A Switching Control Law for Extended Chained Forms

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Abstract

In this paper, the two-input extended chained form is investigated. A switching control law is developed and ultimate exponential stabilization is achieved.

1 Introduction

In this paper, we address the feedback stabilization problem for the following system,

$$\dot{x}^1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x^1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 \qquad (1)$$

$$\dot{x}^2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & v_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & v_1 & 0 \end{bmatrix} x^2 + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 \quad (2)$$

where,

$$x = \begin{bmatrix} x^1 & x^2 \end{bmatrix}^T = \begin{bmatrix} v_1 & x_1 & | & v_2 & x_2 & \cdots & x_n \end{bmatrix}^T$$

This system is actually the chained form (first proposed in [1]) extended with integrators. We call it here the extended chained form. Since this system models the dynamic equations of a large class of nonholonomic systems (e.g. unicycle-type vehicles and car-like vehicles), it has potential significance for application.

System (1)(2) is clearly a nonlinear system with drift term, which consists of two single input subsystems, i.e., LTI (linear time invariant) subsystem (1) and LTV (linear time-varying) subsystem (2) if v_1 is taken as a time function. Note that, when $v_1 \to 0$, subsystem (2) will erode into being uncontrollable. Here, we use the σ process [2] of the following rational discontinuous coordinate transformation

$$z = [z_1 \cdots z_n]^T = T(x) = \left[v_2 \; x_2 \; \frac{x_3}{v_1} \cdots \frac{x_n}{v_1^{n-2}} \right]^T \quad (3)$$

to subsystem (2), we get the transformed system of subsystem (2) as

$$\dot{z} = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & 0 \\ & 1 & \omega(t) & & \\ & & \ddots & \ddots & \\ & 0 & & 1 & (n-2)\omega(t) \end{bmatrix} z + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_2 \quad (4)$$

with

$$\omega(t) = -u_1(t)/v_1(t)$$

The benefits of the above transformation is that, when $v_1 \to 0$, subsystem (4) remains completely controllable as long as parameter $\omega(t)$ is well defined. In [3], we have shown that, using the linear control law

$$u_1 = -(\lambda + \bar{\omega})v_1 - \lambda\bar{\omega}x_1 \qquad \forall \; \lambda > \bar{\omega} > 0$$

into subsystem (1), if $v_1(t_0) + \lambda x_1(t_0) \neq 0$, parameter $\omega(t)$ will get into the open connected invariant set

$$\Omega = \{\omega \in R: \; \omega_{low} < \omega < \omega_{up} \forall -\infty < \omega_{low} < \bar{\omega} < \omega_{up} \leq \lambda\}$$

in a finite time. This region Ω actually defines an invariant manifold $\mathcal{M}$ of configuration x as

$$\mathcal{M} = \{x \in R^{n+2} : \; v_1 \neq 0 \text{ and } \omega(t) \in \Omega\}$$

It sets the condition of a switching control law to take system (1)(4) and hence system (1)(2) exponentially converge to the origin by the following Algorithm.

Algorithm 1 *Consider system (1)(2). Apply following switch control law C1 to stabilize subsystem (1):*

C11: $u_1 = U(x)$, if $\|x\| > \varepsilon$ and $v_1(0) + \lambda x_1(0) = 0$ and $0 \leq t < t_U$.

C12: $u_1 = -(\lambda + \bar{\omega})v_1 - \lambda\bar{\omega}x_1$, elsewhere.

where, $\lambda > \bar{\omega} > 0$. The constant $\varepsilon > 0$ is a given stabilization tolerance (i.e., if $\|x\| < \varepsilon$, no further control effort is required). t_U is a chosen finite instant and $U(x)$ is designed such that $v_1(t_U) + \lambda x_1(t_U) \neq 0$.

Then, subsystem (2) is stabilized by the switch control law C2 as follows.

C21: If $\|x\| > \varepsilon$ and $x \notin \mathcal{M}$, a control law u_2, which maintains the boundedness of subsystem (2), is applied.

C22: If $\|x\| > \varepsilon$ and $x \in \mathcal{M}$, a control law u_2, which exponentially stabilize subsystem (4) and hence subsystem (2), is applied.

C23: If $\|x\| \leq \varepsilon$, a control law $u_2 = 0$ is applied. ∎

The effectiveness of Algorithm 1 is proved in [3], where we also show that control law C11 may be simply realized by the linear feedback

$$U(x) = -\lambda_1(v_1 - \bar{v}) \qquad (5)$$

where $\lambda_1 > 0$ and $\bar{v}$ are constants. In this paper, we focus on the control law C2.

