An Optimal Control Approach to Identify the Worst-Case Cascading Failures in Power Systems

Chao Zhai, Hehong Zhang, Gaoxi Xiao, and Tso-Chien Pan

***Abstract*—Cascading failures in power systems normally occur as a result of initial disturbances or faults on electrical elements, closely followed by situational awareness errors of human operators. It remains a great challenge to systematically trace the source of cascading failures in power systems. In this paper, we develop a mathematical model to describe the cascading outages of transmission lines in power networks. In particular, the direct current (DC) power flow equation is employed to calculate the power flow on the branches. By regarding the disturbances on branches as the control inputs, the problem of identifying the initial disruptive disturbances is formulated with optimal control theory, which provides a systematic approach to explore the most disruptive disturbances that give rise to changes of branch admittance in addition to direct branch outages. Moreover, an iterative search algorithm is proposed to look for the optimal solution leading to the worst-case cascading failures. Theoretical analysis guarantees the asymptotic convergence of the iterative search algorithm. Finally, numerical simulations are carried out on the IEEE test systems to validate the proposed approach.**

***Index Terms*—Cascading failure, optimal control, DC power flow equation, disturbance, power systems.**

Nomenclature

P_{ij}	power flow on the branch that connects Bus i to Bus j.
c_{ij}	power flow threshold of the branch that connects Bus i to Bus j.
Y_p^k	branch admittance vector at the k-th cascading step.
$y_{p,i}^k$	admittance on the i-th branch at the k-th cascading step.
A	branch-bus incidence matrix.
Y_b^k	nodal admittance matrix at the k-th cascading step.
u_k	control input or disturbance at the k-th cascading step.
θ^k	vector of voltage phase angle on buses at the k-th cascading step.
S_i	the i-th subnetwork in power system.
V_i	set of Bus IDs in the i-th subnetwork.
e_i	unit vector with the i-th element being 1 and 0 otherwise.
P^k	vector of power injected on buses at the k-th cascading step.
E_{i_k}	diagonal matrix to select the i_k-th branch at the k-th cascading step.
m	number of buses in power system.
n	number of transmission lines in power system.
h	number of cascading steps in power system.

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I. Introduction

The stability and secure operation of power grids have a great impact on other interdependent critical infrastructure systems such as energy systems, transportation systems, finance systems and communication systems. Nevertheless, contingencies on vulnerable components of power systems and situational awareness errors of human operators [1]could trigger a chain reaction of circuit breakers, leading to a large blackout of power networks. For instance, the North America cascading blackout on August 14, 2003 resulted in a power outage affecting 50 million people [2]. The misoperation of a German operator in November 2006 triggered a chain reaction of power grids that caused 15 million Europeans to lose access to power [3]. Recently, a relay fault near the Taj Mahal in India gave rise to a severe cascading blackout on July 31, 2012 affecting 600 million people. It is vital to identify the worst possible attacks or initial disturbances on the critical electrical elements in advance and develop effective protection strategies to alleviate cascading blackouts in power systems.

A cascading blackout in power systems is defined as a sequence of component outages that include at least one triggering component outage caused by initial contingencies [4], [5] and subsequent tripping component outages due to the overload of transmission lines and situational awareness errors of human operators [6]–[8]. Besides line failures, multiple factors contribute to the occurrence of cascading blackouts, such as weather condition, human factors (e.g. misoperations of operators), the equipment aging and so on. Note that a cascading failure does not necessarily lead to a cascading blackout or load shedding. The existing cascading models basically fall into 3 categories [6]. The first type of models only reveals the topological properties, but ignores physics of power grids. As a result, these models are unable to accurately describe the cascading evolution of power networks in practice [9]–[11]. The second type of models focuses on the quasi-steady-state of power systems and computes the power flow on branches by solving the DC or alternating current (AC) power flow equations [5], [12], [13]. The third one resorts to

[1]the errors of human operators due to the inadequate situational awareness, which includes three components: perception of the environment elements, comprehension of the situation, and projection of future status [1]

the dynamic transient-related modeling of cascading failures in order to allow for the effects of component dynamics [14]–[16]. For example, a dynamic multiprocess-integrated model of cascading failure was presented to deal with the interdependencies of different mechanisms with the transient dynamics of generators and protective relays [15]. It is suggested the transient dynamical behaviors in power systems play a crucial role in the emergence of cascading failures [16].

Actually, many factors (e.g. temperature, frequency, short circuit, poor contactor, etc) may contribute to the changes of branch impedance other than line outages. Some disturbances or faults on the transmission lines of power grids can be described by impedance or admittance changes [17], [18]. As a special case, the outage of a transmission line can lead to infinite impedance or zero admittance between two relevant buses. Linear or nonlinear programming is normally employed to formulate the problem of determining the changes of branch admittance related to disruptive disturbances. [17] presents two different optimization formulations to analyze the vulnerability of power grids. Specifically, nonlinear programming is adopted to address the voltage disturbance, and nonlinear bilevel optimization is employed to deal with the power adjustment. Nevertheless, there is still a lack of mathematical framework and theoretical results for investigating the effects of initial disturbances on the ongoing dynamics of cascading failures. Previous optimization formulations are not sufficient to describe the outage sequence of transmission lines in practice, because the final configuration of a power network strongly depends on the evolution process of transmission lines (i.e. the line outage sequence) in addition to initial conditions.

In this paper, we will develop a cascading model of power networks to describe the changes of branch admittance of the outage sequence. The dynamics of the cascade process includes the sequence of line outages and the redistribution of power flow on branches. Moreover, the problem of identifying initial disturbances causing the worst disruption of cascades is formulated and solved in the framework of optimal control theory by treating the disruptive disturbances as control inputs in the optimal control system. The proposed approach provides a new insight into tracing disruptive disturbances on vulnerable components of power grids. It helps to find the changes of branch admittance related to the most disruptive disturbances in addition to direct branch outages. Since it is difficult to directly identify the initial disturbances that cause the worst-case cascades of real power systems, this work provides a theoretical approach to search for the worst-case cascades in a simplified model. Moreover, it is expected to determine the most disruptive disturbances at the early stage of cascades in practice.

The remainder of this paper is organized as follows. Section II presents the cascading model of power systems and the optimal control approach. Section III provides theoretical results for the problem of identifying disruptive disturbances, followed by simulations and validation on the IEEE test systems in Section IV. Finally, we conclude the paper and discuss future work in Section V.

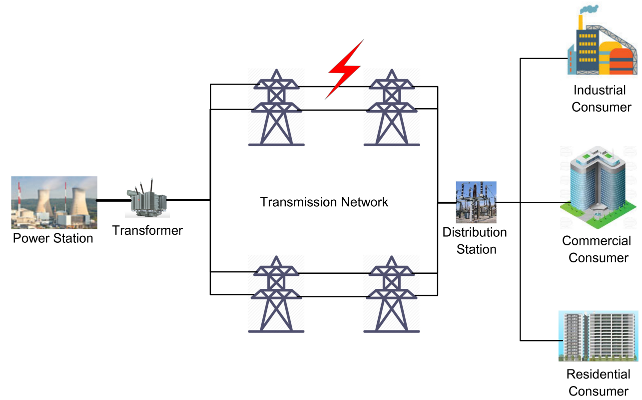

Fig. 1: Schematic diagram of power systems suffering from the lightning on transmission lines.

