Opinion Formation on Multiplex Scale-free Networks

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Abstract – Most individuals, if not all, live in various social networks. The formation of opinion systems is an outcome of social interactions and information propagation occurring in such networks. We study the opinion formation with a new rule of pair-wise interactions in the novel version of the well-known Deffuant model on multiplex networks composed of two layers, each of which is a scale-free network. It is found that in a duplex network composed of two identical layers, the presence of the multiplexity helps either diminish or enhance opinion diversity depending on the relative magnitudes of tolerance ranges characterizing the degree of openness/tolerance on both layers: there is a steady separation between different regions of tolerance range values on two network layers where multiplexity plays two different roles respectively. Additionally, the two critical tolerance ranges follow a *one-sum* rule; that is, each of the layers reaches a complete consensus only if the sum of the tolerance ranges on the two layers is greater than a constant approximately equaling 1, a double of the critical bound on a corresponding isolated network. A further investigation of the coupling between constituent layers quantified by a link overlap parameter reveals that as the layers are loosely coupled, the two opinion systems co-evolve independently, but when the inter-layer coupling is sufficiently strong, a monotonic behavior is observed: an increase in the tolerance range of a layer causes a decline in the opinion diversity on the other layer regardless of the magnitudes of tolerance ranges associated with the layers in question.

Introduction. – Opinion formation is an interesting topic in the studies on complex networks [1, 2]. It helps to predict opinion distribution in a population after a temporal process of spreading through social interactions [3, 4]. Such studies may reveal that local pair-wise interactions of individuals eventually form a global equilibrium. Numerous opinion spreading models have been proposed to reveal the properties of opinion systems by introducing various rules of communication between individuals. Some of the most well-known models include Voter model [5, 6], major rule model [7], Sznajd model [2], etc.

Proposed by Deffuant *et al.* in 2000, the Deffuant model [8], as one of the most popular bounded confidence models, has grabbed much attention over the past decades.

The model employs the parameter of tolerance range. Such a parameter to a certain extent reflects the concept of *selective exposure*, which describes one's tendency to involve himself/herself in social interactions that support his/her opinions [9, 10, 11].

It is found that the tolerance bound (also interpreted as confidence bound, tolerance range, or uncertainty threshold [12, 13, 14]) plays an important role in opinion formation processes. The numerical experiments reveal an interesting result, that is, the number of coexisting opinion clusters in the steady state is approximately $1/(2 * \textit{tolerance range})$ [8, 15]. A notion of critical confidence bound is also captured to refer to a threshold beyond which an opinion system eventually converges to a single cluster sharing the same opinion that equals the average of *first impression* [8, 16, 17]. Some analytical frameworks have been developed using the mean-field approach [18, 19, 20].

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The model is also deliberately modified by replacing continuous-valued opinion with discretized values [21].

Though the studies on opinion formation have achieved remarkable results, most of the existing works have been focusing on non-interacting isolated networks. Meanwhile, it has been widely realized that many real-life systems, including most of the social networks, are often composed of multiple layers coupled together. Such networks are termed as *multiplex networks* [22, 23, 24].

There has been a wide range of studies on the influence of interplay between constituent layers on topological structures and dynamic processes upon multiplex networks, e.g. diffusion dynamics, random walk and epidemics spreading [25, 26, 27]. In computational sociology, some first-step work has also been proposed to discover new properties of opinion formation on multiplex networks. The authors in [28] developed an analytical framework examining the Deffuant model on a one-dimensional multiplex network in which each layer is a network on $\mathbb{Z}$ domain. The work was rigorously conducted to draw an interesting conclusion that under a predefined configuration of multiplexity, the existence of interconnected layers impedes the convergence. However, it is not exhaustive to obtain a universal conclusion in general cases where network structures are diverse. For instance, another consensus model validated in allocation problems; in contrast, concludes that more dimensions of opinions lead to better chances for consensus [29].

We study opinion dynamics under the inter-layer interplay on duplex scale-free networks. On each layer, the agents hold an opinion on a certain issue associated with the underlying layer. The regime of inter-layer interactions mimics the compromise phenomenon in real-life social networks where one's viewpoint on a subject may be influenced by his/her opinion on other subjects, resulting in their decisions on compromising or neglecting others' views. It is found that in duplex networks of identical layers, a weakened-enhanced opinion diversity transition exists and varies as the function of the magnitudes of the tolerance ranges of both layers in a non-monotonic way. An extensive investigation is also carried out to examine the impact of the inter-layer coupling on the opinion formation.

