PI/PD/PID tuning rules for IPTD (integral plus time delay) processes and their realized GPM (gain and phase margins)

—The paper [2] derives PI/PD/PID tuning formulas with a specified GPM as well as the formulas for estimating the GPM attained by a given PI/PD/PID controller for an IPTD process. It provides general analytic solutions for the PID tuning problems. In this supplementary material, we apply the GPM formulas to calculate the GPM achieved by each relevant PI/PD/PID tuning rule that has been collected in the book [1].

Consider the system that can be described by the following equations and Fig. 1.

IPTD model: $G_m(s) = \dfrac{K_m e^{-s\tau_m}}{s}$. Ideal PI controller: $G_c(s) = K_c\left(1 + \dfrac{1}{T_i s}\right)$.

Ideal PD controller: $G_c(s) = K_c\left(1 + T_d s\right)$. Ideal PID controller: $G_c(s) = K_c\left(1 + \dfrac{1}{T_i s} + T_d s\right)$.

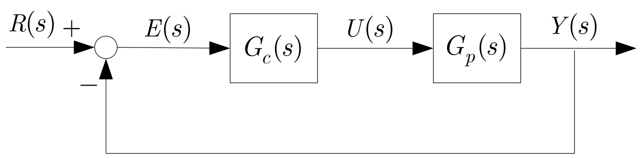

Fig. 1. Control system loop.

The tables basically come from the book [1], pages 76-82 and 259-261. The newly-added columns are marked in grey color, which give the "A_m rank No." and GPM for each of the PI/PD/PID tuning formulas, respectively. The "A_m rank No." is defined as the position of A_m of a tuning formula where the A_m's of all the tuning formulas in the class of PI or PD or PID tuning are sorted in an increasing order of their values. For example, if there are totally four PI tuning formulas leading to A_m values {4, 2.5, 5, 1.6}, then their A_m rank No.'s are {3, 2, 4, 1} respectively. The GPM values are calculated by the function "margin" of MATLAB R2006a. Such calculations are feasible since for the control system as described above, if its PI/PD/PID parameters are expressed in the form of the *PI/PD/PID tuning formulas* [2], the *GPM-PI/PD/PID formulas* [2] indicate that its GPM will be of constant values, regardless of the process parameters K_m and τ_m (Refer to the MATLAB codes presented after the tables for the details of such calculations.).

Among the PI, PD and PID tuning rules collected in [1], there are respectively 25, 1 and 11 of them within the same framework as that of the tuning formulas derived in [2]. The other 15, 1 and 3 formulas which do not belong to the same framework have been omitted from the following tables. Fig. 2-4 illustrate the comparisons between the estimated GPM and the numerically obtained true GPM values. Comparisons on the crossover frequencies are presented in these figures. As we can see, the proposed *GPM-PI/PD/PID formulas* achieve satisfactory accuracy in estimating GPM and the crossover frequencies. For the PID tuning rules listed in Table 3, their respective values of the scalar $k(:= T_d / T_i)$'s are presented in Fig. 5, confirming that they are within the common range of $k \in (0,\ 0.5]$.

Note: *i)* For the PI case, the GPM values are only calculated for the rules with $\theta := \tau_m / T_i < 1$ for ensuring the existence of positive β's [2]. *ii)* The two PI tuning rules marked with yellow color are found to be questionable: the former one turns out to be unable to give a stable system (The original reference of this rule is however out of our reach.); the latter one gives effective tuning formulas yet inaccurate formulas for estimating GPM. We demonstrate for the former case the divergence of an exemplary system response in Fig. 6. For the latter case, the observation is verified by GPM calculated by the *GPM-PI formula*. *iii)* In Table 3, we find that it is hard to obtain the effective gain margins of three PID tuning formulas (marked in light-blue color), which all hold negative phase margins. Although MATLAB can give numerical results of them, the obtained gain margins appear to be not reliable: we solve the original GPM equations numerically and find that the results are sensitive to the initial searching point and cannot attain a consistent but reliable solution. Due to this reason, these three cases have been omitted in the later figures.

