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Automatica 39 (2003) 517–524

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Brief paper

Observer-based stabilization of switching linear systems☆

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Received 19 August 1998; received in revised form 29 August 2002; accepted 30 October 2002

Abstract

In this paper, we present a "deep pole assignment method" to study the observer-based stabilization of switching linear systems where the dynamics of each mode are known a priori but the switching times of modes are arbitrary. The design can be used for both finite and infinite switched linear systems. We emphasize our paper on the case where the switchings of the observer and controller do not coincide with those of the system.

Keywords: Switching linear system; Observer; Controller; Pole assignment

1. Introduction

A switching linear system is one where the system dynamics is linear, but time varying, and switches among a finite number of modes (Fu & Barmish, 1986; Hilhorst, Amerongen, Lohnberg, & Tulleken, 1994; Ezzine & Haddad, 1989). In order to obtain fast and accurate responses of these processes, appropriate controllers and observers should be designed and stored for the different modes. At any moment, the controller and the observer corresponding to the right mode should be used. These kinds of systems is called switching linear systems and it can be used to model synchronously switching linear systems, networks with periodically varying switches and systems subject to failures (Ezzine & Haddad, 1989).

The stability analysis and stabilization of switching systems have been studied by a number of researchers. Michel and Hu (1998) have considered the stability of switched systems via a comparison theory, which is also very useful for the stability analysis of other discontinuous systems. Ezzine and Haddad (1989) studied the problems of controllability, observability and stability of periodic switched linear systems. Wicks, Peleties, and De-carlo (1998) designed a switching law for the stabilization of a class of switched systems, which is an NP-hard problem (Blondel & Tsitsiklis, 1997). Fu and Barmish (1986) have studied the stabilization of the finite switching linear systems successfully. However, the method in Fu and Barmish (1986) cannot be used to consider the stabilization of infinite switching linear systems. It should also be noted that some basic problems have been outlined in Liberzon and Morse (1999). However, the observer-based stabilization of switching linear systems with infinite switching times has not been considered yet, especially in the case where the switchings of the observer and controller do not coincide exactly with those of the system.

In this paper, we overcome the potential problem by introducing a deep pole assignment method. The pole assignment method is used to develop an observer and a controller for each "frozen" system. These observers and controllers form a switching observer and a switching controller for the whole switching control systems where the number of switchings involved can be infinite or finite, and these observers and controllers are stored. Because the "frozen" system cannot be known at any moment, we need to identify the switching instances of the system and the next "frozen" system. This can be done by checking the observer error. Once the next "frozen" system is known, the controller and the observer should be switched to the ones corresponding to the next "frozen" system. It is well known that the poles of a controllable (or observable) "frozen" system can be assigned to any

☆This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Hassan Khalil under the direction of Editor Tamer Basar.

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PII: S0005-1098(02)00267-4

given values (Chen, 1984). If the real parts of poles are negative enough, then the Lyapunov function $V(X(t)) = \|X(t)\|^2$ will decrease along each "frozen" system with any given decay rate even in the case where there are overlaps between the switchings of the system and the switchings of the controllers and the observers. This ensures that $V(X(t))$ decreases along the whole system $\dot{X}(t) = A(m(t))X(t)$ with any given decay rate. Thus the system $\dot{X}(t) = A(m(t))X(t)$ is stable in the sense of Lyapunov. We require that the interval between any two successive switchings has a lower bound ΔT as in Narendra and Balakrishnan (1993). However, our ΔT is only required to be known and it can be arbitrarily small. The controller uses the estimate of the state by the observer for feedback rather than the state of the system. The controller stabilizes the system in the sense of Lyapunov. This is very important because with Lyapunov stability, we can get a handle on the types of "overshoot" behavior. Further, the switchings of the controller and the observer do not need to coincide exactly with the switchings of the system. The result of this paper shows that for any desired decay rate, the closed-loop switching system would yield a stable system with the required decay rate.

The rest of the paper is organized as follows. In the following section, problem formulation is given. The main result is derived in Section 3. In Section 4, a numerical example is given to illustrate the application of the main results. Concluding remarks are given in Section 5.

2. Problem formulation

In this paper, we shall design an observer-based controller for the stabilization of the following switching single input single output linear system:

$$\begin{aligned} \dot{X}(t) &= A(m(t))X(t) + b(m(t))u(t), \\ y(t) &= c(m(t))X(t), \end{aligned} \tag{1}$$

where $\dot{X}(t)$ is the system state vector of dimension r, $u(t)$ is the control input, $y(t)$ is the output, and $m(t)$ is the 'mode index' which is a piecewise constant taking values in the finite index set $\bar{M} = \{1, 2, \ldots, n\}$. We shall refer the ith mode of the system as continuous variable dynamic system i (CVDS i). The switching laws are defined by one of the following two methods:

(I) The mode will automatically switch from mode i to mode j, if the duration time of mode i is $\Delta\tau_i$ (Ezzine & Haddad, 1989).

(II) The mode will switch from mode i to mode j, if the state of mode i is in a given switching conditional set $S_{ij}(X(t))$, which is of the following form (Pettersson & Lennartson, 1996):

$$S_{ij}(X(t)) = \{X(t) | h_{ij}(X(t)) = 0\}. \tag{2}$$

We let t_k^s denote the kth switching instance of the system. Further, we make the following assumption about (1).

Assumption 1.

$$(A(i), c(i)) \quad \text{is observable,} \tag{3}$$

$$(A(i), b(i)) \quad \text{is controllable,} \tag{4}$$

$$\Delta T = \inf_k \{t_{k+1}^s - t_k^s\} > 0, \tag{5}$$

where ΔT is known but can be arbitrarily small.

