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A Closed-Form Solution of Discrete Time Optimal Control and Its Convergence

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Abstract: The convergence of a new closed-form solution for the discrete time optimal control is presented. First, a new time optimal control law with simple structure is constructed in the form of the state feedback for a discrete-time double-integral system by using the state backstepping approach. The control signal sequence in this approach is determined by the linearized criterion according to the position of the initial state point on the phase plane. This closed-form non-linear state feedback control law clearly shows that time optimal control in discrete time is not necessarily the bang-bang control. Second, the convergence of the time optimal control law is proved by demonstrating the convergence path of the state point sequence driven by the corresponding control signal sequence. Finally, numerical simulation results demonstrate the effectiveness of this new discrete time optimal control law.

1 Introduction

Time optimal control (TOC) originated from servo control design problems in the 1950s [1][2][3], and has drawn significant interests in different fields [4][5][6][7]. In particular, the time optimal control for the double-integral system has received considerable attention [8][9]. For the double-integral system defined as

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, |u| \leq r \end{cases} \tag{1}$$

where $x(t) = [x_1(t), x_2(t)]^T \in R^2$, the resulting feedback control law that drives the state from any initial point to the origin in the shortest time is [10][11]

$$u(x_1, x_2, r) = -r\text{sign}(x_1 + \frac{x_2|x_2|}{2r}) \tag{2}$$

and the switching curve function is

$$\Gamma(x_1, x_2) = x_1 + \frac{x_2|x_2|}{2r}. \tag{3}$$

There are many advantages of this time optimal control law over linear controllers [12][13]: 1) any initial state can be driven back to the steady state in the shortest and finite time; 2) it is immunized from disturbance and has maximum accuracy in command following and minimal disturbance recovery time.

However, from an engineering perspective, the applications of time optimal control (TOC) prove to be challenging [14][15]. A nagging issue is the chattering problem in the control signal, that is, the instant switching between extreme values in the control signals required by TOC. It is often neither feasible, because of the physical limits on how fast a control signal can change, nor desirable, because of the stress it puts on the control actuators [16][17]. A number of modifications of the control law in (2) have been proposed to ease the implementation [18][19][20], including adding a linear zone to reduce the chattering of control signals around the origin, replacing the sign function in (2) with the saturation function and so forth. These modifications, however, can only make the solution to be suboptimal because the chattering problem still exists.

Further, with the great developments of computer control technology, most control algorithms are implemented in discrete time domain today. Direct digitization of continuous TOC solution proves to be problematic in practice because of the high frequency chattering of the control signals [21][22]. It was demonstrated that TOC for discrete systems is not a bang-bang control in [23][24], but digitized bang-bang control has been used as an approximation for the discrete time optimal control (DTOC) problems. In [25][26][27], minimum-time feedback control laws for different applications are proposed and algorithms are given to obtain facial descriptions of admissible set. In [23][24], a closed-form solution of DTOC for a discrete-time double-integral system is attained by comparing the position of the initial state with isochronic region obtained by non-linear boundary transformation. However, the structure of the synthetic control function is complex with non-linear calculations including square roots calculation, which makes the synthetic control function difficult to be applied into engineering applications. Further, the convergence of DTOC law was never attained. Facing with the problems above, we present a new DTOC law with simple structure and prove its convergence. The main contributions in this paper are as follows

i: The mathematical derivation of a new closed-form discrete optimal control law for the discrete form of system (1) is presented. This control law avoids the overshoot and the high frequency chattering of the control signals that exist in digitized bang-bang control. Compared with the synthetic control function proposed in [23][24], the discrete optimal control law is determined by the linearized criterion according to the position of the initial state point on the phase plane, which equips the control law with a simpler structure, making it much easier to be applied in the practical engineering.

ii: The convergence of the discrete time optimal control law is proved by showing the convergence path of the state point sequence driven by the derived control law.

The paper is organized as follows: the discrete time optimal control (DTOC) law is constructed in Section 2. In Section 3, the convergence of DTOC law is proved. Comparison experiments are conducted to illustrate the effectiveness of the proposed DTOC law in Section 4, followed by concluding remarks in Section 5.

2 Discrete Time Optimal Control Law

Consider a discrete time double-integral system

$$x(k+1) = Ax(k) + Bu(k), |u(k)| \le r \tag{4}$$

where $A = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ h \end{bmatrix}$ and $x(k) = [x_1(k), x_2(k)]^T$. The objective here is to derive a time optimal control law directly in discrete time domain. The problem is defined as follows:

DTOC Law: Given the system (4) and its initial state $x(0)$, determines the control signal sequence, $u(0), u(1), ..., u(k)$, such that the state $x(k)$ is driven back to the origin in a minimum and finite number of steps, subject to the constraint of $|u(k)| \le r$. That is, finding $u(k^*)$, $|u(k)| \le r$, such that $k^* = \min\{k|x(k+1) = 0\}$.

