

ADI-FDTD Method with Fourth Order Accuracy in Time

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Abstract— This letter presents an unconditionally stable alternating direction implicit finite-difference time-domain (ADI-FDTD) method with 4th order accuracy in time. Analytical proof of unconditional stability and detailed analysis of numerical dispersion are presented. Compared to 2nd order ADI-FDTD and 6-steps SS-FDTD, the 4th order ADI-FDTD generally achieves lower phase velocity error for sufficiently fine mesh. Using finer mesh gridding also reduces the phase velocity error floor, which dictates the accuracy limit due to spatial discretization errors when the time step size is reduced further.

Index Terms—Alternating direction implicit finite-difference time-domain (ADI-FDTD), split-step approach, numerical dispersion, higher order ADI-FDTD

I. INTRODUCTION

Since the development of unconditionally stable alternating direction implicit finite-difference time-domain (ADI-FDTD) method [1],[2], there has been considerable interest in improving its temporal accuracy. In [3],[4], some techniques have been introduced to achieve higher order temporal accuracy for various unconditionally stable methods including the split step (SS) approach. One such technique is based on a sequence of time stepping coefficients determined via a systematic procedure [5]. However, the actual updating procedures and numerical performance of higher order methods have not been discussed in detail. For instance, it is not clear how to incorporate the seven coefficients of [3, Table 1] in SS4-FDTD method to achieve 4th order accuracy in time (or one may need 9 steps in practice?). It is also not obvious about the order of 6-steps SS-FDTD approach proposed in [6], which was claimed to be of higher order accuracy. Moreover, the stability and numerical dispersion of various higher order methods remain to be ascertained and investigated further.

In this letter, we present an unconditionally stable ADI-FDTD method with 4th order accuracy in time. In Section II, the updating procedure of 4th order ADI-FDTD method is developed. Section III presents the analytical proof of unconditional stability and the detailed analysis of numerical dispersion. Comparisons of numerical phase velocity errors are made among 4th order ADI-FDTD, 2nd order ADI-FDTD as well as 6-steps SS-FDTD.

II. ADI-FDTD METHOD WITH 4TH ORDER ACCURACY IN TIME

For simplicity, consider a 2D TM wave propagation in a lossless, isotropic and source-free medium with permittivity ϵ and permeability μ . The conventional ADI-FDTD method calls for the following two steps as [1],[2]

$$\left(I - \frac{\Delta t}{2} A\right) u^{n+\frac{1}{2}} = \left(I + \frac{\Delta t}{2} B\right) u^n \quad (1a)$$

$$\left(I - \frac{\Delta t}{2} B\right) u^{n+1} = \left(I + \frac{\Delta t}{2} A\right) u^{n+\frac{1}{2}} \quad (1b)$$

where I is the 3 by 3 identity matrix,

$$u = [E_z \ H_x \ H_y]^T \quad (2a)$$

$$A = \begin{bmatrix} 0 & 0 & \frac{\partial}{\epsilon \partial x} \\ 0 & 0 & 0 \\ \frac{\partial}{\mu \partial x} & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -\frac{\partial}{\epsilon \partial y} & 0 \\ -\frac{\partial}{\mu \partial y} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2b)$$

and u^n, u^{n+1} are the fields at integer time steps. It is well known that this method is second order accurate in time due to the presence of third order error terms $O(\Delta t^3)$ in its overall updating procedure.

To achieve 4th order accuracy in time, (1) can be extended into 6 steps as follows:

$$\left(I - x_1 \frac{\Delta t}{2} A\right) u^{n,1} = \left(I + x_1 \frac{\Delta t}{2} B\right) u^n \quad (3a)$$

$$\left(I - x_1 \frac{\Delta t}{2} B\right) u^{n,2} = \left(I + x_1 \frac{\Delta t}{2} A\right) u^{n,1} \quad (3b)$$

$$\left(I - x_0 \frac{\Delta t}{2} A\right) u^{n,3} = \left(I + x_0 \frac{\Delta t}{2} B\right) u^{n,2} \quad (3c)$$

$$\left(I - x_0 \frac{\Delta t}{2} B\right) u^{n,4} = \left(I + x_0 \frac{\Delta t}{2} A\right) u^{n,3} \quad (3d)$$

$$\left(I - x_1 \frac{\Delta t}{2} A\right) u^{n,5} = \left(I + x_1 \frac{\Delta t}{2} B\right) u^{n,4} \quad (3e)$$

$$\left(I - x_1 \frac{\Delta t}{2} B\right) u^{n+1} = \left(I + x_1 \frac{\Delta t}{2} A\right) u^{n,5} \quad (3f)$$

Here, $u^{n,m}$ represent the auxiliary fields at intermediate steps.

