Initial Shift Issues on Discrete-Time Iterative Learning Control With System Relative Degree

Mingxuan Sun and Danwei Wang

***Abstract*—This note deals with the initial shift problem that arises from discrete-time iterative learning control. A unified learning scheme is considered for a class of nonlinear systems with well-defined relative degree, which adopts the error data with anticipation in time and provides wider freedom for the updating law formation. The sufficient convergence condition is derived to enable the system to possess asymptotic tracking capability and the converged output trajectory can be assessed by the initial condition. The tracking performance is improved further by the introduction of initial rectifying action and the complete tracking is achieved over a specified interval.**

***Index Terms*—Discrete-time, initial condition problem, learning control, nonlinear systems, relative degree.**

I. Introduction

Iterative learning control overcomes imperfect knowledge about the dynamics structure and/or parameters to achieve the complete tracking through repetition [1]. For execution, the system is moved to an initial position. Then it starts, runs, stops, and resets to the same initial position. In the published literature, the developed learning schemes for nonlinear discrete-time systems use measurable state variables [2], two-step successive error difference [3] or multi-step error data [4]. In [5]–[7], only the output error is required. Analyses for the convergence and robustness have been presented with the aid of discrete-time λ-norm [3]–[6] and by the analysis technique without applying such a norm [2], [7]. These theoretical results are restricted to the requirement for convergence that the initial condition at each cycle should be reset to the initial condition corresponding to the desired trajectory. In the practical implementation, perturbed initial conditions would degrade the tracking performance which motivates researchers to consider the case where the system does not reset the initial condition at each cycle to the desired one. Instead, there exist initial shifts. This study is inherent to improve the tracking performance and is, thus, meaningful in itself [8]. Several researchers addressed themselves to the initial shift problem for linear time-invariant (LTI) systems [9] and nonlinear continuous-time systems [10]–[12]. Very recently, there is certain interest in the same problem for discrete-time systems. In [13], one adjustment scheme of initial state was proposed where the repositioning mechanism is needed to reset the initial state at each cycle to the resultant one by the scheme. The technique of suitably reducing sampling rate, presented in [14], was shown effective to achieve better tracking. The result, however, is restricted to LTI systems. In [15], the complete tracking was achieved over a specified interval by the learning scheme using only output error at each cycle.

In this note, the initial shift problem arisen from the iterative learning control for a class of nonlinear discrete-time systems with well-defined relative degree is addressed. The learning scheme adopts the output errors with anticipation in time and provides wider freedom for the updating law formation in a unified manner, which was primarily studied in [4] and its higher order version was considered in [16]. The learning scheme undertaken will be shown robust with respect to initial shifts by the developed analysis approach, which enables the systems to possess asymptotic tracking capability and the converged output trajectory can be assessed by the initial condition. The tracking performance is shown to be improved by the introduction of initial rectifying action so that the complete tracking with specified transient is achieved.

Manuscript received December 21, 2001; revised June 12, 2002. Recommended by Associate Editor A. Bemporad.

M. Sun is with the Center for Mechanics of Micro-Systems, School of Mechanical and Production Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798, Singapore (e-mail: mmxsun@ntu.edu.sg).

D. Wang is with the School of Electrical and Electronic Engineering, Nanyang Technological University, Nanyang Avenue, Singapore 639798, Singapore (e-mail: edwwang@ntu.edu.sg).

Digital Object Identifier 10.1109/TAC.2002.806668

II. Problem Formulation and Preliminaries

Consider the class of nonlinear discrete-time systems described by

$$x(t+1) = f(x(t), u(t)) \tag{1}$$
$$y(t) = g(x(t)) \tag{2}$$

where t is the discrete-time index, $x \in R^n$, $u \in R^1$, and $y \in R^1$ denote the state, the scalar control input, and the scalar output of the system, respectively. The nonlinear functions $f(\cdot,\cdot) \in R^n$ and $g(\cdot) \in R^1$ are smooth in their domain of definition, which are known about certain properties only. The iterative learning control problem will be handled in terms of system relative degree. Thus, the following notations and definition for relative degree are introduced. Similar concepts of the relative degree of nonlinear discrete-time systems can be found in [17] and [18]. Let $\bar{f}(x)$ be the undriven state dynamics $f(x,0)$ and $\bar{f}^j$ the j-times recursive compositions of $\bar{f}$ in the sense that $\bar{f}^j(x) = f(\bar{f}^{j-1}(x))$ with $\bar{f}^0(x) = x$.

Definition 2.1: Systems (1) and (2) are said to have relative degree μ at (x^0, u^0), if

i) $(\partial/\partial u)\left[g \circ \bar{f}^j(f(x,u))\right] = 0, 0 \le j \le \mu - 2$ and for all (x,u) in a neighborhood of (x^0, u^0);

ii) $(\partial/\partial u)\left[g \circ \bar{f}^{\mu-1}(f(x^0,u^0))\right] \neq 0$.