For driving the initial state back to the origin in the continuous system in (1), the control signal switches between its two extreme values around the switching curve $\Gamma(x_1, x_2)$ in (3). That is, when the initial state is located over the switching curve, the control signal takes on the extreme values, i.e., $u = -r$; otherwise the control signal takes on $u = +r$. The control signal switches the sign after reaching the switching curve. For a continuous system, the control signal can switch instantaneously. For a discrete time system, however, the process of sign switching of the control signal will take place within the sampling period h. During the process, corresponding state sequences would locate in a certain region (denoted as Ω) near the switching curve. The control signals for the state sequences in the region Ω are determined by the linearized criterion. The control signal varies from a certain positive (negative) value to a negative (positive) value when the control signal u passes from one side of the region Ω to the other. All initial state sequences located outside the region Ω when the control signal takes on extreme value, i.e., $u = +r$ or $u = -r$, would locate at certain curves, referred as boundary curves Γ_A and Γ_B. The region Ω is surrounded by these boundary curves.

The basic idea in deriving the DTOC law is to find the control signal sequence for any initial state point $x(0) \in \Omega$ or $x(0) \notin \Omega$. The whole task is divided into two parts:

i: Determine the boundary curves of the region Ω based on state backstepping approach, i.e., the representation of the initial condition $x(0) = [x_1(0), x_2(0)]^T$ in term of h and r, from which the state can be driven back to the origin in $(k+1)$ steps.

ii: For any given initial condition $x(0) \in \Omega$ or $x(0) \notin \Omega$, find the corresponding control signal sequence as a function of $x(0)$.

2.1 *Determination of Boundary Curves*

For any initial state sequence, there is at least one admissible control sequence, $u(0), u(1), ..., u(k)$ that can make the solution of (4) satisfy $x(k+1) = 0$. The solution of (4) with the initial condition $x(0)$ is

$$x(k+1) = A^{k+1}x(0) + \sum_{i=0}^{k} A^{k-i}Bu(i) \tag{5}$$

where $x(0) = [x_1(0), x_2(0)]^T$ and $i = 0, 1, 2, ..., k$. It manifests that $x(k+1) = 0$. Therefore, the initial condition satisfies

$$x(0) = \sum_{i=0}^{k} \begin{bmatrix} (i+1)h^2 \\ -h \end{bmatrix} u(i). \tag{6}$$

Based on the state backstepping approach above, we determine the two boundary curves denoted as Γ_A and Γ_B in the followings.

We first determine the boundary curve Γ_A. Suppose that $\{a_{+k}\}$, $\{a_{-k}\}$ are the sets of any $x(0)$ that can be driven back to the origin with the control signal sequence $u(i) = +r$ and $u(i) = -r$, $i = 0, 1, 2, ..., k$, respectively. Denote that all initial states in the set $\{a_{+k}\}$ consist of Γ_A^+ and all initial states in the set $\{a_{-k}\}$ consist of Γ_A^-.

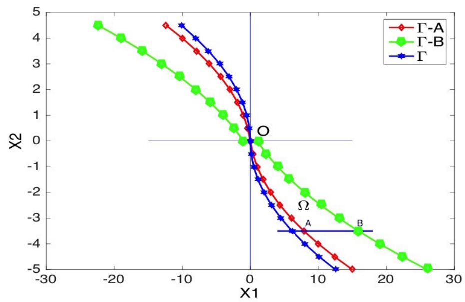

Fig. 1: Illustration of the region Ω, the two boundary curves $\mathit{\Gamma}_A$, $\mathit{\Gamma}_B$, the switching curve Γ and two intersection points A, B that intersected by an auxiliary line and $\mathit{\Gamma}_A$, $\mathit{\Gamma}_B$.

For the set $\{a_{+k}\}$, the following result can be obtained when the control signal sequence takes on $u(i) = +r$ according to (6)

$$x(0) = r\sum_{i=0}^{k} \begin{bmatrix} (i+1)h^2 \\ -h \end{bmatrix}. \tag{7}$$

Then there exist $x_1(0) = rh^2(\frac{k^2}{2} + \frac{3k}{2} + 1)$ and $x_2(0) = -rh(k+1) < 0$. Simplifying $x(0)$ into x and eliminating the variable k, the boundary curve Γ_A^+ is $x_1 = \frac{x_2^2}{2r} - \frac{1}{2}hx_2$, where $x_2 < 0$. Similarly, we can get the boundary curve Γ_A^-: $x_1 = -\frac{x_2^2}{2r} - \frac{1}{2}hx_2$, where $x_2 > 0$. Therefore, the whole boundary curve Γ_A is

$$\Gamma_A : x_1 + \frac{x_2|x_2|}{2r} + \frac{1}{2}hx_2 = 0. \tag{8}$$

We then determine the boundary curve Γ_B. Suppose that $\{b_{+k}\}$, $\{b_{-k}\}(k > 2)$ are the sets of any initial state $x(0)$ that can be driven back to the origin when the control signal takes on $u(0) = -r$ or $u(0) = +r$ in the first step, and from the second step on, the control sequence takes on $u(i) = +r$ or $u(i) = -r, i = 1, 2, ..., k$. Similarly, the boundary curve Γ_B consists of Γ_B^+ and Γ_B^-.