The coefficients x_0 and x_1 are given by [5]

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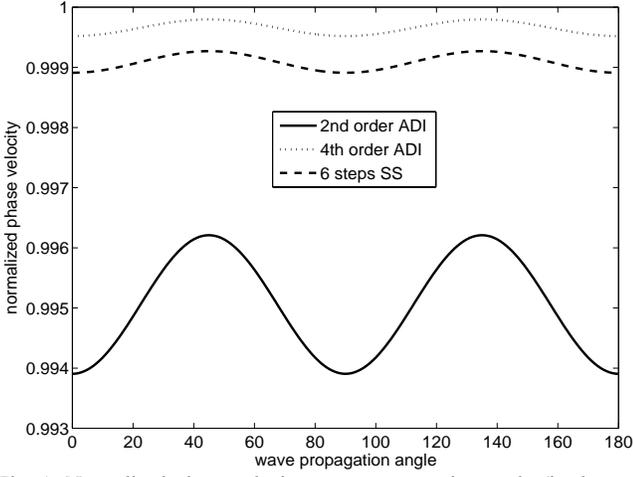


Fig. 1. Normalized phase velocity versus propagation angle (in degrees). CPW=100 and CFLN=6.

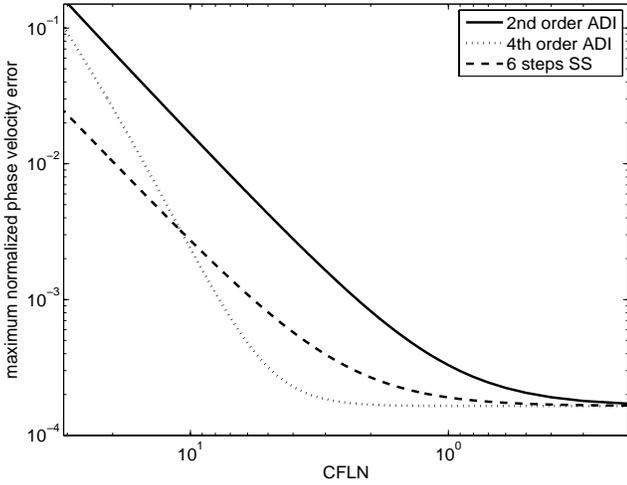


Fig. 2. Maximum normalized phase velocity error versus CFLN. CPW=100.

$$x_0 = \frac{-2^{1/3}}{2-2^{1/3}}, \quad x_1 = \frac{1}{2-2^{1/3}} \quad (4)$$

which have been determined such that

$$x_0 + 2x_1 = 1, \quad x_0^3 + 2x_1^3 = 0 \quad (5)$$

(all third order error terms will then be cancelled). Since the even order error terms are also zero, only the 5th order error terms $O(\Delta t^5)$ onwards are present in (3) and thus it is 4th order accurate in time.

III. STABILITY AND DISPERSION ANALYSIS

Using the Fourier analysis in conjunction with second order spatial central difference on Yee cell, the updating procedure of (3) can be represented in matrix form as

$$u^{n+1} = M_{x_1} \cdot N_{x_1} \cdot M_{x_0} \cdot N_{x_0} \cdot M_{x_1} \cdot N_{x_1} \cdot u^n = \Lambda \cdot u^n \quad (6)$$

where for $\alpha = x_0, x_1$,

$$M_\alpha = \begin{bmatrix} \frac{\mu\varepsilon}{\mu\varepsilon + \alpha^2 W_y^2} & \frac{j\mu\alpha W_y}{\mu\varepsilon + \alpha^2 W_y^2} & \frac{-j\mu\alpha W_x}{\mu\varepsilon + \alpha^2 W_y^2} \\ \frac{j\varepsilon\alpha W_y}{\mu\varepsilon + \alpha^2 W_y^2} & \frac{\mu\varepsilon}{\mu\varepsilon + \alpha^2 W_y^2} & \frac{\alpha^2 W_x W_y}{\mu\varepsilon + \alpha^2 W_y^2} \\ -j\alpha W_x / \mu & 0 & 1 \end{bmatrix}$$