Remark 2.1: The system output at the instant $t+j$, $j \ge 1$ is, in general, written as

$$y(t+j) = g(f(\ldots, f(f(x(t),u(t)), u(t+1)), \ldots), u(t+j-1)). \tag{3}$$

If (1) and (2) have relative degree μ, the output can be evaluated in the following simple form:

$$y(t+j) = g \circ \bar{f}^j(x(t)), \qquad 1 \le j \le \mu - 1 \tag{4}$$
$$y(t+\mu) = g \circ \bar{f}^{\mu-1}(f(x(t), u(t))) \tag{5}$$

which implies that μ is exactly the steps of delay in the output $y(t)$ in order to have the control input $u(t)$ appearing. Equation (4) still holds for $j=0$. The output $y(t+j), 0 \le j \le \mu-1$, is thus independent of the input variables at the instant t, $u_k(t)$.

Given a desired trajectory $y_d(t), 0 \le t \le N+\mu$, for system (1) and (2) with relative degree μ, the control objective is to find an input profile $u(t)$, $0 \le t \le N$, so that the resultant output trajectory $y(t)$, $0 \le t \le N+\mu$, follows the desired trajectory as closely as possible in the presence of initial shifts. Throughout this note, by the term *initial shift* we mean the initial condition which may be reset to some finite point in the state space, shifting from the initial condition corresponding to the desired trajectory. Let S denote the mapping from $(x(0), u(t), 0 \le t \le N)$ to $x(t), 0 \le t \le N+1$, and O the mapping from $(x(0), u(t), 0 \le t \le N)$ to $y(t)$, $0 \le t \le N+\mu$. We denote by $|a|$ the absolute value of a if a is a scalar or the norm $|a| = \max_{1\le i\le n}|a_i|$ if a is an n-dimensional vector, $a = [a_1, \ldots, a_n]^T$. The following properties for (1) and (2) are assumed.

A1) The mappings S and O are one to one.

0018-9286/03$17.00 © 2003 IEEE

A2) The system has relative degree μ for $x(t)$ and $u(t)$, $0 \le t \le N$.

A3) The functions $f(\cdot,\cdot)$ and $g(\cdot)$ are Lipschitz in their arguments, i.e., there exist positive constants l_f and l_g such that $|f(x',u') - f(x'',u'')| \le l_f(|x'-x''| + |u'-u''|)$ and $|g(x') - g(x'')| \le l_g|x'-x''|$ for all $x'(t), x''(t), u'(t)$ and $u''(t)$, $0 \le t \le N$.

A4) The function $(\partial/\partial u)\left[g \circ \bar{f}^{\mu-1}(f(\cdot,\cdot))\right]$ is bounded for all $x(t)$ and $u(t)$, $0 \le t \le N$.

A5) Initial shifts exist at each cycle in the sense that $|x_0 - x_k(0)| \le c_{x_0}$ for any fixed x_0 and some positive constant c_{x_0}.

Remark 2.2: Note the fact that $a \circ b(\cdot)$ is Lipschitz if $a(\cdot)$ and $b(\cdot)$ are Lipschitz in their arguments. Assumption A3) implies that $g \circ \bar{f}^j(\cdot)$, $0 \le j \le \mu - 1$, is Lipschitz in x. That is, there exists some positive constant l_{gf} such that $|g \circ \bar{f}^j(x') - g \circ \bar{f}^j(x'')| \le l_{gf}|x' - x''|$.

III. Main Results

A. Asymptotic Tracking

The updating law undertaken is

$$u_{k+1}(t) = u_k(t) + \gamma_k(t)\sum_{j=0}^{\mu} a_j(y_d(t+j) - y_k(t+j)) \quad (6)$$

where $0 \le t \le N$, k indicates the number of operation cycle, $\gamma_k(t) \in R^1$ represents the learning gain chosen to be bounded and a_j, $0 \le j \le \mu$, are the design parameters. Here, we assume that $a_\mu = 1$. This learning scheme is in a unified form, which provides wider freedom for the updating law formation by the choice of a_j, $0 \le j \le \mu - 1$. The robust designs for numerical differentiation could be used to calculate the term $\sum_{j=0}^{\mu} a_j(y_d(t+j) - y_k(t+j))$. Other available designs are those for predictive filters. We shall show that the learning scheme leads to the trajectory specified by the initial condition, which can follow the desired trajectory asymptotically as time increases.

