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# Initial condition issues on iterative learning control for non-linear systems with time delay

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*Most of the available results on iterative learning control address trajectory tracking problem for systems without time delay. The role of the initial function in tracking performance of iterative learning control for systems with time delay is not yet fully understood. In this paper, asymptotic properties of a conventional learning algorithm are examined for a class of non-linear systems with time delay in the presence of initial function errors. It is shown that a non-zero initial function deviation can cause a lasting tracking error on the entire operation. Impulsive action is one method to eliminate such lasting tracking error but it is not a practical approach. As an alternative, an initial rectifying action is introduced in the learning algorithm. The initial rectifying action is finite and used over a specified interval. It is shown to be effective in the improvement of tracking performance, in particular robustness and uniform convergence. The results are further extended to systems with multiple time delays. An example is given and computer simulations are presented to demonstrate the performance of the proposed approach.*

## 1. Introduction

Iterative learning control is a trajectory tracking improvement technique for systems performing a prescribed task repeatedly, which is characterized by repositioning, input updating and zero-error tracking in the presence of unmodeled dynamics and/or parameters uncertainties (Bien and Xu 1998, Moore 1998, Sun and Huang 1999). A common assumption in iterative learning control is that the initial condition at each cycle is reset to the desired initial condition, or inside a neighborhood of the desired initial condition (Arimoto *et al*. 1984, Hauser 1987, Arimoto 1990, Heinzinger *et al*. 1992, Saab *et al*. 1997). This requirement was relaxed in Lee and Bien (1996), Wang and Cheah (1998) and Sun *et al*. (1998a) so that the initial condition at each cycle remains the same but different from the desired initial condition or within a neighborhood of any fixed point, under which asymptotic tracking is ensured. To eliminate the effect caused by the initial condition shifting, initial impulsive action is needed in a learning algorithm (Porter and Mohamed 1991). The learning algorithm enables zero-error tracking on the entire operation interval. However, the use of an impulsive action is not practical.

Up to now, most works focus on systems without time delay. However, delays are inherent in many applications such as batch processes, and remote controlled robots, vehicles and man-machine systems. Because of inaccuracy in estimation and/or uncertainty of time delay, feedback controls are usually unsatisfactory, especially in transient responses. This motivates researches on iterative learning control for systems with time delay (Sun *et al*. 1994, 1988b, Hideg 1995, Park *et al*. 1998). The convergence issues were investigated for LTI systems with time delay (Hideg 1995, Park *et al*. 1998). In Sun *et al*. (1994), a higher-order learning algorithm was studied for a class of non-linear systems with time delay. However, the considered initial condition is simple but somewhat obscure. Recently, Sun *et al*. (1998b) showed that under certain conditions the output error is asymptotically bounded when the initial function at each cycle is deviated from the desired initial function within an admissible level. If the deviations are eliminated, uniform convergence of the system output to the desired trajectory can be guaranteed.

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This paper aims to examine asymptotic properties of iterative learning control for a class of non-linear systems with time delay in the case where initial function at each cycle need not be close to the desired initial function as required in the published literature. First, we consider the case where the initial function at each cycle remains the same but different from the desired initial function. It is shown that a conventional learning algorithm will lead to a constant tracking error, similar to the case for systems with no time delay (Lee and Bien 1996). Then, we focus on the case where the initial function varies about a fixed function. An initial rectifying action is introduced in the learning algorithm to improve tracking performance. A proof is provided to analyse the robustness of the proposed learning algorithm with respect to such initial function errors. Compared with the initial impulsive approach (Porter and Mohamed 1991), the initial rectifying action is finite and implementable. These results are also extended to systems with multiple time delays. Finally, numerical simulations are given to illustrate the theoretical results.

## 2. Problem formulation

Consider a class of non-linear systems with time delay described by the state space equations

$$\dot{x}_k(t) = f(x_k(t), x_k(t-\tau), t) + B(x_k(t), x_k(t-\sigma), t)u_k(t) \qquad (1)$$

$$y_k(t) = g(x_k(t), t), \qquad (2)$$

where $t$ is the time in the operation interval $[0, T]$ and $k$ is the number of operation cycles. For $t \in [0, T]$ and for all $k$, $x_k(t) \in R^n$, $u_k(t) \in R^r$ and $y_k(t) \in R^m$ are the state, control input and output of the system, respectively. Both $\tau > 0$ and $\sigma > 0$ are constant time delays. For $t \in [-\mu, 0], \mu = \max\{\tau, \sigma\}$, $x_k(t) = \psi_k(t)$ and $\psi_k(t)$ is the initial function of the system.

Given a desired trajectory $y_d(t), t \in [0, T]$, the objective is to find a control input such that the system output follows the desired trajectory. A conventional learning algorithm takes the form of

$$u_{k+1}(t) = u_k(t) + L(y_k(t), t)(\dot{y}_d(t) - \dot{y}_k(t)), \qquad (3)$$

where the learning gain $L(\cdot, \cdot)$ is piecewise continuous and bounded on $R^m \times [0, T]$. It was shown (Sun *et al.* 1994, 1998b) that if $L(\cdot, \cdot)$ is chosen such that

$$\|I - L(g(x(t), t), t)g_x(x(t), t)B(x(t), x(t-\sigma), t)\| \le \rho < 1, t \in [0, T] \qquad (4)$$

and

$$\psi_k(t) = \psi_d(t), t \in [-\mu, 0], \quad k = 0, 1, 2, \ldots, \qquad (5)$$

where $\psi_d(t)$ is the desired initial function, then the system output $y_k(t)$ converges uniformly to $y_d(t)$ on $[0, T]$ as $k \to \infty$. Furthermore, if the initial function at each cycle is allowed to deviate from $\psi_d(t)$ such that

$$\|\psi_d(t) - \psi_k(t)\| \le c_{\psi d}, t \in [-\mu, 0], \quad k = 0, 1, 2, \ldots, \qquad (6)$$

then the asymptotic bound of the output error $y_d(t) - y_k(t)$ is a class-$K$ function of $c_{\psi d}$.

This paper allows larger initial function deviations, $\|\psi_d(t) - \psi_k(t)\| \ge c_{\psi d}$. However, the initial function $\psi_k(t)$ at each cycle aligns with a given function $\psi^*(t)$, namely,

$$\psi_k(t) = \psi^*(t), t \in [-\mu, 0], \quad k = 0, 1, 2, \ldots \qquad (7)$$

or within a ball centered at $\psi^*(t)$, i.e.

$$\|\psi^*(t) - \psi_k(t)\| \le c_\psi, t \in [-\mu, 0], \quad k = 0, 1, 2, \ldots, \qquad (8)$$

We shall analyse the effect due to the initial function errors on the converged system output and propose an approach to eliminate such effect.

The following assumptions on the system (1–2) are imposed.

- A1. The desired trajectory $y_d(t)$ is differentiable on $[0, T]$.
- A2. The functions $f: R^n \times R^n \times [0, T] \to R^n$ and $B: R^n \times R^n \times [0, T] \to R^{n\times r}$ are piecewise continuous in $t$; $g: R^n \times [0, T] \to R^m$ is differentiable in $x$ and $t$ with partial derivatives $g_x(\cdot, \cdot)$ and $g_t(\cdot, \cdot)$.
- A3. The functions $f(\cdot, \cdot, \cdot)$ and $B(\cdot, \cdot, \cdot)$ are uniformly globally Lipschitz in $x$ on $[0, T]$, i.e. $\|a(x_1(t), x_1(t-\theta), t) - a(x_2(t), x_2(t-\theta), t)\| \le l_a(\|x_1(t) - x_2(t)\| + \|x_1(t-\theta) - x_2(t-\theta)\|)$, for $t \in [0, T]$, $\theta \in \{\tau, \sigma\}$ and some finite constant $l_a > 0$, $a \in \{f, B\}$. The function $B(\cdot, \cdot, \cdot)$ is uniformly bounded on $R^n \times R^n \times [0, T]$ with the norm bound $c_B$.
- A4. The functions $g_t(\cdot, \cdot)$, $g_x(\cdot, \cdot)$ are uniformly globally Lipschitz in $x$ on $[0, T]$, i.e. $\|a(x_1(t), t) - a(x_2(t), t)\| \le l_a\|x_1(t) - x_2(t)\|$, for $t \in [0, T]$ and some finite constant $l_a > 0$, $a \in \{g_t, g_x\}$. The function $g_x(\cdot, \cdot)$ is uniformly bounded on $R^n \times [0, T]$ with the norm bound $c_{gx}$.
- A5. The input–output coupling matrix $g_x(\cdot, \cdot)B(\cdot, \cdot, \cdot)$ is of full column rank.

