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Impulsive control for the stabilization and synchronization of Lorenz systems

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Abstract

This Letter derives some sufficient conditions for the stabilization and synchronization of Lorenz systems via impulsive control with varying impulsive intervals. Compared with the existing results, these conditions are less conservative in that the Lyapunov function is only required to be nonincreasing along a subsequence of switchings, instead of the whole sequence of switchings. Moreover, a larger upper bound of impulsive intervals for the stabilization and synchronization can be obtained. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In practice, there exist many examples of impulsive control systems (see [1,5,10]). Recently, impulsive control has been widely used to stabilize and synchronize chaotic systems (see [2–4,8,9,11–13]). For example, Schweizer and Kennedy in [9] proposed an impulsive control scheme with varying impulsive intervals for chaotic systems; Yang and Chua in [11] derived some sufficient conditions for the stabilization and synchronization of Chua's oscillators via impulsive control; Yang et al. in [12,13] studied, respectively, the stabilization and synchronization of a class of chaotic systems called Lorenz systems. The importance of impulsive control is that in many cases impulsive control may give an efficient method to deal with systems, which cannot endure continuous disturbance.

In this Letter, we first consider the stabilization of Lorenz systems via impulsive control with varying impulsive intervals. Some conditions are derived under which the impulsively controlled Lorenz system is asymptotically stable. Then, impulsive synchronization of two Lorenz systems are studied, and some conditions are also obtained for the asymptotic stability. Compared with the existing results obtained in [12,13], these conditions for the stabilization and synchronization of Lorenz systems are less conservative in that the Lyapunov function is only required to be nonincreasing along a subsequence of switchings, instead of the whole sequence of switchings. Moreover, a greater upper bound of impulsive intervals can be obtained.

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The rest of the Letter is organized as follows. In Section 2, we introduce some basic definitions and supporting results. Sections 3 and 4 discuss, respectively, the stabilization and synchronization of Lorenz systems. Finally, concluding remarks are given in Section 5.

2. Supporting results

An impulsive differential system [5] is described by

$$\begin{cases} \dot{X}(t) = f(t,X(t)), \quad t \neq \tau_i, \\ \Delta X(t) \overset{\Delta}{=} X(t^+) - X(t^-) = U_i(X), \quad t = \tau_i, \quad i = 1,2,\cdots, \end{cases} \tag{1}$$

where $X \in \mathbb{R}^n$ is the state variable, $f:\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $U_i: R^n \to R^n$ is the change of the state variable at time instant τ_i, $\tau_i^+ = \lim_{\epsilon \to 0^+}(\tau_i + \epsilon)$ and $\tau_i^- = \lim_{\epsilon \to 0^+}(\tau_i - \epsilon)$. In words, τ_i^+ and τ_i^- denote respectively the instants just after τ_i and just before τ_i. $\{\tau_i : i = 1,2,\cdots,\infty\}$ satisfy

$$0 < \tau_1 < \tau_2 < \cdots < \tau_i < \tau_{i+1} < \cdots,$$

$$\tau_i \to \infty \quad \text{as} \quad i \to \infty.$$

To derive sufficient conditions for the stability of system (1), we first introduce the following definitions.

Definition 1. *[1] Let $V:\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, then V is said to belong to Class $\mathscr{V}_0$ if*

1. *V is continuous in $(\tau_{i-1},\tau_i] \times \mathbb{R}^n$ and for each $X \in \mathbb{R}^n$, $i = 1,2,\cdots$,*

$$\lim_{(t,Y)\to(\tau_i^+,X)} V(t,Y) = V(\tau_i^+,X) \tag{2}$$

exists;

2. *V is locally Lipschitzian in X.*

Definition 2. *[1] For $(t,X) \in (\tau_{i-1},\tau_i] \times \mathbb{R}^n$, we define*

$$D^+V(t,X) \overset{\Delta}{=} \limsup_{h\to 0} \frac{1}{h}\left[V(t+h,X+hf(t,X)) - V(t,X)\right]. \tag{3}$$

We also need the definition of a comparison system, which plays an important role for the stability analysis of impulsive differential systems.