Theorem 3.1: Let (1) and (2) satisfy assumptions A1)–A5) and updating law (6) be applied. If the learning gain is chosen such that for all k and $0 \le t \le N$, for all $x_k(t) \in R^n$, for all $\bar{u}_k(t) \in R$

$$\left|1 - \gamma_k(t)\frac{\partial}{\partial u}\left[g \circ \bar{f}^{\mu-1}(f(x_k(t), \bar{u}_k(t)))\right]\right| \le \rho < 1 \quad (7)$$

and the trajectory $y^*(t)$ is realizable, where $y^*(t)$ is given by

$$y^*(t) = y_d(t) - e^*(t) \quad (8)$$

and $e^*(t)$ satisfies the following difference equation:

$$\sum_{j=0}^{\mu} a_j e^*(t+j) = 0 \quad (9)$$

with $e^*(0) = g(x_d(0)) - g(x_0)$, $e^*(1) = g \circ \bar{f}(x_d(0)) - g \circ \bar{f}(x_0), \ldots$, and $e^*(\mu-1) = g \circ \bar{f}^{\mu-1}(x_d(0)) - g \circ \bar{f}^{\mu-1}(x_0)$, then asymptotic bound of the error $y^*(t) - y_k(t)$ is proportional to c_{x_0} for $\mu \le t \le N + \mu$ as $k \to \infty$. Furthermore, the error converges to zero for $\mu \le t \le N + \mu$ whenever c_{x_0} tends to zero.

Remark 3.1: Clearly, for an appropriate x_0 and a realizable trajectory $y^*(t)$, assumption A1) guarantees that there exists a unique control input $u^*(t)$ that will generate the trajectory. Namely, $y^*(t) = g(x^*(t))$, where $x^*(t)$ is the corresponding state satisfying $x^*(t+1) = f(x^*(t), u^*(t))$ and $x^*(0) = x_0$.

Proof of Theorem 3.1: In view of $y^*(t)$ defined in (8) and (9), updating law (6) can be written as

$$u_{k+1}(t) = u_k(t) + \gamma_k(t)\sum_{j=0}^{\mu} a_j(y^*(t+j) - y_k(t+j))$$

which leads to, denoting $\Delta u_k^*(t) = u^*(t) - u_k(t)$

$$\begin{aligned}\Delta u_{k+1}^*(t) =& \Delta u_k^*(t) - \gamma_k(t) \\ &\times \sum_{j=0}^{\mu-1} a_j\left[g \circ \bar{f}^j(x^*(t)) - g \circ \bar{f}^j(x_k(t))\right] \\ &- \gamma_k(t)\left[g \circ \bar{f}^{\mu-1}(f(x^*(t), u^*(t)))\right. \\ &\left.- g \circ \bar{f}^{\mu-1}(f(x_k(t), u_k(t)))\right].\end{aligned}$$

By the mean value theorem, there exists $\bar{u}_k(t) = \xi u^*(t) + (1-\xi)u_k(t)$, $\xi \in [0,1]$, such that

$$\begin{aligned}g \circ \bar{f}^{\mu-1}&(f(x_k(t), u^*(t))) - g \circ \bar{f}^{\mu-1}(f(x_k(t), u_k(t))) \\ &= \frac{\partial}{\partial u}\left[g \circ \bar{f}^{\mu-1}(f(x_k(t), \bar{u}_k(t)))\right]\Delta u_k^*(t)\end{aligned}$$

which results in

$$\begin{aligned}\Delta u_{k+1}^*(t) =& \left(1 - \gamma_k(t)\frac{\partial}{\partial u}\left[g \circ \bar{f}^{\mu-1}(f(x_k(t), \bar{u}_k(t)))\right]\right) \\ &\times \Delta u_k^*(t) - \gamma_k(t) \\ &\times \sum_{j=0}^{\mu-1} a_j\left[g \circ \bar{f}^j(x^*(t)) - g \circ \bar{f}(x_k(t))\right] \\ &- \gamma_k(t)\left[g \circ \bar{f}^{\mu-1}(f(x^*(t), u^*(t)))\right. \\ &\left.- g \circ \bar{f}^{\mu-1}(f(x_k(t), u^*(t)))\right].\end{aligned}$$

Taking norms on both sides and using assumption A3) yields

$$\begin{aligned}|\Delta u_{k+1}^*(t)| \le& \rho|\Delta u_k^*(t)| + c_\gamma \\ &\times \sum_{j=0}^{\mu-1}|a_j| l_{gf}|\Delta x_k^*(t)| + c_\gamma l_{gf} l_f |\Delta x_k^*(t)| \\ \le& \rho|\Delta u_k^*(t)| + c_1|\Delta x_k^*(t)| \quad (10)\end{aligned}$$

where $\Delta x_k^*(t) = x^*(t) - x_k(t)$, c_γ is the norm bound for $\gamma_k(t)$ and $c_1 = \max\{l_f, \sum_{j=0}^{\mu-1}|a_j|\}c_\gamma l_{gf}$.