Table 1. PI tuning rules for IPTD processes and their realized GPM (25 rules in total, with 44 sets of particular GPM)

A_m rank No.	Rule	K_c	T_i	Comment	GPM, $(A_m,\ \phi_m)$
		Process reaction			
6	Ziegler and Nichols (1942). *Model: Method 2*	$\frac{0.9}{K_m\tau_m}$	$3.33\tau_m$	Quarter decay ratio	$(1.47,\ 18.25°)$
16	Two constraints method – Wolfe (1951). *Model: Method 2*	$\frac{0.6}{K_m\tau_m}$	$2.78\tau_m$	Decay ratio = 0.4	$(2.09,\ 23.18°)$
10		$\frac{0.87}{K_m\tau_m}$	$4.35\tau_m$	Decay ratio is as small as possible	$(1.60,\ 24.19°)$
		Minimum error integral (regulator mode).			
15	Åström and Hägglund (1995) – *page 13*. *Model: Method 1*	$\frac{0.63}{K_m\tau_m}$	$3.2\tau_m$	Ultimate cycle Ziegler-Nichols equivalent	$(2.07,\ 26.06°)$
30	Hay (1998) – *page 188*.	$\frac{0.42}{K_m\tau_m}$	$5.8\tau_m$	*Model: Method 3*	$(3.43,\ 43.26°)$
		Minimum performance index: regulator tuning			
5	Minimum IAE – Shinskey (1988) – *page 123*.	$\frac{0.9524}{K_m\tau_m}$	$4\tau_m$	*Model: Method 1*	$(1.44,\ 19.42°)$
7	Minimum IAE – Shinskey (1994) – *page 74*.	$\frac{0.9259}{K_m\tau_m}$	$4\tau_m$	*Model: Method 1*	$(1.48,\ 20.53°)$
1	Minimum ISE – Hazebroek and Van der Waerden (1950).	$\frac{1.5}{K_m\tau_m}$	$5.56\tau_m$	*Model: Method 2*	$(0.96,\ \text{-}3.34°)$
20	Minimum ITAE – Poulin and Pomerleau (1996). *Model: Method 1*	$\frac{0.5264}{K_m\tau_m}$	$4.5804\tau_m$	Process output step load disturbance	$(2.66,\ 36.51°)$
18		$\frac{0.5327}{K_m\tau_m}$	$3.8853\tau_m$	Process input step load disturbance	$(2.56,\ 32.79°)$
		Minimum performance index: other tuning			
41	Skogestad (2001). *Model: Method 1*	$\frac{0.28}{K_m\tau_m}$	$7\tau_m$	$M_{\max} = 1.4$	$(5.24,\ 47.48°)$
31	Skogestad (2003). *Model: Method* 1	$\frac{0.404}{K_m\tau_m}$	$7\tau_m$	$M_{\max} = 1.7$	$(3.63,\ 47.05°)$