2 Design of Control Law C2

Theorem 1 and Theorem 2 are used to present control law C21 and control law C22 respectively.

Theorem 1 *Consider the following linear change*

$$\chi_n = x_n$$
$$\vdots$$
$$\chi_i = k_{i-1}\chi_{i+2} + L_h\chi_{i+1} \qquad 2 \le i \le n-2$$
$$\vdots$$
$$\chi_1 = v_2$$

where, $h = [0\ x_2\ \cdots\ x_{n-1}]^T$, $k_i > 0$, $L_h\chi_i = \frac{\partial \chi_i}{\partial x_2}h$ is the Lie derivative. Subsystem (2) is converted to

$$\dot{\chi}^1 = \dot{\chi}_1 = u_2 \qquad (6)$$
$$\dot{\chi}^2 = K\left(S(v_1)\chi^2 + bw\right) \qquad (7)$$

where, $\chi^1 = \chi_1$, $\chi^2 = [\chi_2\ \cdots\ \chi_n]^T$, K and $S(v_1)$ are $(n-1)$-dimensional matrices, b is an $(n-1)$-vector. Suppose $|v_1(t)|$ and $|\dot{v}_1(t)|$ are bounded. Then, control

$$u_2 = -k_\alpha(\chi_1 - \alpha) + \frac{\partial \alpha}{\partial \chi^2}K\left(S(v_1)\chi^2 + bw\right) - \frac{\partial V\left(\chi^2\right)}{\partial \chi^2}Kb$$

where,

$$V\left(\chi^2\right) = \frac{\chi_n^2}{2} + \frac{\chi_{n-1}^2}{2k_{n-2}} + \frac{\chi_{n-2}^2}{2k_{n-2}k_{n-3}} + \cdots + \frac{\chi_2^2}{2\prod\limits_{i=1}^{n-2}k_i}$$
$$w = k_1v_1\chi_3 + (L_h\chi_2)v_1 + \chi_1$$
$$\alpha = -k_w|v_1|\chi_2 + \chi_1 - w$$

with k_α and k_w being positive constants, globally asymptotically stabilizes system (6)(7) and hence subsystem (2) to the origin, if $|v_1(t)|$ does not asymptotically tend to zero.

Since change (6) is an isomorphism, the idea of the proof is to show that the dynamics of system (6)(7) is asymptotically stable by the aid of Lyapunov-like function

$$V_a(\chi, t) = V\left(\chi^2\right) + \frac{1}{2}(\chi^1 - \alpha)^2$$

and recursively using the extended version of Barbalat's Lemma.

Theorem 2 *Define*

$$\alpha_n(z, t) = 0$$
$$\alpha_{n-1}(z, t) = -(n-2)\omega(t)z_n - c_n z_n \qquad (8)$$
$$\alpha_k(z, t) = -c_{k+1}\left(z_{k+1} - \alpha_{k+1}(z, t)\right)$$
$$\quad - (k-1)\omega(t)z_{k+1} - z_{k+2}$$
$$\quad + \alpha_{k+2}(z, t) + \dot{\alpha}_{k+1}(z, t)$$

where, $1 \le k \le n-2$, and c_i $(2 \le i \le n)$ is a positive constant. Then, control law

$$u_2(z, t) = -c_1(z_1 - \alpha_1) - z_2 + \alpha_2(z, t) + \dot{\alpha}_1(z, t) \quad (9)$$

where c_1 is a positive constant, globally uniformly exponentially stabilize system (4) to origin.

The idea of the proof is to show that the positive definite, decrescent and radially unbounded function

$$V(z, t) = \frac{1}{2}z_n^2 + \frac{1}{2}\sum_{k=0}^{n-2}\left(z_{k+1} - \alpha_{k+1}(z, t)\right)^2 \qquad (10)$$

is a Lyapunov function whose time derivative is negative definite and satisfies

$$\dot{V}(z, t) = -c_n z_n^2 - \sum_{k=1}^{n-1}c_k\left(z_k - \alpha_k\right)^2 \le -2cV(z, t) \le 0$$

where, $c = \min\{c_1, \cdots, c_n\}$. Then, the Gronwall-Bellman inequality is applied to show the exponential convergence rate.

3 Conclusions

The extended chained form possesses different properties inside and outside a defined manifold. Therefore, for the system locating inside and outside the manifold, a switching law is applied to stabilize the extended chained form. The advantage of the proposed switching control law is that it may produce not only exponential convergent rate but also natural stabilizing locus.

References

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[3] G. Xu and D. Wang, "Exponential stabilization of extended chained form," in *Proceedings of the American Control Conference*, (Chicago, US), pp. 3259–3263, June 2000.