$$N_\alpha = \begin{bmatrix} \frac{\mu\varepsilon}{\mu\varepsilon + \alpha^2 W_x^2} & \frac{j\mu\alpha W_y}{\mu\varepsilon + \alpha^2 W_x^2} & \frac{-j\mu\alpha W_x}{\mu\varepsilon + \alpha^2 W_x^2} \\ j\alpha W_y / \mu & 1 & 0 \\ \frac{-j\varepsilon\alpha W_x}{\mu\varepsilon + \alpha^2 W_x^2} & \frac{\alpha^2 W_x W_y}{\mu\varepsilon + \alpha^2 W_x^2} & \frac{\mu\varepsilon}{\mu\varepsilon + \alpha^2 W_x^2} \end{bmatrix}$$

$W_\xi = (\Delta t / \Delta \xi) \sin(k_\xi \Delta \xi / 2)$. The eigenvalues of Λ are

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3^* = \frac{X + jY}{Z} \quad (7)$$

where $X = Z - 2P$, $Y = \sqrt{Z^2 - X^2}$

$$Z = (x_0^2 W_x^2 + \mu\varepsilon)(x_1^2 W_x^2 + \mu\varepsilon)^2 (x_0^2 W_y^2 + \mu\varepsilon)(x_1^2 W_y^2 + \mu\varepsilon)^2$$

$$P = [x_0 x_1^2 P_{x_1} - (\mu\varepsilon)^2][x_0 x_1^2 P_{x_0} P_{x_1} - P_0 (\mu\varepsilon)^2]$$

$$P_\alpha = (W_x^2 + W_y^2)\mu\varepsilon + W_x^2 W_y^2 \alpha^2, \quad \alpha = 0, x_1, x_0$$

Note that X, Y, Z are real, and all the eigenvalues have the magnitude of one. Thus, the ADI-FDTD of (3) with 4th order accuracy in time is proven to be unconditionally stable.

Next we analyze the numerical dispersion relation of the 4th order ADI-FDTD, which can be derived as

$$\sin^2(\omega\Delta t / 2) = P / Z \quad (8)$$

Fig. 1 shows the phase velocity v normalized to $c = 1/\sqrt{\mu\varepsilon}$ versus propagation angle (in degrees). Uniform mesh size is adopted, and the cell per wavelength (CPW) is selected to be 100. The time step size is specified relative to the Courant limit time step Δt_{CFL} via $CFLN = \Delta t / \Delta t_{CFL}$, where $CFLN=6$ for the present case. It is evident that the phase velocity error is lowest for the present 4th order ADI-FDTD. Note that although the SS-FDTD of [6] yields more accurate phase velocity compared to 2nd order ADI-FDTD, its temporal accuracy is only of second order. This can be ascertained by the presence of third order error terms $O(\Delta t^3)$ in the overall updating procedure of [6].

To investigate the numerical dispersion for other time step sizes, Fig. 2 plots the maximum normalized phase velocity error versus CFLN. It can be seen that the 4th order ADI-FDTD outperforms the 2nd order one over the entire range of CFLN considered. For non excessive time step size, it also features lower phase velocity error than the SS-FDTD of [6].

In Fig. 2, we also observe that when CFLN or Δt is reduced further, there exists a phase velocity error floor for all methods. This error floor represents the minimum error that can be achieved by reducing the time step size. It depends solely on the mesh size and can be found from the numerical

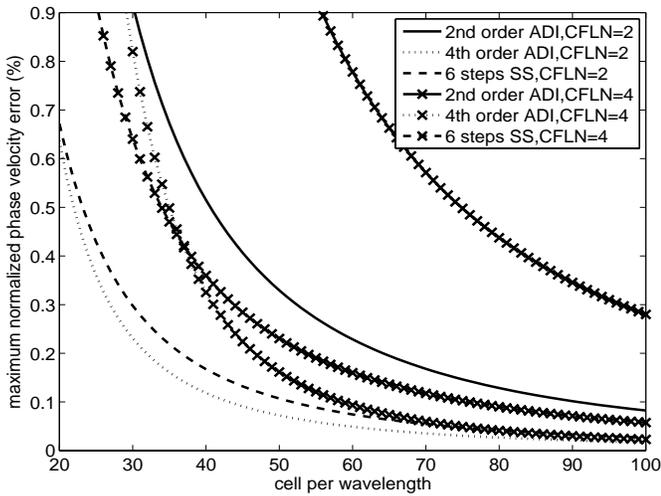


Fig. 3. Maximum normalized phase velocity error versus CPW. CFLN=2,4.