Because of the boundedness of $g_x(\cdot, \cdot)$, assumption (A4) implies that $g(\cdot, \cdot)$ is uniformly globally Lipschitz in $x$ on $[0, T]$. Assumption (A5) guarantees that there exits a bounded $L(\cdot, \cdot)$ satisfying (4). In particular, let $L = \alpha[(g_x B)^T g_x B]^{-1}(g_x B)^T$. We can find $\alpha \in (0, 2)$ so that $\rho = |1 - \alpha| < 1$. The boundedness of $L(\cdot, \cdot)$ can be concluded by boundedness of $B(\cdot, \cdot, \cdot)$ and $g_x(\cdot, \cdot)$.

In the sequel, $\|\cdot\|$ is the vector norm defined as $\|a\| = \max_{1\le i\le n}|a_i|$ for an $n-$dimensional vector $a = [a_1, \ldots, a_n]^{\mathrm{T}}$. For a matrix $A = \{a_{ij}\} \in R^{m\times n}$, $\|A\|$ is the norm induced from the vector norm, $\|A\| = \max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}|$. The following $\lambda-$norm is used for the analysis purpose.

**Definition 2.1:** The $\lambda$-norm for a vector-valued function $b(t) \in R^n, t \in [0, T]$, is defined as

$$\|b\|_\lambda = \sup_{t\in[0,T]}\{e^{-\lambda t}\|b(t)\|\}$$

where $\lambda > 0$.

**Definition 2.2:** The $\infty$-norm for a vector-valued function $b(t) \in R^n, t \in [0, T]$, is defined as

$$\|b\|_\infty = \sup_{t\in[0,T]}\|b(t)\|.$$

From both definitions, note that $\|b\|_\lambda \le \|b\|_\infty \le e^{\lambda T}\|b\|_\lambda$. The $\lambda-$ norm is thus equivalent to the $\infty-$norm.

## 3. Conventional ILC and its constant tracking error

In this section, by exerting the control inputs generated by the updating law (3), the system output is shown to converge to a trajectory that is different from the desired trajectory by a constant, and this constant is determined by the error between the initial function and the desired initial function at time $t = 0$.

**Theorem 3.1:** *Given a desired trajectory $y_d(t), t \in [0, T]$, let the system (1–2) satisfy assumptions (A1–5), and the updating law (3) be applied. Define a trajectory*

$$y^*(t) = y_d(t) - (y_d(0) - g(\psi^*(0), 0)), \tag{9}$$

*with the initial function $\psi^*(t), t \in [-\mu, 0]$, being realizable. If the learning gain is selected such that (4) holds and the initial function at each cycle satisfies the alignment condition (7), the system output $y_k(t)$ converges uniformly to $y^*(t)$ on $[0, T]$ as $k \to \infty$.*

**Proof:** The proof can be found in appendix A. ∎

Theorem 3.1 shows that the converged output trajectory follows $y^*(t)$ that shifts from $y_d(t)$ with a fixed error, $y_d(0) - g(\psi^*(0), 0)$, for all $t \in [0, T]$. The initial function over the interval $[-\mu, 0)$ has no effect on the converged output trajectory. This property implies that the convergence of the updating law (3) can be guaranteed if $g(\psi^*(0), 0) = y_d(0)$ and the initial function on the interval $[-\mu, 0)$ keeps the same at each repetition.

To examine the implications of Theorem 3.1, consider the linear systems with time-delay described by

$$\dot{x}_k(t) = Ax_k(t) + A_1x_k(t-\tau) + Bu_k(t) \tag{10}$$

$$y_k(t) = Cx_k(t). \tag{11}$$

The updating law (3) becomes

$$u_{k+1}(t) = u_k(t) + L(\dot{y}_d(t) - \dot{y}_k(t)), \tag{12}$$

and the condition (4) reduces to

$$\|I - LCB\| \le \rho < 1. \tag{13}$$

The converged trajectory will be

$$y^*(t) = y_d(t) - (y_d(0) - C\psi^*(0)). \tag{14}$$

Note that similar convergence result is obtained for linear systems without time-delay in Lee and Bien (1996). As an extension, however, Theorem 3.1 implies that by using the same learning algorithm, the convergence of the learning algorithm is independent of the time delay in the state variable, and the converged trajectory does not depend on the initial function over the interval $[-\mu, 0)$.

## 4. ILC with initial rectifying action

To overcome the deviated convergence shown in Theorem 3.1, a rectifying action at $t = 0$ is added as the third term in the updating law (3), in the following form

$$\begin{aligned} u_{k+1}(t) = u_k(t) &+ L(y_k(t), t)(\dot{y}_d(t) - \dot{y}_k(t)) \\ &+ \delta_h(t)L(y_k(t), t)(y_d(0) - y_k(0)), \end{aligned} \tag{15}$$

where $\delta_h{:}[0, T] \to R$ is defined as

$$\delta_h(t) = \begin{cases} \frac{2}{h}\left(1 - \frac{t}{h}\right) & t \in [0, h] \\ 0 & t \in (h, T] \end{cases} \tag{16}$$

with

$$\int_0^h \delta_h(s)\,\mathrm{d}s = 1$$

and $h$ is a design parameter.

In the following, when the updating law (15) is applied to the system (1–2), we are going to consider a more realistic case where the initial function $\psi_k(t)$ varies within a ball centered at $\psi^*(t)$. The following theorem specifies asymptotic properties of the learning algorithm.

**Theorem 4.1:** *Given a desired trajectory $y_d(t), t \in [0, T]$, let the system (1–2) satisfy assumptions (A1–5) and the updating law (15) be applied. Define a trajectory*

$$y^*(t) = y_d(t) + \int_h^t \delta_h(s)\mathrm{d}s(y_d(0) - g(\psi^*(0), 0)), \tag{17}$$

*with the initial function $\psi^*(t), t \in [-\mu, 0]$, being realizable. If the learning gain is selected such that (4) holds and the initial function at each cycle satisfies (8), the asymptotic bound of the output error $y^*(t) - y_k(t)$ is a class-K function of $c_\psi$ on $[0, T]$ as $k \to \infty$.*

**Proof:** As in the proof of Theorem 3.1 given in appendix A, for the initial condition $x^*(t) = \psi^*(t)$, $t \in [-\mu, 0]$, let $u^*(t), t \in [0, T]$ be the control input which generates the trajectory $y^*(t), t \in [0, T]$ and $x^*(t), t \in [0, T]$. The notations in the proof of Theorem 3.1 are also adopted here. Using $y^*(t)$ defined in (17), (15) can be written as

$$\begin{aligned} u_{k+1} &= u_k + L_k(\dot{y}^* - \dot{y}_k) + L_k(\dot{y}_d - \dot{y}^*) \\ &\quad + \delta_h L_k(g(\psi^*(0), 0) - y_k(0)) + \delta_h L_k(y_d(0) \\ &\quad - g(\psi^*(0), 0)) \\ &= u_k + L_k(\dot{y}^* - \dot{y}_k) + \delta_h L_k(g(\psi^*(0), 0) - y_k(0)), \end{aligned}$$

which implies

$$\begin{aligned} \Delta^*_{u_{k+1}} &= \Delta u^*_k - L_k[g^*_t + g^*_x(f^* + B^* u^*) - g_{tk} \\ &\quad - g_{xk}(f_k + B_k u_k)] \\ &\quad - \delta_h L_k(g(\psi^*(0), 0) - y_k(0)) \\ &= (I - L_k g_{xk} B_k)\Delta u^*_k - L_k\{g^*_t - g_{tk} \\ &\quad + (g^*_x - g_{xk})(f^* + B^* u^*) \\ &\quad + g_{xk}[(f^* - f_k) + (B^* - B_k)u^*]\} \\ &\quad - \delta_h L_k(g(\psi^*(0), 0) - y_k(0)). \end{aligned}$$