To be simple, here we consider multiplex networks composed of two scale-free networks (due to their ubiquity in natural and social networks), while validation of the results can also be supported by Erdos Renyi (ER) random networks.

Duplex Opinion Dynamics Model. – We construct a duplex network composed of 2 layers. The duplex network is a pair $G = (V, E)$ consisting of 2 layers G_1 and G_2, each of which is a scale-free network $G_i = (V, E_i)$, where V is the set of vertices representing individuals in a population and $E_i \subseteq V \times V$ is the set of edges on layer i, $i = 1, 2$. The edge set of G, denoted by E, consists of 2 sets of edges on the two layers respectively, implying two types of social relationships: $E = E_1 \cup E_2$. The artificial scale-free networks are constructed using the generative model [30]. Given an average degree $\langle k \rangle$, the scaling exponent of the power-law function can be obtained by numerically solving the following equation:

$$\sum_k kp(k) = \langle k \rangle\,, \tag{1}$$

where

$$p(k) = \frac{k^{-\gamma}}{\zeta(\gamma, k_{min}) - \zeta(\gamma, k_{max})}, \tag{2}$$

where k_{min} and k_{max} are the minimum and maximum degrees respectively and $\zeta(\gamma, k)$ is the Hurwitz ζ function [31, 32].

The connectivity in a layer is independent of that in the other layer but the overlap of the links may have significant effects on the universal opinion formation. We quantify this inter-layer interplay as a link overlap parameter denoted by η which can be obtained as follows:

$$\eta = \frac{\mid E_1 \cap E_2 \mid}{\mid E_1 \mid}. \tag{3}$$

In order to make a comparison focusing on effects of the overlap parameter value, we deliberately keep the degree of all nodes on both the layers to be the same. Specifically, given a network G_1, a scheme for constructing a layer G_2 that meets a desired η is described as follows: in case of $\eta = 1$, the two layers are of exactly the same topology; otherwise, perform the following rewiring process: starting with an initial $G_2 \equiv G_1$; randomly choosing nodes $A, B, C, D \in V$ where links $\{A, B\}, \{C, D\} \in E_2$ and $\{A, C\}, \{B, D\} \notin E_2$, then replacing the links $\{A, B\}, \{C, D\}$ with $\{A, C\}, \{B, D\}$, respectively in E_2. Repeat such rewiring operations until the pre-specified η is met. All nodes in the resulting layer G_2 have the same nodal degree as their corresponding nodes have in G_1.

We here introduce a rule of pair-wise communication between individuals on each layer (termed as *intralayer* interaction [33]) under the influence of the multiplexity. Firstly, we recall the rule of compromise making in a single network following the Deffuant model: initially, each node in the network is assigned a continuous-valued opinion drawn from a uniform distribution on the region [0,1]. At time step t, if a randomly selected edge $e = \{u, v\}$ is active, the opinions held by u and v at time step $t+1$ are determined by the following rule: if $|o(u,t) - o(v,t)| \leq d$, where $o(u,t)$ and $o(v,t)$ denote the opinions held by the nodes u and v respectively at time step t, and d is the tolerance bound homogeneous across the population, then the two nodes make consensus:

$$\begin{aligned} o(u, t+1) &= o(u,t) + \mu(o(v,t) - o(u,t)), \\ o(v, t+1) &= o(v,t) + \mu(o(u,t) - o(v,t)), \end{aligned} \tag{4}$$

where $\mu \in (0, 1/2]$ is convergence parameter; otherwise, they maintain their current opinions. According to most of the existing studies, μ only affects the convergence time [8, 11, 34]. We thus fix $\mu = 1/2$ in the remainder of this letter.

