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Iterative Learning Control Design for Uncertain Dynamic Systems with Delayed States

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Abstract. Most available results on iterative learning control address trajectory tracking problem for systems without time delay. The effect of time delay on tracking performance is not yet fully understood. This paper is concerned with iterative learning control design for the systems with delayed states, where the system class under consideration is confined by a defined relative degree. A learning control scheme is proposed to overcome the uncertainties in delay times and/or in system parameters. Robustness of the learning control is established in the presence of initial function errors. Furthermore, uniform convergence of system outputs to the desired trajectory is ensured if the initial function at each cycle is set to align with the desired initial function. Validity of the results is demonstrated through numerical simulations.

Keywords: learning control, robustness, relative degree, time delay, tracking systems

1. Introduction

Iterative learning control (ILC) is characterized by repositioning, input updating and resultant zero-error tracking over the entire operation interval, by which the imperfect knowledge of the dynamics structure and/or parameter values are overcome. Even though the knowledge of system dynamics is totally unknown, better tracking performance can be still expected by using *trial and error* approach to find out the desired input profile iteratively [2]. For quite a long period, this control methodology has received a great deal of attention from researchers [4], [7], [16], [21]. Typically, researchers dealt with convergence and robustness issues of learning controls by developing different analysis techniques, see, e.g., [3], [5], [8], [10], [11], [15], [19], [24], [25].

To learning control design, it is logical that next input action is updated based on the action and its produced results in the previous cycle. The key is to determine the pair of related cause and effect of the system under consideration. Obviously, the control input and the system output can be such a pair when the system has direct transmission term. This fundamental characteristic has been fully investigated in [20]. In particular, analysis for linear systems with relative degree higher than one has been delineated, and the output derivative with the order being equal to relative degree of the system, is used to update control input. Recently, ILC for a class of nonlinear systems with higher relative degree has been reported in [1], [14].

Up to now, most works focus on systems without time delay. However, delays are inherent in many applications, such as batch processes, and remote controlled robots, vehicles or man-machine systems. Conventional controls are usually found to be unsatisfactory because of inaccuracy in estimation and/or uncertainty of time delays [18]. The main difficulty in dealing with the kind of systems is that the state should be properly considered in an infinite dimensional space. It is natural to use iterative learning control to overcome such difficulties. Convergence issue of the learning control was investigated for LTI systems with delayed state in [12] and delayed input in [18]. A higher-order learning algorithm for a class of nonlinear systems with delayed states was studied in [6], where the initial condition was considered to be somewhat obscure. Recently, the role of delayed states in the learning process was characterized in [22]. It was observed that the effect of delayed states on the learning process lies in the requirement on initial function over entire initial interval. Results in the above mentioned papers are confined to systems with relative degree one. None of the papers, however, have considered time delay systems with higher relative degree, and not even for the linear situation. It should be noted that there have been several attempts to address the linearization problem of nonlinear time delay systems [9], [17], [23].

This paper aims to apply iterative learning control methodology to systems with delayed states. Extended relative degree of the systems is defined for the design purpose and learning control method is proposed based on the pair of action taken and its resulting variable. Analyses are provided for a class of nonlinear systems. We shall show that under certain conditions, uniform convergence of output trajectories to the desired one can be guaranteed in the absence of initial function errors. If the initial function at each cycle is deviated from the desired initial function within an admissible level, the output errors will be asymptotically bounded. Numerical simulation is presented to illustrate the theoretical results.

2. System Dynamics and Extended Relative Degree

Consider the class of nonlinear systems with delayed states described by

$$\dot{x}(t) = f(x(t), x(t-\tau), \ldots, x(t-l\tau)) + B(x(t))u(t) \tag{1}$$

$$y(t) = g(x(t)) \tag{2}$$

where $x \in R^n$, $u = [u_1, \ldots, u_r]^T \in R^r$, and $y = [y_1, \ldots, y_m]^T \in R^m$ denote the state, control input, and output of the system, respectively. The functions $f \in R^n$, $B = [b_1, \ldots, b_r] \in R^{n \times r}$ and $g = [g_1, \ldots, g_m]^T \in R^m$ are smooth in their domains of definition which are only known of certain properties. Each delay of the system takes an integral multiple of the fixed delay time τ. For $t \in [-l\tau, 0]$, $x(t) = \psi(t)$ and $\psi(t)$ is the initial function. The systems perform operations repeatedly over the finite interval $[0, T]$. For each fixed $\psi(t)$, S denotes a mapping from $(\psi(t), t \in [-l\tau, 0], u(t), t \in [0, T])$ to $(x(t), t \in [0, T])$ and O a mapping from $(\psi(t), t \in [-l\tau, 0], u(t), t \in [0, T])$ to $(y(t), t \in [0, T])$. In these notations, $x(t) = S(\psi(t), u(t))$ and $y(t) = O(\psi(t), u(t))$.

Let σ denote the pure time delay operator, introduced in [17], that shifts the time t into $t - \tau$, which is defined as, for a function $a(\cdot)$ on the interval $[t - \tau, t]$,

$$\sigma a(t) = a(t - \tau)$$

and for a function $a_1(\cdot)$ which is a function of $a_2(\cdot)$ defined on the interval $[t - \tau, t]$,

$$\sigma a_1(a_2(t)) = a_1(a_2(t - \tau))$$

Obviously, pure time delay operator has the following properties, for integer $i \geq 1$,

$$\sigma^i = \sigma \sigma^{i-1}$$

where $\sigma^0 = 1$, and for integers $i_1, i_2 > 0$,

$$\begin{aligned} \sigma^{i_1}\sigma^{i_2} &= \sigma^{i_2}\sigma^{i_1} \\ \sigma^{i_1+i_2} &= \sigma^{i_1}\sigma^{i_2} \end{aligned}$$

Then system (1)–(2) can be rewritten as

$$\dot{x}(t) = \bar{f}(x, \sigma) + B(x(t))u(t) \tag{3}$$

$$y(t) = g(x(t)) \tag{4}$$

where $\bar{f}(x, \sigma) = f(x(t), x(t - \tau), \ldots, x(t - l\tau))$.