Definition 3. *[1] Let $V \in \mathscr{V}_0$ and assume that*

$$D^+V(t,X) \le g(t,V(t,X)), \quad t \neq \tau_i, \tag{4}$$

$$V(t,X+U_i(X)) \le \psi_i(V(t,X)), \quad t = \tau_i, \tag{5}$$

where $g:\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous, and $\psi_i:\mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing. Then the following system:

$$\begin{cases} \dot{w} = g(t,w),\ t \neq \tau_i, \\ w(\tau_i^+) = \psi_i(w(\tau_i)), \\ w(t_0^+) = w_0 \ge 0, \end{cases} \tag{6}$$

is the comparison system of system (1).

A continuous function $\gamma:\mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathscr{K}$ function if it is strictly increasing and $\gamma(0) = 0$. Let $S_\rho = \{X \in R^n \mid \|X\| < \rho\}$, where $\|\cdot\|$ denotes the Euclidean norm. To support our analysis in later sections, the following existing results are presented.

Lemma 1. *[1] Suppose that for system (1), f(t,0) = 0 and $U_i(0) = 0$ ($i = 1,2,\cdots,\infty$), and for system (6), g(t,0) = 0 and $\psi_i(0) = 0$ ($i = 1,2,\cdots,\infty$). If the following conditions are satisfied, then the stability properties of the trivial solution w = 0 of system (6) imply the corresponding stability properties of the trivial solution X = 0 of system (1).*

i) *$V:\mathbb{R}_+ \times S_\rho \to \mathbb{R}_+, \rho > 0, V \in \mathscr{V}_0$, and $D^+V(t,X) \le g(t,V(t,X)), t \neq \tau_i$;*

ii) *there exists a $\rho_0 > 0$ such that $X \in S_{\rho_0}$ implies that $X + U_i(X) \in S_{\rho_0}$ for all i and $V(t,X+U_i(X)) \le \psi_i(V(t,X))$, $t = \tau_i, X \in S_{\rho_0}$;*

iii) *$\beta(\|X\|) \le V(t,X) \le \alpha(\|X\|)$ on $\mathbb{R}_+ \times S_\rho$, where $\alpha,\beta \in \mathscr{K}$.*

The following lemma gives sufficient conditions for the stability of system (1).

Lemma 2. *[6] For system (6), let $g(t,w) = \dot{\lambda}(t)w$, $\lambda \in C^1[R_+,R_+]$, and $\psi_i(w) = d_i w$, $d_i > 0$, $i =$*

1,2, ⋯ ,∞. Then, the origin of system (1) is asymptotically stable if the following conditions hold.

1. $\sup_i\{d_i \exp(\lambda(\tau_{i+1})-\lambda(\tau_i))\}<\infty$;
2. There exists a $\xi>1$ such that
$$\lambda(\tau_{2i+1})+\ln(\xi d_{2i}d_{2i-1})\le\lambda(\tau_{2i-1}); \tag{7}$$
3. $\lambda(t)$ satisfies that
$$\dot{\lambda}(t)\ge 0. \tag{8}$$

Remark 1. *Condition 1 implies that the state variable of system (1) is bounded within any impulsive interval, while condition 2 implies that the Lyapunov function $V(t,X)=X^{\mathrm{T}}X$ is only required to be nonincreasing along an odd subsequence of switchings.*

3. Impulsive stabilization of Lorenz systems

The state equations of Lorenz system [7] are given by

$$\begin{cases}\dot{x}=-\sigma x+\sigma y,\\ \dot{y}=rx-y-xz,\\ \dot{z}=xy-bz,\end{cases} \tag{9}$$

where $\sigma,r,b\in\mathbb{R}_+$. System (9) is chaotic if $\sigma=10, r=28$ and $b=\frac{8}{3}$. Letting $X^{\mathrm{T}}=[x,y,z]$, then we can rewrite system (9) into the following matrix form

$$\dot{X}=AX+\Phi(X), \tag{10}$$

where

$$A=\begin{bmatrix}-\sigma & \sigma & 0\\ r & -1 & 0\\ 0 & 0 & -b\end{bmatrix},\quad \Phi(X)=\begin{bmatrix}0\\ -xz\\ xy\end{bmatrix}. \tag{11}$$