To proceed, $\Delta x_k^*(t)$ in (10) is written as, for $1 \le t \le N+1$

$$\Delta x_k^*(t) = f(x^*(t-1), u^*(t-1)) - f(x_k(t-1), u_k(t-1)).$$

Taking norms on both sides gives

$$|\Delta x_k^*(t)| \le l_f(|\Delta x_k^*(t-1)| + |\Delta u_k^*(t-1)|), \qquad 1 \le t \le N+1$$

which leads to

$$|\Delta x_k^*(t)| \le \sum_{j=0}^{t-1} l_f^{t-1-j}|\Delta u_k^*(j)| + l_f^t|\Delta x_k^*(0)|. \quad (11)$$

Substituting (11) into (10) and using A5) gives rise to, for $1 \le t \le N+1$

$$|\Delta u_{k+1}^*(t)| \le \rho|\Delta u_k^*(t)| + c_2\sum_{j=0}^{t-1}|\Delta u_k^*(j)| + c_2 c_{x_0} \quad (12)$$

where $c_2 = c_1\varpi$, and $\varpi = \max\left\{1, l_f, \ldots, l_f^N\right\}$. The control input errors can be evaluated based on induction under the condition $\rho < 1$. Define $\vartheta = \max_{0 \le t \le N}\{|\Delta u_0^*(t)|\}$, $\alpha = (c_2/1-\rho)c_{x_0}$ and $\beta = (c_2/1-\rho) + 1$. For the first instant $t = 0$, substituting $t = 0$ into (10) produces

$$\begin{aligned}|\Delta u_k^*(0)| \le& |\Delta u_0^*(0)| + \frac{c_2}{1-\rho}c_{x_0} \le \vartheta + \alpha, \qquad k \ge 1 \\ \limsup_{k\to\infty}|\Delta u_k^*(0)| \le& \frac{c_2}{1-\rho}c_{x_0} = \alpha.\end{aligned}$$

For the second instant $t=1$, substituting $t=1$ into (12) produces

$$\begin{aligned}|\Delta u_k^*(1)| \le& |\Delta u_0^*(1)| + \frac{c_2}{1-\rho}(\vartheta+\alpha)\\ &+ \frac{c_2}{1-\rho}c_{x_0} \le (\vartheta+\alpha)\beta, \qquad k\ge 1\\ \limsup_{k\to\infty}|\Delta u_k^*(1)| \le& \frac{c_2}{1-\rho}\alpha + \frac{c_2}{1-\rho}c_{x_0} = \alpha\beta.\end{aligned}$$

Now, assume the validity for the instant $t=i-1$

$$\begin{aligned}|\Delta u_k^*(i-1)| \le& (\vartheta+\alpha)\beta^{i-1}, \qquad k\ge 1\\ \limsup_{k\to\infty}|\Delta u_k^*(i-1)| \le& \alpha\beta^{i-1}.\end{aligned}$$

We wish to show that the result is true for the instant $t=i$. From (12) and the aforementioned inductive hypothesis, we obtain

$$\begin{aligned}|\Delta u_k^*(i)| \le& |\Delta u_0^*(i)| + \frac{c_2}{1-\rho}\\ &\times\left[\vartheta+\alpha+\cdots+(\vartheta+\alpha)\beta^{i-1}\right]\\ &+\frac{c_2}{1-\rho}c_{x_0}\\ \le& (\vartheta+\alpha)\\ &\times\left[1+(\beta-1)(1+\beta+\cdots+\beta^{i-1})\right]\\ =& (\vartheta+\alpha)\beta^i, \qquad k\ge 1\\ \limsup_{k\to\infty}|\Delta u_k^*(i)| \le& \frac{c_2}{1-\rho}\left[\alpha+\cdots+\alpha\beta^{i-1}\right]+\frac{c_2}{1-\rho}c_{x_0}\\ =& \alpha\left[1+(\beta-1)(1+\beta+\cdots+\beta^{i-1})\right]\\ =& \alpha\beta^i.\end{aligned}$$

Therefore, for $0\le t\le N$, we have

$$|\Delta u_k^*(t)| \le (\vartheta+\alpha)\beta^t, \qquad k\ge 1 \tag{13}$$

$$\limsup_{k\to\infty}|\Delta u_k^*(t)| \le \alpha\beta^t. \tag{14}$$

The error $\Delta x_k^*(t)$, $1\le t\le N+1$, can be evaluated from (11). The result for the error $y^*(t)-y_k(t)$, $\mu\le t\le N+\mu$, follows from (5) and assumption A4). This completes the proof. ■

Remark 3.2: Control input $u_k(t)$ has no effect on outputs $y_k(t+j)$, $0\le j\le \mu-1$. The boundedness of $y_k(t)$, $0\le t\le\mu-1$, is ensured by the resetting requirement. Namely, $|y^*(t)-y_k(t)|\le|g\circ\bar f^t(x_0)-g\circ\bar f^t(x_k(0))|\le l_{gf}c_{x_0}$, $0\le t\le\mu-1$.

Remark 3.3: From (8) and (9), the converged output trajectory depends on the choice of parameters a_j, $0\le j\le\mu-1$. Obviously, the suitable choice of these parameters leads to that the converged output trajectory to track the desired one asymptotically as time increases.