The following notation of derivatives and definition are also needed. The derivative of a scalar function $g(x)$ along the vector $\bar{f}(x, \sigma)$ is defined as, for $x \in R^n$,

$$L_{\bar{f}} g(x, \sigma) = \frac{\partial g(x)}{\partial x} \bar{f}(x, \sigma)$$

Furthermore, the derivative of a scalar function $\bar{h}(x, \sigma) = h(x(t), x(t-\tau), \ldots, x(t-l'\tau))$ along the vector $\bar{f}(x, \sigma)$ is defined as

$$L_{\bar{f}} \bar{h}(x, \sigma) = \sum_{i=0}^{l'} \frac{\partial \bar{h}(x, \sigma)}{\partial \sigma^i x} \sigma^i \bar{f}(x, \sigma)$$

The repeated derivatives along the vector $\bar{f}(x, \sigma)$ are denoted as

$$L_{\bar{f}}^j g(x, \sigma) = \sum_{i=0}^{(j-1)l} \frac{\partial L_{\bar{f}}^{j-1} g(x, \sigma)}{\partial \sigma^i x} \sigma^i \bar{f}(x, \sigma),\ j = 1, 2, \ldots$$

Definition 2.1 Extended relative degree of systems with delayed states (1)–(2) is the vector $\mu = \{\mu_1, \ldots, \mu_m\}$ that satisfies, for $x \in R^n$,

$$\begin{aligned} L_{b_p} L_{\bar{f}}^i g_q(x, \sigma) &= 0,\ 0 \leq i \leq \mu_q - 2 \\ \frac{\partial L_{\bar{f}}^{\mu_q - 1} g_q(x, \sigma)}{\partial \sigma^i x} \sigma^i b_p(x) &= 0,\ 1 \leq i \leq (\mu_q - 1)l \end{aligned}$$

where $1 \le p \le r$, $1 \le q \le m$, and the $m \times r$ matrix

$$D(x) = \begin{bmatrix} \frac{\partial L_{\bar{f}}^{\mu_1 - 1} g_1(x,\sigma)}{\partial x} b_1(x) & \cdots & \frac{\partial L_{\bar{f}}^{\mu_1 - 1} g_1(x,\sigma)}{\partial x} b_r(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial L_{\bar{f}}^{\mu_m - 1} g_m(x,\sigma)}{\partial x} b_1(x) & \cdots & \frac{\partial L_{\bar{f}}^{\mu_m - 1} g_m(x,\sigma)}{\partial x} b_r(x) \end{bmatrix}$$

has full column rank.

Remark 2.1 If a system described by (1)–(2) has extended relative degree $\mu = \{\mu_1, \ldots, \mu_m\}$, the derivatives of system output can be written as, for $1 \le q \le m$,

$$y_q^{(i)} = L_{\bar{f}}^{i} g_q(x,\sigma), 0 \le i \le \mu_q - 1$$

$$y_q^{(\mu_q)} = L_{\bar{f}}^{\mu_q} g_q(x,\sigma) + \left[\frac{\partial L_{\bar{f}}^{\mu_q - 1} g_q(x,\sigma)}{\partial x} b_1(x), \ldots, \frac{\partial L_{\bar{f}}^{\mu_q - 1} g_q(x,\sigma)}{\partial x} b_r(x) \right] u \tag{5}$$

It implies that μ_q is the minimum order of time derivative of the qth output to which a directly transmission is established from at least one component of the control input u, and the level of the output related to the input but decoupled from the delayed-input variables.

Remark 2.2 Definition 2.1 allows the number of outputs greater than the number of inputs. If there exists no delayed state in system dynamics, Definition 2.1 reduces to that given in [13] for the case where number of outputs is same as number of inputs.

Remark 2.3 As a special case, we consider the following linear systems with delayed states

$$\dot{x}(t) = \sum_{i=0}^{l} A_i x(t - i\tau) + Bu(t) \tag{6}$$

$$y(t) = Cx(t) \tag{7}$$

where $A \in R^{n \times n}$, $B = [b_1, \ldots, b_r] \in R^{n \times r}$ and $C = [c_1^T, \ldots, c_m^T]^T \in R^{m \times n}$. If the system has relative degree $\{\mu_1, \ldots, \mu_m\}$, we have, for $1 \le p \le r$, $1 \le q \le m$,

$$\begin{aligned} c_q b_p &= 0, \\ c_q A_{i_1} b_p &= 0, 0 \le i_1 \le l, \\ &\vdots \\ c_q A_{i_1} \cdots A_{i_{\mu_q - 2}} b_p &= 0, 0 \le i_1, \ldots, i_{\mu_q - 2} \le l \\ c_q A_{i_1} \cdots A_{i_{\mu_q - 1}} b_p &= 0, 0 \le i_1, \ldots, i_{\mu_q - 1} \le l (i_1 + \cdots + i_{\mu_q - 1} \ne 0) \end{aligned}$$

and the $m \times r$ matrix

$$D = \begin{bmatrix} c_1 A_0^{\mu_1 - 1} b_1 & \cdots & c_1 A_0^{\mu_1 - 1} b_r \\ \vdots & \vdots & \vdots \\ c_m A_0^{\mu_m - 1} b_1 & \cdots & c_m A_0^{\mu_m - 1} b_r \end{bmatrix}$$

has full column rank. The derivatives of the qth component of $y(t)$ will take the form of

$$\begin{aligned}
\dot{y}_q(t) &= \sum_{i_1=0}^{l} c_q A_{i_1} \sigma^{i_1} x(t), \\
\ddot{y}_q(t) &= \sum_{i_1=0}^{l} \sum_{i_2=0}^{l} c_q A_{i_1} A_{i_2} \sigma^{i_1+i_2} x(t), \\
&\vdots \\
y_q^{(\mu_q-1)}(t) &= \sum_{i_1=0}^{l} \cdots \sum_{i_{\mu_q-1}=0}^{l} c_q A_{i_1} \cdots A_{i_{\mu_q-1}} \sigma^{i_1+\cdots+i_{\mu_q-1}} x(t) \\
y_q^{(\mu_q)}(t) &= \sum_{i_1=0}^{l} \cdots \sum_{i_{\mu_q}=0}^{l} c_q A_{i_1} \cdots A_{i_{\mu_q}} \sigma^{i_1+\cdots+i_{\mu_q}} x(t) \\
&\quad + \sum_{i_1=0}^{l} \cdots \sum_{i_{\mu_q-1}=0}^{l} c_q A_{i_1} \cdots A_{i_{\mu_q-1}} B \sigma^{i_1+\cdots+i_{\mu_q-1}} u(t) \\
&= \sum_{i_1=0}^{l} \cdots \sum_{i_{\mu_q}=0}^{l} c_q A_{i_1} \cdots A_{i_{\mu_q}} \sigma^{i_1+\cdots+i_{\mu_q}} x(t) + c_q A_0^{\mu_q-1} B u(t)
\end{aligned}$$