Let I denote the identity matrix and $\rho(A)$ denote the spectral radius of A. We now introduce the impulsive control for system (10) as follows:

$$\begin{cases}\dot{X}=AX+\Phi(X), & t\ne\tau_i,\\ U_i(X)=BX, & t=\tau_i,\quad i=1,2,\cdots,\end{cases} \tag{12}$$

where B is a symmetric matrix satisfying $\rho(I+B)\le 1$, $\{\tau_i : i=1,2,\cdots,\infty\}$ are varying but satisfy

$$\Delta_1=\sup_{1\le j<\infty}\{\tau_{2j+1}-\tau_{2j}\}<\infty, \tag{13}$$

$$\Delta_2=\sup_{1\le j<\infty}\{\tau_{2j}-\tau_{2j-1}\}<\infty, \tag{14}$$

and for a given constant ϵ,

$$\tau_{2j+1}-\tau_{2j}\le\epsilon(\tau_{2j}-\tau_{2j-1}),\forall j\in\{1,2,\cdots,\infty\}. \tag{15}$$

Remark 2. *Conditions (13) and (14) imply that the number of switchings is infinite, while Condition (15) implies that impulsive intervals may not be equidistant as required in [12,13].*

Then, the following stability result for the impulsively controlled Lorenz system (12) can be obtained.

Theorem 1. *The origin of system (12) is asymptotically stable if there exists a $\xi>1$ such that*

$$0\le q\le-\frac{\ln(\xi d^2)}{(1+\epsilon)\Delta_2}, \tag{16}$$

where q is the largest eigenvalue of $(A+A^{\mathrm{T}})$ and $d=\rho^2(I+B)$.

Proof. Let $V(t,X)=X^{\mathrm{T}}X$. Then from the proof of Theorem 3 in [12] and Lemma 1, we can get the following comparison system

$$\begin{cases}\dot{w}=qw,\ t\ne\tau_i,\\ w(\tau_i^+)=dw(\tau_i),\\ w(t_0^+)=w_0\ge 0,\end{cases}$$

We now consider the conditions in Lemma 2. Since

$$\sup_i\{d\exp(q\tau_{i+1}-q\tau_i)\}$$
$$=d\exp(q\max\{\Delta_1,\Delta_2\})<\infty,$$

thus condition 1 in Lemma 2 is satisfied. Furthermore,

$$\begin{aligned}q\tau_{2i+1}-q\tau_{2i-1}&=q(\tau_{2i+1}-\tau_{2i}+\tau_{2i}-\tau_{2i-1})\\ &\le q(\Delta_1+\Delta_2)\\ &\le q(1+\epsilon)\Delta_2\\ &\le-\ln(\xi d^2),\end{aligned}$$

where the last inequality holds from (16). Thus, condition 2 in Theorem 2 is also satisfied. Therefore, it follows from Lemma 2 that the origin of system (12) is asymptotically stable.

Remark 3. *For $\epsilon < 1$ and any $\xi > 1$ satisfying $0 < \xi d < 1$, if we choose that*

$$\Delta_2 = -\frac{2\ln(\xi d)}{q(1+\epsilon)},$$

then condition (16) holds. That is the trivial solution of system (12) is asymptotically stable. Note that $0 < \xi d < 1$ is also required by Theorem 3 in [12]. However, Δ_2 is greater than the upper bound obtained in (22) of [12]. Thus, a larger upper bound of impulsive intervals can be obtained through our approach.

Remark 4. *Condition (16) implies that V(t,X) is only required to be nonincreasing along an odd subsequence of switchings, instead of the whole sequence of switchings needed by [12]. Thus, our result is less conservative.*

Example 1. We consider Experiment 1 of [12]. For ease of comparison, in this example we choose the same parameters as those in that experiment. That is $\sigma = 10, r = 28, b = \frac{8}{3}$ and

$$B = \begin{bmatrix} k & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is easy to find that $d = (k+1)^2$, $q = 28.051$, and $\rho(I+B) \le 1$ when $k \in (-2,0)$.
For any $\xi > 1$ satisfying $0 < \xi d < 1$, if we choose that $\epsilon = 1$,

$$\Delta_1 = \Delta_2 = -\frac{\ln(\xi d)}{q},$$

and for all $j = 1,2,\cdots,\infty$, let $\tau_{j+1} - \tau_j = \delta \le \Delta_1$, then the situation is the same as that of Experiment 1 in [12].

