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Brief Paper

Convergence and Robustness of Discrete Time Nonlinear Systems with Iterative Learning Control*

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Key Words—learning control; discrete time; robustness; nonlinear systems.

Abstract—In this paper, iterative learning control design of a class of discrete time nonlinear dynamic systems with disturbances are considered. An iterative learning control law is proposed to overcome the uncertainties in system parameters and disturbances. It is shown that the system outputs, states and control inputs can be guaranteed to converge to desired trajectories in the absence of state, output disturbances and repeatability uncertainty. In the presence of these disturbances and initial state uncertainty, the tracking errors will be bounded. Experiment is carried out to verify the theory and results are presented. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

The objective of iterative learning control designs is to overcome the imperfect knowledge of the dynamics structure and/or parameter values to improve tracking performance through repetition. These controllers are applied to the dynamic systems working on a same task repetitively over a fixed operation cycle. The learning laws are updated off line after each operation using the error measurements in the previous cycle. The iterative learning control schemes are well received in the applications of robotic manipulators and disk drive systems which are mostly designed for repetitive tasks. Researchers tackle the convergence and robustness of iterative learning controllers for robotic manipulators in Arimoto (1990), Arimoto *et al.* (1984), Atkeson and McIntyre (1986), Bondi *et al.* (1988), Craig (1984) and Kuc *et al.* (1992). The iterative learning control has been extended to applications such as robots with constraint on end-effector in Wang *et al.* (1995) and with flexible joints in de Luca and Ulivi (1992) and Wang (1995). Recently, iterative learning control has been successfully applied to steel processes which is a periodic system (Manayathara *et al.*, 1996).

The research of iterative learning control has been focusing on continuous time dynamic systems. For general dynamic systems, iterative learning control theory has been developed for certain classes of nonlinear systems. In Hauser (1987), Heinziner *et al.* (1992) and Messner *et al.* (1991), the convergence and robustness issues have been investigated for some learning control schemes. On the other hand, discrete time iterative learning controllers are proposed based on the approximations of the velocity and acceleration of the continuous cases in Tso and Ma (1992) and Wang *et al.* (1993). They are shown to ensure the convergence of input, state to desired trajectories with bounded errors and robustness against certain types of disturbances.

However, the development of iterative learning control for discrete time systems has been restricted to linear systems as can be seen in Kurek (1993), Saab (1995a), Togai and Yamano (1985), Tomizuka *et al.* (1989) and Yi and Park (1996). In Saab (1995b), an iterative learning control is designed to deal with a class of the systems with nonlinear state equations and linear output equations. The given learning control law takes a two-step tracking error difference and thus is equivalent to a numerical approximation of D-type controller in continuous time case (Tso and Ma, 1992).

In this paper, the iterative learning control problem is investigated for a class of time-varying discrete-time dynamic systems in the presence of disturbances. The proposed learning law uses only one-step tracking error with anticipation in time advance for off-line computation. The convergence and robustness issues are investigated using a newly defined α-norm and an inequality. We shall show that the system outputs, states and control inputs can be guaranteed to converge to desired trajectories with bounded tracking errors. Furthermore, in the absence of state, output disturbances and repeatability uncertainty, the tracking errors approach zero.

2. *Main results*

In this paper, we use the following notations. R^n is the n-dimensional Euclidean space with norm $\|x\| = (x^T x)^{1/2}$ for $x \in R^p$. $C \in R^{p\times m}$ indicates C is an $(p \times m)$-dimensional matrix with real elements and we use $\|C\| = \sqrt{\lambda_{max}(C^T C)}$ as the norm for matrices, where $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue for symmetric matrix. Let N be the set of positive integers $\{0, 1, 2, \ldots, n\}$.

Consider a class of time-varying discrete-time nonlinear dynamic systems described by the following difference equations:

$$x_i(k+1) = f(x_i(k), k) + B(x_i(k), k)u_i(k) + w_i(k), \quad (1)$$

$$y_i(k) = A(k)x_i(k) + v_i(k), \quad (2)$$

where i indicates the number of operation cycle, and k is the discrete time index running from $k = 0$ to $k = n$ to complete a cycle. For all $k \in N$, $x_i(k) \in R^p$, $u_i(k) \in R^r$ and $y_i(k) \in R^m$, $w_i(k) \in R^p$ and $v_i(k) \in R^m$ are the states, inputs, outputs, state disturbances and output disturances, respectively. The vector/matrix functions $f: R^p \times N \to R^p$ and $B: R^p \times N \to R^{p\times r}$ and $A(k): N \to R^{m\times p}$ satisfy the properties and bounds stated as follows.