We propose the following r-dimensional observer for mode i:

$$\begin{aligned} \dot{X}^*(t) &= A(i)X^*(t) + b(i)u(t) + \bar{L}(i)(y(t) - y^*(t)), \\ X^*(t_0) &= X_0^*, \\ y^*(t) &= c(i)X^*(t). \end{aligned} \tag{6}$$

Based on the estimate $X^*(t)$, we develop a controller of the form

$$u(t) = -K(i)X^*(t). \tag{7}$$

The requirement of the design is that the closed-loop switching linear system (1) satisfies a prescribed decay rate. Thus, the observer-based stabilization problem can be formulated as follows:

Consider the switching system (1) with switching law (I) or (II), and satisfies Assumption 1. Given any decay rate $\lambda < 0$, design an r-dimensional observer of the form (6), a feedback controller of the form (7) and the switching laws of the controller and the observer, such that the closed-loop system satisfies

$$\lim_{t\to\infty} e^{-\lambda t}\|X(t)\| = 0.$$

The solution to this problem is composed of two steps:

Step 1: Design an observer and a controller for each subsystem.

Step 2: Define a switching law for these observers and controllers. Generally, the controller corresponding to the active subsystem should be used. However, we cannot know the initial subsystem and the subsequent subsystems in advance. Thus, we need to impose some small delay on the switchings of the controllers so that we can identify the initial subsystems and the subsequent active subsystems. Then, the controller corresponding to the active subsystem is switched into action.

Note that there will be overshoot in the interval before the right observer and controller are activated. Thus, we need to present a method to constrain the overshoot and to ensure sufficient decay of the overshoot during the interval when the observer and controller corresponding to the active subsystem are used. This is possible because the poles of each observable and controllable subsystem can be assigned arbitrarily. The whole process will be shown in detail in the next section.

3. Main results

In this section, we shall first present some preliminaries to illustrate the process of pole assignment for the observer and the controller for each subsystem.

Lemma 1 (Wilkinson, 1965). *For any given degenerate rate λ and $\lambda_1,\ldots,\lambda_r$ with $\lambda_i \neq \lambda_j$ when $i \neq j$, let*

$$Q(\lambda,\lambda_1,\ldots,\lambda_r)=\begin{bmatrix} 1 & \cdots & 1 \\ \lambda+\lambda_1 & \cdots & \lambda+\lambda_r \\ (\lambda+\lambda_1)^2 & \cdots & (\lambda+\lambda_r)^2 \\ \vdots & \ddots & \vdots \\ (\lambda+\lambda_1)^{r-1} & \cdots & (\lambda+\lambda_r)^{r-1} \end{bmatrix},$$

$$f(s,r)=(s-\lambda-\lambda_1)\cdots(s-\lambda-\lambda_r) = s^r+d_{r-1}s^{r-1}+\cdots+d_1s+d_0,$$

$$E(\lambda,\lambda_1,\ldots,\lambda_r)=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{r-1} \end{bmatrix},$$

$$D(\lambda,\lambda_1,\ldots,\lambda_r)=\begin{bmatrix} \lambda+\lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda+\lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda+\lambda_r \end{bmatrix}.$$

Then

$$Q^{-1}(\lambda,\lambda_1,\ldots,\lambda_r)E(\lambda,\lambda_1,\ldots,\lambda_r)Q(\lambda,\lambda_1,\ldots,\lambda_r) = D(\lambda,\lambda_1,\ldots,\lambda_r). \quad (8)$$

Lemma 2. *Let*

$$g_i(s)=|sI-A(i)|=s^r+a_{r-1}(i)s^{r-1}+\cdots+a_1(i)s+a_0(i)$$

$$P(i)=\begin{bmatrix} A^{r-1}(i)b(i) \\ \vdots \\ A(i)b(i) \\ b(i) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{r-1}(i) & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_1(i) & a_2(i) & a_3(i) & \cdots & a_{r-1}(i) & 1 \end{bmatrix},$$

$$\mu(i)=\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_{r-1}(i) & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ a_1(i) & a_2(i) & a_3(i) & \cdots & a_{r-1}(i) & 1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} c(i)A^{r-1}(i) \\ \vdots \\ c(i)A(i) \\ c(i) \end{bmatrix}.$$

If $(A(i),b(i))$ is controllable and $(A(i),c(i))$ is observable, then there exists $K(i)$ and $\bar{L}(i)$ given by

$$K(i)=[d_0^{\mathrm{c}}-a_0(i),\ldots,d_{r-1}^{\mathrm{c}}-a_{r-1}(i)]P^{-1}(i), \quad (9)$$

$$\bar{L}(i)=\mu^{-1}(i)[d_0^{\mathrm{o}}-a_0(i),\ldots,d_{r-1}^{\mathrm{o}}-a_{r-1}(i)] \quad (10)$$

such that

$$\tilde{A}(i)=P(i)Q(\lambda,\lambda_1^{\mathrm{c}},\ldots,\lambda_r^{\mathrm{c}})D(\lambda,\lambda_1^{\mathrm{c}},\ldots,\lambda_r^{\mathrm{c}}) Q^{-1}(\lambda,\lambda_1^{\mathrm{c}},\ldots,\lambda_r^{\mathrm{c}})P^{-1}(i), \quad (11)$$

$$\hat{A}(i)=\mu^{-1}(i)(Q^{\mathrm{T}})^{-1}(\lambda,\lambda_1^{\mathrm{o}},\ldots,\lambda_r^{\mathrm{o}})D(\lambda,\lambda_1^{\mathrm{o}},\ldots,\lambda_r^{\mathrm{o}}) Q^{\mathrm{T}}(\lambda,\lambda_1^{\mathrm{o}},\ldots,\lambda_r^{\mathrm{o}})\mu(i) \quad (12)$$

where

$$\tilde{A}(i)=A(i)-b(i)K(i); \quad \hat{A}(i)=A(i)-\bar{L}(i)c(i).$$

Proof. It is straightforward by using Lemma 1 and the pole assignment method (Chen, 1984). □