However, if we choose that $\epsilon = 0.5$ and for all $j = 1,2,\cdots,\infty$,

$$\tau_{2j+1} - \tau_{2j} = \Delta_1 = -\frac{\ln(\xi d)}{1.5q},$$

$$\tau_{2j} - \tau_{2j-1} = \Delta_2 = -\frac{2\ln(\xi d)}{1.5q}. \tag{17}$$

We know that the origin of system (12) with above parameters is asymptotically stable from Theorem 1. But, from (17) we have

$$\int_{\tau_{2j}}^{\tau_{2j+1}} q\,dt + \ln(\xi d) = -\tfrac{1}{3}\ln(\xi d) > 0.$$

Thus, Theorem 3 in [12] cannot be used to deal with this case.

4. Impulsive synchronization of Lorenz systems

In this section, we study the impulsive synchronization of Lorenz systems. In an impulsive synchronization configuration, the driving system is given by (10), while the driven system is given by

$$\dot{\tilde{X}} = A\tilde{X} + \Phi(\tilde{X}), \tag{18}$$

where $\tilde{X} = (\tilde{x},\tilde{y},\tilde{z})^{\mathrm{T}}$ is the state variable of the driven system, A and Φ are defined in (11).

At discrete instants, τ_i, $i = 1,2,\cdots$, the state variables of the driving system are transmitted to the driven system and then the state variables of the driven system are subjected to jumps at these instants. In this sense, the driven system is modeled by the following impulsive equations

$$\begin{cases} \dot{\tilde{X}} = A\tilde{X} + \Phi(\tilde{X}), & t \ne \tau_i, \\ \Delta\tilde{X}|_{t=\tau_i} = -Be, & i = 1,2,\cdots, \end{cases} \tag{19}$$

where B is a symmetric matrix satisfying $\rho(I+B) \le 1$, $\{\tau_i : i = 1,2,\cdots,\infty\}$ satisfy (13), (14) and (15), and $e^{\mathrm{T}} = (e_x,e_y,e_z) = (x-\tilde{x}, y-\tilde{y}, z-\tilde{z})$ is the synchronization error. Let

$$\Psi(X,\tilde{X}) = \Phi(X) - \Phi(\tilde{X}) = \begin{bmatrix} 0 \\ -(xz - \tilde{x}\tilde{z}) \\ xy - \tilde{x}\tilde{y} \end{bmatrix}, \tag{20}$$

then the error system of the impulsive synchronization is given by

$$\begin{cases} \dot{e} = Ae + \Psi(X,\tilde{X}), & t \neq \tau_i, \\ \Delta e|_{t=\tau_i} = U_i(e) = Be, & i = 1,2,\cdots, \end{cases} \tag{21}$$

Note that there exists a positive number M for chaotic system (12) such that $|y(t)| \leq M$ and $|z(t)| \leq M$ for all t. Then, similar to the stabilization of Lorenz systems, we have the following result.

Theorem 2. *The impulsive synchronization of two Lorenz systems, given in (21), is asymptotically stable if there exists a $\xi > 1$ such that*

$$0 \leq (q + 2M) \leq -\frac{\ln(\xi d^2)}{(1+\epsilon)\Delta_2}, \tag{22}$$

where q is the largest eigenvalue of $(A + A^T)$ and $d = \rho^2(I + B)$.

Proof. Let $V(t,e) = e^T e$. Then from the proof of Theorem 3 in [13] and Lemma 1, we can get the following comparison system

$$\begin{cases} \dot{w} = (q+2M)w, t \neq \tau_i, \\ w(\tau_i^+) = dw(\tau_i), \\ w(t_0^+) = w_0 \geq 0. \end{cases}$$

Then, similar to the proof of Theorem 1, we can show that the origin of system (21) is asymptotically stable.