21	Skogestad (2001). *Model: Method 1*	$\frac{0.49}{K_m \tau_m}$	$3.77\tau_m$	$M_{\max} = 2.0$	$(2.77, 32.82°)$
		Direct synthesis: time domain criteria			
27	Tyreus and Luyben (1992). *Model: Method 1 or 9*	$\frac{0.487}{K_m \tau_m}$	$8.75\tau_m$	Max closed loop log modulus = $2dB$; $T_{CL} = \tau_m \sqrt{10}$	$(3.06, 48.49°)$
--		$0.31K_u$	$2.2T_u$		--
23	Fruehauf *et al*. (1993).	$\frac{0.5}{K_m \tau_m}$	$5\tau_m$	*Model: Method 2*	$(2.83, 38.87°)$
9	Rotach (1995). *Model: Method 5*	$\frac{0.75}{K_m \tau_m}$	$2.41\tau_m$	Damping factor for oscillations to a disturbance input = 0.75.	$(1.58, 15.67°)$
4	Cluett and Wang (1997). *Model: Method 1*	$0.9588/K_m \tau_m$	$3.0425\tau_m$	$T_{CL} = \tau_m$	$(1.34, 14.17°)$
17		$0.6232/K_m \tau_m$	$5.2586\tau_m$	$T_{CL} = 2\tau_m$	$(2.29, 36.47°)$
28		$0.4668/K_m \tau_m$	$7.2291\tau_m$	$T_{CL} = 3\tau_m$	$(3.15, 46.28°)$
34		$0.3752/K_m \tau_m$	$9.1925\tau_m$	$T_{CL} = 4\tau_m$	$(3.98, 52.08°)$
38		$0.3144/K_m \tau_m$	$11.1637\tau_m$	$T_{CL} = 5\tau_m$	$(4.80, 55.96°)$
43		$0.2709/K_m \tau_m$	$13.1416\tau_m$	$T_{CL} = 6\tau_m$	$(5.61, 58.75°)$
2	Chidambaram and Sree (2003). *Model: Method 1*	$\frac{1.1111}{K_m \tau_m}$	$4.5\tau_m$	--	$(1.26, 14.02°)$
42	Huba and Žáková (2003). *Model: Method 1*	$\frac{0.23}{K_m \tau_m}$	$2.914\tau_m$	--	$(5.54, 25.00°)$
37		$\frac{0.281}{K_m \tau_m}$	$3.555\tau_m$	--	$(4.77, 31.32°)$
--	Skogestad (2003), (2004b). *Model: Method 1*	$\frac{1}{K_m\left(T_{CL}+\tau_m\right)}$	$4\xi^2\left(T_{CL}+\tau_m\right)$	*Suggested* $\xi = 0.7$ *or 1*	--
25		$\frac{0.5}{K_m \tau_m}$	$8\tau_m$	*'good' robustness* $T_{CL} = \tau_m,\ \xi = 1$	$(2.96, 46.86°)$
		Direct synthesis: frequency domain criteria			
14	Chidambaram (1994), Srividya and Chidambaram	$\frac{0.67075}{K_m \tau_m}$	$3.6547\tau_m$	*Model: Method 6*; $A_m = 2$	$(2.00, 28.01°)$

	(1997).				
--		$\frac{\omega_p}{A_m K_m}$	$\frac{1}{\omega_p\left(0.5\pi - \omega_p \tau_m\right)}$	--	--
	Gain and phase margin – Kookos *et al.* (1999). *Model: Method 1*	Representative results			
8		$0.942/K_m \tau_m$	$4.510\tau_m$	$A_m = 1.5$; $\varphi_m = 22.5°$	$(1.48, 21.70°)$
13		$0.698/K_m \tau_m$	$4.098\tau_m$	$A_m = 2$; $\varphi_m = 30°$	$(1.97, 29.51°)$
26		$0.491/K_m \tau_m$	$6.942\tau_m$	$A_m = 3$; $\varphi_m = 45°$	$(2.99, 45.00°)$
35		$0.384/K_m \tau_m$	$18.710\tau_m$	$A_m = 4$; $\varphi_m = 60°$	$(4.00, 59.94°)$
22	Cheng and Yu (2000). *Model: Method 1*	$0.5236/K_m \tau_m$	$8\tau_m$	$A_m = 2.83$; $\varphi_m = 46.1°$	$(2.83, 46.11°)$
--	O'Dwyer (2001a). *Model: Method 1*	$\frac{x_1}{K_m \tau_m}$	$x_2 \tau_m$	--	--
		Representative coefficient values	GPM in the " <> " are the ones given by the rule "O'Dwyer (2001a)", presented in the book [1].		
		x_1	x_2	$< A_m, \ \phi_m >$	$(A_m, \ \phi_m)$
3		0.558	1.4	$< 1.5, \ 46.2° >$	$(1.26, \ 3.15°)$
11		0.484	1.55	$< 2.0, \ 45.5° >$	$(1.72, \ 7.68°)$
24		0.458	3.35	$< 3.0, \ 59.9° >$	$(2.89, \ 30.28°)$
33		0.357	4.3	$< 4.0, \ 60.0° >$	$(3.89, \ 36.95°)$
39		0.305	12.15	$< 5.0, \ 75° >$	$(4.97, \ 57.31°)$
		Robust			
--	Ogawa (1995). *Model: Method 1; Coefficients of* K_c *and* T_i *deduced from graphs*	$x_1/K_m \tau_m$	$x_2 \tau_m$	--	
		x_1	x_2	--	
29		0.45	11	20% uncertainty in process parameters	$(3.35, 52.51°)$
32		0.39	12	30% uncertainty in process parameters	$(3.88, 55.36°)$
36		0.34	13	40% uncertainty in process parameters	$(4.47, 57.59°)$
40		0.30	14	50% uncertainty in process parameters	$(5.07, 59.30°)$
44		0.27	15	60% uncertainty in process parameters	$(5.65, 60.59°)$
--	Smith (2002). *Model: Method 1*	$\frac{1}{K_m \tau_m}$	τ_m	--	--
		Other methods			
19	Penner (1988). *Model: Method 1*	$\frac{0.58}{K_m \tau_m}$	$10\tau_m$	Max. closed loop gain =1.26	$(2.59, 46.65°)$

