closed adaptive system are bounded for any bounded initial conditions, bounded set-points and extraneous disturbances. The answer to this question is given in Theorem 5.1.

We study the ith loop of the adaptive system and take modelling uncertainty $m_i(t)$ into account in the system stability analysis. From (62) and (69), we can get

$$\phi_i(t+1) = \bar{A}^i(t)\phi_i(t) + B_1 e_i(t+1) + B_2 r_i(t+1), \tag{73}$$

where

$$\overline{A^i}(t) = \begin{pmatrix} -\hat{a}_1^i(t) & -\hat{a}_2^i & \cdots & -\hat{a}_n^i & \hat{b}_1^i(t) & \cdots & & \hat{b}_{n_i}^i(t) \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ & & & 0 & 0 & \cdots & 0 & 0 \\ & & 1 & 0 & 0 & \cdots & 0 & 0 \\ -\hat{p}_1^i & & \cdots & -\hat{p}_{n_i}^i(t) & -l_1^i(t) & \cdots & & -\hat{l}_{n_i}^i(t) \\ 0 & & \cdots & 0 & 1 & \cdots & 0 & 0 \\ 0 & & \cdots & 0 & & & & 0 \\ 0 & & \cdots & 0 & & & 1 & 0 \end{pmatrix}, \tag{74}$$

$$B_i^T = [1, 0, \ldots, 0], \tag{75}$$

$$B_2^T = [0, \ldots, 0, 1, \ldots, 0], \tag{76}$$

$$r_i(t+1) = \hat{P}_i(t)y_i^*(t+1). \tag{77}$$

By analyzing the above system, we can establish

Theorem 5.1. There exists (or exist) a constant ϵ^* (or constants ϵ_i^*) such that for all $0 \leq \epsilon \leq \epsilon^*$ (or $0 \leq \epsilon_i \leq \epsilon_i^*$, $i = 1, \ldots, l$), decentralized adaptive controllers given in (61), (62), (69)–(72), implemented in plant (55) or (56) under Assumptions 5.1–5.2, give a globally bounded input bounded state stable feedback system, i.e. $\|\phi_i(t)\| \; \forall t$ for all i, and arbitrarily bounded initial conditions, bounded set points and disturbances.

Proof. As in the proof of Theorem 4.1, we divide the time interval Z_+ into two subsequences for each controller

$$Z_1^i = \{t \in Z_+ \mid \|\phi_i(t)\| > M_0\}, \tag{78}$$

$$Z_1^i = \{t \in Z_+ \mid \|\phi_i(t)\| \leq M_0\}. \tag{79}$$

An induction proof is used. Clearly, for any given bounded initial conditions $\phi_i(0)$ and reference $r_i(t)$ defined in (77), there exists a constant M_0 such that $\|\phi_i(0)\| \leq M_0$, $\|r_i(t)\|_\infty \leq M_0$ and $d/M_0 \leq \delta$ for all $i = 1, \ldots, l$ and sufficiently small δ. As $M^2 = \bar{k}_1 M_0^2 + \bar{k}_2 > M_0^2$, then $\|\phi_i(0)\| \leq M$ for all i. Now we can assume that $\|\phi_i(\tau)\| \leq M$ for $\tau = 0, \ldots, t-1$, $t \geq 1$ and all $i = 1, \ldots, l$. Thus the result is proved if we can show that $\|\phi_i(t)\| \le M$ for each i. Actually, from the definition of Z_1^i and Z_2^i, is it only necessary to prove that $\|\phi_i(\tau_1)\| \le M$ for $\tau_1 \in Z_1^i$ for each i. So we can confine ourselves in Z_1^i and choose t_{0_i} such that $t_{0_i}, \ldots, t-1 \in Z_1^i$ and $t_{0_i} - 1 \in Z_2^i$.

The proof now proceeds very closely to that of Theorem 4.1. The assumptions of Lemma 5.1 are satisfied. We establish the same properties for $\bar{A}^i(t)$ in (7.3) as $\bar{A}(t)$ in (30) had.

Comments 5.1.

(1) Having established (60) which is the same as (9), it is clear that the stability proof for each loop basically mimics the earlier single loop case. Interactions are easily handled in (59) and (60) by assuming $\|\phi_i(\tau)\| \le M$ for $\tau = 0, 1, \ldots, t-1$.

(2) For each individual loop, t_{0_i} may be different even though we assume $\|\phi_i(\tau)\| \le M$ for $\tau = 0, 1, \ldots, t-1$ and all l loops.