At the beginning, on each of the layers, say l, each node v holds an opinion value, denoted by $o_l(v)$, drawn from a uniform distribution over the region [0,1]. At time step t, each of the two layers is active with an equal probability 0.5. On the active layer, randomly choose a node u and then pick a neighbor node v at random. The rule of inter-layer compromise making is described as follows: if u and v have a connection only on the active layer, e.g. layer 1, they behave the same as that in a single-layer network. In other words, if they are not connected on the other layer, i.e. layer 2, their opinions on layer 2 do not interfere in the opinion exchange process occurring on the current layer. Otherwise, they make consensus on layer 1 if and only if the following condition holds:

$$\frac{|o_1(u,t) - o_1(v,t)| + |o_2(u,t) - o_2(v,t)|}{2} \leq \frac{d_1 + d_2}{2}$$
$$\Leftrightarrow |o_1(u,t) - o_1(v,t)| + |o_2(u,t) - o_2(v,t)| \leq d_1 + d_2, \quad (5)$$

where $d_i, i \in \{1,2\}$, is the confidence bound on layer i which is independent of each other. The opinion updating process proceeds until both layers reach a quasi-steady state at which all agents either maintain their opinions or change their opinions by an amount less than a small fluctuation range (set at 10^{-3} in our study) in any future contacts. The condition holds when on each layer, all pairs of neighboring nodes either share approximately identical opinions or would never make local consensus according to the compromise rule as defined.

Equation (5) reflects a simple case where connected individuals weigh the total differences between their opinions on a few different issues to decide whether they would like to compromise to make a consensus. In the present model, component layers are supposed to play the same role, i.e., they are un-weighted layers. Therefore, the opinion spreading processes on both layers occur concurrently and similarly. For this reason, we mainly discuss the opinion dynamics on layer 1 under the effects of layer 2's opinion system in comparison to the cases of those in isolated single-layer networks.

Results. – To evaluate the degree of consensus as well as the opinion diversity, we employ the quantity of the number of opinion clusters coexisting at the equilibrium state. Such a number of clusters on each layer is calculated by firstly removing links connecting dissenters and then computing the number of disconnected components in each layer by using the Dulmage-Mendelsohn decomposition algorithm [35]. Note that the term *global consensus* is defined in the strictest sense, implying that in the final state, the whole population finally concurs and the number of surviving cluster, therefore, equals 1. Generally, for some cases, even when a giant fraction of the population already converges to a major opinion cluster, several minor groups, though of small sizes sometimes containing

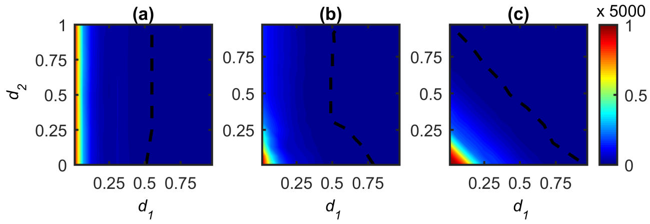

Fig. 1: Number of coexisting opinion clusters on layer 1 at steady state as a function of tolerance ranges d_1 and d_2 when $\eta = 0$ (a), 0.5 (b), and 1 (c), respectively. The number of clusters is shown in color scale. Each layer is a scale-free network with a size $N = 5000$, an average degree $\langle k \rangle = 10$, a minimum degree of 4, and a cutoff degree of 70. Dash lines are boundaries of critical regions in which a single opinion cluster is formed on layer 1.

only a single node (called isolated agents or outliers in some existing studies [21, 36]), still exist in the final state, and maintain their opinions. The corresponding critical confidence bound, hence, may vary according to criteria for global consensus.

To get a first insight, we plot the number of opinion clusters co-existing on layer 1 in the steady state as a function of d_1 and d_2 in different cases of η. When $\eta = 0$, no pair of the nodes concurrently has connections on both the layers. The cluster formation on the layer 1, therefore, is not influenced by d_2 as observed in Fig. 1(a). For $\eta = 1$, the layers are fully coupled. As a result, d_2 has a significant effect on the formation of opinion clusters on layer 1. To be specific, the number of clusters declines as either d_1 or d_2 increases but roughly remains unchanged along the lines $d_1 + d_2 =$ constant. The phenomenon becomes much more complicated when $\eta = 0.5$ that will be discussed in detail later. Here we proceed to get deeper insights by starting with a duplex network of identical layers.