12		$\frac{0.8}{K_m \tau_m}$	$5.9\tau_m$	Max. closed loop gain = 2.0	$(1.81,\ 31.47°)$

Table 2. PD tuning rules for IPTD processes and their realized GPM (1 rule in total, with 3 sets of particular GPM)

A_m rank No.	Rule	K_c	T_d	Comment	GPM, $(A_m,\ \phi_m)$
		Minimum performance index: servo tuning			
--	Visioli (2001). *Model: Method 1*	$x_1 / K_m \tau_m$	$x_3 \tau_m$	--	
		Coefficient values			
1		1.03	0.49	Minimum ISE	$(1.52,\ 51.95°)$
2		0.96	0.45	Minimum ITSE	$(1.70,\ 54.61°)$
3		0.90	0.45	Minimum ISTSE	$(1.81,\ 57.49°)$

Table 3. PID tuning rules for IPTD processes and their realized GPM (11 rules in total, with 29 sets of particular GPM)

A_m rank No.	Rule	K_c	T_i	T_d	Comment	GPM, $(A_m,\ \phi_m)$
		Process reaction				
3	Ford (1953). *Model: Method 3*	$\frac{1.48}{K_m \tau_m}$	$2\tau_m$	$0.37\tau_m$	Decay ratio 2.7:1	$(1.23,\ 16.06°)$
11	Åström and Hägglund (1995) – *page 139*.	$\frac{0.94}{K_m \tau_m}$	$2\tau_m$	$0.5\tau_m$	*Model: Method 1*	$(1.80,\ 32.60°)$
		Ultimate cycle Ziegler-Nichols equivalent				
20	Hay (1998) – *page 188.*	$\frac{0.4}{K_m \tau_m}$	$3.2\tau_m$	$0.8\tau_m$	*Model: Method 3*	$(2.96,\ 44.33°)$
--	Hay (1998) – *page 199. Model: Method 1; K_c, T_d deduced from graphs*	10.0	$3.2 K_m \tau_m^{\ 2}$	$0.55\tau_m$	$K_m \tau_m = 0.1$	$(*,\ -48.05°)$
--		4.0		$0.30\tau_m$	$K_m \tau_m = 0.2$	$(*,\ -21.19°)$
--		2.5		$0.25\tau_m$	$K_m \tau_m = 0.3$	$(*,\ -5.35°)$
8		2.0		$0.25\tau_m$	$K_m \tau_m = 0.4$	$(1.57,\ 5.40°)$
9		1.8		$0.25\tau_m$	$K_m \tau_m = 0.5$	$(1.73,\ 12.53°)$
7		1.8		$0.25\tau_m$	$K_m \tau_m = 0.6$	$(1.55,\ 15.73°)$
		Minimum performance index: regulator tuning				