II. Problem Formulation

The power system is basically composed of power stations, transformers, power transmission networks, distribution stations, and consumers (see Fig. 1). In this work, we are interested in identifying disruptive disturbances (e.g. lightning, storm, temperature fluctuation [19], etc) on transmission lines that trigger chain reactions and cause cascading blackouts in power grids. The disturbances give rise to admittance changes on transmission lines, which results in the redistribution of power flow in power grids. The focus of this work is on the identification of initial disruptive disturbances that cause the worst-case cascades. In practice, most initial disturbances can be modeled as the changes of branch admittance, as can be observed in major blackouts in history [2], [20], [21]. Theoretically, it is reasonable to model a branch disturbance as a change of branch admittance [17]–[19], [22]. The overload of transmission lines causes certain circuit breakers to sever the corresponding branches and readjust the power network topology. The above process does not stop until the power grid reaches a new steady state and transmission lines are not overloaded any more. In this section, we propose a model to describe the cascading outage process of transmission lines, where the DC power flow equation is solved to obtain the power flow on each branch. More significantly, the mathematical formulation based on optimal control is presented by regarding the disruptive disturbances of power grids as the control inputs in the optimal control system.

A. Cascading model

The existing cascading failure models are mainly concerned about the direct branch outages, which ignores the effects of continuous changes of branch admittance. For this reason, we make a cascading model that describes the changes of branch admittance during the cascades. To characterize the connection state of the transmission line, we introduce the state function of the transmission line that connects Bus i and Bus j as follows:

$$g(P_{ij},c_{ij}) = \begin{cases} 0, & |P_{ij}| \geq \sqrt{c_{ij}^2 + \frac{\pi}{2\sigma}}; \\ 1, & |P_{ij}| \leq \sqrt{c_{ij}^2 - \frac{\pi}{2\sigma}}; \\ \frac{1-\sin\sigma(P_{ij}^2 - c_{ij}^2)}{2}, & \text{otherwise.} \end{cases} \tag{1}$$

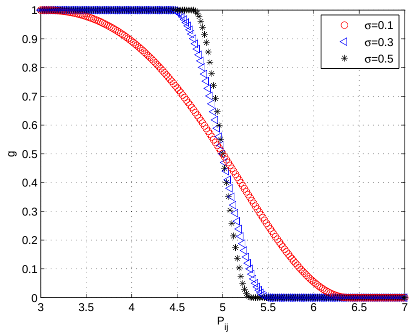

Fig. 2: The state function $g(P_{ij},c_{ij})$ of the transmission line connecting Bus i to Bus j with the power threshold $c_{ij}=5$.

where $i,j\in I_m=\{1,2,...,m\}$, $i\neq j$ and m is the total number of buses in the power system. σ is a tunable positive parameter. P_{ij} refers to the steady-state power flow on the transmission line that links Bus i and Bus j, and c_{ij} denotes its power flow threshold. The state function $g(P_{ij},c_{ij})$ is differentiable with respect to P_{ij}, and more closely resembles a step function as σ increases (see Fig. 2). The transmission line is in good condition when $g(P_{ij},c_{ij})=1$, while $g(P_{ij},c_{ij})=0$ implies that the transmission line has been severed by the circuit breaker. Essentially, σ quantifies the approximation level of $g(P_{ij},c_{ij})$ to a step function. By properly tuning the parameter σ, the function $g(P_{ij},c_{ij})$ is able to reflect the system characteristic of branch outage while guaranteeing its differentiability with respect to P_{ij}, which is indispensable to derive the necessary condition for optimality using optimal control theory.

Remark II.1. *The value of $g(P_{ij},c_{ij})$ is fractional when P_{ij}^2 is in the interval $(c_{ij}^2-\pi/2\sigma, c_{ij}^2+\pi/2\sigma)$. Clearly, the interval length is π/σ, and it is sufficiently small when σ is large enough. As a result, we can increase σ to eliminate the fractional state when it occurs in numerical simulations. Alternatively, the post-processing method is adopted to replace the fractional state with values 0 and 1 respectively and then opt for the one with the smaller cost function value.*

The cascading model of power networks at the k-th cascading step can be presented as:

$$Y_p^{k+1}=G(P_{ij}^k,c_{ij})\cdot Y_p^k+E_{i_k}u_k,\quad k=0,1,2,...h-1 \tag{2}$$

where $Y_p^k=(y_{p,1}^k,y_{p,2}^k,...,y_{p,n}^k)^T$ is the admittance vector for the n transmission lines or branches at the k-th step, and $u_k=(u_{k,1},u_{k,2},...,u_{k,n})^T$ denotes the control inputs or disturbances on transmission lines. h is the total number of cascading steps in power networks. $G(P_{ij}^k,c_{ij})$ and E_{i_k} are the diagonal matrixes defined as $G(P_{ij}^k,c_{ij})=diag(\mathfrak{g}_1^k,\mathfrak{g}_2^k,...,\mathfrak{g}_n^k)$ with $\mathfrak{g}_s^k=g(P_{i_sj_s}^k,c_{i_sj_s}), s\in I_n=\{1,2,...,n\}$ and

$$E_{i_k}=diag(\underbrace{0,...,0,1}_{i_k},0,...,0)\in R^{n\times n},$$

respectively. Here E_{i_k} is a predetermined matrix. By assigning 1 to the i_k-th diagonal element in E_{i_k}, we can add the control input onto the i_k-th branch and thus change the admittance of the selected branch. The cascading model (2) is formulated based on the steady-state power flow on branches, and it mainly allows for the branch outage caused by persistent branch overloads. The branch admittance becomes zero and remains unchanged once the branch is severed. In fact, Equation (2) can be rewritten in the element-wise form (i.e. $y_{p,s}^{k+1}=g(P_{i_sj_s}^k,c_{i_sj_s})\cdot y_{p,s}^k+e_s^TE_{i_k}u_k$, $s\in I_n$).

Compared to existing models, the proposed model enables to identify the disruptive disturbances that cause the changes of branch admittance not restricted to the direct branch outage, by using optimal control theory. Meanwhile, it is flexible to incorporate various factors (e.g. protective relays, load variation, generation control, etc) that affect the cascades. In addition, it is able to reflect the physical characteristics of cascades at the early stage as well as to avoid the non-convergence of numerical algorithms (see Section 1 and Section 2 in Supplementary Materials for more details [23]).