We are now at the position to formulate the control problem to be solved as follows. Given a realizable trajectory $y_d(t)$, $t \in [0, T]$ and a tolerance error bound $\varepsilon > 0$, find a control input $u(t)$, $t \in [0, T]$, by applying an iterative learning control technique, so that the error between the system output $y(t)$ and the desired trajectory $y_d(t)$ is within the tolerance error bound, i.e.,

$$\|y_d(t) - y(t)\| < \varepsilon, \quad t \in [0, T]$$

where $\|\cdot\|$ is the vector norm defined as $\|a\| = \max_{1\le i\le n} |a_i|$ for an $n-$dimensional vector $a = [a_1, \ldots, a_n]^T$. Throughout the paper, for a matrix $A = \{a_{ij}\} \in R^{m\times n}$, the matrix norm is defined as the induced norm by the vector norm, i.e., $\|A\| = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}|$. The $\lambda-$norm for a vector-valued function $b(t) \in R^n$ is defined as

$$\|b(\cdot)\|_\lambda = \sup_{t\in[0,T]} \{e^{-\lambda t}\|b(t)\|\}, \quad \lambda > 0.$$

3. Robust ILC

To solve the formulated problem, we use the learning control described by the following updating law

$$u_{k+1}(t) = u_k(t) + \Gamma_k(t)(y_d^{(\mu)}(t) - y_k^{(\mu)}(t)) \tag{8}$$

where $t \in [0, T]$, k indicates the number of operation cycle, $y_d^{(\mu)}(t) = [y_{1,d}^{(\mu_1)}(t), \ldots, y_{m,d}^{(\mu_m)}(t)]^T$, $y_k^{(\mu)}(t) = [y_{1,k}^{(\mu_1)}(t), \ldots, y_{m,k}^{(\mu_m)}(t)]^T$ and $\Gamma_k(t) \in R^{r\times m}$ is the learning gain to be designed.

Remark 3.1 For a learning control design, it is logical that next input action is updated based on actions and their produced results in the previous operation cycle. In view of (5), $\{u(t), y^{(\mu)}(t)\}$ is a pair of algebraically related cause and effect. This straightforward and strong coupling relation may lead to faster convergence rate and easier way for the learning gain selection. This observation is lent to constitute the updating law (8).

The following assumptions on system (1)–(2) are imposed.

(A1) The mappings S and O are one to one.

(A2) The system has extended relative degree $\mu = \{\mu_1, \ldots, \mu_m\}$ for $x \in R^n$.

(A3) The functions $\bar{f}$, B, g, $L_{\bar{f}}^{\mu_q} g_q$, $1 \le q \le m$ and $\frac{\partial L_{\bar{f}}^{\mu_q-1} g_q}{\partial x} b_p$, $1 \le p \le r$, $1 \le q \le m$ are Lipschitz in their arguments with Lipschitz constants l_f, l_B, l_g, l_1 and l_2, respectively.

(A4) The operator B is bounded for $x \in R^n$.

(A5) For a desired trajectory $y_d(t) = [y_{1,d}(t), \ldots, y_{m,d}(t)]^T$, $t \in [0, T]$, $y_{i,d}(t)$ is μ_i times continuously differentiable.

For any realizable trajectory $y_d(t)$, $t \in [0, T]$, (A1) implies that there exists a unique control input $u_d(t)$ which drives the system output to follow the desired trajectory so that

$$y_d(t) = g(x_d(t)) \tag{9}$$

$$\dot{x}_d(t) = \bar{f}(x_d, \sigma) + B(x_d(t))u_d(t) \tag{10}$$

where $x_d(t)$ is the corresponding state with $x_d(t) = \psi_d(t)$, $t \in [-l\tau, 0]$.

For the practical implementation, we would like to know the effect of delayed states on the learning process, in contrast to systems without delayed state for which there have been a number of efforts toward robustness of the learning algorithms. The following theorem specifies that the learning control (8) can be robust against initial function errors when applied to system (1)–(2) with higher relative degree.

THEOREM 3.1 *Let system (1)–(2) satisfy assumptions (A1)–(A5) and the desired trajectory* $y_d(t)$, $t \in [0, T]$ *be realizable. If updating law (8) is applied with the learning gain being designed such that*

$$\|I - \Gamma_k(t)D(x_k(t))\| \le \rho < 1 \tag{11}$$

and, at the beginning of each cycle,

$$\|\psi_d(t) - \psi_k(t)\| \le c_\psi, t \in [-l\tau, 0] \tag{12}$$

the errors $u_d(t) - u_k(t)$, $x_d(t) - x_k(t)$ *and* $y_d(t) - y_k(t)$ *converge asymptotically into the specified bounds being class-K functions of* c_ψ *as* $k \to \infty$.

Proof: For simplicity, the argument t is dropped in the following proof where confusion will not occur. It follows from (5) and (8) that, denoting by $\Delta u_k = u_d - u_k$ and $c(x) = [L_{\bar{f}}^{\mu_1} g_1(x,\sigma), \ldots, L_{\bar{f}}^{\mu_m} g_m(x,\sigma)]^T$,

$$\begin{aligned} \Delta u_{k+1} &= \Delta u_k - \Gamma_k (y_d^{(\mu)} - y_k^{(\mu)}) \\ &= \Delta u_k - \Gamma_k (c(x_d) + D(x_d) u_d - c(x_k) - D(x_k) u_k) \\ &= (I - \Gamma_k D(x_k)) \Delta u_k - \Gamma_k [c(x_d) - c(x_k) + (D(x_d) - D(x_k)) u_d] \end{aligned}$$