--	Visioli (2001). *Model: Method 1*	$x_1/K_m\tau_m$	$x_2\tau_m$	$x_3\tau_m$	--	
		Coefficient values				
1		1.37	1.49	0.59	Minimum ISE	$(1.16,\ 27.82°)$
2		1.36	1.66	0.53	Minimum ITSE	$(1.23,\ 26.55°)$
4		1.34	1.83	0.49	Minimum ISTSE	$(1.28,\ 26.32°)$
--		***Minimum performance index: other tuning***				
--	Åström and Hägglund (2004). *Model: Method 1*	$x_1/K_m\tau_m$	$x_2\tau_m$	$x_3\tau_m$	--	--
		Coefficient values				
		x_1	x_2	x_3	$M_{\max}$	$(A_m,\ \phi_m)$
29		0.139	76.9	0.346	1.1	$(12.71,\ 79.43°)$
28		0.261	23.3	0.365	1.2	$(6.72,\ 71.09°)$
25		0.367	12.2	0.378	1.3	$(4.76,\ 64.13°)$
22		0.460	7.85	0.389	1.4	$(3.79,\ 58.04°)$
21		0.543	5.78	0.400	1.5	$(3.21,\ 53.09°)$
18		0.616	4.58	0.410	1.6	$(2.83,\ 48.88°)$
17		0.681	3.82	0.418	1.7	$(2.56,\ 45.28°)$
16		0.740	3.28	0.426	1.8	$(2.35,\ 42.07°)$
14		0.793	2.89	0.434	1.9	$(2.20,\ 39.29°)$
13		0.841	2.61	0.440	2.0	$(2.07,\ 36.92°)$
		Direct synthesis				
15	Leonard (1994). *Model: Method 1*	$\frac{0.74}{K_m\tau_m}$	$12.2\tau_m$	$0.41\tau_m$	OS (step input) < 10%; Minimum IAE (disturbance ramp).	$(2.31,\ 58.11°)$
--		$0.47K_u$	$3.05T_u$	$0.10T_u$		--
		K_u, T_u *deduced from graph*				
--	Cluett and Wang (1997). *Model: Method 1*	$x_1/K_m\tau_m$	$x_2\tau_m$	$x_3\tau_m$	$T_{CL} = x_4\tau_m$	--
		Coefficient values				
		x_1	x_2	x_3	x_4	$(A_m,\ \phi_m)$
12		0.9588	3.0425	0.3912	1	$(1.86,\ 36.91°)$
19		0.6232	5.2586	0.2632	2	$(2.91,\ 46.17°)$
23		0.4668	7.2291	0.2058	3	$(3.83,\ 51.83°)$
24		0.3752	9.1925	0.1702	4	$(4.69,\ 55.70°)$
26		0.3144	11.1637	0.1453	5	$(5.53,\ 58.52°)$
27		0.2709	13.1416	0.1269	6	$(6.35,\ 60.66°)$
6	Rotach (1995). *Model: Method 5*	$\frac{1.21}{K_m\tau_m}$	$1.60\tau_m$	$0.48\tau_m$	--	$(1.45,\ 24.33°)$
--		Damping factor for oscillations to a disturbance input = 0.75.				

5	Chidambaram and Sree (2003).	$\frac{1.2346}{K_m\tau_m}$	$4.5\tau_m$	$0.45\tau_m$	*Model: Method 1*	$(1.37,\, 36.53°)$
10	Sree and Chidambaram (2005b).	$\frac{0.896}{K_m\tau_m}$	$2.5\tau_m$	$0.55\tau_m$	*Model: Method 1*	$(1.76,\, 41.32°)$