B. DC power flow equation

In this work, we care about the outage sequence of transmission lines in power systems and thus compute the DC power flow to deal with the overloading problem. Specifically, the DC power flow equation is given by

$$P_i=\sum_{j=1}^{m}Y_b(i,j)(\theta_i-\theta_j) \tag{3}$$

where P_i and θ_i refer to the injection power and voltage phase angle of Bus i, respectively. $Y_b(i,j)$ represents the mutual susceptance between Bus i and Bus j, where $i,j\in I_m$. Equation (3) can be rewritten in matrix form [24]: $P=Y_b\theta$, where $P=(P_1,P_2,...,P_m)^T$, $\theta=(\theta_1,\theta_2,...,\theta_m)^T$, and Y_b is the nodal admittance matrix of power networks with its (i,j) entry $Y_b(i,j)$. The nodal admittance matrix Y_b^k at the k-th cascading step can be obtained as

$$Y_b^k=A^T diag(Y_p^k)A$$

where A denotes the branch-bus incidence matrix [25], and $Y_p^k=(y_{p,1}^k,y_{p,2}^k,...,y_{p,n}^k)^T$ with

$$y_{p,i}^k=-\frac{1}{Im(z_{p,i}^k)},\quad i\in I_n \tag{4}$$

$z_{p,i}^k$ denotes the impedance of the i-th branch at the k-th step. Then the DC power flow equation at the k-th step is given by

$$P^k=Y_b^k\theta^k \tag{5}$$

where $P^k=(P_1^k,P_2^k,...,P_m^k)^T$ and $\theta^k=(\theta_1^k,\theta_2^k,...,\theta_m^k)^T$. During the cascades, the power network may be divided into several subnetworks (*i.e.*, islands), which can be identified by analyzing the nodal admittance matrix Y_b^k. Suppose Y_b^k is composed of q isolated components or subnetworks denoted by $S_i, i\in I_q=\{1,2,...,q\}$ and each subnetwork S_i includes k_i buses. Let $V_i=\{v_1,v_2,...,v_{k_i}\}$ denote the set of bus IDs in the subnetwork S_i, where v_1, v_2,..., v_{k_i} denote the bus IDs and $\sum_{i=1}^q k_i=m$. Notice that Bus v_1 in Subnetwork S_i is designated as the reference bus, which is normally a generator bus connected to the largest generating station. Thus, the power variation of reference bus accounts for a small percentage of its generating capacity. When there is no generator buses in the subnetwork, the power flow on each branch of this subnetwork is zero. In addition, the injected power on the reference bus can be adjusted to balance the power supply and consumption in the subnetwork. Moreover, the nodal admittance matrix of the i-th subnetwork can be computed as

$$Y_{b,i}^k = \mathscr{E}_i^T Y_b^k \mathscr{E}_i, \quad i \in I_q$$

where $\mathscr{E}_i = (e_{i_1}, e_{i_2}, \ldots, e_{i_{k_i}})$. For simplicity, we introduce two operators $*$ and -1^* to facilitate the analytical expression and theoretical analysis of solving the DC power flow equation.

Definition II.1. *Given the nodal admittance matrix Y_b^k, the operators $*$ and -1^* are defined by*

$$\left(Y_b^k\right)^* = \sum_{i=1}^{q} \mathscr{E}_i \left(\begin{array}{c|c} 0 & 0_{k_i-1}^T \\ \hline 0_{k_i-1} & I_{k_i-1} \end{array}\right) Y_{b,i}^k \left(\begin{array}{c|c} 0 & 0_{k_i-1}^T \\ \hline 0_{k_i-1} & I_{k_i-1} \end{array}\right) \mathscr{E}_i^T$$

and

$$\left(Y_b^k\right)^{-1^*} = \sum_{i=1}^{q} \mathscr{E}_i \begin{pmatrix} 0_{k_i-1}^T \\ I_{k_i-1} \end{pmatrix} \left(\mathscr{Y}_{b,i}^k\right)^{-1} \begin{pmatrix} 0_{k_i-1} & I_{k_i-1} \end{pmatrix} \mathscr{E}_i^T,$$

respectively, where

$$\mathscr{Y}_{b,i}^k = \begin{pmatrix} 0_{k_i-1} & I_{k_i-1} \end{pmatrix} Y_{b,i}^k \begin{pmatrix} 0_{k_i-1}^T \\ I_{k_i-1} \end{pmatrix}.$$

I_{k_i-1} is the $(k_i-1)\times(k_i-1)$ identity matrix and 0_{k_i-1} denotes the (k_i-1) dimensional column vector with zero elements.

In this work, the DC power flow equation is solved with the operator -1^* directly without using any power system simulators.

Remark II.2. *The power network represented by the nodal admittance matrix Y_b^k can be decomposed into q isolated subnetworks, and each subnetwork is described by a submatrix $Y_{b,i}^k$, $i \in I_q$. By using breadth-first search or depth-first search, the identification of subnetworks can be completed in linear time in terms of the numbers of nodes and branches in the power network [27]. The operators $*$ and -1^* replace all the elements in the $1st$ row and the $1st$ column of $Y_{b,i}^k$ with 0. Moreover, the operator -1^* also replaces the remaining part of $Y_{b,i}^k$ with its inverse matrix. According to algebraic graph theory, the rank of the nodal admittance matrix $Y_{b,i}^k$ is k_i-1 since each subnetwork S_i, $i \in I_q$ is connected [26]. Thus it is guaranteed that the matrix $\mathscr{Y}_{b,i}^k$ has full rank k_i-1, and hence it is invertible.*

Remark II.3. *For the branch with resistance R and reactance X, the dependence of branch impedance on the temperature can be described by the formulas [28]: $R(T_c) = R(T_0)\cdot[1+\alpha(T_c-T_0)]$ and $X(T_c,\omega) = X(T_0,\omega)\cdot[1+\beta(T_c-T_0)]$, where α and β refer to the temperature coefficients of resistivity and reactance, respectively. T_0 denotes the reference temperature (usually 20°C), and T_c represents the conductor temperature. $R(T_0)$ and $X(T_0,\omega)$ are determined at the reference temperature T_0, and ω is the operating angular frequency.*

C. Optimization formulation

The cascading dynamics of power system are composed of the cascading model defined by Equation (2) and the DC power flow equation described by Equation (5). These two components, coupled with each other, characterize the cascading process of power grids after suffering from disruptive disturbances. The optimal control algorithm allows to obtain the disruptive disturbances by treating the disturbances as the control inputs in the optimal control system (see Fig. 3). Specifically, the cascading model describes the admittance changes of branch outage and updates the branch admittance with the latest power flow, which is provided by the DC power flow equation. Meanwhile, the DC power flow equation is solved with the up-to-date admittance of branches from the cascading model. The above two processes occur iteratively in describing the evolution of admittance and power flow on transmission lines. Moreover, the cascading dynamics of power system exactly function as the state equation of the optimal control system. In this way, optimal control theory enables us to gain the optimal control inputs that lead to the worst-case cascading failures in power systems.