Figure 2 provides a more detailed look on the diversity of opinion sytems as d_2 changes. Similarly to the case of monoplex networks, the number of opinion groups declines as d_1 increases. The formation of opinion clusters on layer 1 is affected significantly by the presence of layer 2 characterized by various d_2. At a very small d_2, e.g. $d_2 = 0.01$, the number of surviving clusters saturates at the size of the network when d_1 approaches 0. However, the fragmentation of the opinion system is severer since when $d_2 \to 0$, two nodes on layer 1 make a local consensus only if $|o_1(u,t) - o_1(v,t)| + |o_2(u,t) - o_2(v,t)| \leq d_1$, meaning the closeness in the nodes' opinions needs to be below a value less than its layer's own tolerance bound d_1, making the individuals harder to make compromise in pair-wise communication sessions and finally diverse to different opinion groups. Note that, though the conclusion above, and all the conclusions reported in the next a few paragraphs, hold in the ER random networks, in an ER network with the same average nodal degree as that of a SF network, the number of clusters tends to be relatively smaller as there are fewer low-degree nodes in the

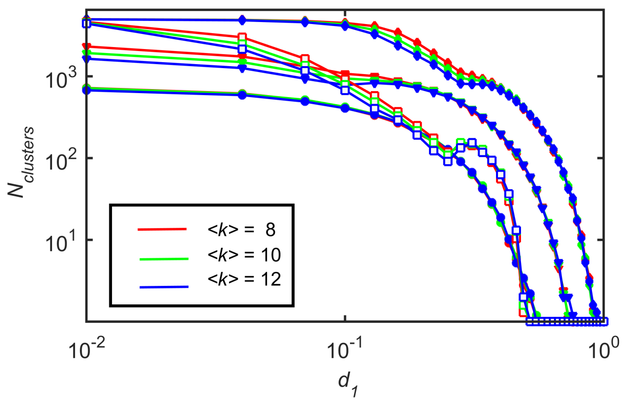

Fig. 2: Number of clusters at steady state as a function of d_1 on duplex network with a size of 5000 and various average nodal degrees in three cases of d_2: 0.01 (diamonds), 0.2 (triangles) and 0.4 (circles). Plot for single network case (squares) is added for reference. The results are averaged over 10 independent realizations.

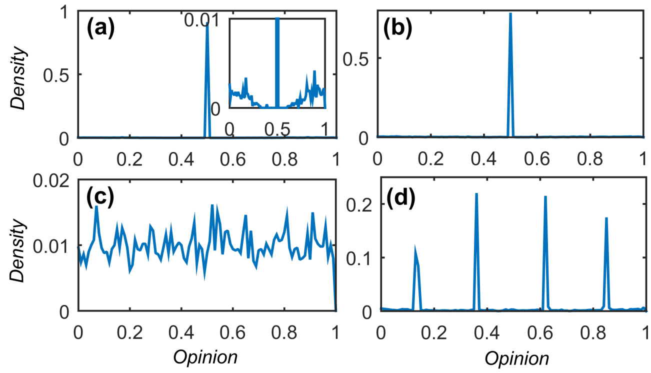

Fig. 4: Opinion distribution on a single network with $d_1 = 0.1$ (d) and layer 1 of a duplex network at steady state with $d_1 = 0.1$ and various d_2: 0.4 (a), 0.2 (b) and 0.01 (c). The network size is 5000 nodes and $\langle k \rangle = 10$. The inset in (a) is the same plot zoomed into small values of opinion density.

ER network than in the SF network.

Noticeably, the consensus-to-polarized transition point (a point on the tolerance range below which the final state of the population's opinion jumps from having a single largest cluster to having two co-existent dominant largest clusters, as termed in [13]) shifts to the left along d_1 axis as d_2 increases and vanishes when d_2 is sufficiently large, e.g. $d_2 \geq 0.4$ (see Fig. 3). It is because under the influence of layer 2 with great openness characterized by a large d_2, a giant cluster on layer 1 containing a large proportion of the population tends to be formed even at small values of d_1. This also causes the extinction of the second largest clusters and the proliferation of microscopic clusters with opinion values distributing over the entire opinion spectrum as seen in Fig. 4(a).