Taking norms and applying the bounds and the Lipschitz conditions, we have

$$\begin{aligned} \|\Delta u_{k+1}\| &\le \|I - \Gamma_k D(x_k)\| \|\Delta u_k\| \\ &\quad + \|\Gamma_k\| (\|c(x_d) - c(x_k)\| + \|D(x_d) - D(x_k)\| \|u_d\|) \\ &\le \rho \|\Delta u_k\| + c_\Gamma (\|c(x_d) - c(x_k)\| + \|D(x_d) - D(x_k)\| c_{u_d}) \end{aligned} \tag{13}$$

where c_Γ is the norm bound for Γ_k, $c_{u_d} = \sup_{t \in [0,T]} \|u_d\|$ and, denoting by $\Delta x_k = x_d - x_k$,

$$\begin{aligned} \|c(x_d) - c(x_k)\| &\le \left\| \begin{bmatrix} \|L_{\bar{f}}^{\mu_1} g_1(x_d,\sigma) - L_{\bar{f}}^{\mu_1} g_1(x_k,\sigma)\| \\ \vdots \\ \|L_{\bar{f}}^{\mu_m} g_m(x_d,\sigma) - L_{\bar{f}}^{\mu_m} g_m(x_k,\sigma)\| \end{bmatrix} \right\| \\ &\le l_1 \left\| \begin{bmatrix} \|\Delta x_k(t)\| + \|\Delta x_k(t-\tau)\| + \cdots + \|\Delta x_k(t - \mu_1 l \tau)\| \\ \vdots \\ \|\Delta x_k(t)\| + \|\Delta x_k(t-\tau)\| + \cdots + \|\Delta x_k(t - \mu_m l \tau)\| \end{bmatrix} \right\| \end{aligned} \tag{14}$$

and

$$\begin{aligned} &\|D(x_d) - D(x_k)\| \\ &\le \left\| \begin{bmatrix} \left\| \frac{\partial L_{\bar{f}}^{\mu_1 - 1} g_1(x_d,\sigma)}{\partial x} b_1(x_d) - \frac{\partial L_{\bar{f}}^{\mu_1 - 1} g_1(x_k,\sigma)}{\partial x} b_1(x_k) \right\| & \cdots & \left\| \frac{\partial L_{\bar{f}}^{\mu_1 - 1} g_1(x_d,\sigma)}{\partial x} b_r(x_d) - \frac{\partial L_{\bar{f}}^{\mu_1 - 1} g_1(x_k,\sigma)}{\partial x} b_r(x_k) \right\| \\ \vdots & \vdots & \vdots \\ \left\| \frac{\partial L_{\bar{f}}^{\mu_m - 1} g_m(x_d,\sigma)}{\partial x} b_1(x_d) - \frac{\partial L_{\bar{f}}^{\mu_m - 1} g_m(x_k,\sigma)}{\partial x} b_1(x_k) \right\| & \cdots & \left\| \frac{\partial L_{\bar{f}}^{\mu_m - 1} g_m(x_d,\sigma)}{\partial x} b_r(x_d) - \frac{\partial L_{\bar{f}}^{\mu_m - 1} g_m(x_k,\sigma)}{\partial x} b_r(x_k) \right\| \end{bmatrix} \right\| \\ &\le l_2 \left\| \begin{bmatrix} \|\Delta x_k(t)\| + \|\Delta x_k(t-\tau)\| + \cdots + \|\Delta x_k(t - (\mu_1 - 1) l \tau)\| & \cdots & \|\Delta x_k(t)\| + \|\Delta x_k(t-\tau)\| + \cdots + \|\Delta x_k(t - (\mu_1 - 1) l \tau)\| \\ \vdots & \vdots & \vdots \\ \|\Delta x_k(t)\| + \|\Delta x_k(t-\tau)\| + \cdots + \|\Delta x_k(t - (\mu_m - 1) l \tau)\| & \cdots & \|\Delta x_k(t)\| + \|\Delta x_k(t-\tau)\| + \cdots + \|\Delta x_k(t - (\mu_m - 1) l \tau)\| \end{bmatrix} \right\| \end{aligned} \tag{15}$$

In order to evaluate the state errors of the right hand sides of (14) and (15), we integrate state equations (3) and (10) to give

$$\begin{aligned}\Delta x_k &= \Delta x_k(0) + \int_0^t (\dot{x}_d - \dot{x}_k)\mathrm{d}s \\ &= \Delta x_k(0) + \int_0^t [\bar{f}(x_d, \sigma) + B(x_d)u_d - (\bar{f}(x_k, \sigma) + B(x_k)u_k)]\mathrm{d}s \\ &= \Delta x_k(0) + \int_0^t [\bar{f}(x_d, \sigma) - \bar{f}(x_k, \sigma) + (B(x_d) - B(x_k))u_d + B(x_k)\Delta u_k]\mathrm{d}s\end{aligned}$$

Taking norms and using their properties, we have

$$\begin{aligned}\|\Delta x_k\| &\le \|\Delta x_k(0)\| + \int_0^t (\|\bar{f}(x_d, \sigma) - \bar{f}(x_k, \sigma)\| + \|B(x_d) - B(x_k)\|\|u_d\| \\ &\quad + \|B(x_k)\|\|\Delta u_k\|)\mathrm{d}s \le c_\psi + \int_0^t [l_{\bar{f}}(\|\Delta x_k(s)\| + \|\Delta x_k(s-\tau)\| \\ &\quad + \cdots + \|\Delta x_k(s - l\tau)\|) + l_B c_{ud}\|\Delta x_k(s)\| + c_B\|\Delta u_k\|]\mathrm{d}s \qquad (16)\end{aligned}$$

Note the facts that, for $t \in [0, \theta]$ with $\theta \in \{\tau, \ldots, l\tau\}$,

$$\int_0^t \|\Delta x_k(s-\theta)\|\mathrm{d}s = \int_{-\theta}^{t-\theta} \|\psi_d(s) - \psi_k(s)\|\mathrm{d}s \le \theta c_\psi$$

and for $t \in (\theta, T]$,

$$\begin{aligned}\int_0^t \|\Delta x_k(s-\theta)\|\mathrm{d}s &= \int_{-\theta}^0 \|\psi_d(s) - \psi_k(s)\|\mathrm{d}s + \int_0^{t-\theta} \|\Delta x_k(s)\|\mathrm{d}s \\ &\le \theta c_\psi + \int_0^{t-\theta} \|\Delta x_k(s)\|\mathrm{d}s \qquad (17)\end{aligned}$$

Combining (17) and (18) yields, for $t \in [0, T]$,

$$\int_0^t \|\Delta x_k(s-\theta)\|\mathrm{d}s \le l\tau c_\psi + \int_0^t \|\Delta x_k(s)\|\mathrm{d}s \qquad (18)$$