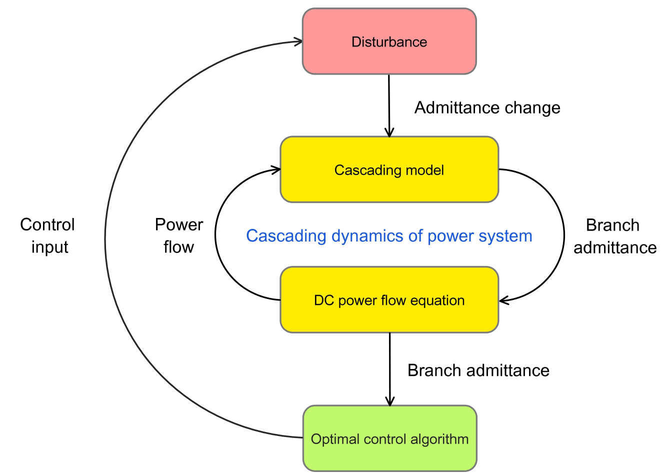

Fig. 3: Optimal control approach to identify disruptive disturbances.

The identification of admittance changes related to disruptive disturbances in power systems can be formulated as the following optimal control problem:

$$\min_{u_k} J(Y_p^h, u_k) \tag{6}$$

with the cost function

$$J(Y_p^h, u_k) = \mathrm{T}(Y_p^h) + \varepsilon \sum_{k=0}^{h-1} \frac{\|u_k\|^2}{\max\{0, \iota - k\}} \tag{7}$$

where ε is a positive weight, and $\|\cdot\|$ represents the 2-norm. As mentioned before, the state equation of the optimal control system consists of Equations (2) and (5). The above cost function includes two terms. Specifically, the first term $\mathrm{T}(Y_p^h)$ denotes the endpoint cost of cascading failures and it is differentiable with respect to Y_p^h. The second term characterizes the control energy at the first ι time steps with the constraint $1 \le \iota \le (h-1)$. In practice, $\mathrm{T}(Y_p^h)$ is designed according to the specific concerns about the worst-case scenario of power systems. For example, it can be designed as $T(Y_p^h) = \|Y_p^h\|^2$ regarding the connectivity of power network or $T(Y_p^h) = \sum_{i=1}^m \sum_{j=1}^m [P_{ij}(Y_p^h)]^2$ in terms of the total power flow. In particular, the parameter ε is set small enough that the first term dominates the cost function. In brief, the idea behind Equation (7) is that the cost function is dominated by its endpoint cost $\mathrm{T}(Y_p^h)$ that quantifies the final disruption of cascades triggered by initial disturbances (i.e. control inputs). Essentially, the minimization of cost function is done over the first ι control inputs (i.e. u_k, $k = 0, 1, ..., \iota - 1$), and the design of the term $\|u_k\|^2/\max\{0, \iota - k\}$ enables us to add the control input u_k at the specified time steps by setting ι. The objective of this work is to minimize $\mathrm{T}(Y_p^h)$ (e.g. to minimize the connectivity of power networks or the power consumption) by adding the appropriate control inputs on the selected branch at the beginning of cascades. Optimal control theory enables us to obtain the optimal control inputs (i.e. the most disruptive disturbances) and the minimum $\mathrm{T}(Y_p^h)$ (i.e. the worst-case cascading failures).

D. Extensibility

The proposed optimal control approach can be extended to allow for multiple factors of cascading failures. Though detailed studies on such extensions are largely out of scope of this paper, we provide some brief discussions in terms of other relays, protective actions (e.g. load shedding and generation control), uncertainties and hidden failures, and power fluctuations as follows.

First of all, the proposed approach can be extended to reflect the physical characteristics of other relays. For example, the time-inverse characteristics of over-current relays can be described by comparing the integral of the branch current over a time interval with the threshold [15]. In addition, the time delay in distance relays can be addressed with the aid of adaptive dynamic programming in optimal control [29].

Protection actions of power system have impacts on the evolution of cascading blackouts [30], [31]. By updating P^k at certain cascading steps, the protection actions such as load shedding and generator tripping can be taken into account without affecting the applicability of the proposed approach, though it may increase the computation burden and complexity of the corresponding optimal control problems (6). By minimizing the change of injected power on buses, a non-linear programming problem can be formulated to allow for protective actions (e.g. load shedding and generation dispatch) against the cascades. It follows from the Karush-Kuhn-Tucker (KKT) conditions that the nonlinear programming problem can be converted into a system of $(7m+6n)$ algebraic equations. By combining it with the system (8), we obtain an extended system of $7m+(6+h)n$ algebraic equations with $7m+(6+h)n$ unknowns. The solutions to the extended system of algebraic equations enable us to compute the disruptive disturbances that cause the worst-case cascading failures of power grids with protective actions. Compared to the system (8), both the number of algebraic equations and the number of unknowns in the extended system increase by $(7m+6n)$ (see Section 3 in Supplementary Materials for more details [23]).

The uncertainties in power systems and hidden failures of the relays are important factors of cascading failures. Optimal control of uncertain systems can be adopted to deal with the uncertainties (e.g. the variations of power generation, consumption, and branch impedance, etc) in power systems [32]. For hidden failures of the relays, the probabilistic model based on Markov chain is more suitable to allow for the stochastic factors [33], and statistics of cascading line outages from utility data can be used as the benchmark to validate the model [34]–[36]. For this case, optimal control of Markov decision processes provides a useful framework to identify the disruptive disturbance by regarding the disturbances in power systems as the policy or actions in Markov decision processes [37], [38].

Other than line failures, there are multiple contributors of cascading failures such as weather condition, human factors, protection actions, voltage instability, device aging, and so on. Considering that the triggering events (i.e. the initial disruptive disturbances) of most major blackouts in history can be modeled by changes of branch admittance [2], [20], [21], this work focuses on the identification of initial disruptive disturbances or faults that change the branch admittance and cause the worst-case cascades. Actually, the proposed approach is also applied to identify the fluctuation of injected power on buses caused by load variation and generation control. In addition, the probabilistic model based on Markov chain is more suitable to allow for the stochastic contributing factors such as device aging and human factors.

III. Theoretical Analysis

In this section, we present some theoretical results on the proposed optimal control problem. The main theoretical results are presented in Theorem III.1, which provides the necessary conditions of optimal solutions to the proposed optimal control problem (i.e. a system of algebraic equations). Specifically, Lemma III.1 contributes to the proof of Lemma III.3, and Lemma III.2 allows to compute the power flow on each branch. Both Lemma III.2 and Lemma III.3 are used to derive the system of algebraic equations in Theorem III.1. First of all, the properties of operators $*$ and -1^* are given by the following lemmas.

Lemma III.1. *For the nodal admittance matrix $Y_b^k \in R^{m\times m}$, the equations*

$$\left(Y_b^k\right)^* \left(Y_b^k\right)^{-1^*} = \left(Y_b^k\right)^{-1^*} \left(Y_b^k\right)^* = \sum_{i=1}^q \mathscr{E}_i \; diag(0, 1_{k_i-1}^T)\mathscr{E}_i^T$$

hold, where $1_{k_i-1} = (1, 1, ..., 1)^T \in R^{(k_i-1)}$.