Taking norms and applying the bounds and the Lipschitz conditions, we have

$$\begin{aligned} \|\Delta u^*_{k+1}\| &\le \|I - L_k g_{xk} B_k\| \|\Delta u^*_k(t)\| + \|L_k\|[\|g^*_t - g_{tk}\| \\ &\quad + \|g^*_x - g_{xk}\| \, \|f^* + B^* u^*\| \\ &\quad + \|g_{xk}\|(\|f^* - f_k\| + \|B^* - B_k\| \, \|u^*\|)] \\ &\quad + \|\delta_h\| \|L_k\| \|g(\psi^*(0), 0) - y_k(0)\| \\ &\le \rho \|\Delta u^*_k\| + c_L[(l_{gt} + l_{gx} c^* + c_{gx} c_1)\|\Delta x^*_k\| \\ &\quad + c_{gx}(l_f \|\Delta x^*_k(t - \tau)\| + l_B c_{u^*} \|\Delta x^*_k(t - \sigma)\|)] \\ &\quad + \frac{2}{h} c_L l_g \|\Delta x^*_k(0)\|, \end{aligned} \tag{18}$$

where $c_L$ is the norm bound for $L(\cdot, \cdot)$; $c^* = \sup_{t \in [0,T]} \|f^* + B^* u^*\|$, $c_{u^*} \triangleq \sup_{t \in [0,T]} \|u^*(t)\|$, and $c_1 = l_f + l_B c_{u^*}$.

For evaluating the state errors of the right hand side of (18), we integrate the state equations to obtain

$$\begin{aligned} \Delta x^*_k &= \Delta x^*_k(0) + \int_0^t [f^* + B^* u^* - (f_k + B_k u_k)] \, \mathrm{d}s \\ &= \Delta x^*_k(0) + \int_0^t [f^* - f_k + (B^* - B_k)u^* + B_k \Delta u^*_k] \, \mathrm{d}s. \end{aligned}$$

Taking norms and using their properties yield

$$\begin{aligned} \|\Delta x^*_k\| &\le \|\Delta x^*_k(0)\| + \int_0^t (\|f^* - f_k\| + \|B^* - B_k\| \|u^*\| \\ &\quad + \|B_k\| \|\Delta u^*_k\|) \, \mathrm{d}s \\ &\le \|\Delta x^*_k(0)\| + \int_0^t (c_1 \|\Delta x^*_k\| + l_f \|\Delta x^*_k(s - \tau)\| \\ &\quad + l_B c_{u^*} \|\Delta x^*_k(s - \sigma)\| + c_B \|\Delta u^*_k\|) \, \mathrm{d}s. \end{aligned} \tag{19}$$

Note the facts that, for $t \in [0, \theta]$ with $\theta \in \{\tau, \sigma\}$,

$$\begin{aligned} \int_0^t \|\Delta x^*_k(s - \theta)\| \, \mathrm{d}s &= \int_{-\theta}^{t-\theta} \|\psi^*(s) - \psi_k(s)\| \, \mathrm{d}s \\ &\le \mu c_\psi, \end{aligned} \tag{20}$$

and for $t \in (\theta, T]$,

$$\begin{aligned} \int_0^t \|\Delta x^*_k(s - \theta)\| \, \mathrm{d}s &= \int_{-\theta}^{0} \|\psi^*(s) - \psi_k(s)\| \, \mathrm{d}s \\ &\quad + \int_0^{t-\theta} \|\Delta x^*_k(s)\| \, \mathrm{d}s \\ &\le \mu c_\psi + \int_0^{t-\theta} \|\Delta x^*_k(s)\| \, \mathrm{d}s. \end{aligned} \tag{21}$$

Combining (20) and (21) produces, for $t \in [0, T]$,

$$\int_0^t \|\Delta x^*_k(s - \theta)\| \, \mathrm{d}s \le \mu c_\psi + \int_0^t \|\Delta x^*_k(s)\| \, \mathrm{d}s. \tag{22}$$

Substituting (22) into (19) gives rise to

$$\|\Delta x^*_k\| \le (1 + c_1 \mu) c_\psi + \int_0^t (2c_1 \|\Delta x^*_k\| + c_B \|\Delta u^*_k\|) \, \mathrm{d}s.$$

Then applying Bellman–Gronwall Lemma, we obtain

$$\|\Delta x^*_k\| \le (1 + c_1 \mu) c_\psi e^{2c_1 t} + \int_0^t e^{2c_1 (t - s)} c_B \|\Delta u^*_k\| \, \mathrm{d}s, \tag{23}$$

which implies

$$\begin{aligned} \|\Delta x^*_k(t - \theta)\| &\le (1 + c_1 \mu) c_\psi e^{2c_1 (t - \theta)} \\ &\quad + \int_0^{t-\theta} e^{2c_1 (t - \theta - s)} c_B \|\Delta u^*_k\| \, \mathrm{d}s, \; t \in (\theta, T]. \end{aligned}$$

Because of $e^{-2c_1 \theta} < 1$,

$$\begin{aligned} \|\Delta x^*_k(t - \theta)\| &\le (1 + c_1 \mu) c_\psi e^{2c_1 t} \\ &\quad + \int_0^t e^{2c_1 (t - s)} c_B \|\Delta u^*_k\| \, \mathrm{d}s, \; t \in (\theta, T], \end{aligned} \tag{24}$$

which is also true for $t \in [0, \theta]$ since $\|\Delta x^*_k(t - \theta)\| = \|\psi^*(t - \theta) - \psi_k(t - \theta)\| \le c_\psi, t \in [0, \theta]$.
Now, substituting (23) and (24) into (18) produces

$$\|\Delta u_{k+1}^*\| \le \rho\|\Delta u_k^*\| + c_L c_2 c_B \int_0^t e^{2c_1(t-s)}\|\Delta u_k^*\|\,\mathrm{d}s + c_L c_2(1+c_1\mu)c_\psi e^{2c_1 t} + \frac{2}{h}c_L l_g c_\psi$$
$$\le \rho\|\Delta u_k^*\| + c_3\int_0^t e^{c_3(t-s)}\|\Delta u_k^*\|\,\mathrm{d}s + \frac{1+e^{c_3 t}}{2}c_3 c_\psi,$$

where $c_2 = l_{gt} + l_{gx}c^* + 2c_{gx}c_1$ and

$$c_3 = \max\left\{2c_1, c_L c_2 c_B, 2c_L c_2(1+c_1\mu), \frac{4}{h}c_L l_g\right\}.$$

Multiplying both sides by $e^{-\lambda t}$ with $\lambda > 0$ gives

$$e^{-\lambda t}\|\Delta u_{k+1}^*\| \le \rho e^{-\lambda t}\|\Delta u_k^*\| + c_3\int_0^t e^{(c_3-\lambda)(t-s)}e^{-\lambda s}\|\Delta u_k^*\|\,\mathrm{d}s + \frac{e^{-\lambda t}+e^{(c_3-\lambda)t}}{2}c_3 c_\psi.$$