Substituting (18) into (16) gives rise to

$$\|\Delta x_k\| \le c_1 c_\psi + \int_0^t (c_2\|\Delta x_k\| + c_B\|\Delta u_k\|)\mathrm{d}s$$

where $c_1 = 1 + l^2 l_{\bar{f}}\tau$ and $c_2 = (l+1)l_{\bar{f}} + l_B c_{ud}$. Then applying Bellman-Gronwall Lemma yields

$$\|\Delta x_k\| \le c_1 c_\psi e^{c_2 t} + c_B \int_0^t e^{c_2(t-s)}\|\Delta u_k\|\mathrm{d}s \qquad (19)$$

It follows immediately from (19) that

$$\|\Delta x_k(t-\theta)\| \le c_1 c_\psi e^{c_2(t-\theta)} + c_B \int_0^{t-\theta} e^{c_2(t-\theta-s)}\|\Delta u_k\|\mathrm{d}s,\ t \in (\theta, T]$$

Because of $e^{-c_2\theta} < 1$, we have

$$\|\Delta x_k(t-\theta)\| \le c_1 c_\psi e^{c_2 t} + c_B \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s,\ t \in (\theta, T] \tag{20}$$

Note that (20) is still true for $t \in [0, \theta]$ since $\|\Delta x_k(t-\theta)\| = \|\psi_d(t-\theta) - \psi_k(t-\theta)\| \le c_\psi$, $t \in [0, \theta]$ and $c_1 > 1$.

Then substituting (19) and (20) into (14) and (15), respectively, and defining $c_3 = l_1(l \max_{1\le q\le m}\{\mu_q\} + 1)$ and $c_4 = r l_2 l \max_{1\le q\le m}\{\mu_q\}$, we obtain

$$\begin{aligned}\|c(x_d) - c(x_k)\| &\le l_1 \left\| \begin{bmatrix} (l\mu_1 + 1)(c_1 c_\psi e^{c_2 t} + c_B \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s) \\ \vdots \\ (l\mu_m + 1)(c_1 c_\psi e^{c_2 t} + c_B \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s) \end{bmatrix} \right\| \\ &\le c_1 c_3 c_\psi e^{c_2 t} + c_3 c_B \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s \end{aligned} \tag{21}$$

and

$$\begin{aligned}&\|D(x_d) - D(x_k)\| \\ &\le l_2 \left\| \begin{bmatrix} l\mu_1(c_1 c_\psi e^{c_2 t} + c_B \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s) \cdots l\mu_1(c_1 c_\psi e^{c_2 t} + c_B \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s) \\ \vdots \quad \vdots \quad \vdots \\ l\mu_m(c_1 c_\psi e^{c_2 t} + c_B \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s) \cdots l\mu_m(c_1 c_\psi e^{c_2 t} + c_B \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s) \end{bmatrix} \right\| \\ &\le c_1 c_4 c_\psi e^{c_2 t} + c_4 c_B \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s \end{aligned} \tag{22}$$

Now, substituting (21) and (22) into (13) produces

$$\|\Delta u_{k+1}\| \le \rho \|\Delta u_k\| + c_5 \int_0^t e^{c_2(t-s)} \|\Delta u_k\| \mathrm{d}s + c_6 c_\psi e^{c_2 t} \tag{23}$$

where $c_5 = c_\Gamma c_B (c_3 + c_4 c_{ud})$ and $c_6 = c_\Gamma c_1 (c_3 + c_4 c_{ud})$. Multiplying both sides of (23) by $e^{-\lambda t} (\lambda > 0)$ and taking supremums for $t \in [0, T]$ result in

$$\begin{aligned} e^{-\lambda t} \|\Delta u_{k+1}(t)\| &\le \rho e^{-\lambda t} \|\Delta u_k(t)\| + c_5 \int_0^t e^{(c_2 - \lambda)(t-s)} e^{-\lambda s} \|\Delta u_k(s)\| \mathrm{d}s + c_6 c_\psi e^{-\lambda t} e^{c_2 t} \\ &\le \rho \sup_{t\in[0,T]} \{e^{-\lambda t} \|\Delta u_k(t)\|\} \\ &\quad + c_5 \sup_{t\in[0,T]} \left\{ \int_0^t e^{(c_2-\lambda)(t-s)} \sup_{s\in[0,T]} \{e^{-\lambda s} \|\Delta u_k(s)\|\} \mathrm{d}s \right\} \\ &\quad + c_6 c_\psi \sup_{t\in[0,T]} \left\{ e^{-\lambda t} e^{c_2 t} \right\} \end{aligned}$$

Noting the facts that, for $\lambda > c_2$,

$$\sup_{t\in[0,T]}\left\{\int_0^t e^{(c_2-\lambda)(t-s)}\sup_{s\in[0,T]}\{e^{-\lambda s}\|\Delta u_k(s)\|\}\mathrm{d}s\right\} \le \frac{1-e^{(c_2-\lambda)T}}{\lambda-c_2}\sup_{t\in[0,T]}\{e^{-\lambda t}\|\Delta u_k(t)\|\}$$

$$\sup_{t\in[0,T]}\{e^{(c_2-\lambda)t}\} \le 1$$

and using the definition of $\lambda-$norm give rise to

$$\|\Delta u_{k+1}\|_\lambda \;\le\; \rho\|\Delta u_k\|_\lambda + c_5\frac{1-e^{(c_2-\lambda)T}}{\lambda-c_2}\|\Delta u_k\|_\lambda + c_6 c_\psi \tag{24}$$

Defining

$$\bar{\rho} = \rho + c_5\frac{1-e^{(c_2-\lambda)T}}{\lambda-c_2}$$

(24) simplifies to

$$\|\Delta u_{k+1}\|_\lambda \le \bar{\rho}\|\Delta u_k\|_\lambda + c_6 c_\psi \tag{25}$$

Due to $0 \le \rho < 1$, it is possible to choose $\lambda > c_2$ large enough such that $0 \le \bar{\rho} < 1$. Then (25) is a contraction in $\|\Delta u_k\|_\lambda$. Iterating k leads to

$$\|\Delta u_k\|_\lambda \;\le\; \bar{\rho}^k\|\Delta u_0\|_\lambda + \frac{1-\bar{\rho}^k}{1-\bar{\rho}}c_6 c_\psi$$