Proof: See Appendix A. ∎

Lemma III.1 indicates that the two operators $*$ and -1^* are commutative for the same square matrix. Given the injection power for each bus $P^k = (P_1^k, P_2^k, ..., P_m^k)^T$ at the k-th time step, the quantitative relationship between Y_p^k and power flow on each branch is presented as follows.

Lemma III.2.

$$P_{ij}^k = e_i^T Y_b^k e_j (e_i - e_j)^T (Y_b^k)^{-1^*} P^k, \; i, j \in I_m$$

where $e_i = (\underbrace{0, ..., 0, 1}_{i}, 0, ...0)^T \in R^m$.

Proof: See Appendix B. ■

Similar to the matrix inversion, the operators $*$ and -1^* satisfy the following equation in terms of the derivative operation.

Lemma III.3.

$$\frac{\partial (Y_b^k)^{-1^*}}{\partial y_{p,i}^k} = -(Y_b^k)^{-1^*}(A^T diag(e_i)A)^*(Y_b^k)^{-1^*}.$$

Proof: See Appendix C. ■

Next, we present theoretical results about the optimal control problem (6), which is actually a special case (*i.e.*, the time invariant case) of the optimal control for the time-varying discrete time nonlinear system in [39]. By applying the approach in [39] to the optimal control problem (6), we obtain the necessary conditions to identify the disruptive disturbance of power systems with the cascading model (2) and the DC power flow equation (5).

Theorem III.1. *The necessary conditions for the optimal control problem (6) are the solutions of the following system of algebraic equations.*

$$Y_p^{k+1} - G(P_{ij}^k, c_{ij})Y_p^k - E_{i_k}u_k = 0_n, \tag{8}$$

where the control input u_k is given by

$$u_k = -\frac{\max\{0, \iota - k\}}{2\varepsilon} E_{i_k} \prod_{s=0}^{h-k-1} \frac{\partial Y_p^{h-s}}{\partial Y_p^{h-s-1}} \cdot \frac{\partial \mathrm{T}(Y_p^h)}{\partial Y_p^h} \tag{9}$$

with $k = 0, 1, ..., h-1$.

Proof: See Appendix D. ■

Remark III.1. *Substituting (9) into (8) yields the system of $h \times n$ algebraic equations with $h \times n$ unknowns (i.e., $y_{p,i}^k$, $i \in I_n$, $k \in I_h$). Thus, the solution to the above system of algebraic equations enables us to obtain the branch admittances at each cascading step. The optimal control input u_k is also available by replacing the unknowns in (9) with the computed branch admittances.*

It is necessary to winnow the solutions to Equation (8), since they just satisfy necessary conditions for optimal control problem (6). Thus, we introduce a search algorithm to explore the optimal control input or initial disturbances. Table I presents the implementation process of the Iterative Search Algorithm (ISA) in detail. First of all, we set the maximum iteration steps i_{max} of the ISA and the initial value of cost function J^*, which is a sufficiently large number $J_{\max}$ and is larger than the maximum value of the cost function. The solution to the system of algebraic equations (8) allows us to obtain the control input u^i from (9). Then we compute the cost function J^i from (7) by adding the control input u^i in power systems. Then J^* and u^* are replaced with J^i and u^i if J^i is less than J^*. Finally, the algorithm goes to the next iteration and solves the system of algebraic equations (8) once again.

Regarding the Iterative Search Algorithm in Table I, we have the following theoretical result.

Theorem III.2. *The Iterative Search Algorithm in Table I ensures that the cost function J^* and control input u^* converge to the optima as the iteration steps $i_{\max}$ go to infinity.*

TABLE I: Iterative Search Algorithm.

Set the maximum iteration steps $i_{\max}$, $i = 0$ and $J^* = J_{\max}$
while ($i <= i_{\max}$)
Solve the system of algebraic equations (8)
Compute the control input u^i from (9)
Validate the control input u^i in (2)
Compute the resulting cost function J^i from (7)
if ($J^i < J^*$)
Set $u^* = u^i$ and $J^* = J^i$
end if
Set $i = i + 1$
end while

Proof: The ISA in Table I indicates that the cost function J^* decreases monotonically as time proceeds. Considering that J^* is the lower bounded (*i.e.*, $J^* \geq 0$), it can be proved that J^* asymptotically converges to the infimum according to monotone convergence theorem in real analysis [40]. For each iteration, the system of algebraic equation (8) is solved with a random initial condition. As a result, the cost function J^* and control input u^* converge to the optima as the iteration steps $i_{\max}$ go to infinity. ■

While the ISA does not guarantee the optimal solution, it provides additional merits compared to greedy algorithms, as explained in the following remarks.

Remark III.2. *The estimated optimum by ISA approximates the optimal solution as the iteration steps increase. Nevertheless, this does not imply that the optimal solution is always available. This is because theoretical results only provide the necessary condition for solutions to the optimal control problem (6). For some special cases, we can guarantee that the estimated optimum is the optimal solution if it results in the smallest possible value of cost function.*

Remark III.3. *Compared with greedy algorithms to identify the worst disruptions, the proposed optimal control approach can provide additional merits. Specifically, the proposed approach provides a theoretical formulation that allows for the dynamics and intermediate steps of cascades, and it is expected to approximate the global optimum with ISA. In contrast, greedy algorithms choose the local optimum at each stage without taking into account intermediate steps. In addition, the proposed approach is able to identify the worst-case cascading failures caused by changes of branch admittance in addition to direct branch outages.*

IV. Simulation and Validation

In this section, we implement the ISA in Table I to search for the disruptive disturbance that is added on each branch of the IEEE 14 Bus System. The numerical results on disruptive disturbances are validated by disturbing the selected branch with the computed magnitude of disturbance. To sever as many branches as possible, we define the terminal constraint in the cost function (7) as follows

$$T(Y_p^h) = \frac{1}{2}\|Y_p^h\|^2$$

and derive its partial derivative with respect to Y_p^h

$$\frac{\partial \mathrm{T}(Y_p^h)}{\partial Y_p^h} = Y_p^h \tag{10}$$

Since the admittance of outage branch is 0, the smaller value of $T(Y_p^h)$ implies the fewer connected branches. Thus, the choice of $T(Y_p^h)$ allows to quantify the connectivity of power networks. By substituting (10) into (9), we obtain the desired control input for the system of algebraic equations (8). In addition, numerical simulations are also conducted on the larger IEEE test systems to validate the scalability of the proposed approach.