Lemma 3. *Suppose that $(c(i),A(i))$ is observable, then for any $L\in R^r$, $(c(i),A(i)-Lc(i))$ is observable.*

For any desired decay rate λ and observer state $X^*(t)$, we denote

$$e(t)=X(t)-X^*(t),$$

$$\Lambda(\lambda_1,\ldots,\lambda_r,t)=\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_r t} \end{bmatrix},$$

$$\Delta_1(i,\lambda,\lambda_1^{\mathrm{c}},\ldots,\lambda_r^{\mathrm{c}},t), = e^{\tilde{A}(i)t}=P(i)Q(i\lambda,\lambda_1^{\mathrm{c}},\ldots,\lambda_r^{\mathrm{c}}t)\Lambda(i,\lambda,\lambda_1^{\mathrm{c}},\ldots,\lambda_r^{\mathrm{c}},t), \quad (13)$$
$$Q^{-1}(i\lambda,\lambda_1^{\mathrm{c}},\ldots,\lambda_r^{\mathrm{c}})P^{-1}(i)$$

$$\Delta_2(i,\lambda,\lambda_1^{\mathrm{o}},\ldots,\lambda_r^{\mathrm{o}},t)=e^{\hat{A}(i)t} =\mu^{-1}(i)(Q^{\mathrm{T}})^{-1}(i\lambda,\lambda_1^{\mathrm{o}},\ldots,\lambda_r^{\mathrm{o}}t)\Lambda(i,\lambda,\lambda_1^{\mathrm{o}},\ldots,\lambda_r^{\mathrm{o}},t)$$
$$Q^{\mathrm{T}}(i,\lambda,\lambda_1^{\mathrm{o}},\ldots,\lambda_r^{\mathrm{o}}t)\mu(i) \quad (14)$$

$$\Delta_3(i,\lambda,\lambda_1^{\rm c},\ldots,\lambda_r^{\rm c},\lambda_1^{\rm o},\ldots,\lambda_r^{\rm o},t_2-t_1)$$
$$=\int_{t_1}^{t_2} e^{\tilde{A}(i)(t_2-t)}b(i)$$
$$[d_0^{\rm o}-a_0(i),\ldots,d_{r-1}^{\rm o}-a_{r-1}(i)]P^{-1}(i)e^{\hat{A}(i)(t-t_1)}\,{\rm d}t. \qquad (15)$$

Then, we have

$$\begin{bmatrix}\dot{X}(t)\\ \dot{e}(t)\end{bmatrix}=\begin{bmatrix}A(i)-b(i)K(i) & -b(i)K(i)\\ 0 & A(i)-\bar{L}(i)c(i)\end{bmatrix}\begin{bmatrix}X(t)\\ e(t)\end{bmatrix},$$
$$\tilde{y}(t)=y(t)-y^*(t)=[0 \quad c(i)]\begin{bmatrix}X(t)\\ e(t)\end{bmatrix}. \qquad (16)$$

Suppose that the mode of the system is i within the interval $[t_j^{\rm s},t_{j+1}^{\rm s}]$, using Lemmas 1 and 2, we have

$$e(t_{j+1}^{\rm s})=\Delta_2(i,\lambda,\lambda_1^{\rm o},\ldots,\lambda_r^{\rm o},t_{j+1}^{\rm s}-t_j^{\rm s})e^{(t_{j+1}^{\rm s}-t_j^{\rm s})\lambda}e(t_j^{\rm s})$$

$$X(t_{j+1}^{\rm s})=\Delta_1(i,\lambda,\lambda_1^{\rm c},\ldots,\lambda_r^{\rm c},t_{j+1}^{\rm s}-t_j^{\rm s})e^{(t_{j+1}^{\rm s}-t_j^{\rm s})\lambda}X(t_j^{\rm s})$$
$$+\Delta_3(i,\lambda,\lambda_1^{\rm c},\ldots,\lambda_r^{\rm c},\lambda_1^{\rm o},\ldots,\lambda_r^{\rm o},t_{j+1}^{\rm s}-t_j^{\rm s})$$
$$e^{(t_{j+1}^{\rm s}-t_j^{\rm s})\lambda}e(t_j^{\rm s}).$$

Thus, we have

$$\|e(t_{j+1}^{\rm s})\|\leqslant\lambda_{\max}^{1/2}(\Delta_2^{\rm T}\Delta_2)e^{\lambda(t_{j+1}^{\rm s}-t_j^{\rm s})}\|e(t_j^{\rm s})\|, \qquad (17)$$

$$\|X(t_{j+1}^{\rm s})\|\leqslant\lambda_{\max}^{1/2}(\Delta_1^{\rm T}\Delta_1)e^{\lambda(t_{j+1}^{\rm s}-t_j^{\rm s})}\|X(t_j^{\rm s})\|$$
$$+\lambda_{\max}^{1/2}(\Delta_3^{\rm T}\Delta_3)e^{\lambda(t_{j+1}^{\rm s}-t_j^{\rm s})}\|e(t_j^{\rm s})\|. \qquad (18)$$