For the set $\{b_{+k}\}$, according to the rule of choosing the control signal sequence above, we can obtain $x_1 = rh^2(\frac{k^2}{2} + \frac{3k}{2} - 1)$ and $x_2 = -rh(k-1) < 0$. After eliminating the variable k, the boundary curve Γ_B^+ is $x_1 = \frac{x_2^2}{2r} - \frac{5}{2}hx_2 + h^2r$ and there exists $x_1 + hx_2 = \frac{1}{2}rh^2k(k+1) > 0$. Similarly, we can obtain the boundary curve Γ_B^-: $x_1 = -\frac{x_2^2}{2r} - \frac{5}{2}hx_2 - h^2r, x_1 + hx_2 < 0$. Therefore, the whole boundary curve Γ_B is

$$\Gamma_B : x_1 - \tilde{s}\frac{x_2^2}{2r} + \frac{5}{2}hx_2 - \tilde{s}h^2r = 0 \tag{9}$$

where $\tilde{s} = \text{sign}(x_1 + hx_2)$. The above two boundary curves of the region Ω are determined with the state backstepping method, and they are shown on the phase plane in Figure 1.

Remark 1. *The whole phase plane is divided into two regions, the region Ω surrounded by the two boundary curves Γ_A and Γ_B, and the rest part $R^2 - \Omega$.*

2.2 *Construction of the Discrete TOC Law*

In this subsection, the DTOC law is obtained constructively based on the boundary curves and regions proposed above. As shown in Figure 1, we assume that for any initial state $M(x_1, x_2)$ in the fourth quadrant $(x_1 > 0, x_2 < 0)$, there is an auxiliary line $x_2 = x_2(M)$ that intersects with the boundary curves at points A and B (in the

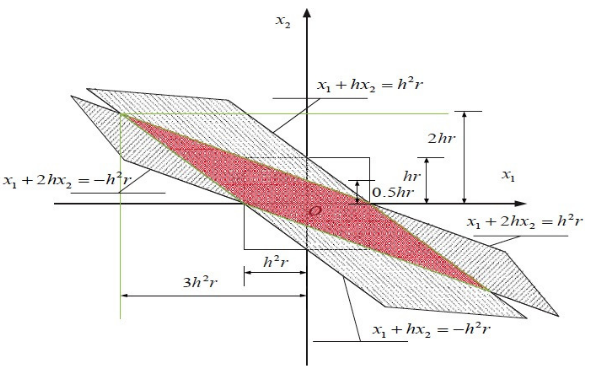

Fig. 2: Illustration of the region Ω_2.

direction of x_1). Their x-axis value x_A and x_B are

$$\begin{cases} x_A = \frac{x_2^2}{2r} + \frac{1}{2}h|x_2| \\ x_B = \frac{x_2^2}{2r} + \frac{5}{2}h|x_2| + h^2 r \end{cases} \tag{10}$$

For any initial state $M(x_1, x_2)$ satisfying $x_1 < x_A$ or $x_1 > x_B$, the control signal is taken on $u = +r$ or $u = -r$. For any initial state $M(x_1, x_2)$ satisfying $x_1 \in [x_A, x_B]$, the control signal can be taken on as follows:

$$u = -r\epsilon \text{sign}(\text{x}_2) \tag{11}$$

where $\epsilon = \frac{x_B + x_A - 2|x_1|}{x_B - x_A}$.

When the initial state $M(x_1, x_2)$ is in the second quadrant, the control signal sequence can be constructed similarly.

However, when the initial state $M(x_1, x_2)$ (located outside the region Ω) is in the first or third quadrant, there are two different cases for choosing the control signal. When $M(x_1, x_2)$ cannot be driven back to the origin within two steps, that is, the initial state does not satisfy the condition $x_1^2 + x_2^2 = 0$, let $u = -r\text{sign}(x_1 + hx_2)$. When $M(x_1, x_2)$ can be driven back to the origin within two steps, the initial state $x(0)$ and the corresponding control signal sequence satisfy (6), i.e.,

$$\begin{cases} x_1(1) = x_1(0) + hx_2(0) \\ x_2(1) = x_2(0) + hu(0) \\ x_1(2) = x_1(1) + hx_2(1) \\ x_2(2) = x_2(1) + hu(1). \end{cases}$$

Furthermore, when $M(x_1, x_2)$ can be driven back to the origin within two steps, the corresponding control signals can be derived as follows:

$$\begin{cases} u(0) = -\frac{x_1(0) + 2hx_2(0)}{h^2} \\ u(1) = \frac{x_1(0) + hx_2(0)}{h^2}. \end{cases} \tag{12}$$

The condition $u(1) \leq r$ is a necessary condition for driving the initial state back to the origin within two steps. If it is not satisfied, the initial state cannot be driven back to the origin within two steps. If it is satisfied, the control signal can take on $u(0)$ and $u(1)$ in (11) to drive the initial state back to the origin.