Taking supremum for $t \in [0,T]$ and $\lambda > c_3$ according to the $\lambda$-norm definition, we get

$$\|\Delta u_{k+1}^*\|_\lambda \le \bar{\rho}\|\Delta u_k^*\|_\lambda + c_3 c_\psi, \tag{25}$$

where $\bar{\rho} = \rho + c_3(1-e^{(c_3-\lambda)T})/(\lambda - c_3)$. Since $\rho < 1$, it is possible to find a $\lambda > c_3$ sufficiently large such that $\bar{\rho} < 1$. Then (25) is a contraction in $\|\Delta u_k^*\|_\lambda$. Iterating $k$ leads to

$$\|\Delta u_k^*\|_\lambda \le \bar{\rho}^k\|\Delta u_0^*\|_\lambda + \frac{1-\bar{\rho}^k}{1-\bar{\rho}}c_3 c_\psi.$$

When the iterations increase, $k \to \infty$, the error $\|\Delta u_k^*\|_\lambda$ is bounded in the sense that, due to $\bar{\rho} < 1$,

$$\|\Delta u_k^*\|_\lambda \le \|\Delta u_0^*\|_\lambda + \frac{c_3}{1-\bar{\rho}}c_\psi, \quad k = 1,2,\ldots, \tag{26}$$

$$\limsup_{k\to\infty}\|\Delta u_k^*\|_\lambda \le \frac{c_3}{1-\bar{\rho}}c_\psi. \tag{27}$$

Furthermore, using (23) and similar manipulations, we have

$$\|\Delta x_k^*\|_\lambda \le (1+c_1\mu)c_\psi + c_B\frac{1-e^{(c_3-\lambda)T}}{\lambda - c_3} \times \left(\|\Delta u_0^*\|_\lambda + \frac{c_3}{1-\bar{\rho}}c_\psi\right), \quad k = 0,1,\ldots \tag{28}$$

$$\limsup_{k\to\infty}\|\Delta x_k^*\|_\lambda \le \left(1 + c_1\mu + c_B\frac{1-e^{(c_3-\lambda)T}}{\lambda-c_3}\frac{c_3}{1-\bar{\rho}}\right)c_\psi. \tag{29}$$

To obtain the result for $y^* - y_k$, we use the fact that $g(\cdot,\cdot)$ is uniformly globally Lipschitz in $x$ on $[0,T]$. Therefore, $\|y^* - y_k\|_\lambda \le l_g\|\Delta x_k^*\|_\lambda$ for some positive constant $l_g$ and thus,

$$\|y^* - y_k\|_\lambda \le l_g(1+c_1\mu)c_\psi + l_g c_B\frac{1-e^{(c_3-\lambda)T}}{\lambda-c_3}\left(\|\Delta u_0{}^*\|_\lambda + \frac{c_3}{1-\bar{\rho}}c_\psi\right), \quad k = 0,1,\ldots \tag{30}$$

$$\limsup_{k\to\infty}\|y^* - y_k\|_\lambda \le l_g\left(1 + c_1\mu + c_B\frac{1-e^{(c_3-\lambda)T}}{\lambda-c_3}\frac{c_3}{1-\bar{\rho}}\right)c_\psi. \tag{31}$$

This completes the proof. ■

Theorem 4.1 implies that a suitable choice of $L(\cdot,\cdot)$ leads to that the system output converges to the trajectory $y^*(t)$ for all $t \in [0,T]$ as $c_\psi$ tends to zero. Based on the definition of $y^*(t)$ in (17), $y^*(t) = y_d(t), t \in (h,T]$. Uniform convergence of the system output to desired trajectory $y_d(t)$ is achieved on $(h,T]$, while the converged output trajectory on $[0,h]$ is specified by the initial rectifying action which can be viewed as a transient from initial position to the desired trajectory. The specified trajectory in the interval $[0,h]$ is for initial rectifying and the later part for trajectory tracking.

It is already known (Sun *et al.* 1994, 1998b) that when the conventional learning algorithm is used, the asymptotic bound of the output error between $y_d(t)$ and $y_k(t)$ is a class-$K$ function of $c_{\psi d}$, the bound on the initial function error $\psi_d(t) - \psi_k(t)$. The asymptotic bound of the output error will be very large when the initial function at each repetition is in the neighborhood of $\psi^*(t)$ and $\|\psi_d(t) - \psi^*(t)\| \gg c_{\psi d}, t \in [-\mu, 0]$. On the other hand, Theorem 4.1 shows that when the proposed initial rectifying action is applied, the tracking error will be a class-$K$ function of $c_\psi$ and thus substantially reduced after $t \ge h$. It indicates that the initial rectifying action in our proposed learning algorithm helps to improve tracking performance.

Note that $\delta_h(t)$ will be the Dirac delta function when $h \to 0$. For this case, the resulting control input contains impulsive action at $t = 0$ so that zero-error tracking is achieved along the whole span of operation interval in the absence of initial function errors. Our work examines the way to avoid impulsive action by introducing the initial rectifying action. In the implementation, the selection of $h$ should be done based on the trade-off among factors such as the resulting control input, transient response and the error bounds given in (26–31).

For $0 \le t < \sigma$, (4) can be rewritten as

$$\|I - L(g(x(t),t),t)g_x(x(t),t)B(x(t),\psi^*(t-\sigma),t)\| \le \rho < 1.$$

This implies that the sufficient condition for robustness and convergence of the learning algorithm (15) depends on the initial function of the system. However, the design of $L(\cdot,\cdot)$ is clearly independent of the time delay $\tau$.

The results in Theorems 3.1 and 4.1 are suitable for the following non-linear systems with measurement delay

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t) \quad (32)$$

$$y(t) = g(x(t-\tau)), \quad (33)$$

and feedback control is used in the manner

$$u(t) = c(y(t), y_d(t), t) + v(t), \quad (34)$$

and $v(t)$ is the learning control part. State equation of the closed-loop system is then given by

$$\dot{x}(t) = \bar{f}(x(t), x(t-\tau), t) + B(x(t))v(t), \quad (35)$$

where

$$\bar{f}(x(t), x(t-\tau), t) = f(x(t)) + B(x(t))c(g(x(t-\tau)), y_d(t), t).$$

Since the learning update is off-line,

$$\bar{y}(t) = y(t+\tau) = g(x(t)) \quad (36)$$

is available after each operation and can be considered as the output signal of the new formed system.

## 5. Extension to systems with multiple time delays

The above results can be extended to a class of non-linear systems with multiple time delays, which is described by

$$\dot{x}_k(t) = f(x_k(t), x_k(t-\tau_1), \ldots, x_k(t-\tau_{n_1}), t) + B(x_k(t), x_k(t-\sigma_1), \ldots, x_k(t-\sigma_{n_2}), t)u_k(t) \quad (37)$$

$$y_k(t) = g(x_k(t), t), \quad (38)$$

where $t \in [0, T], \tau_i > 0, i = 1, \ldots, n_1$ and $\sigma_i > 0, i = 1, \ldots, n_2$ are constant time delays. For $t \in [0, T]$ and for all $k$, $x_k(t) \in R^n$, $u_k(t) \in R^r$ and $y_k(t) \in R^m$. For $t \in [-\mu, 0]$, $\mu = \max\{\tau_i, i = 1, \ldots, n_1; \sigma_i,\ i = 1, \ldots, n_2\}$, $x_k(t) = \psi_k(t)$.