B. Initial Rectifying Action

Introducing initial rectifying action into (6), the following updating law is achieved:

$$\begin{aligned}u_{k+1}(t) =& u_k(t)+\gamma_k(t)\sum_{j=0}^{\mu}a_j(y_d(t+j)-y_k(t+j))\\ &-\gamma_k(t)\sum_{j=0}^{\mu-1}\sum_{s=0}^{\mu-1}\theta(t+j-s)a_j(y_d(s)-y_k(s))\end{aligned} \tag{15}$$

where $\theta(t)$ is a scalar function defined as

$$\theta(t)=\begin{cases}1 & t=0\\ 0 & t\ne 0.\end{cases} \tag{16}$$

The added term will ensure the convergence of the system output to the desired trajectory in the sense that $y_k(t)\to y_d(t)$, $\mu\le t\le N+\mu$, as $k\to\infty$. The merging occurs at the moment $t=\mu$. The following theorem presents such a converged output trajectory.

Theorem 3.2: Let system (1) and (2) satisfy assumptions A1)–A5) and updating law (15) be applied. If the learning gain is chosen such that (7) holds and the trajectory $y^*(t)$ is realizable, where $y^*(t)$ is given by

$$y^*(t)=y_d(t)-\sum_{j=0}^{\mu-1}\theta(t-j)(y_d(j)-g\circ\bar f^j(x_0)) \tag{17}$$

then asymptotic bound of the error $y^*(t)-y_k(t)$ is proportional to c_{x_0} for $\mu\le t\le N+\mu$ as $k\to\infty$. Furthermore, the error converges to zero for $\mu\le t\le N+\mu$ whenever c_{x_0} tends to zero.

Remark 3.4: From (17), the converged output trajectory satisfies

$$y^*(t)=g\circ\bar f^t(x_0), \qquad 0\le t\le\mu-1 \tag{18}$$

$$y^*(t)=y_d(t), \qquad t\ge\mu. \tag{19}$$

Proof of Theorem 3.2: By (16), (18), and (19), the following equality is satisfied:

$$\begin{aligned}&\sum_{j=0}^{\mu}a_j(y_d(t+j)-y^*(t+j))\\ &\qquad-\sum_{j=0}^{\mu-1}\sum_{s=0}^{\mu-1}\theta(t+j-s)a_j(y_d(s)-y^*(s))=0.\end{aligned}$$

Thus

$$\begin{aligned}u_{k+1}(t)=&u_k(t)+\gamma_k(t)\sum_{j=0}^{\mu}a_j(y^*(t+j)-y_k(t+j))\\ &-\gamma_k(t)\sum_{j=0}^{\mu-1}\sum_{s=0}^{\mu-1}\theta(t+j-s)a_j(y^*(s)-y_k(s))\end{aligned}$$

which implies that

$$\begin{aligned}\Delta u_{k+1}^*(t)=&\Delta u_k^*(t)-\gamma_k(t)\\ &\times\left[g\circ\bar f^{\mu-1}(f(x^*(t),u^*(t)))\right.\\ &\left.-g\circ\bar f^{\mu-1}(f(x_k(t),u_k(t)))\right]\\ &-\gamma_k(t)\sum_{j=0}^{\mu-1}a_j\left[g\circ\bar f^j(x^*(t))-g\circ\bar f^j(x_k(t))\right]\\ &+\gamma_k(t)\sum_{j=0}^{\mu-1}\sum_{s=0}^{\mu-1}\theta(t+j-s)a_j\\ &\times\left[g\circ\bar f^s(x_0)-g\circ\bar f^s(x_k(0))\right].\end{aligned}$$

From the mean value theorem, there exists $\bar u_k(t)=\xi u^*(t)+(1-\xi)u_k(t)$, $\xi\in[0,1]$, such that

$$\begin{aligned}\Delta u_{k+1}^*(t)=&\left(1-\gamma_k(t)\frac{\partial}{\partial u}\left[g\circ\bar f^{\mu-1}(f(x_k(t),\bar u_k(t)))\right]\right)\\ &\times\Delta u_k^*(t)-\gamma_k(t)\\ &\times\left[g\circ\bar f^{\mu-1}(f(x^*(t),u^*(t)))\right.\\ &\left.-g\circ\bar f^{\mu-1}(f(x_k(t),u^*(t)))\right]\\ &-\gamma_k(t)\sum_{j=0}^{\mu-1}a_j\left[g\circ\bar f^j(x^*(t))-g\circ\bar f(x_k(t))\right]\\ &+\gamma_k(t)\sum_{j=0}^{\mu-1}\sum_{s=0}^{\mu-1}\theta(t+j-s)a_j\\ &\times\left[g\circ\bar f^s(x_0)-g\circ\bar f^s(x_k(0))\right].\end{aligned}$$

Performing the norm operation for both sides of the previous equation gives rise to

$$\begin{aligned}|\Delta u_{k+1}^*(t)| \le& \rho\,|\Delta u_k^*(t)| + c_\gamma l_{gf} l_f\,|\Delta x_k^*(t)| \\ &+ c_\gamma \sum_{j=0}^{\mu-1} |a_j| l_{gf}\,|\Delta x_k^*(t)| \\ &+ c_\gamma \sum_{j=0}^{\mu-1}\sum_{s=0}^{\mu-1} \theta(t+j-s)|a_j|\,|\Delta x_k^*(0)| \\ \le& \rho\,|\Delta u_k^*(t)| + c_1\left(|\Delta x_k^*(t)| + |\Delta x_k^*(0)|\right) \qquad (20)\end{aligned}$$

where c_γ is the norm bound for $\gamma_k(t)$ and $c_1 = \max\left\{l_f, \sum_{j=0}^{\mu-1}|a_j|, \sum_{j=0}^{\mu-1}\mu|a_j|\right\} c_\gamma l_{gf}$.