Also seen in Fig. 2, for a certain value of d_2, there exists a corresponding value of d_1, below (above) which the number of opinion clusters on layer 1 is smaller (larger) than that in the single network. We term such a point as weaken-enhance transition on the $d_1 - d_2$ plane. As shown in Fig. 5, the *weakened diversity* region implies that the multiplexity facilitates the consensus formation; in other words, the inter-layer interplay declines the diversity of the opinion system compared to the case in single-layer networks. *Enhanced diversity* indicates the region where the multiplexity makes the number of clusters larger than that in a single network. Lastly, *global consensus* shows the region where the duplex network reaches a global consensus, i.e, only a single giant cluster of opinion survives. The figure shows that for small values of d_2, the multiplexity always increases the opinion diversity. As d_2 increases, the weaken-enhance transition shifts to the right along the d_1-axis. Noticeably, as d_1 increases beyond 0.25, the transition point starts to plunge until d_1 reaches around 0.4. It is because in duplex networks, the consensus-to-polarized transition tends to vanish at large values of d_2 as aforementioned, while in single networks, this point clearly exists at $d_1 \approx 0.25$, causing a surge in the number of opinion clusters since the second largest cluster is broken into minor parts [13]. Note that in multiple rounds of numerical simulations, the boundaries between different regions appear to be largely steady, with differences of only small fluctuations in between.

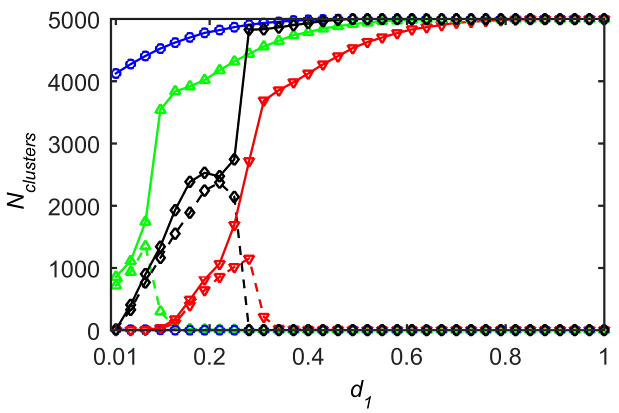

Fig. 3: Sizes of biggest (solid lines) and second biggest (dash lines) clusters in the final state on single network (black) and layer 1 of duplex network with $d_2 = 0.01$ (red), 0.2 (green) and 0.4 (blue), respectively. All networks have a size of 5000 and an average degree of 10.

A question comes naturally that given a layer 2 with a certain degree of openness quantified by d_2, at which confidence bound the layer 1 reaches a complete consensus. In single networks, the critical threshold d^c is 0.5 [16]. In duplex networks, the phenomenon becomes more complex since the tolerance range of one layer may interfere in the opinion formation on the other layer coupled with it. d_1^c, therefore, may change accordingly. We proceed to quantitatively determine the critical bound d_1^c for various d_2. A trial-and-error approach is adopted. Specifically, for each d_2, d_1^{lower} and d_1^{upper} are set at 0.01 and

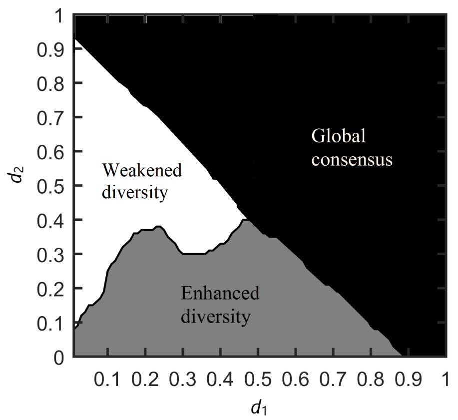

Fig. 5: The effect of multiplexity on the diversity of opinion clusters in comparison to the opinion dynamics on single networks.

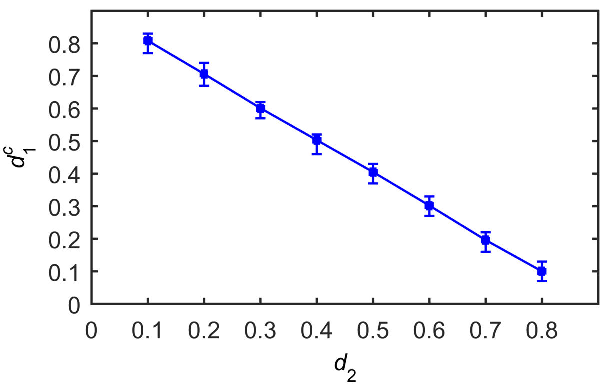

Fig. 6: Critical confidence bound on layer 1 as a function of d_2. For each d_2, the corresponding d_1^c is the average of 10 realizations. Errors bars represent 95% confidence intervals.