A. Numerical simulations

The ISA has been implemented on each branch of the IEEE 14 Bus System to trace the initial disturbances that result in the worst-case blackout of power network. The relevant data and network topology of the IEEE 14 Bus System are presented in Matpower [41], where Bus 1, Bus 2, Bus 3, Bus 6, and Bus 8 are designated as generator buses, and all others are load buses. It is worth noting that Bus 1 also acts as the reference bus (slack bus) while computing the DC power flow at the initial step. The net injection power on each bus is equal to the difference between the consumption power and the generation power. Per unit values are adopted with the base value of power 100 MVA. Moreover, the solver “fsolve” in Matlab is employed to solve the desired system of algebraic equations (8). The iteration number in ISA is related to the size of power networks and the initial condition of numerical algorithm. Normally, the iteration number increases as the network size gets larger. In addition, a proper selection of the initial condition of numerical algorithm allows to obtain the satisfactory solution with a relatively small number of iterations. For simplicity, it is assumed that the injection power on each bus remains constant during the numerical computation. In addition, the power threshold of circuit breakers on each branch is set to 0.3 except for $c_{23} = 0.4$ and $c_{34} = 0.7$. Such settings of power threshold guarantee the normal operation of the IEEE 14 Bus System when there are no disruptive disturbances. The initial admittance of each branch can be computed according to Equation (4). Other parameters are given as follows: $\sigma = 5 \times 10^4$, $\varepsilon = 10^{-4}$, $\iota = 1$, $i_{\max} = 10$, $J_{\max} = 10^6$, and $h = 10$. Figure 4 presents the computed control input on each branch and the resulted cost level at the final step. As we can see, the computed control inputs or disturbances on Branch 3, Branch 6, Branch 9, Branch 10, and Branch 15 lead to rather small values of cost function. Moreover, the computed disturbances on Branch 1, Branch 2, Branch 3, Branch 4, Branch 11, Branch 13, and Branch 16 exactly result in the outage of their respective branches. Of all the computed disturbances, we can observe that the disturbance on Branch 6 leads to the least value (34.87) of cost function, which implies the worst-case blackout of power network. The iterative search process for the control inputs on Branch 6 and the least cost values are illustrated in Fig. 5. We can observe that the computed control inputs are 0 in the first 5 iteration steps or rounds with the maximum values of cost function. At the 6-th round, the ISA succeeds in finding the optimal control input 1.95 on Branch 6 causing the minimum cost value in the 10 rounds.

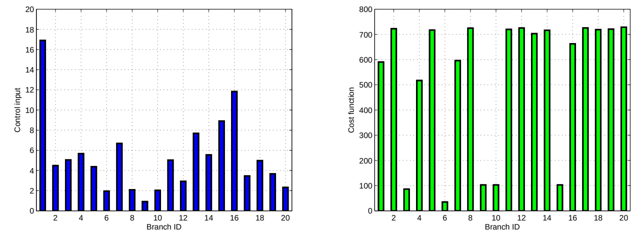

Fig. 4: The computed control inputs with the ISA and the resulted cost on each branch.

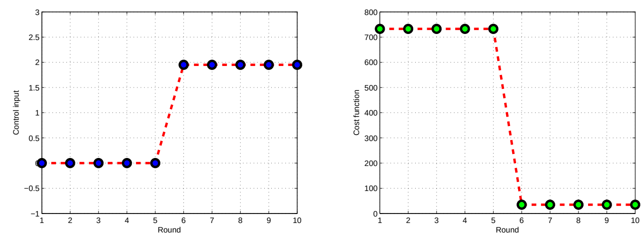

Fig. 5: Time evolution of the computed control input and the resulted cost on Branch 6 during the ISA.

B. Cascading validation

The cascading process triggered by the initial admittance change of 1.95 on Branch 6 is shown in Fig. 6. In particular, this computed disturbance does not sever Branch 6 (red link) at the 1st cascading step, but it leads to the outage of Branch 3 at the 2nd step, which initiates the chain reaction of cascading failures. Branch 6 is severed at the 3rd step, and Branch 1, Branch 2, Branch 4, Branch 5 and Branch 7 are removed from the power network by circuit breakers at the 4th step. Subsequently, Branch 8, Branch 14 and Branch 15 are cut off at the 5th step. The process ends up with 2 connected subnetworks and 8 isolated buses after 6 cascading steps. The subnetwork with one generator bus (Bus 6) and 3 load buses (Bus 5, Bus 12 and Bus 13) is still in operation, while the other one with two load buses (Bus 9 and Bus 14) stops running due to the lack of power supply.

The validation results on the IEEE 14 Bus System demonstrate that power networks can be completely destroyed by disruptive disturbances on certain branches. In the simulations, the convergence rate of the ISA depends on the initial condition of the Matlab solver while solving the system of algebraic equation (8) in each iteration.

C. Scalability

To validate the scalability of the optimal control approach, the ISA is implemented in larger power networks including the IEEE 24, 39, 57 and 118 Bus Systems [41]. The parameter setting is the same as that for the IEEE 14 Bus System. And the power flow threshold on each branch is 10% larger than the normal power flow on the corresponding branch

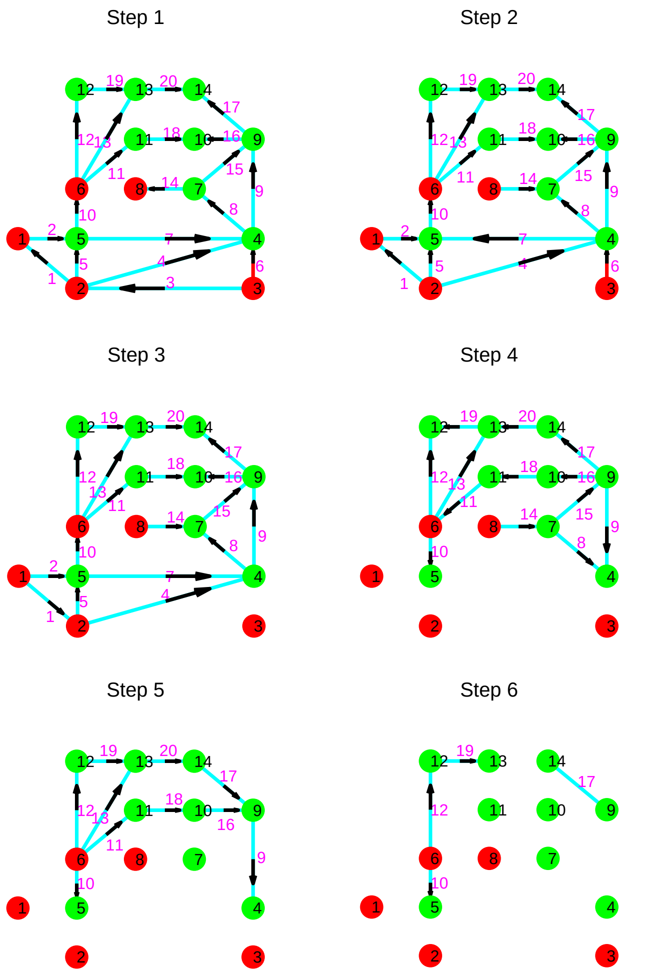

Fig. 6: Cascading process of the IEEE 14 Bus System under the computed initial disturbances on Branch 6. Red balls represent the generator buses, and green ones denote the load buses. Bus ID and Branch ID are marked with black and magenta numbers, respectively. The arrow on each branch refers to the power flow, and it disappears if there is no transmission power on the branch.

of power systems without any disturbances. In addition, the normalized index γ is introduced to quantify the disruptive level of cascading failures compared to the normal state. It is defined as $\gamma = J(Y_p^h, u)/J(Y_p^h, 0)$, where a smaller γ indicates a worse cascading failure of power system. For each bus system, it is demonstrated that the ISA can identify the worst-case cascading failures and their respective disturbances and branch IDs in terms of the index γ as summarized in Table II. Note that the identified disturbances on the IEEE 14, 24 and 57 Bus Systems (i.e. 1.95, 10.056 and 13.809) do not lead to the direct outage of their respective branches (i.e. Branch 6, Branch 7 and Branch 41) at the initial step. In contrast, the identified disturbances on the IEEE 39 and 118 Bus Systems (i.e. 71.416 and 27.290) directly sever their respective branches (i.e. Branch 35 and Branch 129).