(3) Theorem 5.1 constitutes the first global stability result for a decentralized indirect adaptive control system. Surprisingly, no complicated estimator modifications are needed. Of course, such modifications may lead to improved performance.

(4) Praly and Trulsson (1986) used an inductive proof in their studies on indirect decentralized adaptive control. However, the stability condition derived depends on the initial state values since the normalizing term in their estimator was not fully exploited when bounding the modelling errors. Referring to Comment 4.1.3, the devices used here enable the stability bound to be made independent of the initial state.

6. CONCLUSIONS

In this paper, we studied a basic adaptive control algorithm consisting of a gradient estimator, subject to parameter projection as the only modification, plus a pole assignment controller. The only *a priori* information required for the implementation of this algorithm is a range that each unknown parameter of the reduced order plant lies in. This is quite reasonable.

We first examined the robustness of this adaptive control algorithm applied to scalar systems with modelling error including unmodelled dynamics and bounded disturbances. It is shown that if the unmodelled dynamics are sufficiently small, then the closed loop adaptive system is bounded input bounded state stable in the sense that all the states in the closed loop are bounded for any bounded initial conditions, set points and external disturbances. We then turned to consider the problem of indirect decentralized adaptive control by applying the algorithm to design local adaptive controllers for isolated reduced order subsystems. It is shown that stable feedback systems can be ensured for those plants that have sufficiently weak interactions between subsystems and small unmodelled dynamics of subsystems. Further, by using the uniform bound device for all the loops in the inductive proof, the problem of handling loop interactions was shown to be essentially the same as that of studying single loop robustness. For both single loop and multiloop systems subject to some constraint on external disturbances and/or set points, we can also obtain small, in the mean, tracking error if appropriate adaptive control schemes (Goodwin and Sin, 1984; Ioannou and Tao, 1987; Middleton and Wang, 1988) are used. It is also clear that those results established in earlier global convergence analysis of ideal situations are preserved if plants to be controlled are completely decoupled and satisfy the "ideal assumptions" (Goodwin and Sin, 1984). In particular, we can achieve perfect tracking in this case.

Our analysis in the paper implies that *a priori* knowledge on system modelling errors are not necessarily required to ensure global stability in the implementation of robust adaptive algorithms involving other modifications (Middleton *et al.*, 1988; Praly, 1983, 1984; Ioannou and Tsakalis, 1986; Ioannou and Kokotovic, 1983; Kreisselmeier and Anderson, 1986). However, such modifications (when the required *a priori* knowledge is available) may and should lead to superior performance.

Our results can be easily extended to include those plants in which the reduced order systems are slowly time varying as in Middleton and Goodwin (1988), Wen and Hill (1990) and Kreisselmeier (1986). But a counterpart version in continuous time systems is not available due to the method of proof via induction used here for discrete-time systems.

REFERENCES

Anderson, B. D. O., R. Bitmead, C. R. Johnson, P. V. Kokotovic, R. L. Kosut, I. Mareels, L. Praly and B. Riedle (1986). *Stability of Adaptive Systems Passivity and Averaging Analysis.* The MIT Press, Cambridge, MA.

Åström, K. J. (1987). Adaptive feedback control. *Proc. IEEE,* **75,** 185–217.

Åström, K. J. and B. Wittenmark (1984). *Computer Controlled Systems—Theory and Design.* Prentice-Hall, Englewood Cliffs, NJ.

Desoer, C. A. and M. Vidyasagar (1975). *Feedback Systems: Input–Output Properties.* Academic Press, New York.

Egardt, B. (1979). *Stability of Adaptive Controllers.* Springer-Verlag, New York.

Gavel, D. T. and D. D. Siljak (1989). Decentralized adaptive control: structural conditions for stability. *IEEE Trans. Aut. Control*, **34**, 413–426.

Goodwin, G. C. and K. S. Sin (1984). *Adaptive Filtering, Prediction and Control.* Prentice-Hall, Englewood Cliffs, NJ.

Goodwin, G. C., P. Ramadge and P. Caines (1980). Discrete time multivariable adaptive control. *IEEE Trans. Aut. Control*, **25**, 449–456.

Hardy, G. H., J. E. Littlewood and G. Polya (1952). *Inequalities.* Cambridge University Press, London.

Hill, D. J., C. Wen and G. C. Goodwin (1988). Stability analysis of decentralized robust adaptive control. *Systems and Control Letters*, **11**, 277–284.