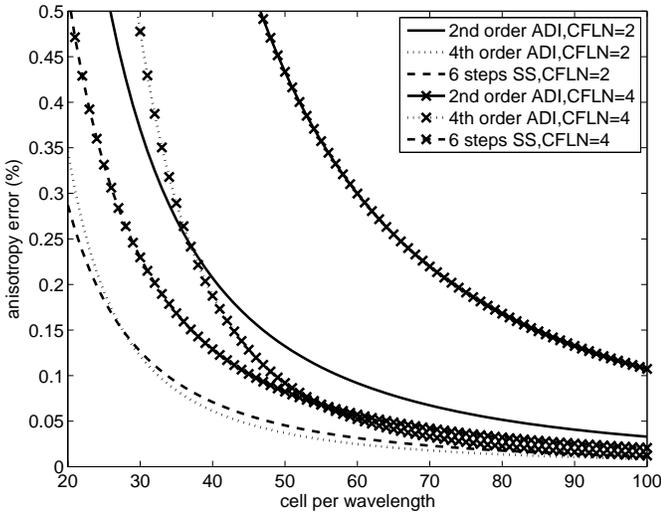


Fig. 4. Anisotropy error versus CPW. CFLN=2,4.

dispersion expression. To that end, we multiply (8) by $\mu\epsilon / \Delta t^2$, and take the limit as $\Delta t \rightarrow 0$:

$$\lim_{\Delta t \rightarrow 0} \sin^2\left(\frac{\omega\Delta t}{2}\right) \frac{\mu\epsilon}{\Delta t^2} = \lim_{\Delta t \rightarrow 0} \frac{P}{Z} \frac{\mu\epsilon}{\Delta t^2} \quad (9)$$

Upon some manipulation, (9) can be simplified as

$$\omega^2 / (4c^2) = V_x^2 + V_y^2 \quad (10)$$

where $V_\xi = (1/\Delta\xi)\sin(k_\xi \Delta\xi / 2)$. It can be seen that (10) dictates the accuracy limit due to spatial discretization errors when the time step size is reduced further. By applying this equation, one can solve for the error floor in terms of CPW. For CPW=100, we obtain an error floor value of 1.645×10^{-4} , which coincides with the level depicted in Fig. 2. Moreover, the error floor may be reduced by increasing CPW, or in other words, using finer mesh gridding.

Fig. 3 shows the maximum normalized phase velocity error versus CPW for CFLN=2 and 4. It is evident that for CFLN=2, the phase velocity error of 4th order ADI-FDTD is the lowest throughout. For CFLN=4, it is still lowest for most

CPW's, except at the small CPW region, the error of 4th order ADI-FDTD may exceed that of 6-steps SS-FDTD. This implies that, to exploit the present ADI-FDTD with 4th order accuracy in time, sufficiently fine mesh should be adopted to avoid the error being overwhelmed by the coarse mesh. This is exemplified by the previous case when CPW=100, CFLN can be as high as 10 and the 4th order ADI-FDTD still outperforms. Besides phase velocity error, the anisotropy error defined as $(v_{\max} - v_{\min}) / v_{\min}$ is also investigated, cf. Fig. 4. Again, the 4th order ADI-FDTD may achieve smaller anisotropy error with finer mesh gridding. Alternatively, to avoid excessive meshes, one may consider increasing the spatial accuracy from the present second order to higher order or parameter optimized ones as [7]-[9].

IV. CONCLUSION

This letter has presented an unconditionally stable ADI-FDTD method with 4th order accuracy in time. Analytical proof of unconditional stability and detailed analysis of numerical dispersion have been presented. Compared to 2nd order ADI-FDTD and 6-steps SS-FDTD, the 4th order ADI-FDTD generally achieves lower phase velocity error for sufficiently fine mesh. Using finer mesh gridding also reduces the phase velocity error floor, which dictates the accuracy limit due to spatial discretization errors when the time step size is reduced further.

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