TABLE II: The Worst-Case Cascading Failures Identified by ISA

IEEE Test Systems	Branch ID	Disturbance	γ
IEEE 14 Bus System	6	1.95	0.024
IEEE 24 Bus System	7	10.056	0.012
IEEE 39 Bus System	35	71.416	0.101
IEEE 57 Bus System	41	13.809	0.076
IEEE 118 Bus System	129	27.290	0.065

V. Conclusions

A cascading model of transmission lines was developed to describe the evolution of branches on power systems under malignant disturbances. With the cascading model and the DC power flow equation, the identification problem of worst-case cascading failures was formulated with the aid of optimal control theory by treating the disturbances as the control inputs. Moreover, the Iterative Search Algorithm was proposed to search for the worst-case cascading failures. Simulation results demonstrated the effectiveness of our approach. The main advantage of the proposed approach over typical cascading failure simulators lies in its capability of identifying the disruptive disturbances that lead to changes of branch admittance in addition to the direct branch outages. It provides a new perspective of designing the corresponding protection strategy to enhance the resilience and stability of power systems and interdependent critical infrastructure systems.

Appendix A
Proof of Lemma III.1

It follows from Definition II.1 that

$$\begin{aligned}
&\left(Y_b^k\right)^* \left(Y_b^k\right)^{-1^*} \\
&= \sum_{i=1}^q \mathscr{E}_i \left(\begin{array}{c|c} 0 & 0_{k_i-1}^T \\ 0_{k_i-1} & I_{k_i-1}\end{array}\right) Y_{b,i}^k \left(\begin{array}{c|c} 0 & 0_{k_i-1}^T \\ 0_{k_i-1} & I_{k_i-1}\end{array}\right) \\
&\quad \cdot \begin{pmatrix} 0_{k_i-1}^T \\ I_{k_i-1}\end{pmatrix} \left(\mathscr{Y}_{b,i}^k\right)^{-1} \begin{pmatrix} 0_{k_i-1} & I_{k_i-1}\end{pmatrix} \mathscr{E}_i^T \\
&= \sum_{i=1}^q \mathscr{E}_i \begin{pmatrix} 0_{k_i-1}^T \\ I_{k_i-1}\end{pmatrix} \mathscr{Y}_{b,i}^k \begin{pmatrix} 0_{k_i-1} & I_{k_i-1}\end{pmatrix} \\
&\quad \cdot \begin{pmatrix} 0_{k_i-1}^T \\ I_{k_i-1}\end{pmatrix} \left(\mathscr{Y}_{b,i}^k\right)^{-1} \begin{pmatrix} 0_{k_i-1} & I_{k_i-1}\end{pmatrix} \mathscr{E}_i^T.
\end{aligned}$$

Moreover, since

$$\mathscr{Y}_{b,i}^k \begin{pmatrix} 0_{k_i-1} & I_{k_i-1}\end{pmatrix} \begin{pmatrix} 0_{k_i-1}^T \\ I_{k_i-1}\end{pmatrix} \left(\mathscr{Y}_{b,i}^k\right)^{-1} = I_{k_i-1}$$

we obtain $\left(Y_b^k\right)^* \left(Y_b^k\right)^{-1^*} = \sum_{i=1}^q \mathscr{E}_i \; diag(0, 1_{k_i-1}^T) \mathscr{E}_i^T$. Likewise, we can prove $\left(Y_b^k\right)^{-1^*} \left(Y_b^k\right)^* = \sum_{i=1}^q \mathscr{E}_i \; diag(0, 1_{k_i-1}^T) \mathscr{E}_i^T$. The proof is thus completed.

Appendix B
Proof of Lemma III.2

It follows from the solution to the DC power flow equations $\theta^k = (Y_b^k)^{-1^*} P^k$ that

$$\begin{aligned}
P_{ij}^k = Y_b^k(i,j)(\theta_i^k - \theta_j^k) &= e_i^T Y_b^k e_j (e_i - e_j)^T \theta^k \\
&= e_i^T Y_b^k e_j (e_i - e_j)^T (Y_b^k)^{-1^*} P^k,
\end{aligned}$$

which completes the proof.

Appendix C
Proof of Lemma III.3

Lemma III.1 allows to obtain

$$(Y_b^k)^* \cdot (Y_b^k)^{-1^*} = \sum_{i=1}^{q} \mathscr{E}_i \ diag(0, 1_{k_i-1}^T)\mathscr{E}_i^T.$$

Since the derivative of the constant is 0, we have

$$\begin{aligned}\frac{\partial\left[(Y_b^k)^* \cdot (Y_b^k)^{-1^*}\right]}{\partial y_{p,i}^k} &= \frac{\partial (Y_b^k)^*}{\partial y_{p,i}^k} \cdot (Y_b^k)^{-1^*} + (Y_b^k)^* \cdot \frac{\partial (Y_b^k)^{-1^*}}{\partial y_{p,i}^k} \\ &= \frac{\partial}{\partial y_{p,i}^k} \sum_{i=1}^{q} \mathscr{E}_i \ diag(0, 1_{k_i-1}^T)\mathscr{E}_i^T \\ &= 0_{m\times m}.\end{aligned}$$

Then it follows from

$$\begin{aligned}&(Y_b^k)^{-1^*}\left[\frac{\partial (Y_b^k)^*}{\partial y_{p,i}^k} \cdot (Y_b^k)^{-1^*} + (Y_b^k)^* \cdot \frac{\partial (Y_b^k)^{-1^*}}{\partial y_{p,i}^k}\right] \\ &= (Y_b^k)^{-1^*} \cdot \frac{\partial (Y_b^k)^*}{\partial y_{p,i}^k} \cdot (Y_b^k)^{-1^*} + diag(0, 1_{n-1}^T) \cdot \frac{\partial (Y_b^k)^{-1^*}}{\partial y_{p,i}^k} \\ &= (Y_b^k)^{-1^*} \cdot \frac{\partial (Y_b^k)^*}{\partial y_{p,i}^k} \cdot (Y_b^k)^{-1^*} + \frac{\partial (Y_b^k)^{-1^*}}{\partial y_{p,i}^k} \\ &= 0_{m\times m}\end{aligned}$$

that

$$\begin{aligned}\frac{\partial (Y_b^k)^{-1^*}}{\partial y_{p,i}^k} &= -(Y_b^k)^{-1^*}\frac{\partial (Y_b^k)^*}{\partial y_{p,i}^k}(Y_b^k)^{-1^*} \\ &= -(Y_b^k)^{-1^*}(A^T diag(\frac{\partial Y_P^k}{\partial y_{p,i}^k})A)^*(Y_b^k)^{-1^*} \\ &= -(Y_b^k)^{-1^*}(A^T diag(e_i)A)^*(Y_b^k)^{-1^*}.\end{aligned}$$

The proof of this lemma is thus completed.