1 respectively; the process of opinion exchanging with a tolerance range $d_1 = \frac{d_1^{lower}+d_1^{upper}}{2}$ starts until the opinion systems on both layers enter the steady state. The number of opinion clusters on layer 1, denoted by $N_{clusters}$, is then calculated: if $N_{clusters} > 1$, d_1^{lower} is reset at d_1; otherwise, d_1^{upper} adopts d_1 as a new boundary. Then the entire process starts over until d_1^{lower} deviates from d_1^{upper} by an amount less than 0.01. The corresponding value of d_1, being equal to $\frac{d_1^{lower}+d_1^{upper}}{2}$, is considered as the critical tolerance range d_1^c.

As shown in Fig. 6, critical bound d_1^c descends monotonically with respect to the increase of d_2. The numerical result shows the sum $d_1^c + d_2$ is consistent and around 0.9. We assure that this sum reaches 1, a double of critical confidence bound in an isolated network, as the network size approaches infinity. It can be understood that the full coupling of two layers strengthens the reciprocal tie of the layers, leading to a manner that a great receptiveness of layer 2 characterized by a large value of d_2 makes layer 1 easier to reach a complete consensus and vice versa.

We here turn to a deeper look at the effects of overlap factor η by examining the survival opinion clusters on layer 1 under fixed values of d_2 for various η. When d_2 is of a small value, e.g. 0.2 in Fig. 7(a), the system behaves in a non-monotonic way: for small values of d_1, a tight inter-layer coupling, i.e. a larger η, reduces the number of opinion clusters on layer 1, but for large values of d_1, the larger η is, the greater the number of clusters becomes; that is, under the influence of layer 2 with small d_2, the inter-layer coupling becomes an enhancing factor of the opinion diversity on layer 1. Extensive simulations carried out on networks with various sizes confirm that when $d_2 \geq 0.5$, this non-monotonic behavior vanishes and the number of opinion clusters decreases with the increase of η for all $d_1 \in (0, 0.5)$ (see Figs. 7(b) and 7(c)). For these cases, it can be explained that a layer with a large tolerance range tends to weaken the opinion diversity on its coupled layer. Furthermore, the stronger the inter-layer coupling is, the stronger the effect becomes, regardless of the confidence bound on the layer in question.

Figure 8 shows the effects of d_2 for different cases of η. In loosely-coupled duplex networks, the probability of two nodes having connections on both layers is low, causing a layer seldom interferes in the opinion formation on the other layer. As a result, the effect of d_2 on the number of opinion clusters on layer 1 is not obvious as seen in Fig. 8(a). However, it is found that when η increases beyond 0.5, the system starts exhibiting a monotonic behavior: an increase of d_2 causes a decline in the diversity of the opinion system irrespective of d_1.

By adopting the same trial-and-error approach, we determine d_1^c in different cases of link overlap. Scale-free networks with a size of 10000 and an average degree of 10 are employed. The results are averaged over 10 different realizations, each corresponding to a randomly generated network. An interesting observation is shown in Fig. 9: for a range of d_2 less than 0.5, the increase in the overlap of the two layers enlarges critical tolerance range in layer 1, meaning that layer 1 requires a higher threshold to reach a global consensus. This can be understood that an increase of η intensifies the impact of layer 2's opinion system on opinion formation on layer 1. Therefore, with a d_2 less than the critical threshold in isolated networks, a

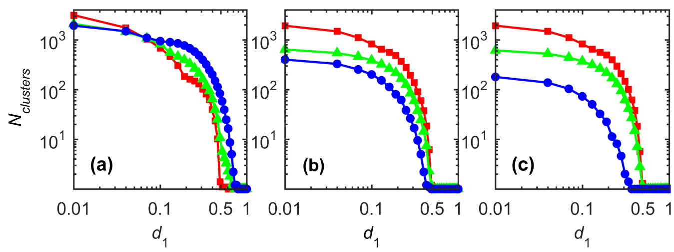

Fig. 7: Number of steady-state opinion clusters on layer 1 of duplex network with a size $N = 5000$ for $d_2 = 0.2$ (a) and $d_2 = 0.5$ (b), and $d_2 = 0.6$ (c). Each subplot shows results in three cases of η: 0.2 (squares), 0.6 (triangles) and 1 (circles).