Since $0 \le \bar{\rho} < 1$, Δu_k is bounded in the sense that

$$\|\Delta u_k\|_\lambda \;\le\; \|\Delta u_0\|_\lambda + \frac{c_6}{1-\bar{\rho}}c_\psi,\, k = 1, 2, \ldots \tag{26}$$

$$\limsup_{k\to\infty}\|\Delta u_k\|_\lambda \;\le\; \frac{c_6}{1-\bar{\rho}}c_\psi \tag{27}$$

Furthermore, using (19) and similar manipulations, we have

$$\|\Delta x_k\|_\lambda \le c_1 c_\psi + c_B\frac{1-e^{(c_2-\lambda)T}}{\lambda-c_2}\|\Delta u_k\|_\lambda$$

which leads to

$$\|\Delta x_k\|_\lambda \le c_B\frac{1-e^{(c_2-\lambda)T}}{\lambda-c_2}\|\Delta u_0\|_\lambda + \left(c_1 + c_B\frac{1-e^{(c_2-\lambda)T}}{\lambda-c_2}\frac{c_6}{1-\bar{\rho}}\right)c_\psi,\, k = 0, 1, \ldots \tag{28}$$

$$\limsup_{k\to\infty}\|\Delta x_k\|_\lambda \le \left(c_1 + c_B\frac{1-e^{(c_2-\lambda)T}}{\lambda-c_2}\frac{c_6}{1-\bar{\rho}}\right)c_\psi \tag{29}$$

From (2) and using Lipschitz condition, the result for output error $y_d - y_k$ is given as

$$\begin{aligned}\|y_d - y_k\|_\lambda \;\le\; & l_g c_B \frac{1 - e^{(c_2-\lambda)T}}{\lambda - c_2} \|\Delta u_0\|_\lambda \\ & + l_g \left(c_1 + c_B \frac{1 - e^{(c_2-\lambda)T}}{\lambda - c_2} \frac{c_6}{1 - \bar{\rho}} \right) c_\psi, \quad k = 0, 1, \ldots \end{aligned} \tag{30}$$

$$\limsup_{k\to\infty} \|y_d - y_k\|_\lambda \;\le\; l_g \left(c_1 + c_B \frac{1 - e^{(c_2-\lambda)T}}{\lambda - c_2} \frac{c_6}{1 - \bar{\rho}} \right) c_\psi \tag{31}$$

This completes the proof. ∎

Remark 3.2 Theorem 3.1 implies that a suitable choice of $\Gamma_k(t)$ leads to uniform convergence of system outputs to the desired trajectory for all $t \in [0, T]$ whenever c_ψ tends to zero. However, condition (11) depends on the delay times and the initial functions unless extended relative degree of the system under consideration is $\{1, \ldots, 1\}$.

Remark 3.3 For linear system (6)–(7), updating law (8) reduces to

$$u_{k+1}(t) = u_k(t) + \Gamma(y_d^{(\mu)}(t) - y_k^{(\mu)}(t)) \tag{32}$$

where learning gain Γ is constant and condition (11) becomes

$$\|I - \Gamma D\| < 1 \tag{33}$$

where D is given in Remark 2.3. It can be seen that condition (33) for uniform convergence of the learning control is independent of the time delays. Effect of the time-delays on the learning process lies in the requirement on initial function described by (12). Therefore, the delay times are not required to be estimated for design of the proposed learning control for the linear systems.

Remark 3.4 Since $D(x_k(t))$ is of full column rank, there exists a constant $\beta > 0$ such that

$$\min_{1\le i\le r} |\lambda_i(D^T(x_k(t))D(x_k(t)))| > \beta$$

where $\lambda_i, 1 \le i \le r$ are eigenvalues of the matrix $D^T(x_k(t))D(x_k(t))$. Therefore, there exists a bounded $\Gamma_k(t)$ such that (11) holds [19]. Condition (11) implies that the learning algorithm allows larger model discrepancies. Let us take the SISO case as an example. If the system parameter $D(x(t))$ is modeled to be $\hat{D}(x(t))$ and we assume that $\hat{D}(x(t)) = \alpha(x(t))D(x(t))$. We choose $\Gamma_k(t) = [\hat{D}^T(x_k(t))\hat{D}(x_k(t))]^{-1}\hat{D}^T(x_k(t))$ so that $\|I - \Gamma_k(t)D(x_k(t))\| = |1 - \alpha^{-1}(x_k(t))|$. Condition (11) reduces to

$$|1 - \alpha^{-1}(x_k(t))| \le \rho < 1$$

which holds for $0.5 < \alpha(x) < \infty$.

4. Simulation Illustrations

In this section, numerical simulations are conducted for a linear and a nonlinear system with delayed states, respectively, to illustrate the theoretical results presented in the previous sections.

Example 1 A simulation is conducted for the linear system with delayed states

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{bmatrix} x(t) + \begin{bmatrix}0 & 0 & 0\\ 0.5 & 0.5 & 0\\ 0 & 0 & 0.5\end{bmatrix} x(t-\tau) + \begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix} u(t)\\ y(t) &= \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

where $\tau > 0$ is the delay time. This system has extended relative degree $\{2, 1\}$. By the condition (33), if the updating law (32) is used and the learning gain is $\Gamma = I$. However, there exists room in this condition for selection of the learning gain to accommodate the inaccurate knowledge of the system parameters. In the simulation the learning gain is chosen as

$$\Gamma = \begin{bmatrix}0.7 & 0\\ 0 & 0.7\end{bmatrix}$$

The desired trajectory is given to be

$$y_d(t) = \begin{bmatrix}4t^3 - 3t^4\\ -4t^3 + 3t^4\end{bmatrix}, t \in [0, 1]$$