The region in which any $x(0)$ can be driven back to the origin within two steps, denotes as Ω_2, is surrounded by two pairs of parallel lines $x_1 + hx_2 = \pm h^2 r$ and $x_1 + 2hx_2 = \pm h^2 r$. As shown in Figure 2, Ω_2 is a parallelogram defined by the four points of $(-h^2 r, 0)$, $(-3h^2 r, 2hr)$, $(h^2 r, 0)$ and $(3h^2 r, -2hr)$.

Now, any initial state $M(x_1, x_2)$ on the $x_1 - x_2$ plane can be driven back to the origin in a minimum and finite number of steps according to the control signal sequence above. The complete DTOC law is described as follows:

Step 1: Setting $z_1 = x_1 + \lambda hx_2, z_2 = z_1 + hx_2$, where $\lambda \in (0, 1]$ is a tuning parameter to determine the different characteristic points x_A, x_B. If $|z_1| > h^2 r$ or $|z_2| > h^2 r$, then $M(x_1, x_2)$ cannot be driven back to the origin within two steps, i.e., $M(x_1, x_2) \notin \Omega_2$, and go to next step; otherwise, go to step 5;

Step 2: If the initial state $M(x_1, x_2)$ satisfies $x_1 x_2 \geq 0$ and $M(x_1, x_2) \notin \Omega_2 \cup \Omega$, then the control signal takes on $u = -r\text{sign}(x_1 + hx_2)$;

Step 3: Determine the boundary of the region Ω, i.e., $x_A = \frac{x_2^2}{2r} + \frac{1}{2}h|x_2|$ and $x_B = \frac{x_2^2}{2r} + \frac{5}{2}h|x_2| + h^2 r$;

Step 4: If $|x_1| \geq x_B$, then the control signal takes on $u = -r\text{sign}(x_1)$; if $|x_1| \leq x_A$, then the control signal takes on $u = r\text{sign}(x_1)$; otherwise, the control signal takes on $u = -r\epsilon\text{sign}(x_2)$ where $\epsilon = \frac{x_B + x_A - 2|x_1|}{x_B - x_A}$;

Step 5: If the initial state $M(x_1, x_2) \in \Omega_2$, then the control signal takes on $u = -\frac{x_1 + 2hx_2}{h^2}$;

Step 6: The algorithm ends.

From the deduction above, the mathematical derivation of a closed-form discrete time optimal control law (**DTOC Law**) as a function of x_1, x_2, r and h, denoted as $u(k) = Ftd(x_1(k), x_2(k), r, h)$, is obtained.

Remark 2. *The DTOC law presented above demonstrates that the control signal u in discrete time does not always take on the extreme value, which is different from its continuous time counterpart. The characteristics of no chattering problem in the control signal makes the new control law advantageous in engineering applications.*

Remark 3. *The essence of DTOC law proposed in [23] [24] is a non-linear boundary transformation, which includes complex non-linear calculations. The control signal of the proposed DTOC law is determined by piecewise linear function according to the relative positions of the initial state and the corresponding x-axis value of the intersection points A and B, which allows the DTOC law to have a simple structure.*

Remark 4. *For the **Step 1** of the algorithm in Subsection 2.2, the different λ can lead to different points x_A, x_B. Different points x_A, x_B will result in different control signal according to (11) when the initial state $M(x_1, x_2)$ is located inside the region Ω. However, the whole algorithm does not need to change.*

3 Convergence proof of time optimal control law

The main result in this section is stated as the following Theorem.

Theorem 1. *Given the system (4) and any initial state $x(0)$, the state $x(k)$ can converge to the origin in the minimum and finite number of steps with the discrete time optimal control signal sequence $u(0)$, $u(1)$, ..., $u(k)$ determined by $u(k) = Ftd(x_1(k), x_2(k), r, h)$, subject to the constraint of $|u(k)| \leq r$. That is, $x_1(k) \to 0$ and $x_2(k) \to 0$ with $u(k^*)$, where $|u(k)| \leq r$ and $k^* = \min[k|x(k+1) = 0]$.*

Proof: Here we will split the proof into two steps according to the position of the initial state on the phase plane.

Step A: Any initial state $x(0)$ located outside the region Ω can converge to the region Ω when the control signal sequence takes on the extreme value.

Consider the following control system when the initial state is located outside the region Ω:

$$\begin{cases} x_1(k+1) = x_1(k) + hx_2(k) \\ x_2(k+1) = x_2(k) + hu(k) \\ u(k) = -r\text{sign}(s(k)) \end{cases} \tag{13}$$

where $s(k)$ is the boundary curve $\Gamma_A : x_1 + \frac{x_2|x_2|}{2r} + \frac{1}{2}hx_2 = 0$.