From the above derivations, we know that the key problem is how to choose $\lambda_j^{\rm c}$ $(j=1,2,\ldots,r)$ and $\lambda_j^{\rm o}$ $(j=1,2,\ldots,r)$ such that for any given decay rate $\lambda<0$, all $t\geqslant\Delta T$ and all $i=1,2,\ldots,n$, we have

$$\lambda_{\max}(\Delta_1^{\rm T}(i,\lambda,\lambda_1^{\rm c},\ldots,\lambda_r^{\rm c},t)\Delta_1(i,\lambda,\lambda_1^{\rm c},\ldots,\lambda_r^{\rm c},t))<\frac{1}{4\beta_4}$$
$$\lambda_{\max}(\Delta_2^{\rm T}(i,\lambda,\lambda_1^{\rm o},\ldots,\lambda_r^{\rm o},t)\Delta_2(i,\lambda,\lambda_1^{\rm o},\ldots,\lambda_r^{\rm o},t))<\frac{1}{4\beta_4}$$
$$\lambda_{\max}(\Delta_3^{\rm T}(i,\lambda,\lambda_1^{\rm c},\ldots,\lambda_r^{\rm c},\lambda_1^{\rm o},\ldots,\lambda_r^{\rm o},t)$$
$$\Delta_3(i,\lambda,\lambda_1^{\rm c},\ldots,\lambda_r^{\rm c},\lambda_1^{\rm o},\ldots,\lambda_r^{\rm o},t))<\frac{1}{4\beta_4},$$

where $\lambda_{\max}(A)$ denotes the maximum eigenvalue of matrix A, and $\beta_4\geqslant 1$ can be chosen to give some flexibility in determining the switching time of the observer and the controller.

This problem will be solved by using a "deep pole assignment method" through the following three lemmas. The proofs are omitted due to the space limitation.

Lemma 4. *Given any desired decay rate* $\lambda<0$ *and* $t>0$, *there exist* $\tilde{\lambda}_j^{\rm c}(i,\lambda,t)<0$ $(j=1,2,\ldots,r)$ *and* $\tilde{\lambda}_j^{\rm o}(i,\lambda,t)<0$ $(j=1,2,\ldots,r)$, *such that*

$$\lambda_{\max}(\Delta_1^{\rm T}(i,\lambda,\tilde{\lambda}_1^{\rm c},\ldots,\tilde{\lambda}_r^{\rm c},t)$$
$$\Delta_1(i,\lambda,\tilde{\lambda}_1^{\rm c},\ldots,\tilde{\lambda}_r^{\rm c},t))<\frac{1}{4\beta_4}$$
$$\lambda_{\max}(\Delta_2^{\rm T}(i,\lambda,\tilde{\lambda}_1^{\rm o},\ldots,\tilde{\lambda}_r^{\rm o},t)\Delta_2(i,\lambda,\tilde{\lambda}_1^{\rm o},\ldots,\tilde{\lambda}_r^{\rm o},t))<\frac{1}{4\beta_4}$$
$$\lambda_{\max}(\Delta_3^{\rm T}(i,\lambda,\tilde{\lambda}_1^{\rm c},\ldots,\tilde{\lambda}_r^{\rm c},\tilde{\lambda}_1^{\rm o},\ldots,\tilde{\lambda}_r^{\rm o},t)$$
$$\Delta_3(i,\lambda,\tilde{\lambda}_1^{\rm c},\ldots,\tilde{\lambda}_r^{\rm c},\tilde{\lambda}_1^{\rm o},\ldots,\tilde{\lambda}_r^{\rm o},t))<\frac{1}{4\beta_4}.$$

Lemma 5. *Given any* $\lambda<0$, *there exist* $\hat{\lambda}_j^{\rm c}(i,\lambda)<0$ $(j=1,2,\ldots,r)$ *and* $\hat{\lambda}_j^{\rm o}(i,\lambda)$ $(j=1,2,\ldots,r)$, *such that*

$$\lambda_{\max}(\Delta_1^{\rm T}(i,\lambda,\hat{\lambda}_1^{\rm c},\ldots,\hat{\lambda}_r^{\rm c},t)\Delta_1(i,\lambda,\hat{\lambda}_1^{\rm c},\ldots,\hat{\lambda}_r^{\rm c},t))<\frac{1}{4\beta_4}$$
$$\lambda_{\max}(\Delta_2^{\rm T}(i,\lambda,\hat{\lambda}_1^{\rm o},\ldots,\hat{\lambda}_r^{\rm o},t)\Delta_2(i,\lambda,\hat{\lambda}_1^{\rm o},\ldots,\hat{\lambda}_r^{\rm o},t))<\frac{1}{4\beta_4}$$
$$\lambda_{\max}(\Delta_3^{\rm T}(i,\lambda,\hat{\lambda}_1^{\rm c},\ldots,\hat{\lambda}_r^{\rm c},\hat{\lambda}_1^{\rm o},\ldots,\hat{\lambda}_r^{\rm o},t)$$
$$\Delta_3(i,\lambda,\hat{\lambda}_1^{\rm c},\ldots,\hat{\lambda}_r^{\rm c},\hat{\lambda}_1^{\rm o},\ldots,\hat{\lambda}_r^{\rm o},t))<\frac{1}{4\beta_4}$$

for $t\in[\Delta T,\Delta\Gamma_i]$; $i\in\{1,2,\ldots,n\}$, *where* $\Delta\Gamma_i$ *is a finite positive number that bounds the switching interval of CVDS i.*

Lemma 6. *Given any desired decay rate* $\lambda<0$, *there exist* $\tilde{\lambda}_j^{\rm c}<0$ $(j=1,2,\ldots,r)$ *and* $\tilde{\lambda}_j^{\rm o}<0$ $(j=1,2,\ldots,r)$, *such that*

