Minimum Dominating Set of Multiplex Networks: Definition, Application and Identification

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***Abstract*—The minimum dominating set (MDS) of network is a node subset of smallest size that every node in the network is either in this subset or is adjacent to one or more nodes of this subset. MDS has found wide applications, ranging from network monitoring, routing, to epidemic control and text processing. However, the majority of existing studies on MDS problem are confined to single networks. In real world, more and more complex systems consist of a set of elements linked up by different types of connections, which are best modeled as multiplex networks with interacting network layers. Though vastly important, the MDS of multiplex networks has not yet been formally defined and its application and identification remain open issues. In this paper, we present the definition of the MDS of multiplex network and show some of its possible applications. For solving the MDS problem of multiplex network, we built a spin glass model and solve it through the belief-propagation equations under the replica symmetry mean field theory. As a consequence, we can predict the relative size of the MDS of multiplex network theoretically and we can propose a belief-propagation-guided decimation algorithm to construct an approximate optimal dominating set in practice. Then the algorithm is improved in both accuracy and efficiency by embedding a novel multiplex network-oriented leaf-removal strategy. The effectiveness of the proposed algorithms is finally verified by comparing with other methods on a number of multiplex network examples.**

***Index Terms*—Minimum dominating set, multiplex network, complex network, message passing.**

I. Introduction

A Dominating set of network is a subset of nodes that every node in the network is either in this subset or adjacent to at least one node of this subset. A dominating set with the minimum number of nodes is called a minimum dominating set (MDS) [1].

The MDS has widespread applications in many fields. For example, it has been widely applied in monitoring large-scale traffic networks and power grids and other transportation systems, controlling epidemic (virus/fake news/disease) propagation and some other network dynamical processes, etc. For wireless networks, the MDS has been utilized to address the topology control, sensor deployment, network routing and power management [2]–[4]. In addition, the MDS can also be used to identify topic sentences for document summarization through the construction of sentence graph [5], [6]. [7]–[9].

While it is easy to state the MDS problem of identifying a MDS for a given network, solving it is notoriously hard since it is a classical hard combinatorial problem [1]. In past decades, much effort has been made to solve such problem and a large number of innovative algorithms have been proposed, including exact (exponential-time) algorithms, greedy algorithms and heuristic algorithms like ant colony optimization, genetic algorithms, simulated annealing, etc [10]–[21].

However, the majority of the achievements published so far are mainly confined to the MDS problem of single networks, which may seem inconsistent with the widely acknowledged fact that multiple types of relationships simultaneously exist among the elements in almost of the social, natural and engineering systems. A great advance in network science that has recently been achieved is the introduction of multiplex networks, formed by several single network layers, each of which contains the same set of nodes yet different intra-layer links and thus represents one type of relationship between the nodes [22]. A lot of real-world networks like biological networks, social networks, transportation networks and engineering networks have been modeled and studied as multiplex networks. The topological structure of multiplex networks and dynamical behavior and process (such as epidemic spreading, evolutionary game, diffusion, percolation and synchronization) on top of them have attracted considerable attention in both empirical and theoretical studies [23]–[46].

Considering the developments in studies on multiplex networks and the MDS problem, some interesting questions naturally pose themselves, which we aim to solve in this paper. Namely, 1) What is a reasonable definition of the MDS of multiplex network? 2) What are the practical applications of the MDS of multiplex network? 3) How to efficiently find the MDS for a given multiplex network? To the best of our knowledge, there are no results on the MDS of multiplex network, and the existing achievements on the MDS of single networks can hardly be directly applied to that in multiplex network.

In present work, we focus on the above three questions. Our main contributions can be summarized as follows:

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1) We present a reasonable definition of the MDS of multiplex network. To the best of our knowledge, this is the first study on the MDS of multiplex network.
2) Three practical applications of the MDS of multiplex network are given, including monitoring epidemic in multiplex network, early detection and controlling of epidemic spreading in multiplex network and constructing extractive text summarization respectively.
3) To tackle the MDS problem of multiplex networks, we built a spin glass model and solve it through the belief-propagation equations under the replica symmetry mean field theory. As a consequence, we can predict the relative size of the MDS of the multiplex network theoretically and we also propose a belief-propagation-guided decimation (BPD) algorithm to construct approximate optimal dominating set in practice.
4) We propose the multiplex network-oriented leaf-removal strategy. To the best of our knowledge, this is the first study on the leaf-removal strategy in multiplex network. When embedding the LR into the BPD, the BPD shows the best performance on MDS problem in both accuracy and efficiency compared with other algorithms on a number of multiplex network examples.