Through the same derivation to arrive at (11) leads to, for $1 \le t \le N+1$

$$|\Delta x_k^*(t)| \le \sum_{j=0}^{t-1} l_f^{t-1-j}\,|\Delta u_k^*(j)| + l_f^t\,|\Delta x_k^*(0)|. \qquad (21)$$

Substituting (21) into (20) and using A5) result in, for $1 \le t \le N$

$$|\Delta u_{k+1}^*(t)| \le \rho\,|\Delta u_k^*(t)| + c_2 \sum_{j=0}^{t-1} |\Delta u_k^*(j)| + c_2 c_{x_0} \qquad (22)$$

where $c_2 = c_1(1+\varpi)$, and $\varpi = \max\left\{1, l_f, \ldots, l_f^N\right\}$.

We can see that (22) corresponds to (12) in the proof of Theorem 3.1. The rest of the proof is exactly the same as that of Theorem 3.1 after (12). ■

C. Design Issue

Denote $d_k(t) = (\partial/\partial u)\left[g \circ \bar{f}^{\mu-1}(f(x_k(t), \bar{u}_k(t))\right]$ and note that $d_k(t)$ includes uncertain variable $\bar{u}_k(t)$. However, the design issue can be argued when $d_1 < d_k(t) < d_2$. According to (7), one convergence range of $\gamma_k(t)$ is $\gamma_k(t) \in (0, 2/d_2)$ if $d_1 > 0$ or $\gamma_k(t) \in (2/d_1, 0)$ if $d_2 < 0$. Thus, the learning gain is chosen as follows:

$$\gamma_k(t) = \begin{cases} \alpha\frac{2}{d_2}, & \text{if } d_1 > 0 \\ \alpha\frac{2}{d_1}, & \text{if } d_2 < 0 \end{cases} \qquad (23)$$

where $\alpha \in (0,1)$ is an adjustable parameter. When the system dynamics described by (1) is affine in $u(t)$, i.e., $x(t+1) = f(x(t)) + b(x(t))u(t)$, and has relative degree μ, the system output expressed by (4) and (5) can be rewritten as $y(t+j) = g \circ f^j(x(t))$, $1 \le j \le \mu-1$, and $y(t+\mu) = g \circ f^{\mu-1}(f(x(t)+b(x(t))u(t))$. Furthermore, provided $(\partial^2/\partial u^2)\left[g \circ f^{\mu-1}(f(x)+b(x)u)\right] = 0$, the term $(\partial/\partial u) g \circ f^{\mu-1}(f(x)+b(x)u)$ will be independent of u and can be denoted as $d(x)$. For this case, the output can be evaluated as $y(t+\mu) = g \circ f^{\mu-1}(f(x(t))+b(x(t))u)|_{u=0} + \int_0^{u(t)} (\partial/\partial u) g \circ f^{\mu-1}(f(x(t))+b(x(t))u)du = g \circ f^\mu(x(t)) + d(x(t))u(t)$ and (7) reduces to $|1-\gamma_k(t)d(x_k(t))| \le \rho < 1$. The learning control design becomes straightforward and one alternative is

$$\gamma_k(t) = \alpha\,\mathrm{sgn}(d(x_k(t))) \qquad (24)$$

and $\alpha \in (0, 2/|d(x_k(t))|)$. In the case of $d_1 < d(x_k(t)) < d_2$, the design of (23) is still applicable.

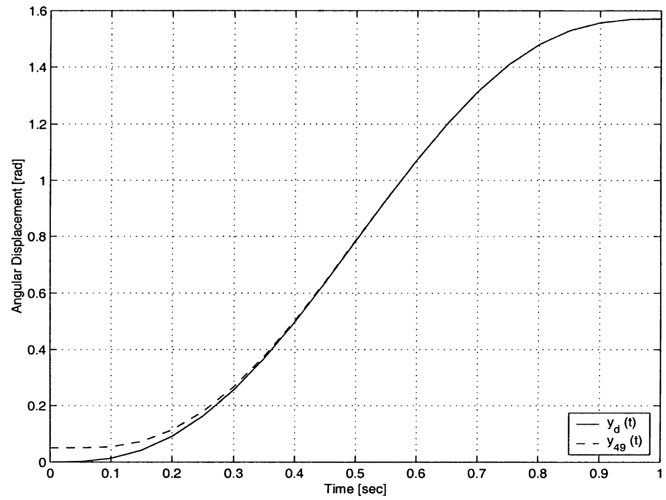

Fig. 1. Resultant output trajectory by (26).