Appendix D
Proof of Theorem III.1

Proof: According to the optimal control of discrete-time nonlinear systems in [39], the necessary conditions for the optimal control problem (6) can be determined as

$$Y_p^{k+1} = G(P_{ij}^k, c_{ij}) \cdot Y_p^k + E_{i_k} u_k \tag{11}$$

$$\left(\frac{\partial Y_p^{k+1}}{\partial u_k}\right)^T \lambda_{k+1} + \frac{\varepsilon}{\max\{0, \iota - k\}} \cdot \frac{\partial \|u_k\|^2}{\partial u_k} = 0 \tag{12}$$

$$\lambda_k = \left(\frac{\partial Y_p^{k+1}}{\partial Y_p^k}\right)^T \lambda_{k+1} + \frac{\varepsilon}{\max\{0, \iota - k\}} \cdot \frac{\partial \|u_k\|^2}{\partial Y_p^k} \tag{13}$$

$$\frac{\partial \mathrm{T}(Y_p^h)}{\partial Y_p^h} - \lambda_h = 0_n \tag{14}$$

where $0_n = (0, 0, ..., 0)^T \in R^n$, and λ_k denotes the costate vector. Solving Equation (12) leads to

$$u_k = -E_{i_k}\frac{\lambda_{k+1}}{2\varepsilon}\max\{0, \iota - k\} \tag{15}$$

and simplifying Equation (13) yields

$$\lambda_k = \left(\frac{\partial Y_p^{k+1}}{\partial Y_p^k}\right)^T \lambda_{k+1} \tag{16}$$

with the final condition $\lambda_h = \frac{\partial \mathrm{T}(Y_p^h)}{\partial Y_p^h}$ being derived from Equation (14). Therefore we have

$$\lambda_{k+1} = \prod_{s=0}^{h-k-1} \frac{\partial Y_p^{h-s}}{\partial Y_p^{h-s-1}} \cdot \frac{\partial \mathrm{T}(Y_p^h)}{\partial Y_p^h}. \tag{17}$$

Combining Equations (15) and (17), we obtain

$$u_k = -\frac{\max\{0, \iota - k\}}{2\varepsilon} E_{i_k} \prod_{s=0}^{h-k-1} \frac{\partial Y_p^{h-s}}{\partial Y_p^{h-s-1}} \cdot \frac{\partial \mathrm{T}(Y_p^h)}{\partial Y_p^h} \tag{18}$$

Substituting (18) into (11) yields the integrated mathematical representation of the necessary conditions (11), (12), (13) and (14) for the optimal control problem (6). Next, we focus on the computation of the matrix

$$\frac{\partial Y_p^{k+1}}{\partial Y_p^k} = \left(\frac{\partial y_{p,l}^{k+1}}{\partial y_{p,s}^k}\right)_{n\times n}, \quad k = 0, 1, ..., h-1 \tag{19}$$

where $y_{p,l}^{k+1} = g(P_{i_l j_l}^k, c_{i_l j_l}) y_{p,l}^k + e_l^T E_{i_k} u_k$, $l \in I_n$. We have

$$\begin{aligned}\frac{\partial y_{p,l}^{k+1}}{\partial y_{p,s}^k} &= \frac{\partial g(P_{i_l j_l}^k, c_{i_l j_l})}{\partial y_{p,s}^k} y_{p,l}^k + g(P_{i_l j_l}^k, c_{i_l j_l})\frac{\partial y_{p,l}^k}{\partial y_{p,s}^k} \\ &= \frac{\partial g(P_{i_l j_l}^k, c_{i_l j_l})}{\partial P_{i_l j_l}^k} \cdot \frac{\partial P_{i_l j_l}^k}{\partial y_{p,s}^k} y_{p,l}^k + g(P_{i_l j_l}^k, c_{i_l j_l})\frac{\partial y_{p,l}^k}{\partial y_{p,s}^k},\end{aligned} \tag{20}$$

for $s, l \in I_n$, where

$$\frac{\partial y_{p,l}^k}{\partial y_{p,s}^k} = \begin{cases} 1, & s = l, \\ 0, & s \neq l. \end{cases} \tag{21}$$

and

$$\frac{\partial g(P_{i_l j_l}^k, c_{i_l j_l})}{\partial P_{i_l j_l}^k} = -P_{i_l j_l}^k \sigma \cos\sigma((P_{i_l j_l}^k)^2 - c_{i_l j_l}^2) \tag{22}$$

when $c_{i_l j_l}^2 - \frac{\pi}{2\sigma} < (P_{i_l j_l}^k)^2 < c_{i_l j_l}^2 + \frac{\pi}{2\sigma}$. It follows from Lemma III.2 and Lemma III.3 that

$$\begin{aligned}&\frac{\partial P_{i_l j_l}^k}{\partial y_{p,s}^k} \\ &= \frac{\partial\left[e_{i_l}^T Y_b^k e_{j_l}\right]}{\partial y_{p,s}^k}(e_{i_l} - e_{j_l})^T (Y_b^k)^{-1^*} P^k \\ &+ e_{i_l}^T Y_b^k e_{j_l}\frac{\partial\left[(e_{i_l} - e_{j_l})^T (Y_b^k)^{-1^*} P^k\right]}{\partial y_{p,s}^k} \\ &= e_{i_l}^T A^T diag\left(\frac{\partial Y_P^k}{\partial y_{p,s}^k}\right) A e_{j_l}(e_{i_l} - e_{j_l})^T (Y_b^k)^{-1^*} P^k \\ &+ e_{i_l}^T Y_b^k e_{j_l}(e_{i_l} - e_{j_l})^T \frac{\partial (Y_b^k)^{-1^*}}{\partial y_{p,s}^k} P^k \\ &= e_{i_l}^T A^T diag(e_s) A e_{j_l}(e_{i_l} - e_{j_l})^T (Y_b^k)^{-1^*} P^k \\ &- e_{i_l}^T Y_b^k e_{j_l}(e_{i_l} - e_{j_l})^T (Y_b^k)^{-1^*} \\ &\cdot (A^T diag(e_s)A)^*(Y_b^k)^{-1^*} P^k.\end{aligned} \tag{23}$$

Thus, each element in Matrix (19) is explicitly expressed by Equation (20), which can be obtained by taking into account Equations (1), (21), (22) and (23). This completes the proof of Theorem III.1. ■

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