The remainder of this paper is organized as follows. In Section II, we briefly review the MDS problem and multiplex networks respectively. Section III presents the definition of the MDS problem of multiplex network and demonstrates three practical applications of the MDS of multiplex network. The spin glass model and message passing theory for MDS problem of multiplex network is given and the BPD algorithm and the leaf-removal algorithm are presented in Section IV. Section V verifies the effectiveness of the BPD on a variety of multiplex networks. Final, conclusions are drawn in Section VI.

II. Related work

A. Minimum dominating set problem of single network

Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a simple undirected and unweighted network where $\tilde{V}$ is the node set and $\tilde{E}$ the edge set. The subset $\tilde{D} \subseteq \tilde{V}$ is called a dominating set if every node $i \in \tilde{V} \backslash \tilde{D}$ is adjacent to one or more nodes of $\tilde{D}$. A node i is said to be occupied if $i \in \tilde{D}$; otherwise it is known as empty. A node is said to be observed if it is either occupied or empty but adjacent to at least one occupied node. The MDS problem aims at identifying in the network the smallest dominating set. The size of the MDS of network $\tilde{G}$ is defined as the domination number of the network and denoted by $\gamma(\tilde{G})$.

The exact (exponential-time) algorithms have been extensively studied for solving the MDS problem. Until 2004, the trivial $O(2^n)$ algorithm was still the fastest algorithm on the MDS problem which just simply searches through all of the possible state combinations of $\tilde{V}$. Then three independent works break the barrier of the $O(2^n)$ algorithm. Fomin, Kratsch and Woeginger first proposed algorithm running in time $O(1.9379^n)$ [10]. By using the matching techniques to restrict the search space, Randerath and Schiermeyer gave an $O(1.8999^n)$ algorithm [11]. Grandoni then reduced the running time to $O(1.8019^n)$ [12]. To our knowledge, the best exact algorithm for MDS problem at present runs in $O(1.4864^n)$ time and polynomial space which is designed through the measure and conquer technique [14]. Obviously, the exact algorithm with exponential running time is only available for relatively small scale networks which thus significantly limits its practical applications.

As an alternative, a large variety of algorithms have been developed in order to gain near optimal solutions for MDS problem, such as greedy algorithms and heuristic algorithms. The straightforward greedy strategy consists of successively selecting nodes which could cover the largest number of unobserved neighbors. Sanchis studied several variations on this greedy strategy, including GreedyRev, GreedyRan, GreedyVote and GreedyVoteGr, which concern more sophisticated procedure for choosing nodes [15]. For example, GreedyRev algorithm iterates through removing nodes from $\tilde{D}$ whose initial state contains every node of the graph. The iteration is stopped when the removal of any node would result in $\tilde{D}$ no longer being a dominating set.

For networks without core, Zhao et al. proved that the MDS problem can be solved exactly through the generalized leaf-removal strategy [16]. They also proposed a message-passing algorithm to construct the near optimal dominating sets for network with any degree distribution by using the replica-symmetric mean field theory. The simulated annealing algorithm has been proved to relatively easily give a solution of dominating set with size $\gamma > \gamma(\tilde{G})$. However, when γ gets close to $\gamma(\tilde{G})$, the search complexity increases dramatically such that no efficient solution can be obtained within finite computational time [17]. Hedar and Ismail propose genetic algorithm that uses a 0-1 variable representation of solutions in searching for the MDS [18], [19]. Besides, the ant colony optimization is also one of the most effective meta-heuristics for MDS problem and has attracted much attentions [20], [21].

B. Multiplex network

Multi-type links or interactions among units widely appear in real-world systems [22]. For example, in social systems, individuals can communicate with each other through many different channels, such as physical interaction of face-to-face and virtual connections including phone calls, WeChat, Facebook, e-mail and so on. In public transportation systems, cities are connected with others through buses, subways and airlines. In biological systems the proteins can have genetic, co-localization, physical or several other kinds of interactions. Obviously, the simple model or abstraction of these systems into a single network of nodes and links of the same type is not sufficient which may lead to inaccurate knowledge of the structure and function of the systems. In this sense, the framework of multiplex networks is proposed to provide a reasonable model for capturing high level complexity of the real-world systems. It is defined as a multiplex network of interacting network layers where each layer contains the same set of nodes yet different type of links [23], [28]–[33].