$$\lambda_{\max}(\Delta_1^{\rm T}(i,\lambda,\tilde{\lambda}_1^{\rm c},\ldots,\tilde{\lambda}_r^{\rm c},t)\Delta_1(i,\lambda,\tilde{\lambda}_1^{\rm c},\ldots,\tilde{\lambda}_r^{\rm c},t))<\frac{1}{4\beta_4}$$
$$\lambda_{\max}(\Delta_2^{\rm T}(i,\lambda,\tilde{\lambda}_1^{\rm o},\ldots,\tilde{\lambda}_r^{\rm o},t)\Delta_2(i,\lambda,\tilde{\lambda}_1^{\rm o},\ldots,\tilde{\lambda}_r^{\rm o},t))<\frac{1}{4\beta_4}$$
$$\lambda_{\max}(\Delta_3^{\rm T}(i,\lambda,\tilde{\lambda}_1^{\rm c},\ldots,\tilde{\lambda}_r^{\rm c},\tilde{\lambda}_1^{\rm o},\ldots,\tilde{\lambda}_r^{\rm o},t)$$
$$\Delta_3(i,\lambda,\tilde{\lambda}_1^{\rm c},\ldots,\tilde{\lambda}_r^{\rm c},\tilde{\lambda}_1^{\rm o},\ldots,\tilde{\lambda}_r^{\rm o},t))<\frac{1}{4\beta_4}$$

for $t\geqslant\Delta T$, $i\in\{1,2,\ldots,n\}$.

Based on the above five lemmas, we can design the subcontroller and subobserver for each subsystem. The initial states for the first subobserver are arbitrary while the initial states for the subsequent subobserver are the same as the end states of the previous subobserver. Meanwhile, we also need to identify the active subsystems, the switching instances of the system such that the appropriate switchings of the controller and the observer can be well defined.

To identify the mode of the system, we note that for any given $A(\hat{i}), \bar{L}(i), K(i), b(i), c(\hat{i})$, there exists a constant $t_i^* \leqslant \beta_1 \Delta T/3$ where $\beta_1 < 1$, such that

$$\frac{\|e^{(A(\hat{i})-\bar{L}(i)c(\hat{i}))t_i^*}\|^2}{e^{2\lambda t_i^*}} \leqslant 2, \quad 1 \leqslant i \neq \hat{i} \leqslant n, \tag{19}$$

$$\frac{\|e^{(A(\hat{i})-b(\hat{i})K(i))t_i^*}\|^2}{e^{2\lambda t_i^*}} \leqslant 2, \quad 1 \leqslant i \neq \hat{i} \leqslant n, \tag{20}$$

$$\frac{\|\int_0^{t_i^*} e^{(A(\hat{i})-b(\hat{i})K(i))(t_i^*-t)} b(\hat{i})K(i)e^{(A(\hat{i})-\bar{L}(i)c(\hat{i}))t}\,\mathrm{d}t\|^2}{e^{2\lambda t_i^*}} \leqslant 2,$$
$$1 \leqslant i \neq \hat{i} \leqslant n. \tag{21}$$

Remark 1. Note that (19)–(21) hold when $t_i^* = 0$, the right sides of (19)–(21) are continuous functions of t_i^*. It follows that there exists a $t_i^* \leqslant \beta_1 \Delta T/3$ such that (19)–(21) hold.

Suppose that the mode of the system is l_j within $[t_{j-1}^{\mathrm{s}}, t_j^{\mathrm{s}}]$. Since $[c(l_j), A(l_j) - \bar{L}(k)c(l_j)]$ is observable for any mode $k \in \bar{M}$, $e(t)$ and $X(t)$ can be obtained by using the value of $\tilde{y}(\tau)$ in (16) when τ is in the interval $[t_j^{\mathrm{s}}, t_j^{\mathrm{s}} + t_{l_j}^*]$, and it is of the following form:

$$e(t) = \left[\int_t^{t+t_{l_j}^*} e^{(A(l_j)-\bar{L}(k)c(l_j))^{\mathrm{T}}(t_{l_j}^*+t-\tau)} c^{\mathrm{T}}(l_j) \times c(l_j)e^{(A(l_j)-\bar{L}(k)c(l_j))(t_{l_j}^*+t-\tau)}\,\mathrm{d}\tau\right]^{-1} \times \int_t^{t+t_{l_j}^*} e^{(A(l_j)-\bar{L}(k)c(l_j))^{\mathrm{T}}(t+t_{l_j}^*-\tau)} c^{\mathrm{T}}(l_j)\tilde{y}(\tau)\,\mathrm{d}\tau$$

$$X(t) = e(t) + X^*(t).$$

Then, we can compute $\hat{y}(\tau) = C(k)X(\tau)$, $\tau \in [t_j^{\mathrm{s}}, t_j^{\mathrm{s}} + t_i^*]$ for a certain $k \neq l_j$. If $\hat{y}(\tau) = y(\tau)$ holds for all $\tau \in [t_j^{\mathrm{s}}, t_j^{\mathrm{s}} + t_{l_j}^*]$, then the mode of the system is k. Otherwise, we need to check other mode. In this way, we can identify the active mode of the system.

Next, we need to identify the switching instance of the system as follows:

$$t_0^{\mathrm{s}} = t_0,$$

$$t_j^{\mathrm{s}} = \sup_t \{t > t_{j-1}^{\mathrm{c}} \mid \|X^*(t - t_{l_j}^*)\|^2 \leqslant \frac{1}{2^j} e^{2\lambda(t-t_{l_j}^*-t_0)}\|X^*(t_0)\|^2\}.$$

Finally, we need to define the switching instances of the observers and the controllers. Suppose that t_j^{o} and t_j^{c} are respectively the jth switching instance of the observer and the controller, we choose $t_j^{\mathrm{c}} = t_j^{\mathrm{o}}$ and it is defined according to the following five cases:

Case 1: $\Omega_1(j) \neq \emptyset$, $\Omega_2(j) \neq \emptyset$ and $\Omega_1(j) \cap \Omega_2(j) \neq \emptyset$.

$$t_j^{\mathrm{o}} \in \Omega_1(j) \cap \Omega_2(j). \tag{22}$$

Case 2: $\Omega_1(j) \neq \emptyset$, $\Omega_2(j) \neq \emptyset$ and $\Omega_1(j) \cap \Omega_2(j) = \emptyset$.