A1. Suppose $w(k)$ and $v(k)$ are both zero for $k \in N$. A target set for iterative learning control design is given as a bounded output sequence $y_d(k)$, $k \in N$, with a bounded state sequence $x_d(k)$ and a unique bounded input sequence $u_d(k)$ that satisfy equations (1) and (2).

A2. The vector function $f(x, k)$ and matrix function $B(x, k)$ are globally uniformly Lipschitz in x on N in the sense of

$$\|f(x_1, k) - f(x_2, k)\| \le c_f \|x_1 - x_2\|,$$

and

$$\|B(x_1, k) - B(x_2, k)\| \le c_B \|x_1 - x_2\|,$$

for positive constants c_f and c_B.

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A3. The matrices $B(x, k)$ and $A(k)$ are defined in $R^p \times N$ with bounds as $\|B(x, k)\| \le b_B$ and $\|A(k)\| \le b_A$, where b_B and b_A are positive constants. Furthermore, the product matrix $A(k+1)B(x, k)$ has full rank for $(x, k) \in R^p \times N$.

A4. The disturbances are bounded in the sense of $\|v_i(k)\| \le b_v$ and $\|w_i(k)\| \le b_w$ for all $i \ge 0$, $k \in N$ and positive constants b_v and b_w.

A5. All operations start within a neighborhood of $x_d(0)$ in the sense of $\|x_d(0) - x_i(0)\| \le b_{x_0}$ for $i \ge 0$ and a positive constant b_{x_0}. This assumption is natural in many repetitive dynamic systems such as the repeatability in robot specifications.

Now let us consider an iterative learning control update law, after ith operation cycle,

$$u_{i+1}(k) = u_i(k) + L(k)e_i(k+1), \tag{3}$$

where $e_i(k) = y_d(k) - y_i(k)$ is the output tracking error, and $L: N \to R^{r \times m}$ is a learning gain matrix which has a bound b_L, i.e. $\|L(k)\| \le b_L$ for all $k \in N$.

Theorem. Consider the time-varying discrete-time system described by equations (1) and (2) with the iterative learning control law (3). Assume that the properties (A1)–(A5) are satisfied and the following inequality:

$$\|I - L(k)A(k+1)B(x, k)\| \le \rho < 1 \tag{4}$$

holds for all $(x, k) \in R^p \times N$. Given a bounded output sequence $y_d(k)$, $k \in N$ and in absence of state disturbance, output disturbance and initial state uncertainties, i.e. $b_w = b_v = b_{x0} = 0$, it is guaranteed that the control $u_i(k)$ converges to $u_d(k)$, the state $x_i(k)$ to $x_d(k)$, and the output $y_i(k)$ to $y_d(k)$, for $k \in N$, as operation cycles increase, $i \to \infty$. In the presence of these disturbances and uncertainty, the above convergence are guaranteed with error bounds that are functions of b_w, b_v and b_{x0} as to be specified in inequalities (15), (18) and (20).

Definition. The α-norm is defined for a positive real function $q: N \to R$ as, with $\alpha \ge 1$,

$$\|q(\cdot)\|_\alpha = \sup_{k \in N} q(k)(1/\alpha)^k.$$

Remark

1. The α-norm is equivalent to the ∞-norm defined as $\|q(\cdot)\|_\infty = \sup_{k \in N} q(k)$ by noting $\|q(k)\|_\alpha \le \|q(k)\|_\infty \le \alpha^n \|q(k)\|_\alpha$. Hence the claims in the theorem are established by showing the convergence in the α-norm.