Ioannou, P. (1986). Decentralized adaptive control of interconnected systems. *IEEE Trans. Aut. Control*, **31**, 291–298.

Ioannou, P. and P. V. Kokotovic (1983). *Adaptive Systems with Reduced Models.* Springer-Verlag, New York.

Ioannou, P. and P. V. Kokotovic (1984). Instability analysis and improvement of robustness of adaptive control. *Automatica*, **20**, 583–594.

Ioannou, P. and P. V. Kokotovic (1985). Decentralized adaptive control of interconnected systems with reduced-order models. *Automatica*, **21**, 401–412.

Ioannou, P. and J. Sun (1988). Theory and design of robust direct and indirect adaptive control schemes. *Int. J. Control*, **47**, 775–813.

Ioannou, P. and G. Tao (1987). Dominant richness and improvement of performance of robust adaptive control. *Report 87-04-01*, USC, CA.

Ioannou, P. and K. Tsakalis (1986). A robust direct adaptive controller. *IEEE Trans. Aut. Control*, **31**, 1033–1043.

Kreisselmeier, G. (1986) Adaptive control of a class slowly time-varying plants. *Systems and Control Letters*, **8**, 97–103.

Kreisselmeier, G. and B. D. O. Anderson (1986). Robust model reference adaptive control. *IEEE Trans. Aut. Control*, **31**, 127–134.

Kreisselmeier, G. and K. S. Narendra (1982). Stable model reference adaptive control in the presence of bounded disturbances. *IEEE Trans. Aut. Control*, **27**, 1169–1175.

Morse, A. S. (1980). Global stability of parameter–adaptive control systems. *IEEE Trans. Aut. Control*, **25**, 433–439.

Middleton, R. H. and G. C. Goodwin (1988). Adaptive control of time varying linear systems. *IEEE Trans. Aut. Control*, **33**, 150–155.

Middleton, R. H. and G. C. Goodwin (1989). *Digital Control and Estimation: A Unified Approach.* Prentice-Hall, Englewood Cliffs, NJ.

Middleton, R. H. and Y. Wang (1988). The internal model principle in the adaptive control of time varying linear systems. *Technical Report EE8830*, University of Newcastle.

Middleton, R. H., G. C. Goodwin, D. J. Hill and D. Q. Mayne (1988). Design issues in adaptive control. *IEEE Trans. Aut. Control*, **3**, 50–58.

Narendra, K. S., Y. H. Lin and L. S. Valavani (1980). Stable adaptive controller design, part II: proof of stability. *IEEE Trans. Aut. Control*, **25**, 440–448.

Ortega, R. and T. Yu (1987). Theoretical results on robustness of direct adaptive controllers: A survey. *Proc. IFAC 10th World Congress on Automatic Control*, Munich, Germany, Vol. 10, pp. 1–15.

Praly, L. (1983). Robustness of indirect adaptive control based on pole-placement design. *Proc. IFAC Workshop on Adaptive Control*, San Francisco, CA.

Praly, L. (1984). Robust model reference adaptive controllers—Part 1: stability analysis. *Proc. 23rd IEEE Conf. Decision Control*, NV.

Praly, L. and E. Trulsson (1986). Decentralized indirect adaptive control. *RAIRO APPI*, **20**, 295–315.

Praly, L. (1990). Almost exact modelling assumption in adaptive control. *Int. J. Control*, **51**, 643–668.

Reed, J. S. and P. A. Ioannou (1988). Discrete-time decentralized adaptive control. *Automatica*, **24**, 419–421.

Rohrs, C. E., L. Valavani, M. Athans and G. Stein (1982). Robustness of adaptive control algorithms in the presence of unmodelled dynamics. *Preprints 21st IEEE Conf. Decision Control*, Orlando, FL.

Wen, C. and D. J. Hill (1989). Robustness of adaptive control without deadzones, data normalization or persistence of excitation. *Automatica*, **25**, 943–947.

Wen, C. and D. J. Hill (1990). Decentralized adaptive control of linear time varying systems. *Proc. IFAC 11th World Congress on Auto. Control*, Tallinn, U.S.S.R.

Yang, W. Y. and G. P. Papavassilopoulos (1985). On a class of decentralized discrete-time adaptive control problems. *Proc. 24th Conf. Decision Control*, Ft. Lauderdale.

Ydstie, B. E. (1989). Stability of discrete model reference adaptive control—revisited. *Systems and Control Letters*, **13**, 429–438.