For the realizable trajectory $y^*(t)$ defined in (17), let $u^*(t)$ and $x^*(t)$ be the control input and the state, respectively. Assume that the functions $f$ and $B$ be uniformly globally Lipschitz in $x$ on $[0, T]$, i.e.

$$\|a(x_1(t), x_1(t-\theta_1), \ldots, x_1(t-\theta_{n_i}), t) - a(x_2(t), x_2(t-\theta_1), \ldots, x_2(t-\theta_{n_i}), t)\| \le l_a[\|x_1(t) - x_2(t)\| + \|x_1(t-\theta_1) - x_2(t-\theta_1)\| + \cdots + \|x_1(t-\theta_{n_i}) - x_2(t-\theta_{n_i})\|],$$

for $t \in [0, T]$; $\theta \in \{\tau, \sigma\}$; $n_i \in \{n_1, n_2\}$ and some finite constant $l_a > 0$, $a \in \{f, B\}$. Performing manipulations similar to those in the proof of Theorem 4.1 yields, in parallel to (19),

$$\|\Delta x_k^*\| \le \|\Delta x_k^*(0)\| + \int_0^t [l_f(\|\Delta x_k^*\| + \|\Delta x_k^*(s-\tau_1)\| + \cdots + \|\Delta x_k^*(s-\tau_{n_1})\|) + l_B c_{u^*}(\|\Delta x_k^*\| + \|\Delta x_k^*(s-\sigma_1)\| + \cdots + \|\Delta x_k^*(s-\sigma_{n_2})\|) + c_B\|\Delta u_k^*\|]\mathrm{d}s.$$

Note the fact that

$$\int_0^t \|\Delta x_k^*(s-\theta)\|\,\mathrm{d}s \le \mu c_\psi + \int_0^t \|\Delta x_k^*(s)\|\,\mathrm{d}s,$$

where $\theta \in \{\tau_i, i = 1, \ldots, n_1, \sigma_i, i = 1, \ldots, n_2\}$, which leads to

$$\|\Delta x_k^*\| \le [1 + (l_f n_1 + l_B c_{u^*} n_2)\mu]c_\psi + \int_0^t [(l_f(n_1+1) + l_B c_{u^*}(n_2+1))\|\Delta x_k^*\| + c_B\|\Delta u_k^*\|]\,\mathrm{d}s.$$

Defining $c_1 = l_f(n_1+1) + l_B c_{u^*}(n_2+1)$ and applying Bellman–Gronwall Lemma give rise to

$$\|\Delta x_k^*\| \le [1 + (l_f n_1 + l_B c_{u^*} n_2)\mu]c_\psi e^{c_1 t} + \int_0^t e^{c_1(t-s)} c_B\|\Delta u_k^*\|\,\mathrm{d}s$$

$$\|\Delta x_k^*(t-\theta)\| \le [1 + (l_f n_1 + l_B c_{u^*} n_2)\mu]c_\psi e^{c_1 t} + \int_0^t e^{c_1(t-s)} c_B\|\Delta u_k^*\|\,\mathrm{d}s.$$

Now we can make the same claims as in Theorem (4.1). For the control design one can choose $L(\cdot,\cdot)$ properly such that

$$\|I - L(g(x(t), t), t)g_x(x(t), t)B(x(t), x(t-\sigma_1), \ldots, x(t-\sigma_{n_2}), t)\| \le \rho < 1,$$

which depends on $\psi(t-\sigma_i)$ as $0 \le t < \sigma_i, i = 1, \ldots, n_2$.

## 6. Simulation illustrations

The following example and simulations are presented to illustrate the theoretical results of this paper. Consider the non-linear system with time delay

$$\begin{bmatrix} \dot{x}_{1,k}(t) \\ \dot{x}_{2,k}(t) \\ \dot{x}_{3,k}(t) \end{bmatrix} = \begin{bmatrix} x_{2,k}(t-\tau) + x_{3,k}(t) \\ x_{1,k}(t) + x_{3,k}(t-\tau) \\ x_{1,k}(t-\tau) + \dfrac{1}{1+|(t-\tau)x_{2,k}(t)|} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \sin((t-\sigma)x_{2,k}(t-\sigma)) & \cos((t-\sigma)x_{1,k}(t-\sigma)) \end{bmatrix} \begin{bmatrix} u_{1,k}(t) \\ u_{2,k}(t) \end{bmatrix}$$

$$\begin{bmatrix} y_{1,k}(t) \\ y_{2,k}(t) \end{bmatrix} = \begin{bmatrix} 2x_{2,k}(t) + \sin(tx_{2,k}(t)) \\ x_{1,k}(t) \end{bmatrix},$$

where $\tau = \sigma = 0.5$, $x_{i,k}(t) = \psi_{i,k}(t)$, $i = 1, 2, 3, t \in [-0.5, 0]$. Let the desired trajectories be given as

$$\begin{bmatrix} y_{1,d}(t) \\ y_{2,d}(t) \end{bmatrix} = \begin{bmatrix} 12t^2(1-t) \\ 12t(1-t)^2 \end{bmatrix}, t \in [0, 1].$$

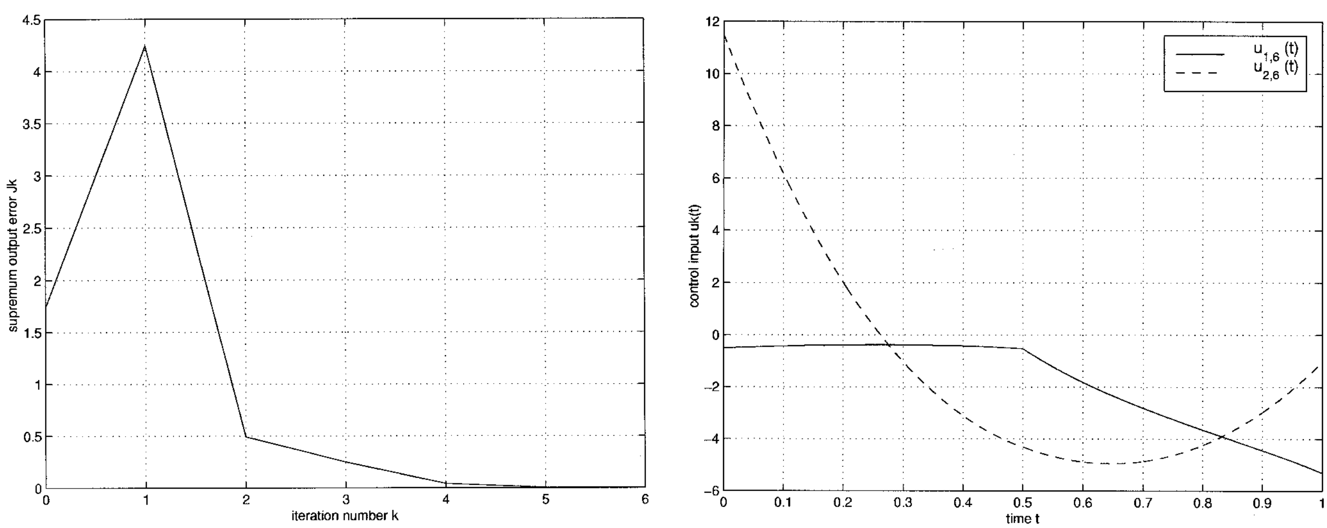
Note that the non-linear functions $1/1 + |tz|, \sin(tz)$, and $\cos(tz)$ are all uniformly globally Lipschitz in $z$ and uniformly bounded for all $t \in [-0.5, 1]$ and for all $z \in R$. It is thus concluded that $g_x$ and $B$ satisfy assumptions (A3) and (A4). Because $g_x B$ is a full rank matrix, the learning gain in (4) is chosen as

$$L = \begin{bmatrix} \dfrac{\alpha}{2 + t\cos(tx_{2,k})} & 0 \\ 0 & \beta \end{bmatrix}.$$

We should select $\alpha \in (0, 2)$ and $\beta \in (0, 2)$ to satisfy $\max\{|1-\alpha|, |1-\beta|\} < 1$. In this example $\alpha = 0.8$ and $\beta = 0.8$ are selected. Simulations are conducted for the following three cases.