It is obvious and well recognized that the multiplex networks has significantly distinct structural and dynamical prop-

erties from that in single networks [22]. Refs. [34], [35] studied the extension of centrality measures of single network to multiplex networks, such as the degree centrality, betweenness centrality, PageRank centrality, eigenvector-like centrality and the closeness centrality, etc. Ref. [36] established the concepts of walk, path and length of multiplex networks. Refs. [37], [38] provided the framework for the study of mesoscopic structures known as communities and motifs of the multiplex networks. Refs. [33], [39] considered the correlations between the roles of the nodes in different layers and between statistical properties of the single layers. Refs. [40], [41] discussed the generative models for multiplex networks, most of which can be classified into two main categories, growing multiplex networks models and multiplex network ensembles.

Zhao et al. proved that the epidemic could spread to a massive scale in the multiplex network even if all the network layers are well below their respective epidemic thresholds. Moreover, the correlations between different network layers will also significantly influence the outbreak of epidemic and its propagation speed and scale [33]. Solé-Ribalta et al. found that the topology of multiplex networks could introduce congestion for network flows while which will be decongested when the individual layers were not interconnected [42]. Gómez-Gardeñes et al. showed that the resilience of cooperative behaviors for extremely large values of the temptation to defect is enhanced by the multiplex structure [43]. Min et al. found that the negatively correlated multiplex networks are vulnerable to random failure, but they can be robust against targeted attack [39]. Jalan and Singh demonstrated that the interactions between nodes in one layer could influence the cluster synchronizability in other layers [44]. Other studies on dynamical process upon multiplex networks can be found in [27], [45]–[48].

III. Definition and application

Considering a multiplex network formed by M different layers, i.e. $G = (G^1, ..., G^M)$, where every layer $G^\alpha = (V, E^\alpha)$ has a same set V of N nodes and a distinct set of edges $E^\alpha = V \times V$, in this section, we define the MDS problem of multiplex network and show its possible applications in three areas.

Definition III.1. A subset of nodes $D \subseteq V$ is called a dominating set of G if it is the dominating set of each network layer simultaneously. That is, every node of $V \backslash D$ needs to be adjacent to one or more nodes of D in each network layer. Identifying a dominating set with the minimum number of nodes for a multiplex network is defined as the MDS problem of multiplex network.

We adopt the same terms as that in single network to define the state of nodes of multiplex network. A node i of multiplex network is said to be occupied if $i \in D$; otherwise it is in empty state. A node is said to be observed if it is either occupied or empty but adjacent to at least one occupied node in every layer. If an empty node is adjacent to occupied nodes in particular layers, we say the node is observed in those layers. Otherwise, if an empty node has no occupied neighboring nodes in particular layers, the node is called unobserved in those layers.

Figure 1 presents a simple example of the dominating set of multiplex network. In what follows, we will show three practical applications which prompt us to introduce such a definition of the MDS of multiplex networks.

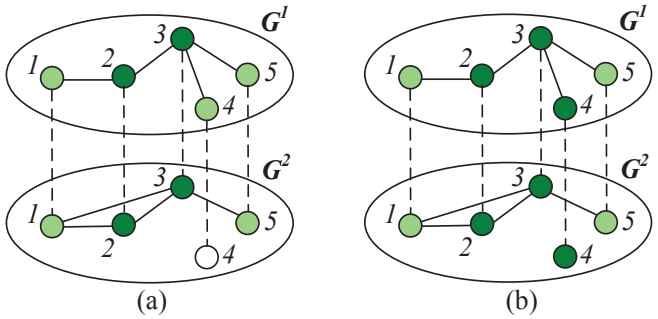

Fig. 1. Illustration of dominating set of multiplex network coupled by two layers, i.e. $G = (G^1, G^2)$. The dark green node is occupied, while the light green node is empty but observed. (a) The set {2,3} is not a dominating set of G since node 4 is not observed in layer G^2. (b) The set {2,3,4} is an MDS of G.

A. Application: Monitoring epidemic in multiplex network

Epidemic spreadings can be found in various areas of human society, like the virus propagation in computer networks, rumor and information spreading in social networks, disease spreading in physical contact networks and failure propagation in engineering networks [47], [49]–[54]. In this context, a challenging problem is to efficiently place observer/sensors to monitor the epidemic status of every node throughout the network, which has broad implications for practical applications including detecting harmful epidemics, recording data for epidemiological studies and reconstructing spreading network etc. Particularly, if the observer nodes can only monitor their neighboring nodes, the MDS-based observer placement which chooses the nodes belonging to the MDS of the network as observer nodes becomes one of the best strategies: every node shall be monitored by placing the smallest number of observer nodes.