When the initial state $M(x_1, x_2)$ is located above the Γ_A, there exists $s(k) > 0$. The following Lyapunov function is constructed

$$
\begin{aligned}
\triangle s(k) &= s(k+1) - s(k) \\
&= hx_2(k) - \frac{1}{2}h^2 r - h(x_2(k) - \frac{1}{2}hr)\text{sign}(x_2(k)) \\
&= h(x_2(k) - |x_2(k)|) - \frac{1}{2}h^2 r(1 - \text{sign}(x_2(k))) \leq 0
\end{aligned} \tag{14}
$$

There are two cases for the value of $\triangle s(k)$.

(1) If $x_2(k) > 0$, then $\triangle s(k) = 0$.

According to (13), the initial state $x_2(k)$ will keep decreasing until it arrives at Γ_A. There exists $x_2(k+1) = x_2(k) - hr$, that is, $x_2(k) = x_2(0) - khr$. Hence there exists a positive constant $k_0 = \frac{x_2(0)}{hr}$ that can make $x_2(k) < 0$ when $k > k_0$. Therefore, any initial state located above Γ_A on upper phase plane can be driven to lower phase plane, i.e., $x_2(k) < 0$.

(2) If $x_2(k) < 0$, then $\triangle s(k) = -2h|x_2(k)| - h^2 r < -h^2 r < 0$.

When $s(k) > 0$, there exists $s(k+1) - s(k) < -h^2 r$, that is, $s(k) < s(0) - kh^2 r$ and clearly there is a positive constant $k_1 = \frac{s(0)}{h^2 r}$ that can guarantee $s(k) < 0$ when $k > k_1$.

The above statement manifests that the initial state $x(0)$ located outside the region Ω can approach the boundary curve Γ_A and converge to the region Ω when the control signal sequence takes on the extreme value described in (13). Similar conclusion can be obtained when the initial state $M(x_1, x_2)$ is located below Γ_A.

Step B: The state sequence located inside the region Ω can be driven into the region Ω_2 with the control signal in (11) in a limited number of steps.

In this step, we only need to prove the convergence of the state sequence on the fourth quadrant because the region Ω is symmetric in the second and fourth quadrant. For simplicity, we may still adopt parametric expressions by introducing the parameters α and β ($\alpha > 0$, $\beta > 0$), where any state point on the phase plane can be expressed as $x_1 = \alpha h^2 r, x_2 = -\beta hr$. Further, for convenience, we may as well adopt dimensionless expression by introducing another parameter θ to ignore the value of h and r.

Based on the above, the boundary curves Γ_A and Γ_B on the fourth quadrant in parametric form are

$$
\begin{cases}
\Gamma_A^+ : \alpha = \frac{\beta}{2}(\beta+1) \\
\Gamma_B^+ : \alpha = \frac{\beta}{2}(\beta+1) + (2\beta+1).
\end{cases} \tag{15}
$$

Any initial state point $M(x_1, x_2)$ located inside the region Ω in dimensionless form can be

$$
\begin{cases}
\alpha = \frac{\beta}{2}(\beta+1) + (2\beta+1)\theta \\
\beta = \beta
\end{cases} \tag{16}
$$

where $0 \leq \theta \leq 1$. Its corresponding control signal sequence according to **step 4** in Section 2.2 is

$$
\begin{aligned}
u &= -r\frac{x_B + x_A - 2|x_1|}{x_B - x_A}\text{sign}(x_2) \\
&= r(1 - 2\theta)
\end{aligned} \tag{17}
$$

where $x_A = \frac{\beta(\beta+1)}{2}h^2 r$ and $x_B = [\frac{\beta(\beta+5)}{2} + 1]h^2 r$.

The state variables at steps k and $(k+1)$ in dimensionless form can be

$$
\begin{cases}
x_1(k) = \alpha(k)h^2 r \\
x_2(k) = -\beta(k)hr \\
x_1(k+1) = \alpha(k+1)h^2 r \\
x_2(k+1) = -\beta(k+1)hr
\end{cases}
$$

Because of (4), there exists

$$
\begin{cases}
\alpha(k+1) = \alpha(k) - \beta(k) \\
\beta(k+1) = \beta(k) - 1 + 2\theta(k)
\end{cases}
$$

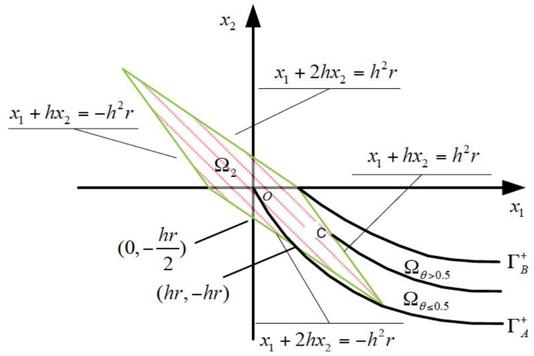

Fig. 3: Illustration of the region Ω_2, $\Omega_{\theta>0.5}$ and $\Omega_{\theta\leq0.5}$.