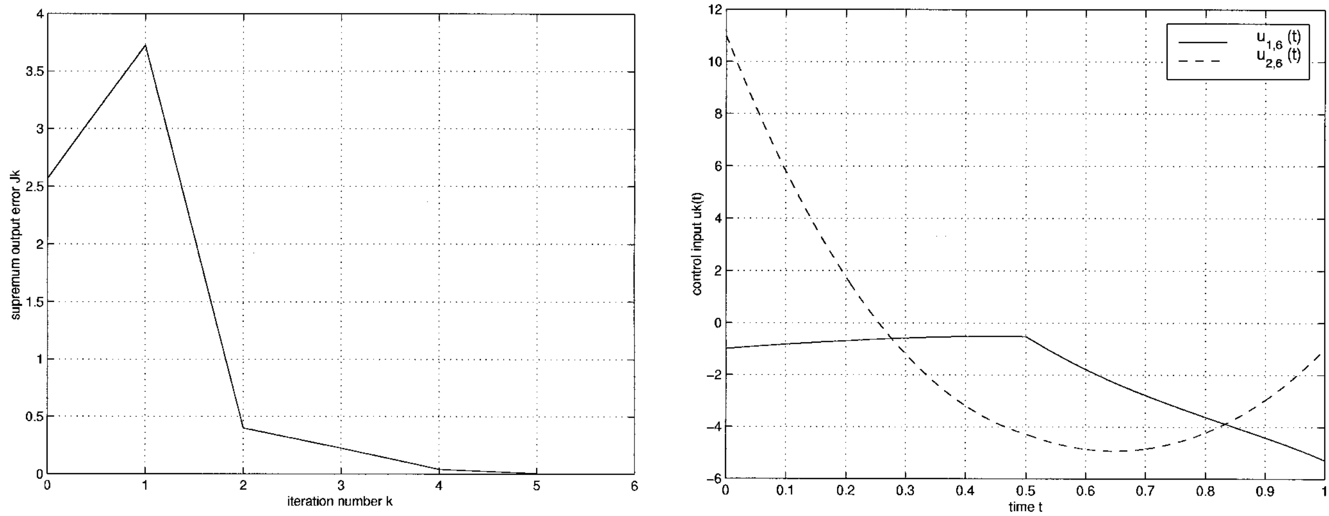
### 6.1. *Convergence*

Let the initial functions be $\psi_{i,k}(t) = t$ and $\psi_{i,k}(t) = 2t, i = 1, 2, 3, t \in [-0.5, 0]$, respectively. The updating law (3) is applied with the initial controls $u_{1,0} = 0$ and $u_{2,0} = 0$ for all $t \in [0, 1]$. Define the performance index $J_k = \sup_{t\in[0,1]} \|y_d(t) - y_k(t)\|_\infty$. The iteration stops when the tracking index $J_k < 0.005$. For both cases, this requirement of tracking performance is achieved at the sixth iteration. Figures 1 and 2 show the tracking histories and the resulting control inputs respectively. The effect of the time delays is clearly shown by the turning points in the control inputs at the time $t = 0.5$, but uniform convergence of the system outputs to the desired trajectories is guaranteed due to the zero initial function errors at $t = 0$.




(*a*) (*b*)

**Figure 1. Responses when the conventional learning algorithm is used with the initial function $\psi_{i,k}(t) = t$. (a) Tracking errors, (b) control input $u_k(t)$ at the sixth iteration.**




(*a*) (*b*)

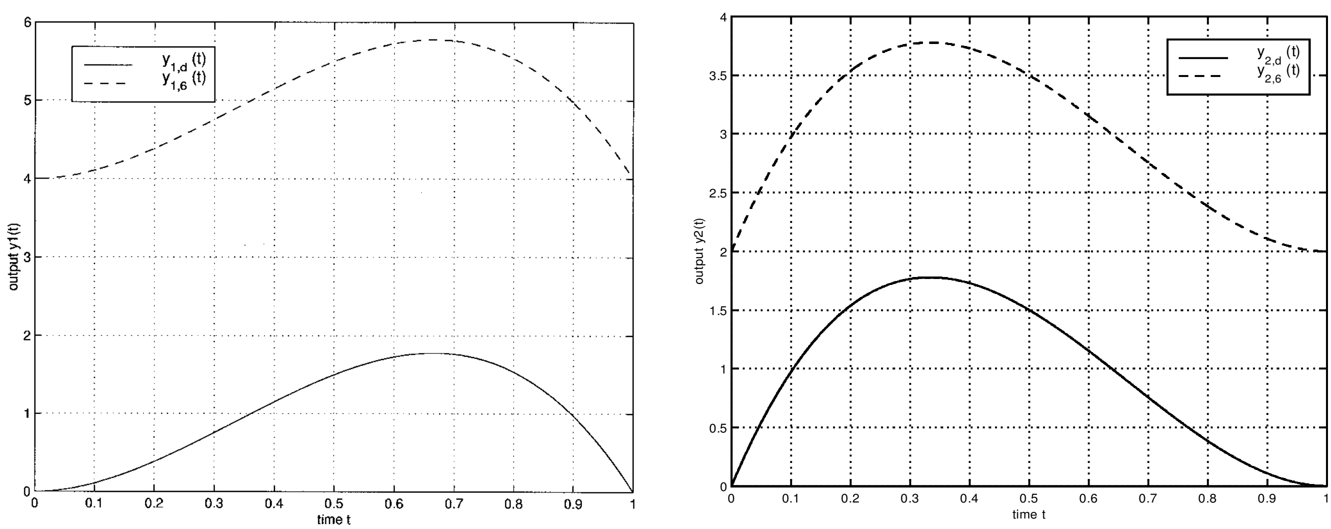
**Figure 2. Responses when the conventional learning algorithm is used with the initial function $\psi_{i,k}(t) = 2t$. (a) Tracking errors, (b) control input $u_k(t)$ at the sixth iteration.**

## 6.2. *Divergence and initial rectifying*

The initial functions at each iteration are chosen as $\psi_{i,k}(t) = 2t + 2, i = 1, 2, 3, t \in [-0.5, 0]$. There exist initial function errors at $t = 0$. Figure 3 shows resulting output trajectories at the sixth iteration when applying the updating law (3), in which the output trajectories track the desired trajectories with the error defined by (9). Figure 4 shows resulting output trajectories at the eighth iteration when applying the updating law (15) with $h = 0.2$. The output trajectories uniformly converge to the desired trajectories on the interval $[0.2, 1]$. Meanwhile, the tracking performance $J_k = \sup_{t\in[0.2,1]} \|y_d(t) - y_k(t)\|_\infty < 0.005$ is achieved at the eighth iteration.

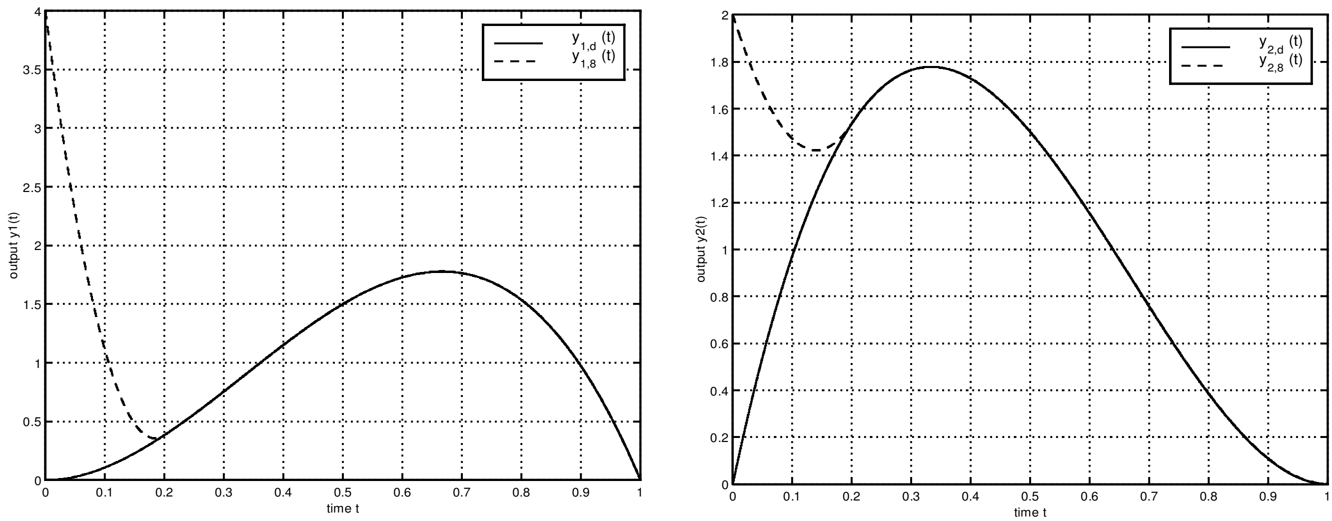
## 6.3. *Robustness*

Let the initial functions be $\psi_{i,k}(t) = t + 1 + 0.01\text{randn}, \ i = 1, 2, 3, t \in [-0.5, 0]$ and $\psi_{i,k}(t) = 2t + 2 + 0.01\text{randn}, \ i = 1, 2, 3, t \in [-0.5, 0]$, respectively.