$$t_j^{\mathrm{o}} = \inf_{t \in \Omega_1(j) \cup \Omega_2(j)} \{t\}. \tag{23}$$

Case 3: $\Omega_1(j) \neq \emptyset$ and $\Omega_2(j) = \emptyset$.

$$t_j^{\mathrm{o}} \in \Omega_1(j). \tag{24}$$

Case 4: $\Omega_1(j) = \emptyset$ and $\Omega_2 \neq \emptyset$.

$$t_j^{\mathrm{o}} \in \Omega_2(j). \tag{25}$$

Case 5: $\Omega_1(j) = \emptyset$ and $\Omega_2(j) = \emptyset$.

$$t_j^{\mathrm{o}} = t_j^{\mathrm{s}} + \frac{2\beta_1 \Delta T}{3}, \tag{26}$$

where

$$\Omega_1(j) = \left\{t \mid t \geqslant t_{j-1}^{\mathrm{o}} + \left(1 - \frac{2\beta_1}{3}\right)\Delta T, |t - t_j^{\mathrm{s}}| \leqslant \frac{2\beta_1 \Delta T}{3}, \frac{1}{2^j} e^{2\lambda(t-t_{l_j}^*-t_0)}\|e(t_0)\|^2 \leqslant \|e(t - t_{l_j}^*)\|^2 \leqslant \frac{\beta_4}{2^j} e^{2\lambda(t-t_j^*-t_0)}\|e(t_0)\|^2\right\}, \tag{27}$$

$$\Omega_2(j) = \left\{t \mid t \geqslant t_{j-1}^{\mathrm{o}} + \left(1 - \frac{2\beta_1}{3}\right)\Delta T, |t - t_j^{\mathrm{s}}| \leqslant \frac{2\beta_1 \Delta T}{3}, \frac{1}{2^j} e^{2\lambda(t-t_{l_j}^*-t_0)}(\|X(t_0)\|^2 + (2j-1)\|e(t_0)\|^2) \leqslant \|X(t - t_j^*)\|^2 \leqslant \frac{\beta_4}{2^j} e^{2\lambda(t-t_{l_j}^*-t_0)}((2j-1)\|e(t_0)\|^2 + \|X(t_0)\|^2)\right\} \tag{28}$$

with $t_0^{\mathrm{o}} = t_0^{\mathrm{c}} = t_0 + \beta_1 \Delta T/3$, and $e(t_0)$ and $X(t_0)$ are obtained by the following equations:

$$e(t_0) = \left[\int_{t_0}^{t_0+(\beta_1\Delta T/3)} e^{(A(l_1)-\bar{L}(l_1)c(l_1))^{\mathrm{T}}(t_0+(\beta_1\Delta T/3)-t)} c^{\mathrm{T}}(l_1) \times c(l_1)e^{(A(l_1)-\bar{L}(l_1)c(l_1))(t_0+(\beta_1\Delta T/3)-t)}\,\mathrm{d}t\right]^{-1} \times \int_{t_0}^{t_0+(\beta_1\Delta T/3)} e^{(A(l_1)-\bar{L}(l_1)c(l_1))^{\mathrm{T}}(t_0+(\beta_1\Delta T/3)-t)} c^{\mathrm{T}}(l_1)y(t)\,\mathrm{d}t$$

$$X(t_0) = e(t_0) + X^*(t_0),$$

where l_1 is the initial mode of the system.

Remark 2. In the sets (27) and (28), the key idea is to restrict $\|X(t_j^{\mathrm{o}} - t_{l_j}^*)\|^2 \leqslant (\beta_4/2^j)e^{2\lambda(t_j^{\mathrm{o}}-t_{l_j}^*-t_0)}((2j-1)\|e(t_0)\|^2 + \|X(t_0)\|^2)$ and $\|e(t_j^{\mathrm{o}} - t_j^*)\|^2 \leqslant (\beta_4/2^j)e^{2\lambda(t_j^{\mathrm{o}}-t_j^*-t_0)}\|e(t_0)\|^2\}$ when $t_j^{\mathrm{o}} - t_j^* > t_j^{\mathrm{s}}$.

With the above switchings for the controller and observer, we have the following result.

Theorem 1 (Observer-based switching control theorem). *Consider the switching system* (1) *with switching laws* (I) *or* (II). *Suppose that the system satisfies Assumption* 1, *then for any given decay rate* $\lambda < 0$, *there exist an r dimensional observer of the form* (6) *and a feedback controller of the form* (7) *for mode i with the switching instances of the controller and the observer defined in* (22), *such that*

$$\lim_{t\to\infty} e^{-\lambda t}\|X(t)\| = 0.$$

Proof. For each subsystem i and any given decay rate λ. Using Lemmas 4–6, we can select $\lambda_1^c,\ldots,\lambda_r^c,\ \lambda_1^o,\ldots,\lambda_r^o$. Based on these eigenvalues, an r dimensional observer of the form (6) and a feedback controller of the form (7) can be designed by using Lemmas 1–2. The design of the switchings of the controller and the observer is given in (22).

Suppose that the switching instances of the system are $t_1^s, t_2^s,\ldots,t_g^s$ and the switching instances of the controller are $t_1^c, t_2^c,\ldots,t_g^c$, and $t > t_g^c$. Note that for finite switching systems, g is finite and $g\to\infty$ for infinite switching systems. Further, we suppose that the mode of the system is l_i within the interval $[t_{j-1}^s, t_j^s]$.

It can be easily proved by the induction method that

(a) For any j, we have

$$t_j^c - t_j^* = t_j^o - t_j^* \geqslant t_j^s. \tag{29}$$

(b) For any j,

$$\|e(t_j^o - t_j^*)\|^2 \leqslant \frac{\beta_4}{2^j}\, e^{2\lambda(t_j^o - t_j^* - t_0)}\|e(t_0)\|^2 \tag{30}$$

holds for both $\Omega_1(j) = \emptyset$ and $\Omega_1(j) \neq \emptyset$.