Remark 5. *Based on similar reasons in Remarks 3 and 4, we can get a greater upper bound of impulsive intervals than that obtained in [13], and the Lyapunov function V(t,e) is also only required to be nonincreasing along an odd subsequence of switchings, instead of the whole sequence of switchings needed by [13].*

Example 2. Similarly, in this example the same parameters as those in Experiment 2 of [13] are chosen as $\sigma = 10, r = 28, b = \frac{8}{3}, M = 50$ and

$$B = \begin{bmatrix} k & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{bmatrix}.$$

We can find that $q = 28.051$, and $\rho(I + B) \leq 1$ when $k \in (-2,0)$. The corresponding impulsive coupling is given by

$$\Delta e|_{t=\tau_i} = \begin{cases} ke_x \\ -0.1e_y \\ -0.1e_z \end{cases}.$$

Then we can get

$$d = \begin{cases} (k+1)^2, & \text{if } (k+1)^2 > 0.81 \\ 0.81, & \text{otherwise} \end{cases}.$$

For any $\xi > 1$ satisfying $0 < \xi d < 1$, if we choose that $\epsilon = 1$,

$$\Delta_1 = \Delta_2 = -\frac{\ln(\xi d)}{q + 2M},$$

and for all $j = 1,2,\cdots,\infty$, let $\tau_{j+1} - \tau_j = \delta \leq \Delta_1$, then the situation is the same as that of Experiment 2 in [13].

However, if we choose that $\epsilon = 0.5$ and for all $j = 1,2,\cdots,\infty$,

$$\tau_{2j+1} - \tau_{2j} = \Delta_1 = -\frac{\ln(\xi d)}{1.5(q+2M)},$$

$$\tau_{2j} - \tau_{2j-1} = \Delta_2 = -\frac{2\ln(\xi d)}{1.5(q+2M)}. \tag{23}$$

We know that the origin of system (21) with above parameters is asymptotically stable from Theorem 2. But, from (23) we have

$$\int_{\tau_{2j}}^{\tau_{2j+1}} (q+2M)\,dt + \ln(\xi d) = -\tfrac{1}{3}\ln(\xi d) > 0.$$

Thus, Theorem 3 in [13] cannot be used to study the stability in this case.

5. Conclusion

This Letter has studied the issue on the stabilization and synchronization of Lorenz systems via an impulsive control with varying impulsive intervals. Through our approach, some less conservative conditions were derived in that the Lyapunov function is only required to be nonincreasing along a subsequence of switchings, instead of the whole sequence of switchings. Moreover, a larger upper bound of impulsive intervals for the stabilization and synchronization of Lorenz systems can be obtained.

References

[1] D.D. Bainov, P.S. Simeonov. Systems With Impulse Effect: Stability, Theory and Applications, Halsted Press, New York, 1989.

[2] L.O. Chua, M. Itoh, L. Kocarev, K. Eckert, J. Circuits, Syst. and Comput. 3 (1993) 93.

[3] E.R. Hunt, G. Johnson, IEEE Spectrum 32 (1993) 32.

[4] M. Itoh, T. Yang, L.O. Chua, Int. J. Bifurc. Chaos 9 (1999) 1393.

[5] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, Theory of Impulsive Differential Equations, World Scientific, Singapore, 1989.

[6] Z.G. Li, C.Y. Wen, Y.C. Soh, IEEE Transactions on Automatic Control, 2000, revised.

[7] E.N. Lorenz, J. Atmospheric Sci. 20 (1963) 130.

[8] A.I. Panas, T. Yang, L.O. Chua, Int. J. Bifurc. Chaos Appl. Sci. Eng. 8 (1998) 639.

[9] J. Schweizer, M.P. Kennedy, Physs Rev. E 52 (1995) 4865.

[10] T. Yang, IEEE Trans. Autom. Control 44 (1999) 1081.

[11] T. Yang, L.O. Chua, IEEE Trans. Circuit Syst. Part I: Fundam. Appl. 44 (1997) 976.

[12] T. Yang, Lin-Bao Yang, Chun-Mei Yang, Physica D: Nonlinear Phenomena 110 (1997) 18.

[13] T. Yang, Lin-Bao Yang, Chun-Mei Yang, Phys. Lett. A 226 (1997) 349.