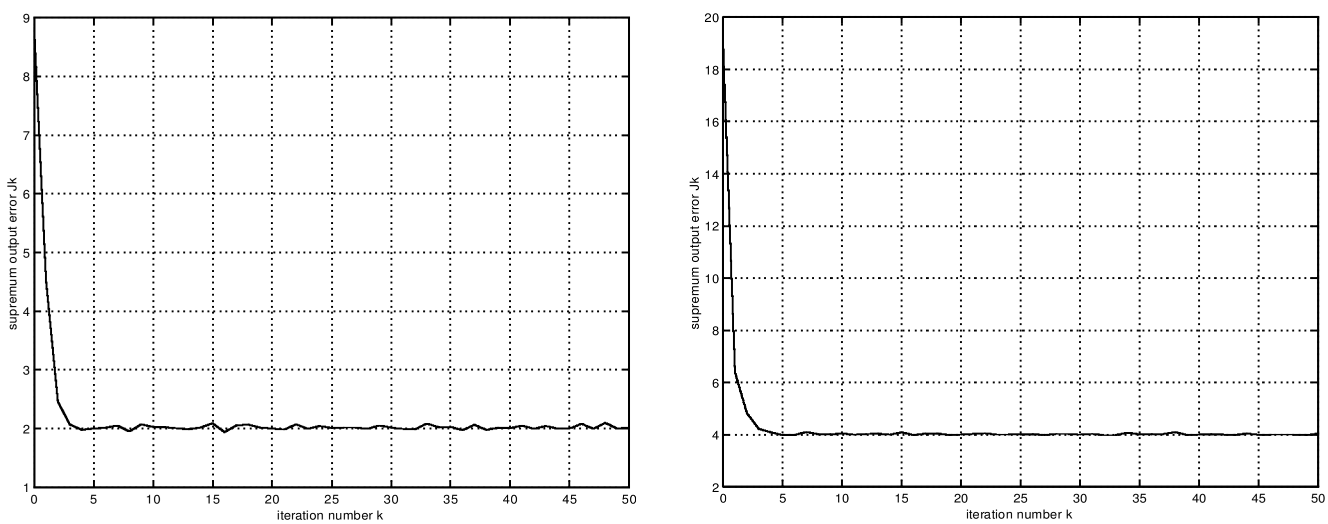

(*a*) (*b*)

**Figure 3. Output trajectories when the conventional learning algorithm is used with the initial function $\psi_{i,k}(t) = 2t + 2$. (a) Output trajectory $y_1(t)$ at the sixth iteration, (b) output trajectory $y_2(t)$ at the sixth iteration.**




(*a*) (*b*)

**Figure 4. Output trajectories when the proposed learning algorithm with initial rectifying action is used with the initial function $\psi_{i,k}(t) = 2t + 2$. (a) Output trajectory $y_1(t)$ at the eighth iteration, (b) output trajectory $y_2(t)$ at the eighth iteration.**



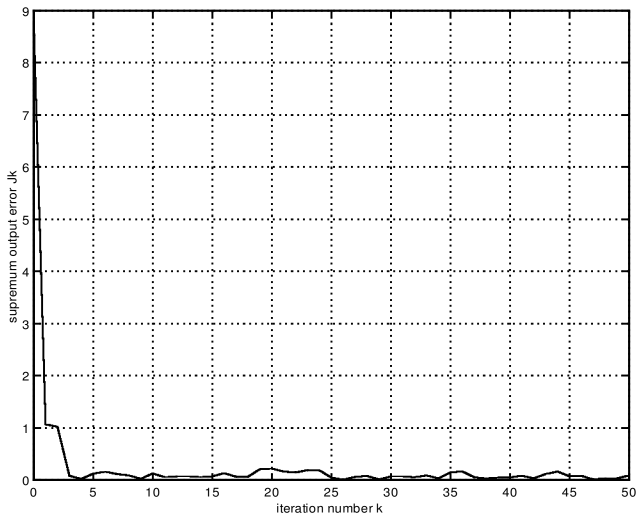

(*a*) (*b*)

**Figure 5. Tracking errors when the conventional learning algorithm is used in the presence of random initial function errors. (a) $\psi_{i,k}(t) = t + 1 + 0.01$randn, (b) $\psi_{i,k}(t) = 2t + 2 + 0.01$randn.**

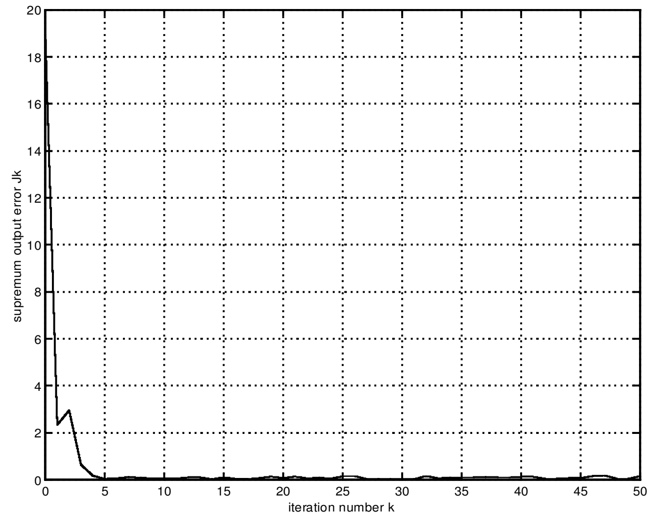
Randn is a generator of random scalar with normal distribution, mean = 0, and variance = 1 (white Gaussian noise). The performance index is still defined as $J_k = \sup_{t\in[0.2,1]} \|y_d(t) - y_k(t)\|_\infty$ and the repetitions are conducted until $k = 50$. It can be observed from figure 5 that the tracking errors generated by using the updating law (3) move with the initial function errors and will become very large when the initial functions at each iteration are far away from the desired initial functions. However, figure 6 indicates that due to the initial rectifying action, better tracking performance is obtained by using the proposed updating law (15) regardless of the tracking on the interval $[0, 0.2]$.

## 7. Conclusion

In this paper, the trajectory tracking problem is formulated and solved using iterative learning control methodology for a class of non-linear systems with time delay. It is shown that the tracking performance can be poor due to an initial function shifting when a conventional learning algorithm is applied. An initial rectifying action




(a)




(b)

**Figure 6. Tracking errors when the proposed learning algorithm with initial rectifying action is used in the presence of random initial function errors. (a) $\psi_{i,k}(t) = t + 1 + 0.01$randn, (b) $\psi_{i,k}(t) = 2t + 2 + 0.01$randn.**

is introduced for performance improvement and a proof is provided for analysing its robustness and convergence against initial function errors. The theoretical and simulation results show that the robustness performance of the learning algorithm can be improved by the initial rectifying action. Whenever the initial function of the system is reset to a fixed function that needs not close to the desired one, uniform convergence of the system output to the desired trajectory is guaranteed also due to the initial rectifying action.