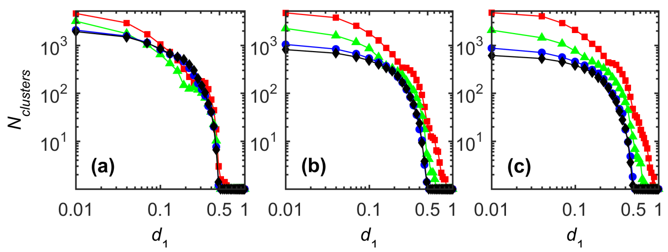

Fig. 8: Number of steady-state opinion clusters on layer 1 of duplex network with a size $N = 5000$ for $\eta = 0.2$ (a), and 0.5 (b), and 0.6 (c). Each subplot shows results in different cases of d_2: 0.01 (squares), 0.2 (triangles), 0.4 (circles), and 0.6 (diamonds).

larger critical bound on layer 1 is needed as a mechanism to compensate potential fragmentation on layer 2 due to a small d_2. However, for $d_2 > 0.5$, d_1^c no longer declines with an increase in d_2. It is found that for $d_2 > 0.5$, d_1^c remains at around 0.5 since there are always a number of minor opinion groups insisting on their different opinions from that in the giant cluster.

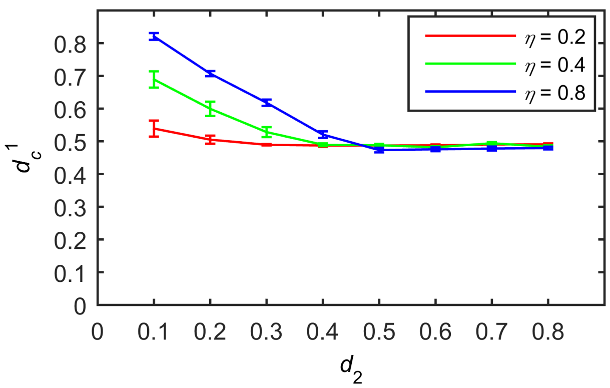

Fig. 9: Critical bound d_1^c as a function of d_2 for various values of link overlap parameter: $\eta = 0.2$ (red), $\eta = 0.4$ (green) and $\eta = 0.8$ (blue). Errors bars represent 95% confidence intervals.

Conclusion. – We studied the dynamics of opinion formation on duplex scale-free networks. It is found that the presence of multiplexity in underlying networks either enhances or weakens the diversity of opinion systems compared to the case on non-interacting monoplex networks. Theses effects depend on the relative magnitudes of tolerance ranges associated with different layers. The global consensus state where the entire population shares the same opinion is reached if the sum of two tolerance ranges is not smaller than 1, a double of the critical range in opinion systems on single networks.

The investigation of the effect of inter-layer coupling reveals that when the link overlap is sufficiently larger, which is 0.5 in our experiments, an increase in tolerance range on a layer decreases the diversity of opinion clusters on the other layer monotonically. The consideration of the global consensus state reveals that the one-sum rule no longer holds when duplex networks are not fully coupled. Following that, an open-minded network layer with a large tolerance cannot complement global consensus formation on the other layer, i.e., a layer can reach a complete consensus state only if its associated confidence bound is sufficiently large, which is at least around 0.5 in duplex scale-free networks.

The two layers are assumed to be active with an equal probability. However, in real-life, relative frequencies of being active may vary from layer to layer. Future investigations may consider such differences as a parameter in the quantification of inter-layer interplay that may reveal new properties in opinion formation.

Though most conclusions reported in this letter hold in both synthetic ER and SF networks, some other network topologies may lead to significant differences in opinion dynamics and cluster formation. For examples, some conclusions may not hold in networks with strong community structures. Investigations on dynamics of opinion formation on other network topologies will be discussed in a separate report.