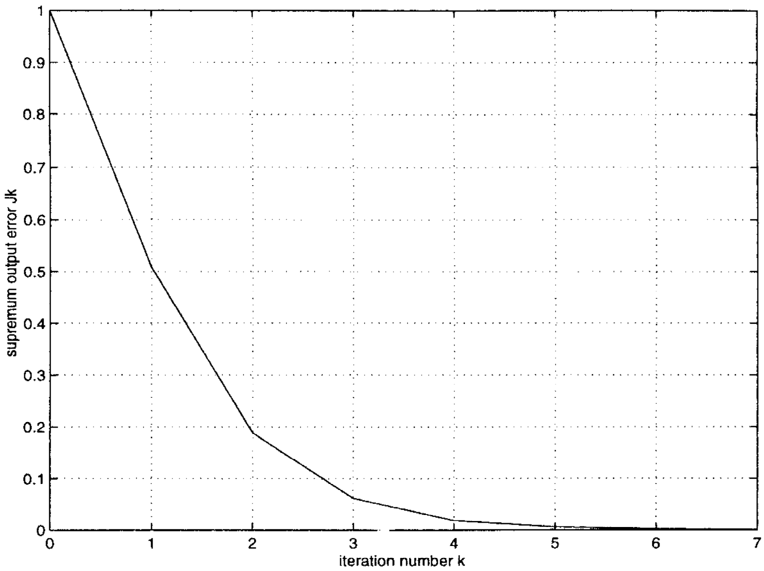
Let the initial functions be $\psi_{i,k}(t) = 0, t \in [-\tau, 0]$. The cases with different τ are simulated. Define the performance index $J_k = \sup_{t\in[0,1]} \|y_d(t) - y_k(t)\|$. The iteration stops if the tracking index $J_k < 0.001$. Figure 1 and Figure 2 show the tracking errors when the performance requirement is achieved and the resultant control inputs, with $\tau = 0.1$ and $\tau = 0.3$, respectively. It is observed that with the same learning gain, uniform convergence over the time interval [0, 1] is achieved for the different delay times.

Example 2 Consider the nonlinear system with delayed states described by

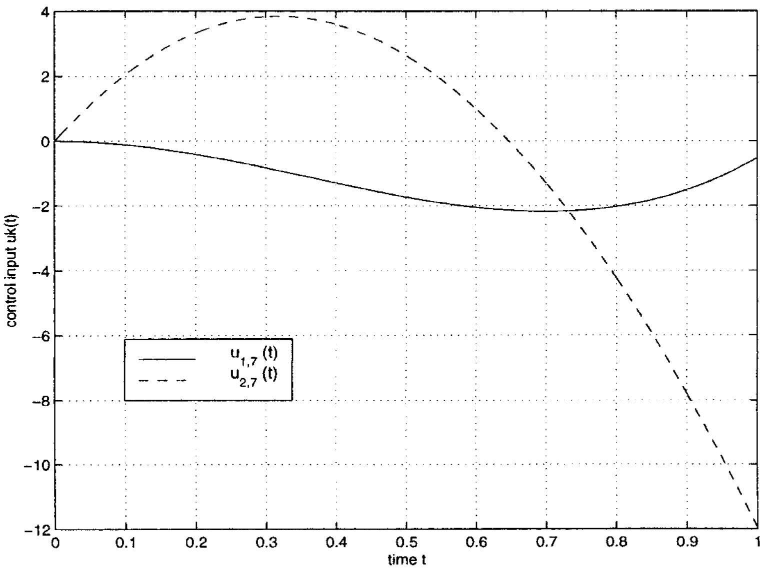
$$\begin{aligned}\dot{x}(t) &= \bar{f}(x, \sigma) + b(x(t))u(t)\\ y(t) &= g(x(t))\end{aligned}$$

where

$$\bar{f}(x, \sigma) = \begin{bmatrix}x_2\\ \sigma x_2 + x_3\\ \cos(x_1 + \sigma^2 x_4)\\ 0\end{bmatrix}, b(x(t)) = \begin{bmatrix}0\\ 0\\ 1\\ \sin(x_3(t))\end{bmatrix}, g(x(t)) = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix} x(t)$$

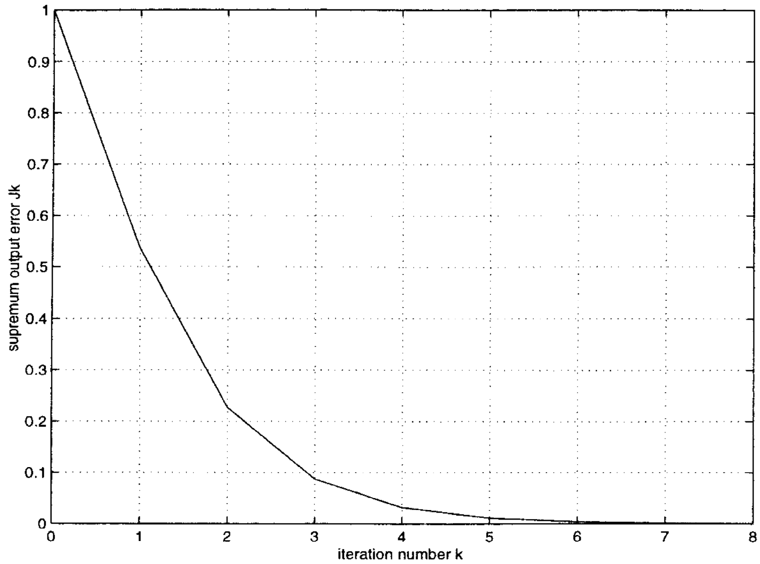


(a)

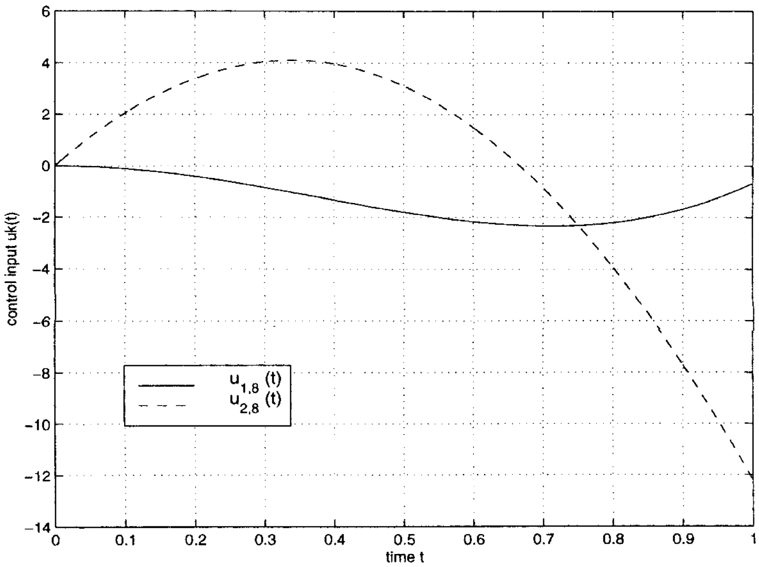


(b)

Figure 1. Learning response for the case $\tau = 0.1$: (a) Tracking errors; (b) Control inputs.