Recently, the study of epidemic dynamics on multiplex networks has received increasing interest. In comparison with the spreading in single networks, a noticeable distinct characteristic of that in multiplex networks is the epidemic could spread between multiple layers, called cross-layers cascade [55], [56]. For example, in multiplex social networks, when a user receives new information via a social medium, like WeChat, the user may spread it again through WeChat or do it over another online media such as Twitter, or both (see Fig. 2). In this sense, in order to efficiently monitor the epidemic in multiplex networks, every node should be monitored in every layer since the states of the same node in different layers may be different. That is, every node should be connected with at least one observer node in each layer. Letting the nodes in the MDS of the multiplex network be observer nodes, we shall then be able to monitor every node in each layer using the smallest number of observers.

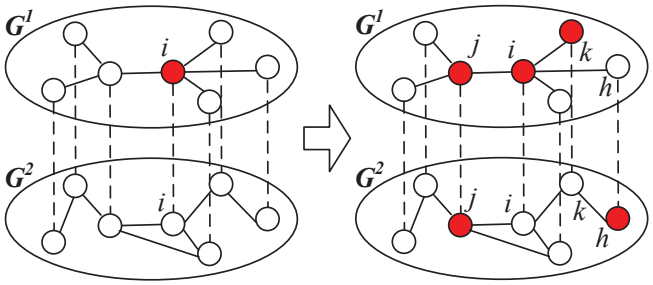

Fig. 2. An example of cross-layers cascade in multiplex social network $G = (G^1, G^2)$. When node i posts a message on G^1, its neighboring nodes may repost it through G^1 (like node k) or do it over G^2 (like node h) or repost it in both two layers (like node j).

B. Application: Early detecting and controlling epidemic in multiplex network

To control epidemic spreading on networks, a novel method of competing spreading through multiplex networks has drawn wide attention recently [57], [58]. An archetypical scenario is that the disease spreading on a network of persistent real contacts will naturally introduce the diffusion of crisis awareness information on a network of virtual social contacts between the same individuals. The diffusion of awareness information, in turn should further restrain the disease spreading, thus forming the competing spreading processes of disease and awareness information on the multiplex network with two layers. [59], [60]. A similar scenario is the competing spreading of virus and patch where patch dissemination depresses the virus propagation while they spread through different channels [54]. However, in such a framework it may be an unrealistic assumption that all nodes have the willingness and ability to identify the infection and send the awareness or patch.

Takaguchi et al. presented an epidemic control mechanism which takes advantage of both traditional network immunization and competing spreading. They considered the deployment of observer nodes on networks and assumed that the observer could immediately recognize its infected neighbors and then immunize its other uninfected neighbors. Though such an approach could monitor all the network nodes and efficiently recognize the infection, it cannot immediately immunize all the network nodes when infection spreading is confirmed. In addition, the immunization may not be easy to be implemented in the same network through which the infection spreads out.

In this sense, based on the definition of the MDS of multiplex network, we propose a network model as shown in Fig. 3 which consists of a two-layer network and a central node. The upper layer corresponds to the immunization distribution network layer (IL). The lower layer represents the epidemic spreading network layer (EL). The central node connects with all the nodes of the MDS of the multiplex network. Under the network model, we have the following approach which could recognize the infection immediately upon its happening as well as immediately immunize all the network nodes by using the smallest number of observer nodes: the nodes belonging to the MDS of the multiplex network are selected to be observer nodes. When one neighbor is infected in EL, the observer could immediately recognize the infection and send the information to the central node. The central node would then distribute the crisis information or patches to all the observers and the observers then spread them to their neighbors in IL. Since every node connects with at least one observer in both IL and EL, the infection could be detected and stopped immediately, while the number of observers is minimized.

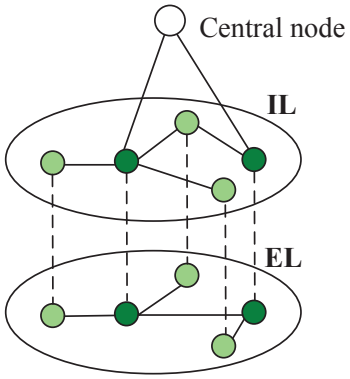

Fig. 3. Network model of early detection and control of infection spreading based on MDS of multiplex network. It consists of a multiplex network coupled by IL and EL and a central node. The set of dark green nodes is the MDS of the two-layers network. The central node only connects with the nodes of MDS.

C. Application: Constructing extractive text summarization

Extractive text summarization is one of the most important branches of natural language processing which constructs a summary by selecting a subset of sentences from the original text [61]–[64]. The summary consists of the most representative sentences of the text and can be used to facilitate readers to quickly understand the main topic of the text.