IV. An Illustrative Example

In this section, we are going to illustrate the theoretical results with a pendulum. The Euler's approximation of the dynamics takes the following state-space representation:

$$\begin{bmatrix} q(t+1) \\ \dot{q}(t+1) \end{bmatrix} = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} + h\begin{bmatrix} \dot{q}(t) \\ -\dot{q}(t) - \sin(q(t)) + u(t) \end{bmatrix} \qquad (25)$$

where $q(t) = q(th)$, $\dot{q}(t) = \dot{q}(th)$, $u(t) = u(th)$ and h is the sampling period. The system has relative degree two if only the angle displacement is available. Let the desired trajectory be given as $q_d(t) = \pi/2(6(th)^5 - 15(th)^4 + 10(th)^3)$ rad, $0 \le t \le N+2$, $N = 18$, and the sampling period be $h = 0.05$ s. The initial conditions at each cycle are set to $q_k(0) = 0.05$ and $\dot{q}_k(0) = 0.01$. The proposed learning schemes are used in the presence of the initial shift. The first applied is the following updating law:

$$u_{k+1}(t) = u_k(t) + \frac{\alpha}{h^2}\left(e_k(t+2) - 1.2e_k(t+1) + 0.36e_k(t)\right) \qquad (26)$$

where $e_k(t) = y_d(t) - y_k(t), y_d(t) = q_d(t), y_k(t) = q_k(t)$, α is an adjustable parameter and chosen as $\alpha = 0.2$. Fig. 1 depicts the resultant trajectory of the 49th cycle versus the discrete-time variable ht, being a solution of (8). The converged trajectory is observed to follow the desired one asymptotically as time increases. In order to achieve the convergence over the specified interval, initial rectifying action is introduced in the updating law (26) as follows:

$$\begin{aligned}u_{k+1}(t) =& u_k(t) + \frac{\alpha}{h^2} \\ &\times\left(e_k(t+2) - 1.2e_k(t+1) + 0.36e_k(t)\right) \\ &- \frac{\alpha}{h^2}\left[0.36\theta(t)e_k(0)\right. \\ &\left.+ (0.36\theta(t-1) - 1.2\theta(t))e_k(1)\right]. \qquad (27)\end{aligned}$$

Define the performance index $J_k = \max_{2\le t\le N+2}|e_k(t)|$. The iteration stops after the 46th cycle as $J_k < 0.001$ and the resultant trajectory is shown in Fig. 2, where the trajectory tracks the desired one after two steps delay only. In our simulation, the robustness in the presence of random initial shifts shown in Theorems 3.1 and 3.2 are examined.

V. Conclusion

The unified learning scheme for the class of nonlinear discrete-time systems with well-defined relative degree has been characterized to ad-

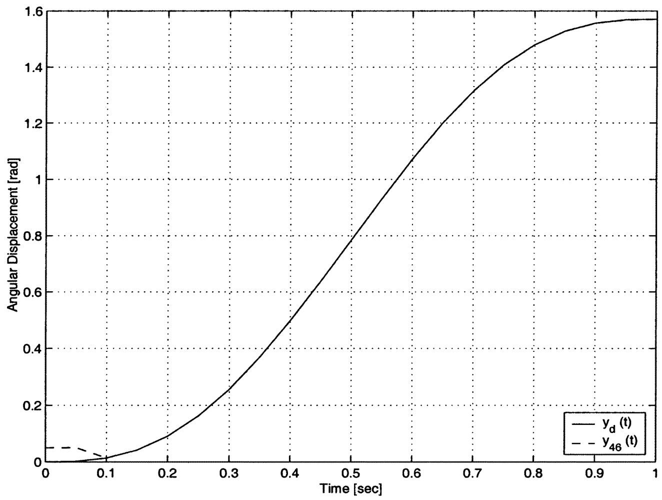

Fig. 2. Resultant output trajectory by (27).

dress the initial shift problem. The convergence and robustness properties of the scheme with respect to initial shifts have been presented by the developed analysis technique. Under certain conditions, the system output is ensured to converge to a neighborhood of the predefined trajectory and the error bound is proportional to the bound on initial shifts. The system undertaken has been shown to possess asymptotic tracking capability and the converged output trajectory can be assessed by the initial condition. The initial rectifying action has been shown effective to improve the tracking performance further, by which the complete tracking with specified transient is guaranteed.

References

[1] S. Arimoto, "Learning control theory for robotic motion," *Int. J. Adapt. Control Signal Processing*, vol. 4, pp. 543–564, 1990.

[2] D.-H. Hwang, Z. Bien, and S.-R. Oh, "Iterative learning control method for discrete-time dynamic systems," *Proc. Inst. Elect. Eng., Part D*, vol. 138, no. 2, pp. 139–144, 1991.

[3] S. S. Saab, "Robustness and convergence rate of a discrete-time learning control algorithm for a class of nonlinear systems," *Int. J. Robust Nonlinear Control*, vol. 9, pp. 559–571, 1999.