Proof of Theorem. Use $u_d(k)$ on both sides of equation (3), then equation (2),

$$\delta u_{i+1}(k) = \delta u_i(k) - L(k)A(k+1)\delta x_i(k+1) + L(k)v_i(k+1),$$

where $\delta z_i = z_d - z_i$ for $z \in \{x, u\}$. Use equation (1) and its desired dynamics,

$$x_d(k+1) = f(x_d(k), k) + B(x_d(k), k)u_d(k),$$

we obtain

$$\begin{aligned}\delta u_{i+1}(k) = {} & [I - L(k)A(k+1)B(x_i(k), k)]\delta u_i(k) \\ & + L(k)[A(k)w_i(k) + v_i(k+1)] \\ & - L(k)\{A(k+1)[f(x_d(k), k) - f(x_i(k), k)] \\ & + [B(x_d(k), k) - B(x_i(k), k)]u_d(k)\}\end{aligned} \tag{5}$$

Take norms on both sides of equation (5) and we get

$$\|\delta u_{i+1}(k)\| \le \rho \|\delta u_i(k)\| + b_1 + h_2 \|\delta x_i(k)\|, \tag{6}$$

where, noting inequality (4) and $\|u_d\| \le b_{ud}$,

$$\|I - L(k)A(k+1)B(x_i, k)\| \le \rho = 1,$$

$$b_1 = b_L(b_A b_w + b_v),$$

$$h_1 = c_f + b_{u_d} c_B,$$

and

$$h_2 = b_L b_A h_1.$$

From equation (1), we have

$$\begin{aligned}\delta x_i(k+1) = {} & f(x_d(k), k) - f(x_i(k), k) \\ & + [B(x_d(k), k) - B(x_i(k), k)]u_d(k). \\ & + B(x_i(k), k)\delta u_i(k) - w_i(k).\end{aligned} \tag{7}$$

Take norms on both sides of equation (7) to yield

$$\|\delta x_i(k+1)\| \le h_1 \|\delta x_i(k)\| + b_B \|\delta u_i(k)\| + b_w. \tag{8}$$

Use the lemma from the Appendix in equation (8) and we have

$$\|\delta x_i(k)\| \le \sum_{j=0}^{k-1} h_1^{k-1-j}[b_B \|\delta u_i(j)\| + b_w] + h_1^k b_{x_0}. \tag{9}$$

Substitute equation (9) into equation (6) to get

$$\begin{aligned}\|\delta u_{i+1}(k)\| \le {} & \rho \|\delta u_i(k)\| + b_1 + h_2 h_1^k b_{x_0} \\ & + h_2 \sum_{j=0}^{k-1} h_1^{k-1-j}[b_B \|\delta u_i(j)\| + b_w].\end{aligned} \tag{10}$$

Multiply both sides of equation (10) by $(1/\alpha)^k$, with $\alpha > \max[1, h_1]$, we have

$$\begin{aligned}\|\delta u_{i+1}(k)\|(1/\alpha)^k \le {} & \rho \|\delta u_i(k)\|(1/\alpha)^k + b_1(1/\alpha)^k + h_2 b_{x_0}(h_1/\alpha)^k \\ & + \frac{h_2}{\alpha}\sum_{j=0}^{k-1}(h_1/\alpha)^{k-1-j}[b_B \|\delta u_i(j)\|(1/\alpha)^j \\ & + b_w(1/\alpha)^j].\end{aligned} \tag{11}$$

In the α-norm, noting that the α-norm of a constant is the constant itself,

$$\begin{aligned}\|\delta u_{i+1}\|_\alpha \le {} & \rho \|\delta u_i\|_\alpha + b_1 + h_2 b_{x_0} \\ & + \frac{h_2(b_w + b_B \|\delta u_i\|_\alpha)}{\alpha}\sum_{j=0}^{k-1}(h_1/\alpha)^{k-1-j}\end{aligned}$$

or

$$\|\delta u_{i+1}\|_\alpha \le \left[\rho + \frac{h_2 b_B(1 - (h_1/\alpha)^n)}{\alpha - h_1}\right]\|\delta u_i\|_\alpha + \varepsilon, \tag{12}$$

where

$$\varepsilon = b_1 + h_2 b_{x_0} + \frac{h_2 b_w}{\alpha - h_1}(1 - (h_1/\alpha)^n). \tag{13}$$

Choose α large enough so that

$$\hat{\rho} = \rho + \frac{h_2 b_B(1 - (h_1/\alpha)^n)}{\alpha - h_1} < 1. \tag{14}$$

Thus, equation (12) is a contraction in $\|\delta u\|_\alpha$. When the operations increase, $i \to \infty$, we get

$$\limsup_{i \to \infty} \|\delta u_i\|_\alpha \le \frac{\varepsilon}{1 - \hat{\rho}}. \tag{15}$$