(a)



(b)

Figure 2. Learning response for the case $\tau = 0.3$: (a) Tracking errors; (b) Control inputs.

and the delay time $\tau = 0.1$. It is easy to check that

$$
\begin{aligned}
L_{\bar{f}} g(x,\sigma) &= x_2 \\
L_{\bar{f}}^2 g(x,\sigma) &= \sigma x_2 + x_3 \\
L_{\bar{f}}^3 g(x,\sigma) &= \cos(x_1 + \sigma^2 x_4) + (\sigma^2 x_2 + \sigma x_3)
\end{aligned}
$$

Thus, this system has extended relative degree three since

$$
\begin{aligned}
L_b g(x,\sigma) &= 0 \\
L_b L_{\bar{f}} g(x,\sigma) &= 0 \\
\frac{\partial L_{\bar{f}}^2 g(x,\sigma)}{\partial \sigma^i x} \sigma^i b(x) &= 0, \qquad 1 \le i \le 4 \\
\frac{\partial L_{\bar{f}}^2 g(x,\sigma)}{\partial x} b(x) &= 1
\end{aligned}
$$

The desired trajectory is given to be

$$y_d(t) = 5t^4 - 4t^5,\ t \in [0,1]$$

Updating law (8) is applied with the learning gain $\Gamma_k(t) = 0.5$. Let the initial functions be $\psi_{i,k}(t) = 0$, $i = 1, 2, 3, 4$, $t \in [-0.2, 0]$. Performance requirement $J_k = \sup_{t\in[0,1]} \|y_d(t) - y_k(t)\|_\infty < 0.001$ is achieved at the 9th iteration and the resultant tracking errors are shown in Figure 3. To examine robustness performance of the learning control, we let the initial functions be $\psi_{i,k}(t) = 0 + 0.01\text{randn}$, $i = 1, 2, 3, 4$, $t \in [-0.2, 0]$. The randn is a generator of random scalar with normal distribution, mean = 0, and variance = 1. The repetitions are conducted until $k = 100$ and the resultant tracking errors are shown in Figure 4. It is observed from Figure 3 and 4 that by using updating law (8), system outputs converge to a neighborhood of the desired one and remain within it in the presence of initial function errors. Moreover, the proposed learning algorithm ensures that system outputs converge to the desired trajectory whenever initial function errors tend to zero.

5. Conclusion

The result of this paper shows that the concept of iterative learning control can be applied to solve trajectory tracking problem of the class of uncertain dynamic systems with delayed states if the operation task is performed repeatedly. When the system is described by the defined relative degree, the proposed learning control scheme ensures convergence of system outputs to the desired one and is robust against the presence of initial function uncertainty. It is made clear that the learning control does not use the delay times explicitly and the effect of delay times on the learning process depends on initial functions.

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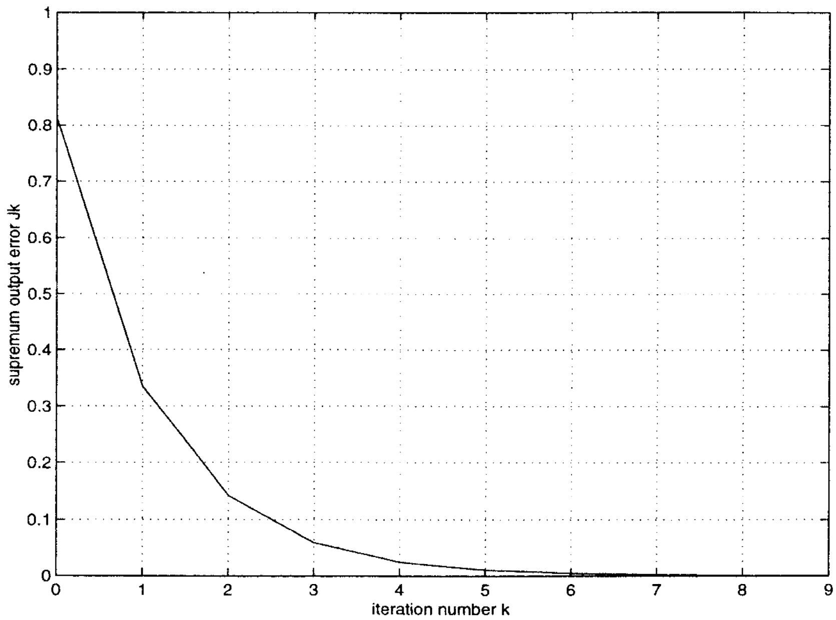

Figure 3. Uniform convergence in absence of initial function errors.

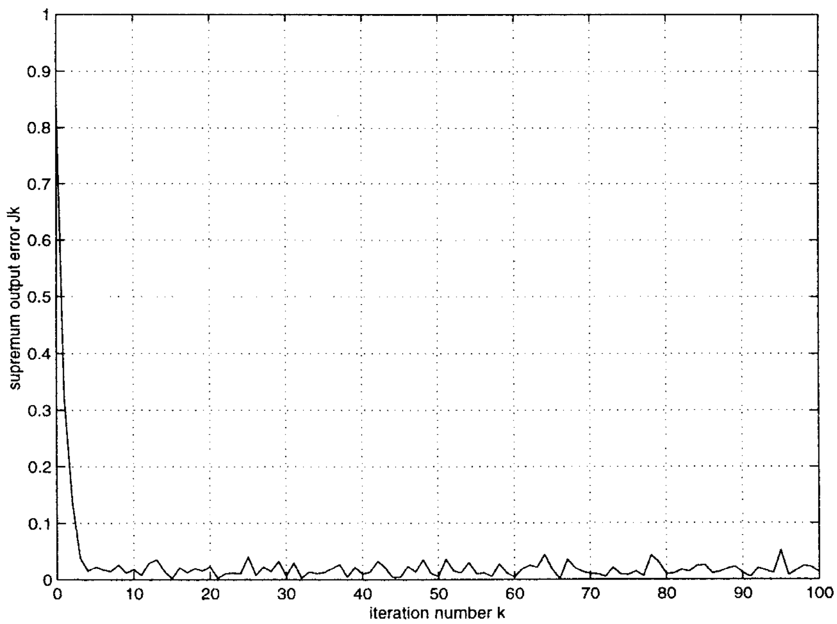

Figure 4. Robustness performance in presence of initial function errors.

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