## A.1. Appendix A

**Proof of Theorem 3.1:** Given the initial condition $x^*(t) = \psi^*(t), t \in [-\mu, 0]$, denote $u^*(t), t \in [0, T]$ as the control input satisfying

$$y^*(t) = g(x^*(t), t) \tag{A.1}$$

$$\dot{x}^*(t) = f(x^*(t), x^*(t-\tau), t) + B(x^*(t), x^*(t-\sigma), t)u^*(t), \tag{A.2}$$

where $x^*(t), t \in [0, T]$ is the corresponding state. For simplicity, the following notations are used: $f^* = f(x^*(t),\ x^*(t-\tau), t)$, $f_k = f(x_k(t),\ x_k(t-\tau),\ t)$, $B^* = B(x^*(t),\ x^*(t-\sigma),\ t)$, $B_k = B(x_k(t), x_k(t-\sigma), t)$, $g_t^* = g_t(x^*(t),\ t)$, $g_{tk} = g_t(x_k(t),\ t)$, $g_x^* = g_x(x^*(t),\ t)$, $g_{xk} = g_x(x_k(t), t)$, $L_k = L(y_k(t), t), \Delta u_k^* = u^*(t) - u_k(t)$, and $\Delta x_k^* = x^*(t) - x_k(t)$. Using the definition of $y^*(t)$ in (9), (3) can be written as

$$\begin{aligned} u_{k+1} &= u_k + L_k(\dot{y}^* - \dot{y}_k) + L_k(\dot{y}_d - \dot{y}^*) \\ &= u_k + L_k(\dot{y}^* - \dot{y}_k), \end{aligned}$$

which implies

$$\begin{aligned} \Delta u_{k+1}^* &= (I - L_k g_{xk} B_k)\Delta u_k^* \\ &\quad - L_k\{g_t^* - g_{tk} + (g_x^* - g_{xk})(f^* + B^* u^*) \\ &\quad + g_{xk}[(f^* - f_k) + (B^* - B_k)u^*]\}. \end{aligned}$$

Taking norms and applying the bounds and the Lipschitz conditions, we have:

$$\begin{aligned} \|\Delta u_{k+1}^*\| &\le \rho\|\Delta u_k{}^*\| + c_L[(l_{gt} + l_{gx}c^* + c_{gx}c_1)\|\Delta x_k^*\| \\ &\quad + c_{gx}(l_f\|\Delta x_k^*(t-\tau)\| + l_B c_{u^*}\|\Delta x_k^*(t-\sigma)\|)], \end{aligned} \tag{A.3}$$

where $c_L$ is the norm bound for $L(\cdot,\cdot)$; $c^* = \sup_{t\in[0,T]}\|f^* + B^* u^*\|$, $c_{u^*} \triangleq \sup_{t\in[0,T]}\|u^*(t)\|$, and $c_1 = l_f + l_B c_{u^*}$.

For evaluating the state errors on the right hand side of (A.3), we integrate both sides of (1) and (A.2) and use (7) to obtain:

$$\Delta x_k^* = \int_0^t [f^* - f_k + (B^* - B_k)u^* + B_k \Delta u_k^*]\, ds.$$

Taking norms and using their properties yield:

$$\begin{aligned} \|\Delta x_k^*\| &\le \int_0^t (c_1\|\Delta x_k^*\| \\ &\quad + l_f\|\Delta x_k^*(s-\tau)\| + l_B c_{u^*}\|\Delta x_k^*(s-\sigma)\| \\ &\quad + c_B\|\Delta u_k^*\|)\, ds. \end{aligned} \tag{A.4}$$

Note that for $t \in [0, \theta]$ with $\theta \in \{\tau, \sigma\}$:

$$\int_0^t \|\Delta x_k^*(s-\theta)\| \,\mathrm{d}s = \int_{-\theta}^{t-\theta} \|\psi^*(s) - \psi_k(s)\| \,\mathrm{d}s = 0, \qquad (A.5)$$

and for $t \in (\theta, T]$:

$$\int_0^t \|\Delta x_k^*(s-\theta)\| \,\mathrm{d}s = \int_{-\theta}^{0} \|\psi^*(s) - \psi_k(s)\| \,\mathrm{d}s + \int_0^{t-\theta} \|\Delta x_k^*(s)\| \,\mathrm{d}s = \int_0^{t-\theta} \|\Delta x_k^*(s)\| \,\mathrm{d}s. \qquad (A.6)$$

Combining (A.5) and (A.6) produces

$$\int_0^t \|\Delta x_k^*(s-\theta)\| \,\mathrm{d}s \le \int_0^t \|\Delta x_k^*(s)\| \,\mathrm{d}s, \qquad (A.7)$$

where $t \in [0, T]$. Substituting (A.7) into (A.4) gives rise to

$$\|\Delta x_k^*\| \le \int_0^t (2c_1 \|\Delta x_k^*\| + c_B \|\Delta u_k^*\|) \mathrm{d}s.$$

Then applying Bellman–Gronwall Lemma, we obtain:

$$\|\Delta x_k^*\| \le \int_0^t e^{2c_1(t-s)} c_B \|\Delta u_k^*\| \,\mathrm{d}s, \qquad (A.8)$$

which implies

$$\|\Delta x_k^*(t-\theta)\| \le \int_0^{t-\theta} e^{2c_1(t-\theta-s)} c_B \|\Delta u_k^*\| \,\mathrm{d}s, \; t \in (\theta, T].$$

Because of $e^{-2c_1\theta} \le 1$,

$$\|\Delta x_k^*(t-\theta)\| \le \int_0^t e^{2c_1(t-s)} c_B \|\Delta u_k^*\| \,\mathrm{d}s, \; t \in (\theta, T], \qquad (A.9)$$

which is also true for $t \in [0, \theta]$ since $\|\Delta x_k^*(t-\theta)\| = \|\psi^*(t-\theta) - \psi_k(t-\theta)\| = 0, t \in [0, \theta]$. Now, substituting (A.8) and (A.9) into (A.3) produces:

$$\|\Delta u_{k+1}^*\| \le \rho \|\Delta u_k^*\| + c_L c_2 c_B \int_0^t e^{2c_1(t-s)} \|\Delta u_k^*\| \,\mathrm{d}s,$$

where $c_2 = l_{gt} + l_{gx} c^* + 2c_{gx} c_1$. Defining $c_3 = \max\{2c_1, c_L c_2 c_B\}$ and multiplying both sides by $e^{-\lambda t}$ $(\lambda > 0)$ lead to:

$$e^{-\lambda t} \|\Delta u_{k+1}^*\| \le \rho e^{-\lambda t} \|\Delta u_k^*\| + c_3 \int_0^t e^{(c_3-\lambda)(t-s)} e^{-\lambda s} \|\Delta u_k^*\| \,\mathrm{d}s.$$

Taking supremum for $t \in [0, T]$ and $\lambda > c_3$ according to the $\lambda$-norm definition, we get:

$$\|\Delta u_{k+1}^*\|_\lambda \le \bar{\rho} \|\Delta u_k^*\|_\lambda, \qquad (A.10)$$

where $\bar{\rho} = \rho + c_3(1 - e^{(c_3-\lambda)T})/(\lambda - c_3)$. Since $\rho < 1$, it is possible to find a $\lambda > c_3$ sufficiently large such that $\bar{\rho} < 1$. Then, (A.10) is a contraction in $\|\Delta u_k^*\|_\lambda$. When the iterations increase, $k \to \infty$, we obtain $\|\Delta u_k^*\|_\lambda \to 0$ so that $u_k \to u^*$ uniformly on $[0, T]$ as $k \to \infty$. Furthermore, from (A.8) and using similar manipulations give

$$\|\Delta x_k^*\|_\lambda \le c_B \frac{1 - e^{(c_3-\lambda)T}}{\lambda - c_3} \|\Delta u_k^*\|_\lambda.$$

Therefore, $x_k$ converges to $x^*$ uniformly on $[0, T]$ as $k \to \infty$. To obtain the result for $y_k$, we use the fact that $g(\cdot, \cdot)$ is Lipschitz in $x$ and the uniform convergence of $\Delta x_k^*$. This completes the proof. ■

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