*** The MATLAB codes for calculating the GPM are as follows.**

a) The PI case: let the PI parameters be

$$K_c = \frac{k_1}{k_m\tau_m},\ T_i = k_2\tau_m. \tag{1}$$

The codes for deriving the realized GPM are as follows:

```
s=tf('s');
Kp=100; Tau=0.2;      % Kp and Tau can be arbitrary positive real numbers, which also applies to the PD/PID cases.
Kc=k1/(Kp*Tau); Ti=k2*Tau;
sys=Kc*Kp*(1+s*Ti)*exp(-s*Tau)/(s^2*Ti);
[A_m, Phi_m, Wp, Wg]=margin(sys) % Note that the gain and phase crossover frequencies in MATLAB
                                 % are defined reversely as those in the normal control theory,
                                 % i.e., Wp, Wg in control theory are Wg, Wp in MATLAB, respectively.
```

b) The PD case: let the PD parameters be

$$K_c = \frac{k_1}{k_m\tau_m},\ T_d = k_2\tau_m. \tag{2}$$

The codes for deriving the realized GPM are:

```
s=tf('s');
Kp=100; Tau=0.2;
Kc=k1/(Kp*Tau); Td=k2*Tau;
sys=Kc*Kp*(1+s*Td)*exp(-s*Tau)/s;
[A_m, Phi_m, Wp, Wg]=margin(sys)
```

c) The PID case: let the PID parameters be

$$K_c = \frac{k_1}{k_m\tau_m},\ T_i = k_2\tau_m,\ T_d = k_3\tau_m. \tag{3}$$

The codes for deriving the realized GPM are:

```
s=tf('s');
Kp=100; Tau=0.2;
Kc=k1/(Kp*Tau); Ti=k2*Tau; Td=k3*Tau;
sys=Kc*Kp*(1+s*Ti+s^2*Ti*Td)*exp(-s*Tau)/(s^2*Ti);
[A_m, Phi_m, Wp, Wg]=margin(sys)
```

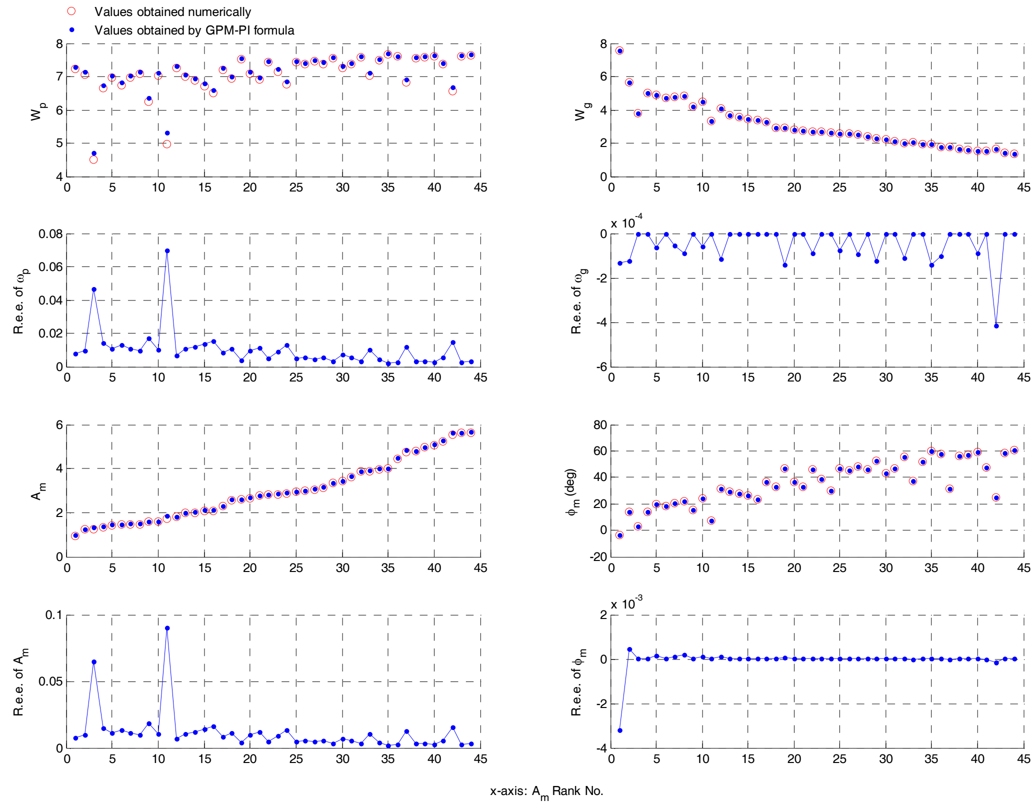

Fig. 2. The realized GPM of the PI tuning formulas in Table 1: comparisons between the GPM numerically obtained by MATLAB function “margin” (taken as the true GPM) and those estimated by the *GPM-PI formula* [2]. The relative estimation error (R.e.e.) is defined as *R.e.e. := (the estimated value - the true value) / the true value.*