Similarly, multiply both sides of equation (9) by $(1/\alpha)^k$,

$$\begin{aligned}\|\delta x_i(k)\|(1/\alpha)^k \le {} & \frac{1}{\alpha}\sum_{j=0}^{k-1}(h_1/\alpha)^{k-1-j}[b_B \|\delta u_i(j)\|(1/\alpha)^j \\ & + b_w(1/\alpha)^j] + (h_1/\alpha)^k b_{x_0}.\end{aligned} \tag{16}$$

In the α-norm, we have

$$\|\delta x_i\|_\alpha \le \frac{b_B[1 - (h_1/\alpha)^n]}{\alpha - h_1}\|\delta u_i\|_\alpha + \frac{b_w[1 - (h_1/\alpha)^n]}{\alpha - h_1} + b_{x_0}. \tag{17}$$

Hence

$$\limsup_{i \to \infty} \|\delta x_i\|_\alpha \le \frac{b_B[1 - (h_1/\alpha)^n]\varepsilon}{(\alpha - h_1)(1 - \hat{\rho})} + \frac{b_w[1 - (h_1/\alpha)^n]}{\alpha - h_1} + b_{x_0}. \tag{18}$$

Finally, from equation (2), we have

$$\|\delta y_i\|_\alpha \le c_g \|\delta x_i\|_\alpha + b_v, \tag{19}$$

and this implies

$$\limsup_{i\to\infty}\|\delta y_i\|_\alpha \le \frac{c_g b_B[1-(h_1/\alpha)^n]\varepsilon}{(\alpha-h_1)(1-\hat{\rho})} + \frac{c_g b_w[1-(h_1/\alpha)^n]}{\alpha-h_1} + b_{x_0}c_g + b_v. \quad (20)$$

In the absence of noises w_i, v_i and with perfect repeatability, i.e. $b_w = 0$, $b_v = 0$ and $b_{x_0} = 0$, then the error bound in equation (13) becomes

$$\varepsilon = 0.$$

Clearly, from the inequalities (15), (18) and (20), we have $\|\delta u_i\|_\alpha \to 0$, $\|\delta x_i\|_\alpha \to 0$ and $\|\delta y_i\|_\alpha \to 0$ as results. □

Remarks (Continued)

2. The update law of $u_{i+1}(k)$ uses the output tracking error $e_i(k+1) = y_d(k+1) - y_i(k+1)$, instead of $e_i(k)$, because in the previous cycle, the output tracking error at $k+1$ is due to the control action $u_i(k)$.
3. Computation of $u_{i+1}(n)$ is not required because $k = n$ is the end of operation cycle. The initial guess u_0 can be chosen zero for convenience.

3. *Experiment*

In the experiment study, we use a mechanism of a DC-motor driving a single rigid link through a gear, as shown in Fig. 1. An optical encoder is mounted on the link side to measure link angle position. All parameters are unknown except we know that the dynamics of the system is governed by the following second-order differential equation:

$$\left(J_m + \frac{J_l}{n^2}\right)\ddot{\theta}_m + \left(B_m + \frac{B_l}{n^2}\right)\dot{\theta}_m + \frac{Mgl}{n}\sin\left(\frac{\theta_m}{n}\right) = u, \quad (21)$$

and the link angle position is related to motor angle as

$$\theta_l = \theta_m/n, \quad (22)$$

where θ_m, J_m, B_m and θ_l, J_l, B_l are the motor and link angles, inertia and damping coefficients, respectively, n is the gear ratio, u is the motor torque, M is the lumped mass and l is the center of mass from the axis of motion. The motor is controlled by a PC with a power amplifier.

With the discrete time interval set to $\Delta = 50$ ms, the operation cycle is $\Gamma = 3$ s or, equivalently, $N = \{0, 1, \ldots, 60\}$. The Euler's approximation takes the form of state difference equation (1) with $x = (x_1, x_2)^T = (\theta_m, \dot{\theta}_m)^T$, $y = (y_1, y_2)^T = (\theta_l, \dot{\theta}_l)^T$ and the following functions:

$$f(x(k), k) = \begin{pmatrix} x_1(k) + \Delta x_2(k) \\ x_2(k) + \dfrac{\Delta}{J_m + J_l/n^2}\left[-(B_m + B_l/n^2)x_2(k) - \dfrac{Mg_l}{n}\sin\left(\dfrac{x_1(k)}{n}\right)\right] \end{pmatrix} \quad (23)$$

$$B(x(k), k) = \begin{pmatrix} 0 \\ \dfrac{\Delta}{J_m + J_l/n^2} \end{pmatrix}. \quad (24)$$

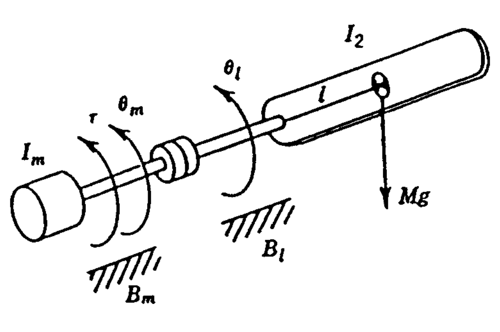

Fig. 1. Single-link mechanism.