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[1] Pluchino A., Latora V. Rapisarda A., *The European Physical Journal B-Condensed Matter and Complex Systems*, **50** (2006) 169.
[2] Sznajd-Weron K. Sznajd J., *International Journal of Modern Physics C*, **11** (2000) 1157.
[3] Castellano C., Fortunato S. Loreto V., *Reviews of Modern Physics*, **81** (2009) 591.
[4] Noah E. Friedkin, Anton V. Proskurnikov, Roberto Tempo Sergey E. Parsegov, *Phys. Rev. Lett.*, **112** (1994) 41301.
[5] Castellano C., Vilone D. Vespignani A., *EPL (Europhysics Letters)*, **63** (2003) 153.
[6] Sood V. Redner S., *Physical review letters*, **94** (2005) 178701.
[7] Galam S., *The European Physical Journal B-Condensed Matter and Complex Systems*, **25** (2002) 403.
[8] Deffuant G., Neau D., Amblard F. Weisbuch G., *Advances in Complex Systems*, **03** (2000) 87.
[9] Weiner I. B., Millon T. Lerner M. J., *Handbook of Psychology, Personality and Social Psychology* Vol. 5 (John Wiley & Sons) 2003.
[10] Weisbuch G., Deffuant G. Amblard F., *Physica A: Statistical Mechanics and its Applications*, **353** (2005) 555.
[11] Meng X., Van Gorder R. A. Porter M. A., *arXiv preprint arXiv:1701.02070*, (2017) .

[12] Deffuant G., *Journal of Artificial Societies and Social Simulation*, **9** (2006) 8.

[13] Kozma B. Barrat A., *Physical Review E*, **77** (2008) 016102.

[14] Mathias J.-D., Huet S. Deffuant G., *Journal of Artificial Societies and Social Simulation*, **19** (2016) 6.

[15] Weisbuch G., Deffuant G., Amblard F. Nadal J.-P., *Complexity*, **7** (2002) 55.

[16] Fortunato S., *International Journal of Modern Physics C*, **15** (2004) 1301.

[17] Jacobmeier D., *International Journal of Modern Physics C*, **17** (2006) 1801.

[18] Ben-Naim E., Krapivsky P. L. Redner S., *Physica D: Nonlinear Phenomena*, **183** (2003) 190.

[19] Lorenz J., *Physica A: Statistical Mechanics and its Applications*, **355** (2005) 217.

[20] Yu Y., Xiao G., Li G., Tay W. P. Teoh H. F., *Chaos: An Interdisciplinary Journal of Nonlinear Science*, **27** (2017) 103115.

[21] Stauffer D., Sousa A. Schulz C., *Journal of Artificial Societies and Social Simulation*, **7** (2004) .

[22] Salehi M., Sharma R., Marzolla M., Magnani M., Siyari P. Montesi D., *IEEE Transactions on Network Science and Engineering*, **2** (2015) 65.

[23] D'agostino G. Scala A., *Networks of networks: the last frontier of complexity* Vol. 340 (Springer) 2014.

[24] Lee K.-M., Kim J. Y., Lee S. Goh K.-I., *Multiplex networks* in *Networks of networks: The last frontier of complexity* (Springer) 2014 pp. 53–72.

[25] Mucha P. J., Richardson T., Macon K., Porter M. A. Onnela J.-P., *Science*, **328** (2010) 876.

[26] Gomez S., Diaz-Guilera A., Gomez-Gardenes J., Perez-Vicente C. J., Moreno Y. Arenas A., *Physical review letters*, **110** (2013) 028701.

[27] Granell C., Gómez S. Arenas A., *Physical review letters*, **111** (2013) 128701.

[28] Shang Y., *Journal of Physics A: Mathematical and Theoretical*, **48** (2015) 395101.

[29] Lorenz J., *European Journal of Economic and Social Systems, EJESS*, (2007) .

[30] Catanzaro M., Boguñá M. Pastor-Satorras R., *Physical Review E*, **71** (2005) 027103.

[31] Clauset A., Shalizi C. R. Newman M. E. J., , (2007) .

[32] Muchnik L., Pei S., Parra L. C., Reis S. D. S., Andrade J. S., Havlin S. Makse H. a., *Scientific reports*, **3** (2013) 1783.

[33] De Domenico M., Solé-Ribalta A., Cozzo E., Kivelä M., Moreno Y., Porter M. A., Gómez S. Arenas A., *Physical Review X*, **3** (2013) 041022.

[34] Lorenz J., *International Journal of Modern Physics C*, **18** (2007) 1819.

[35] Pothen A. Fan C.-J., *ACM Transactions on Mathematical Software (TOMS)*, **16** (1990) 303.

[36] Weisbuch G., Deffuant G., Amblard F. Nadal J.-P., *Heterogenous agents, interactions and economic performance*, (2003) 225.