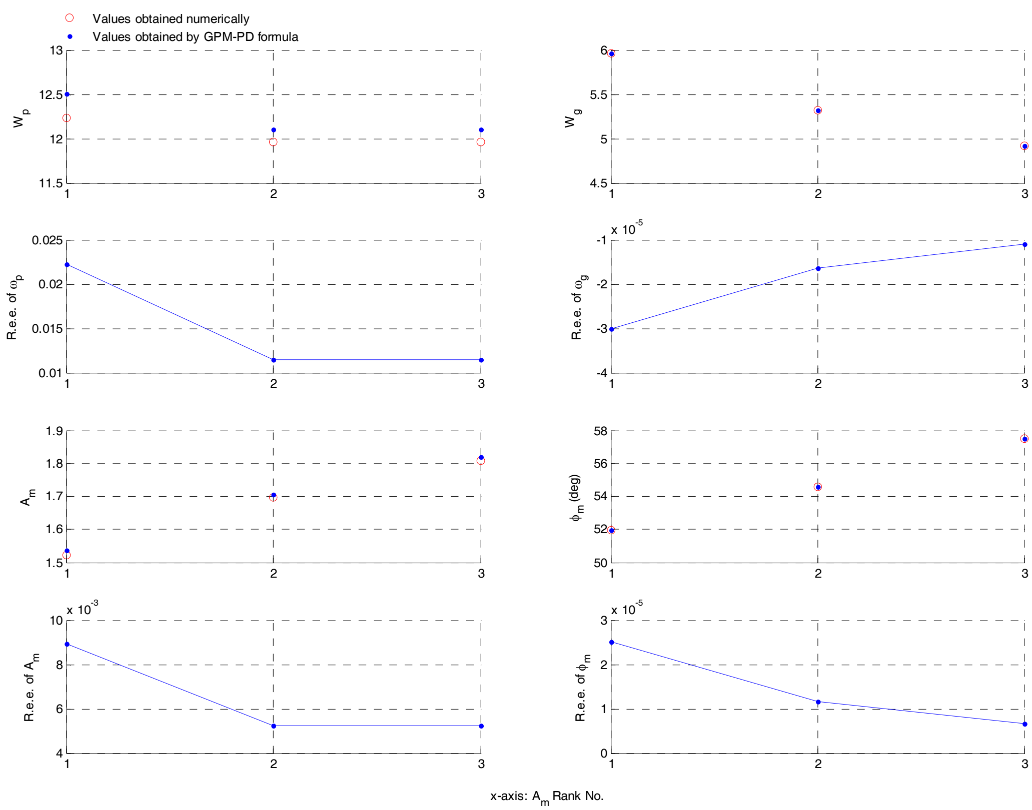

Fig. 3. The realized GPM of the PD tuning formulas in Table 2: comparisons between the GPM numerically obtained by MATLAB function “margin” (taken as the true GPM) and those estimated by the *GPM-PD formula* [2]. R.e.e. is defined the same as that in Fig. 2.