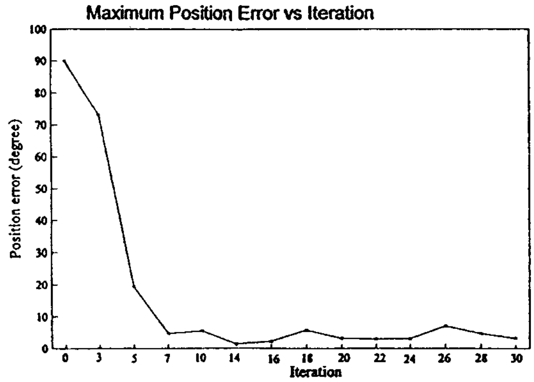

Fig. 2. Convergence of tracking error.

The disturbances $w(k)$ include the frictions and amplifier circuit uncertainties. The output equations include the discrete time version of equation (22) as well as the numerical differentiation for the link angular velocity in the form of

$$\dot{\theta}_l(k) = \dot{\theta}_m(k)/n. \quad (25)$$

The output equation takes the form of equation (2) with coefficient matrix as

$$A(k) = \frac{1}{n}I. \quad (26)$$

The output noises $v(k)$ include the sensor noise and error due to the numerical differentiation. The learning control gain matrix has two gains as $L = [l_1, l_2]$. The controller takes the form of equation (3), i.e.

$$u_{i+1}(k) = u_i(k) + l_1(\dot{\theta}_{ld}(k+1) - \dot{\theta}_{li}(k+1)) + l_2(\theta_{ld}(k+1) - \theta_{li}(k+1)). \quad (27)$$

The convergence condition (4) becomes $|1 - l_2 n\Delta/nJ_m + J_l| < \rho < 1$. It is easy to choose a learning gain to satisfy the inequality. In the experiment, $l_1 = l_2 = 2$ were chosen. The initial input is set to zero and no on-line feedback control is used, i.e. $u_0(k) = 0$, for $k \in N$.

The desired trajectory is given as

$$\theta_{ld}(k) = \frac{\pi\Delta^2}{6}k^2 - \frac{\pi\Delta^3}{27}k^3, \text{ rad.} \quad (28)$$

The starting position (0°) of the link is vertical upwards and the ending position (90°) is horizontally pointing out where the gravity effect is the most.

The experimental results are given in Fig. 2. In particular, the maximum link position tracking errors converge nicely as the operation repeats only a few times. Thus, the experimental results verify the theory developed in this paper.

4. *Conclusions*

The results show that iterative learning control can be applied to a class of time-varying discrete-time nonlinear dynamic systems. It is shown that the iterative learning control uses a tracking error with one-step ahead for time anticipation effect and it can ensure the convergence of inputs, states and outputs to their desired trajectories within bounds and robust against the presence of state, output disturbances and initial uncertainty. Furthermore, if these disturbances tend to zero, the convergence of input, state and output errors can be ensured to reduce to zero. The experimental results show that the convergence is satisfactory.

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Appendix

Lemma. Given a difference inequality

$$z(k+1) \leq \beta(k) + hz(k), \tag{A.1}$$

where $z(\cdot)$ and $\beta(\cdot)$ are scalar functions of $k \geq 0$, and h is a positive constant. Then, for $k \geq 1$,

$$z(k) \leq \sum_{j=0}^{k-1} h^{k-1-j}\beta(j) + h^k z(0). \tag{A.2}$$

The proof is established by induction.

Compared with the discrete Bellman–Gronwall Lemma in Desoer and Vidyasagar (1975, p. 254), the inequality (A.1) in the Lemma takes a special form and inequality (A.2) gives a tighter upper bound. Furthermore, inequalities (A.1) and (A.2) appear in a form that is more convenient for the development in this paper.