According to (16), we can derive $\alpha(k+1) = \frac{\beta(k+1)[\beta(k+1)+1]}{2} + [2\beta(k+1)+1]\theta(k+1)$. Therefore, the expressions of $\alpha(k+1)$, $\beta(k+1)$ and $\theta(k+1)$ can be derived as follows:

$$
\begin{cases}
\alpha(k+1) = \frac{\beta(k+1)[\beta(k+1)+1]}{2} + [2\beta(k+1)+1]\theta(k+1) \\
\beta(k+1) = \beta(k) + 2\theta(k) - 1 \\
\theta(k+1) = \frac{2\theta(k)[1-\theta(k)]}{2\beta(k)-1+4\theta(k)}.
\end{cases} \tag{18}
$$

According to (17), there exists a curve (denoted as Γ_C) between the boundary curves Γ_A and Γ_B. It can be determined by choosing $\theta = 0.5$ as follows.

Suppose that $\{c_{+k}\}$ and $\{c_{-k}\}$ $(k > 2)$ are the sets of any initial state $x(0)$ that can be driven back to the origin when the control signal takes on $u(0) = 0$ beginning in the first step, then the control sequence takes on $u(i) = +r$ or $u(i) = -r, i = 1, 2, ..., k$. Similarly, the boundary curve Γ_C consists of Γ_C^+ and Γ_C^-.

For set $\{c_{+k}\}$, there exists $x_1 = \frac{1}{2}rh^2(k^2 + 3k + 2)$ and $x_2 = -rhk < 0$ according to the rule for choosing the control signal sequence, as shown above. By eliminating the variable k, we have that the control characteristic curve Γ_C^+ is $x_1 = \frac{x_2^2}{2r} - \frac{3}{2}hx_2 + \frac{1}{2}h^2 r$. Similarly we can obtain the control characteristic curve Γ_C^-: $x_1 = -\frac{x_2^2}{2r} - \frac{3}{2}hx_2 + \frac{1}{2}h^2 r$. Therefore, the entire control characteristic curve Γ_C is $x_1 + \frac{x_2|x_2|}{2r} + \frac{3}{2}hx_2 - \frac{1}{2}h^2 r = 0$.

We assume that the curve Γ_C and the region Ω_2 intersect at point C, which can be determined by their simultaneous equations:

$$
\begin{cases}
x_1 - \frac{x_2^2}{2r} + \frac{3}{2}hx_2 - \frac{1}{2}h^2 r = 0 \\
x_1 + hx_2 = h^2 r
\end{cases} \tag{19}
$$

Solving the above equation, we can get the intersection point $(\frac{\sqrt{5}+1}{2}h^2 r, -\frac{\sqrt{5}-1}{2}hr)$.

The region Ω is divided into two regions (denoted as $\Omega_{\theta>0.5}$ and $\Omega_{\theta\leq0.5}$) by the curve Γ_C. The definitions of these two regions are given as follows and shown in Figure 3.

Definition 3.1. $\Omega_{\theta>0.5} = \{M(\alpha, \beta) | M \in \Omega, M \notin \Omega_2, \theta > 0.5\}$.

Definition 3.2. $\Omega_{\theta\leq0.5} = \{M(\alpha, \beta) | M \in \Omega, M \notin \Omega_2, \theta \leq 0.5\}$.

From the definitions above, the region $\Omega - \Omega_2$ is divided into two different regions. What we should prove is that any initial state $M(\alpha, \beta) \in \Omega - \Omega_2$ can be driven into the region Ω_2 with the control signal sequence described in (17). The whole proof can be split into two steps.

Step 1: When the initial state satisfies $M(x_1, x_2) \in \Omega_{\theta>0.5}$ as shown in Figure 4, there exists $\beta(0) \geq 0, \theta > 0.5$.

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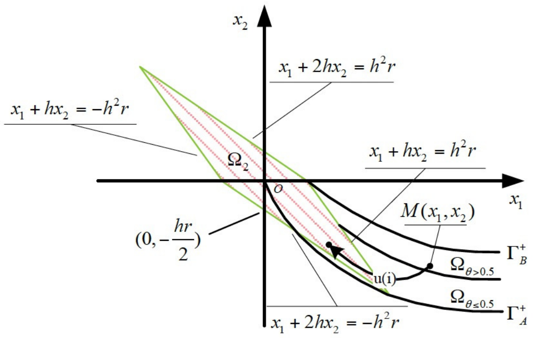

Fig. 4: Convergence path for any initial state M located in the region $\Omega_{\theta>0.5}$.