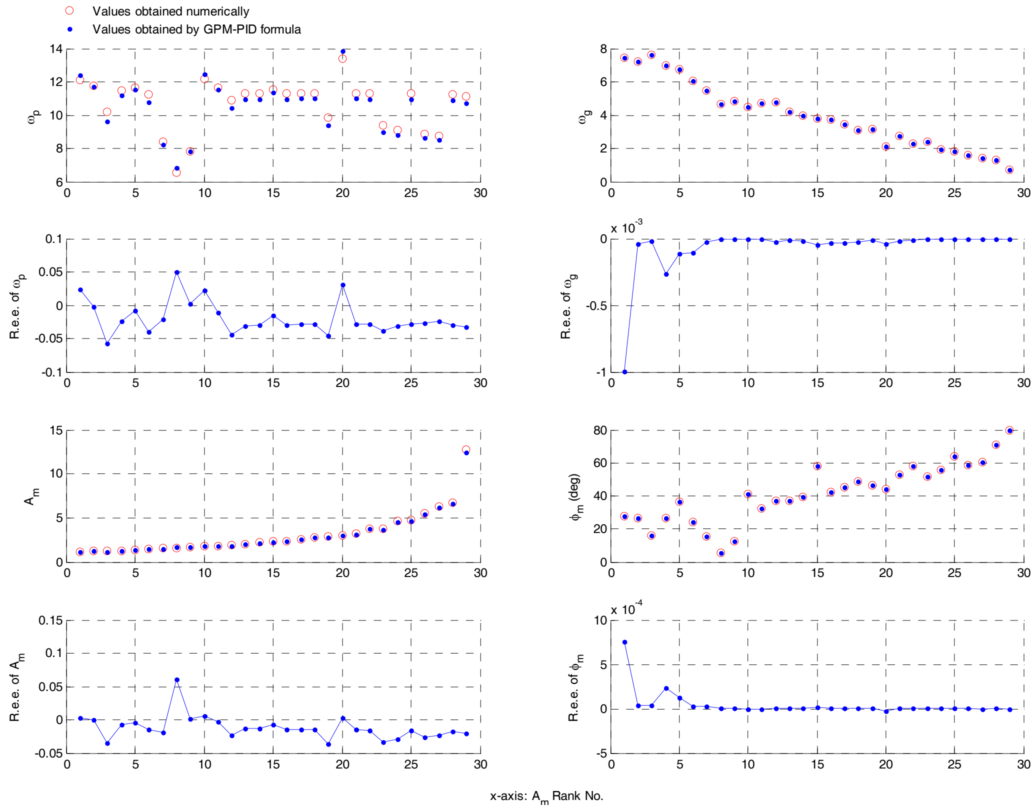

Fig. 4. The realized GPM of the PID tuning formulas in Table 3: comparisons between the GPM numerically obtained by MATLAB function "margin" (taken as the true GPM) and those estimated by the *GPM-PID formula* [2]. R.e.e. is defined the same as that in Fig. 2.

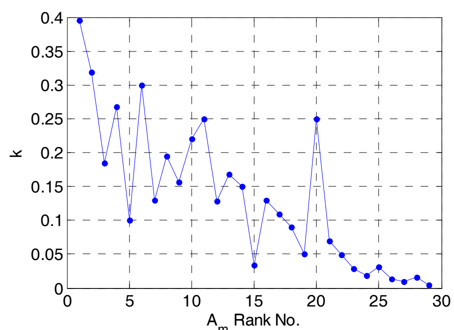

Fig. 5. The values of scalar k 's ($k := T_d / T_i$) used in the PID tuning formulas in Table 3.

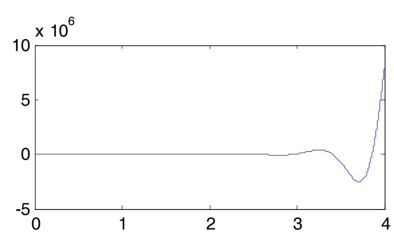

Fig. 6. Step response of an exemplary PI control system: the IPTD process is $G_p(s) = 100e^{-0.2s} / s$ and the controller is tuned by the rule "Minimum ISE – Hazebroek and Van der Waerden (1950)" as presented in Table 1. The divergent response indicates the instability of the rule, justifying its realized gain margin of $A_m < 1$.

References

[1] O'Dwyer, *Handbook of PI and PID Controller Tuning Rules*, 2nd Edition, Imperial College Press, 2006.

[2] W. Hu, G. Xiao, and X. Li, "An Analytical Method for PID Controller Tuning with Specified Gain and Phase Margins for Integral plus Time Delay Processes," *Submitted*, 2010.