[4] T.-J. Jang, H.-S. Ahn, and C.-H. Choi, "Iterative learning control for discrete-time nonlinear systems," *Int. J. Syst. Sci.*, vol. 25, no. 7, pp. 1179–1189, 1994.

[5] C. J. Chien, "A discrete iterative learning control for a class of nonlinear time-varying systems," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 748–752, May 1998.

[6] D. Wang, "Convergence and robustness of discrete time nonlinear systems with iterative learning control," *Automatica*, vol. 32, no. 11, pp. 1445–1448, 1998.

[7] J.-X. Xu, "Analysis of iterative learning control for a class of nonlinear discrete-time systems," *Automatica*, vol. 33, no. 10, pp. 1905–1907, 1997.

[8] H.-S. Lee and Z. Bien, "Study on robustness of iterative learning control with nonzero initial error," *Int. J. Control*, vol. 64, no. 3, pp. 345–359, 1996.

[9] K.-H. Park and Z. Bien, "A generalized iterative learning controller against initial state error," *Int. J. Control*, vol. 73, no. 10, pp. 871–881, 2000.

[10] Y. Chen, C. Wen, Z. Gong, and M. Sun, "An iterative learning controller with initial state learning," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 371–376, Feb. 1999.

[11] M. Sun and D. Wang, "Iterative learning control with initial rectifying action," *Automatica*, vol. 38, no. 7, pp. 1177–1182, 2002.

[12] ——, "Closed-loop iterative learning control for nonlinear systems with initial shifts," *Int. J. Adapt. Control Signal Processing*, vol. 16, pp. 515–538, 2002.

[13] C.-H. Choi and G.-M. Jeong, "Perfect tracking for maximum-phase nonlinear systems by iterative learning control," *Int. J. Syst. Sci.*, vol. 32, no. 9, pp. 1177–1183, 2001.

[14] S. Hillenbrand and M. Pandit, "An iterative learning controller with reduced sampling rate for plants with variations of initial states," *Int. J. Control*, vol. 73, no. 10, pp. 882–889, 2000.

[15] M. Sun and D. Wang, "Analysis of nonlinear discrete-time systems with higher-order iterative learning control," *Dyna. Control*, vol. 11, no. 1, pp. 81–96, 2001.

[16] ——, "Robust discrete-time iterative learning control: Initial shift problem," in *Proc. 40th IEEE Conf. Decision Control*, Orlando, FL, Dec. 2001, pp. 1211–1216.

[17] S. Monaco and D. Normand-Cyrot, "Minimum phase nonlinear discrete-time systems and feedback stabilization," in *Proc. 26th IEEE Conf. Decision Control*, Los Angeles, CA, Dec. 1987, pp. 979–986.

[18] H. Nijmeijer and A. J. van der Schaft, *Nonlinear Dynamical Control Systems*. New York: Springer-Verlag, 1990.

Joint Optimization of Communication Rates and Linear Systems

Lin Xiao, Mikael Johansson, Haitham Hindi, Stephen Boyd, and Andrea Goldsmith

***Abstract*—We consider a linear control system in which several signals are transmitted over communication channels with bit rate limitations. With the coding and medium access schemes of the communication system fixed, the achievable bit rates are determined by the allocation of communications resources such as transmit powers and bandwidths, to different communication channels. We model the effect of bit rate limited communication channels by uniform quantization and the quantization errors are modeled by additive white noises whose variances depend on the achievable bit rates. We optimize the stationary performance of the linear system by jointly allocating resources in the communication system and tuning parameters of the controller.**

***Index Terms*—Communication systems, control over networks, convex optimization, quantization noise, resource allocation.**

I. Introduction

We consider a linear system in which several signals are transmitted over wireless communication channels, as illustrated in Fig. 1. All signals are vector-valued: w is a vector of exogenous signals (such as disturbances or noises acting on the system); z is a vector of performance signals (including error signals and actuator signals); and y and

Manuscript received June 26, 2001; revised April 10, 2002 and August 13, 2002. Recommended by Associate Editor B. Chen. This work was supported in part by the Air Force Office of Scientific Research under Grant F49620-01-1-0365, in part by the Defense Advanced Research Projects Agency (DARPA) under Contract F33615-99-C-3014, in part by the National Science Foundation under Grant CCR-9714002, in part by the Caltrans PATH program under Contract MOV4212, in part by the Swedish Foundation for International Cooperation in Higher Education (STINT), and in part by a Stanford University Graduate Fellowship.

L. Xiao is with the Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305 USA (e-mail: lxiao@stanford.edu).

M. Johansson is with the Department of Signals, Sensors, and Systems, Royal Institute of Technology, SE-100 44 Stockholm, Sweden (e-mail: mikaelj@s3.kth.se).

H. Hindi, S. Boyd, and A. Goldsmith are with the Department of Electrical Engineering, Stanford University, Stanford, CA 04305 USA (e-mail: hhindi@stanford.edu; boyd@stanford.edu; andrea@systems.stanford.edu).

Digital Object Identifier 10.1109/TAC.2002.806669

0018-9286/03$17.00 © 2003 IEEE