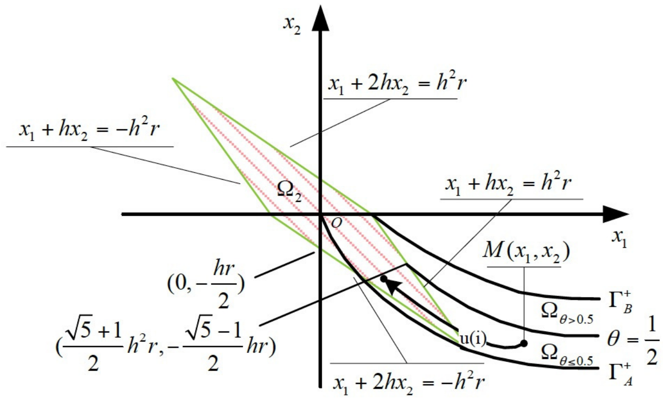

Fig. 5: Convergence path for any initial state M located in the region $\Omega_{\theta\leq 0.5}$.

Suppose that $\theta(0) = 0.5 + \delta$ where $0 < \delta < 0.5$. According to (18) we can derive

$$\begin{aligned} 0 \leq \theta(1) &= \frac{2\theta(0)[1-\theta(0)]}{2\beta(0)+4\theta(0)-1} \\ &= \frac{\frac{1}{2}}{2\beta(0)+4\delta+1} \leq \frac{1}{2}\cdot\frac{1}{2\beta(0)+1} \leq \frac{1}{2}. \end{aligned} \tag{20}$$

However, this contradicts with the condition that the initial state satisfies $\theta > 0.5$. Therefore we can derive the conclusion that any initial state can be driven into the region Ω_2 or the region $\Omega_{\theta\leq 0.5}$ with the control signal in one step, and the control signal is $u(1) = r(1-2\theta(1)) \geq 0.5r$ according to (16).

Step 2: When the initial state satisfies $M(x_1, x_2) \in \Omega_{\theta\leq 0.5}$ shown in Figure 5, there exists $x_2(k) = -\beta(k)hr < -\frac{\sqrt{5}-1}{2}hr$, i.e., $\beta(0) > \frac{\sqrt{5}-1}{2}$.

If the initial state cannot be driven out of the region $\Omega_{\theta\leq 0.5}$ with the corresponding control signal in one step, according to (18), there exists

$$\begin{aligned} \theta(0) &= \frac{2\theta(0)[1-\theta(0)]}{2\beta(0)+4\theta(0)-1} \\ &= \frac{2\theta(0)[1-\theta(0)]}{2\beta(1)+1} < \frac{1}{2}\cdot\frac{1}{2\beta(1)+1} < \frac{1}{2\sqrt{5}}. \end{aligned} \tag{21}$$

From (16) and (20), $u(1) = r(1-2\theta(1)) > (1-\frac{1}{\sqrt{5}})r$ can be obtained.

According to **Step 1** and **Step 2** above, where the initial state is in the region $\Omega_{\theta>0.5}$ or $\Omega_{\theta\leq 0.5}$, the control signal satisfies $u > \frac{1}{2}r$.

From (5), we can obtain that

$$x_2(k+1) = x_2(0) + h\sum_{i=0}^{k} u(i) > x_2(0) + (1-\frac{1}{\sqrt{5}})(k-2)hr - \frac{1}{2}hr. \tag{22}$$

Further, we have

$$\beta(k+1) < \beta(0) - (1-\frac{1}{\sqrt{5}})(k-2) - \frac{1}{2}. \tag{23}$$

From (23), we know $\beta(k+1)$ is a monotonic function. However, for any initial state located in the region $\Omega_{\theta\leq 0.5}$, the condition $\beta > 0$ holds. Therefore, the state must be driven out of the region $\Omega_{\theta\leq 0.5}$ with the corresponding control signal in a limited number of steps.

Suppose that the state $M(\alpha(k), \beta(k))$ located in the region $\Omega_{\theta\leq 0.5}$ becomes $M(\alpha(k+1), \beta(k+1))$ located outside the region $\Omega_{\theta\leq 0.5}$ by adding the control signal in one step, we need to prove that the state $M(\alpha(k+1), \beta(k+1))$ is in the region Ω.

From (18), we can obtain

$$\begin{cases} \alpha(k+1) = \frac{\beta^2(k)}{2} - \frac{\beta(k)}{2} + 2\theta(k)\beta(k) + \theta(k) \\ \beta(k+1) = \beta(k) + 2\theta(k) - 1. \end{cases} \tag{24}$$

Further, since $\beta(k) > \frac{\sqrt{5}-1}{2}$ and $0 \leq \theta \leq 1$, the following equations can be obtained

$$\begin{cases} -2\theta^2(k) - \beta(k) + \frac{1}{2} < 0 \\ -2\theta^2(k) + 2\theta(k) \geq 0. \end{cases} \tag{25}$$

According to (25), the state $M(\alpha(k+1), \beta(k+1))$ satisfies the condition that $\beta(k+1)$ is in the region surrounded by Γ_A^+ and Γ_C; meanwhile, the state is not in the region $\Omega_{\theta\leq 0.5}$. Thus it can be concluded that the state must locate in the region Ω_2.