(c) For any j,

$$\|X(t_j^o - t_j^*)\|^2 \leqslant \frac{\beta_4}{2^j}\, e^{2\lambda(t_j^o - t_j^* - t_0)}((2j-1)\|e(t_0)\|^2 + \|X(t_0)\|^2) \tag{31}$$

holds for both $\Omega_2(j) = \emptyset$ and $\Omega_2(j) \neq \emptyset$.

We shall now prove that for any $t_{g+1}^s > t > t_g^o$, we have

$$\|e(t)\|^2 \leqslant M_1 e^{2\lambda(t-t_g^o)}\|e(t_g^o)\|^2,$$

$$\|X(t)\|^2 \leqslant M_1 e^{2\lambda(t-t_g^o)}(\|e(t_g^o)\|^2 + \|X(t_g^o)\|^2).$$

Using Lemmas 4–6, we have

$$\begin{aligned}
\|e(t)\|^2 &= e^{\mathrm{T}}(t_g^o)e^{\hat{A}^{\mathrm{T}}(l_g)(t-t_g^o)}e^{\hat{A}(l_g)(t-t_g^o)}e(t_g^o)\\
&= e^{2\lambda(t-t_g^o)}e^{\mathrm{T}}(t_g^o)\Delta_2^{\mathrm{T}}(l_g,\lambda,\lambda_1^o,\ldots,\lambda_r^o,t-t_g^o)\\
&\quad\times\Delta_2(l_g,\lambda,\lambda_1^o,\ldots,\lambda_r^o,t-t_g^o)e(t_g^o)\\
&\leqslant \lambda_{\max}(\Delta_2^{\mathrm{T}}(l_g,\lambda,\lambda_1^o,\ldots,\lambda_r^o,t-t_g^o)\\
&\quad\times\Delta_2(l_g,\lambda,\lambda_1^o,\ldots,\lambda_r^o,t-t_g^o))e^{2\lambda(t-t_g^o)}\|e(t_g^o)\|^2\\
&\leqslant M_1 e^{2\lambda(t-t_g^o)}\|e(t_g^o)\|^2.
\end{aligned}\tag{32}$$

Similarly,

$$\|X(t)\|^2 \leqslant M_1 e^{2\lambda(t-t_g^o)}(\|X(t_g^o)\|^2 + \|e(t_g^o)\|^2),$$

where if the bound of the interval is known, then

$$M_1 = \max_{1\leqslant j\leqslant 3}\ \max_{1\leqslant i\leqslant n}\ \max_{0\leqslant t\leqslant \Delta t_i}\{\zeta_j(i,t)\},$$

else

$$M_1 = \max_{1\leqslant j\leqslant 3}\ \max_{1\leqslant i\leqslant n}\ \max_{0\leqslant t\leqslant \infty}\{\zeta_j(i,t)\},$$

where

$$\zeta_1(i,t) = \lambda_{\max}(\Delta_1^{\mathrm{T}}(i,\lambda_1^c,\ldots,\lambda_r^c,t)\Delta_1(i,\lambda_1^c,\ldots,\lambda_r^c,t)),$$

$$\zeta_2(i,t) = \lambda_{\max}(\Delta_2^{\mathrm{T}}(i,\lambda_1^o,\ldots,\lambda_r^o,t)\Delta_2(i,\lambda_1^o,\ldots,\lambda_r^o,t)),$$

$$\begin{aligned}\zeta_3(i,t) = \lambda_{\max}(&\Delta_3^{\mathrm{T}}(i,\lambda_1^c,\ldots,\lambda_r^c,\lambda_1^o,\ldots,\lambda_r^o,t)\\ &\Delta_3(i,\lambda_1^c,\ldots,\lambda_r^c,\lambda_1^o,\ldots,\lambda_r^o,t)).\end{aligned}$$

Finally, we shall prove that the required result holds. Let

$$\begin{aligned}M_2 = \max_{0\leqslant t\leqslant \beta_1\Delta T}\ \max_{i\neq j}\max\Big\{&\|e^{(A(i)+b(i)K(j))t}\|^2 e^{-2\lambda\beta_1\Delta T},\\
&\|e^{(A(i)+\tilde{L}(j)c(i))t}\|^2 e^{-2\lambda\beta_1\Delta T}\\
&\times\left\|\int_0^t e^{(A(i)+b(i)K(j))(t-\tau)}b(i)K(j)e^{(A(i)+\tilde{L}(j)c(i))\tau}\,\mathrm{d}\tau\right\|^2\\
&e^{-2\lambda\beta_1\Delta T}\Big\}.\end{aligned}$$

It follows that when $t_{g+1}^o > t \geqslant t_{g+1}^s$, we have

$$\|e(t)\|^2 \leqslant M_2 e^{2\lambda(t_{g+1}^s)}\|e(t_{g+1}^s)\|^2, \tag{33}$$

$$\|X(t)\|^2 \leqslant M_2 e^{2\lambda(t_{g+1}^s)}(\|e(t_{g+1}^s)\|^2 + \|X(t_{g+1}^s)\|^2). \tag{34}$$

Using inequalities (30) and (32), and when $t_{g+1}^s \geqslant t > t_g^o$, we have

$$\begin{aligned}\|e(t)\|^2 &\leqslant M_1 e^{2\lambda(t-t_g^o)}\|e(t_g^o)\|^2\\
&\leqslant 2M_1 e^{2\lambda(t-t_g^s+t_g^*)}\|e(t_g^s - t_g^*)\|^2\\
&< \frac{\beta_4 M_1}{2^{g-1}}\, e^{2\lambda(t-t_0)}\|e(t_0)\|^2.\end{aligned}$$