However, we prove that the state sequence located inside of the region Ω can be driven into the region Ω_2 with the corresponding control signal sequence in a minimum and finite number of steps. Once the initial state enters the region Ω_2, it can be driven back to the origin according to (12). Combining **Step A** and **Step B**, we prove the convergence of the time optimal control law described in **Theorem 1**.

4 Numerical Simulations

We may as well denote the DTOC law derived in this paper as $u(k) = Ftd(x_1(k), x_2(k), r, h)$, where r is the quickness factor and h is the sampling period. In this section, numerical simulations will be conducted to compare the performance of the proposed DTOC law with the law (denoted by $Fhan$) proposed by Han [23][24].

Firstly, we illustrate the state trajectory under two different algorithms, $Fhan$ and Ftd. From Figure 6, we can see that the state

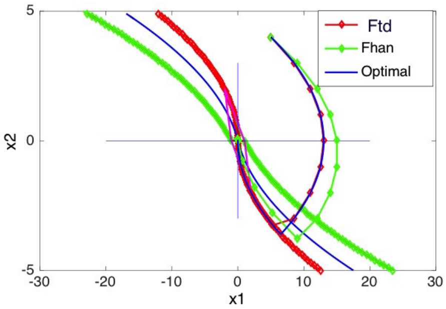

Fig. 6: Illustration of state trajectory under two different algorithms.

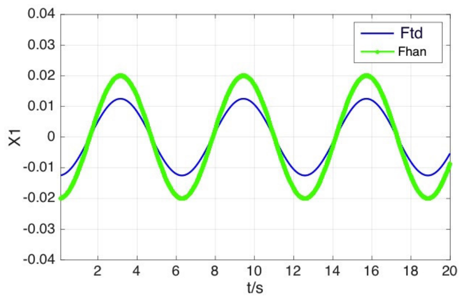

Fig. 7: Comparison of tracking errors between $Fhan$ and Ftd algorithms.

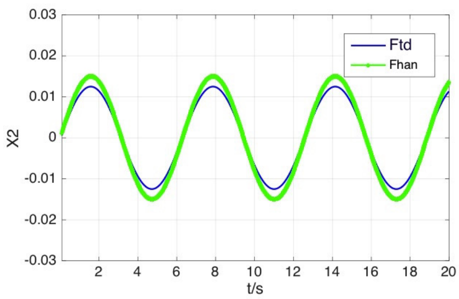

Fig. 8: Differentiation of signal errors compared between $Fhan$ and Ftd algorithms.

trajectory for Ftd is in accordance with the optimal trajectory while the state trajectory for $Fast$ lags the optimal trajectory.

Based on the discrete time optimal feedback control, i.e., $u(k) = Ftd(x_1(k), x_2(k), r, h)$, we can easily construct the following time optimal system-based tracking differentiator by replacing x_1 in (4) with $x_1 - v$.

$$\begin{cases} u(k) = Ftd(x_1(k) - v(k), x_2(k), r, c_0 h) \\ x_1(k+1) = x_1(k) + hx_2(k) \\ x_2(k+1) = x_2(k) + hu(k), k = 0, 1, 2, ... \end{cases} \tag{26}$$

where r is the quickness factor, c_0 is the filtering factor, and h is the sampling period. We then demonstrate the errors of tracking and differentiation of signals between the algorithm $Fhan$ and Ftd by using the input signal sequence $V(t) = \sin(2\pi t)$. In the simulations, the parameters, such as, the sampling step $h = 0.01$, the quickness factor $r_0 = 100$, and the filtering factor $c_0 = 1.5$ are chosen by means of a trial-and-error approach to make the algorithm $Fhan$ track input signal quickly and derive excellent derivative signal as much as possible. We use the same parameters for the proposed algorithm Ftd.

The simulation results are plotted in Figures 6 and 7. The algorithm Ftd quickly track an input signal without overshooting and chattering, while also obtaining an excellent differentiation of the input signal. This algorithm is more accurate in its tracking and differentiation compared with the other algorithm.

5 Conclusion

In this paper, the convergence of the discrete time optimal control law has been proved. A new discrete time optimal control law is determined by linearized criterion for a discrete-time double-integral system, which equips the discrete time optimal control law with simple structure. This closed-form non-linear state feedback control law avoids the problem associated with the time optimal control of instant switching between extreme values in the control signal. The control law proposed in this paper is ready for implementation in digital control systems, much more practical than the bang-bang control law. The numerical simulation results show that the proposed DTOC law Ftd is effective and noise tolerance. Some issues remain for future work, and they include accuracy analysis and convergence proof of the discrete time-optimal system-based tracking differentiator.

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