Using inequality (33), and when $t_{g+1}^o > t > t_{g+1}^s$, we have

$$\begin{aligned}\|e(t)\|^2 &\leqslant M_2 e^{2\lambda(t-t_{g+1}^s)}\|e(t_{g+1}^s)\|^2\\
&< \frac{M_2}{2^{g+1}}\, e^{2\lambda(t-t_0)}\|e(t_0)\|^2.\end{aligned}$$

Thus

$$\|e(t)\|^2 \leqslant \max\left\{\beta_4 M_1, \frac{M_2}{4}\right\}\frac{1}{2^{g-1}}\, e^{2\lambda(t-t_0)}\|e(t_0)\|^2.$$

Similarly, we have

$$\|X(t)\|^2 < \max\left\{\beta_4 M_1, \frac{M_2}{4}\right\}\frac{1}{2^{g-1}}\, e^{2\lambda(t-t_0)}((2g+1)\|e(t_0)\|^2 + \|X(t_0)\|^2).$$

Note that $g \to \infty / t \to \infty$ and

$$\lim_{g\to\infty} \max\left\{\beta_4 M_1, \frac{M_2}{4}\right\} \frac{1}{2^{g-1}} ((2g+1)\|e(t_0)\|^2 + \|X(t_0)\|^2) = 0.$$

It follows that

$$\lim_{t\to\infty} e^{-\lambda t}\|X(t)\| = 0. \qquad \square$$

4. Numerical example

Example 1. Consider a switching linear system which is composed of two modes:

Mode 1:

$$A(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad b(1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c(1) = [1 \quad 0].$$

Mode 2:

$$A(2) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad b(2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c(2) = [0 \quad 1].$$

Due to space limitation, we only consider switching law I. The durations for modes 1 and 2 are 3 and 4 s, respectively. The initial state of the system is $X(0) = [100\ 200]^{\mathrm{T}}$, the initial state of the observer is $X^*(0) = [50\ 50]^{\mathrm{T}}$ and the initial mode for the system is mode 1. We choose the desired decay rate as $\lambda = 1$.

Let the controllers be

$$K(1) = [25 \quad 10] \quad \text{and} \quad K(2) = [10 \quad 25]$$

and the observers be

$$\bar{L}(1) = \begin{bmatrix} 10 \\ 25 \end{bmatrix} \quad \text{and} \quad \bar{L}(2) = \begin{bmatrix} 25 \\ 10 \end{bmatrix}.$$

The time responses of $e^t \times X(t)$ and control input of the switching control system are given in Figs. 1–3. Clearly, the states converge exponentially to zero.

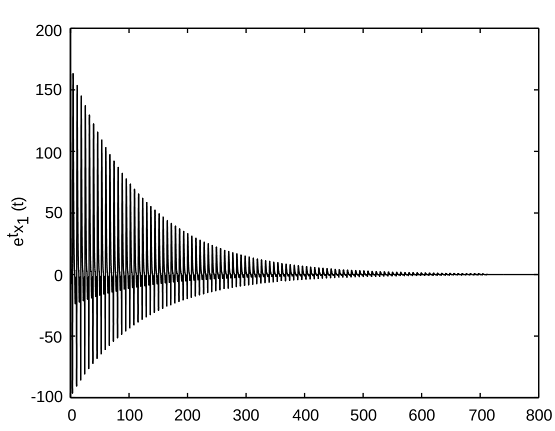

Fig. 1. The responses $e^t \times X_1(t)$ of the system.

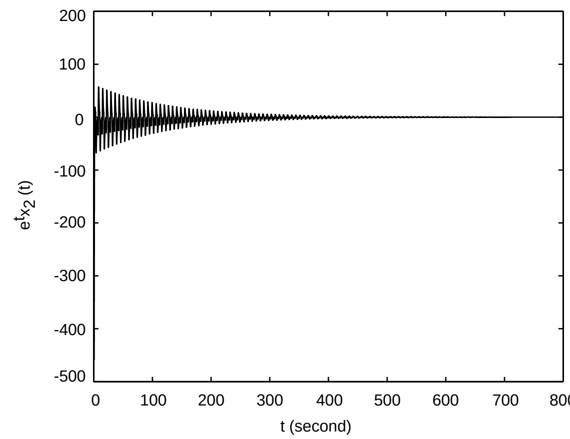

Fig. 2. The responses $e^t \times X_2(t)$ of the system.

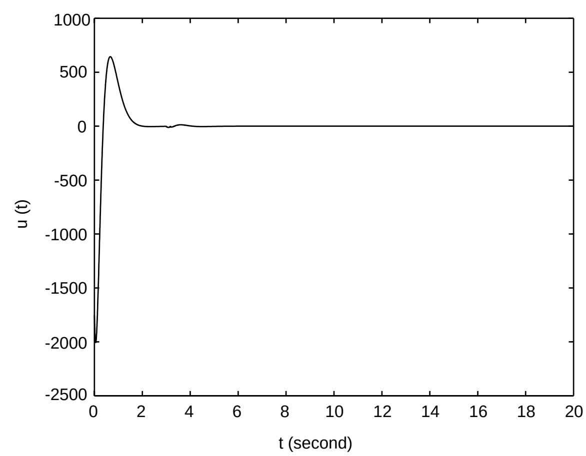

Fig. 3. The control input $u(t)$ of the system.

5. Conclusion

In this paper, we have used the pole assignment method to design an observer-based controller for both the finite and infinite switching linear systems. The switching of the controller and the observer do not need to coincide exactly with the switchings of the system. We have reconstructed the state of the system and the error between the state and the estimate of the system by the output of the system in some interval. The state and the error have been used to determine the switching instances of the controller and the observer. It has been shown that the proposed method can achieve any prescribed decay rate.

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