Recently, graph-based summarization has attracted increasing attention from researchers [6], [65], [66]. Specifically, the graph-based method represents the text as a graph where sentence of the text is regarded as the vertex of the graph and edge between two vertices represents how similar of two sentences are. A commonly used technique to link two sentences is to measure their similarity and if the similarity is larger than a default threshold then the two sentences are linked. Shen and Li introduced the MDS of network for solving the summarization problem, which is a typical graph-based summarization method [65]. They claimed that the MDS of the sentence graph can directly act as the summary: it is representative since each sentence is either in the MDS or connected to at least one sentence of this set; and it is with the minimal redundancy since this set has the minimum number of elements.

Many different similarity indexes have been applied to construct the sentence graph, such as cosine similarity, Jaccard similarity, Tanimoto coefficient, dependency tree kernels similarity and semantics-based similarity etc. [6], [65], [66]. However, most of the graph-based methods only use one similarity measurement to construct the sentence graph; their performances may change when the similarity measurement varies. With the definition of MDS of multiplex network,

we may propose a novel graph-based summarization method which could simultaneously consider multiple similarity measures and thus may produce more robust summary.

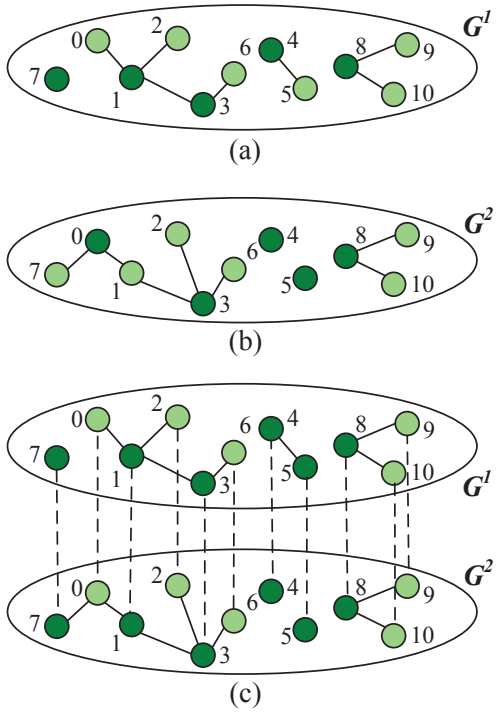

Fig. 4. Applications of MDS of network on text summarization. Each network layer corresponds to a sentence graph constructed based on one similarity measure. The set of dark green nodes is the MDS of the network and is used to represent the summary.

Taking a subset of documents included in DUC 2004, also completely shown in [66], for example, when using the cosine similarity which defines the similarity of two sentences $\overrightarrow{S_i} = (S_{i_1}, S_{i_2}, \ldots, S_{i_n})$ and $\overrightarrow{S_j} = (S_{j_1}, S_{j_2}, \ldots, S_{j_n})$ as

$$\text{cosine}(\overrightarrow{S_i}, \overrightarrow{S_j}) = \frac{\sum_k S_{i_k} S_{j_k}}{\sqrt{\sum_k S_{i_k}^2}\sqrt{\sum_k S_{j_k}^2}} \tag{1}$$

and setting the threshold at 0.17, we have the sentence graph as Fig. 4(a). The MDS-based summary is the sentence set $\{1, 3, 4, 7, 8\}$. When we use the Tanimoto coefficient-based similarity index which is given by

$$\text{Tanimoto}(\overrightarrow{S_i}, \overrightarrow{S_j}) = \frac{\sum_k S_{i_k} S_{j_k}}{\sqrt{\sum_k S_{i_k}^2} + \sqrt{\sum_k S_{j_k}^2} - \sum_k S_{i_k} S_{j_k}} \tag{2}$$

and set the threshold be 0.7, we have the sentence graph as shown in Fig. 4(b) which has the same number of edges as that in the Fig.4(a). The corresponding MDS summary is $\{0, 3, 4, 5, 8\}$.

We can see that the MDS method produces different summaries when different similarity measures are considered. To overcome this limitation, we can construct multiplex network where each layer corresponds to a sentence graph constructed based on a similarity measure (see Fig. 4(c)). By such a way, we can obtain a more robust summary based on the MDS of multiplex network since it considers multiple similarity measures simultaneously.

IV. Identification

Statistical physics theory has been successfully used for solving several combinatorial optimization problems [67]. The replica method and the cavity method of spin glass mean field theory has been successfully used to solve the travelling salesman problem, the K-satisfiability problem, the feedback vertex set problem and the vertex cover problem, etc. [16], [68]–[70].

In Ref. [16], Zhao et al. proposed a message passing algorithm and leaf removal strategy which are very efficient for solving the MDS problem of single network. However, it cannot be directly applied to the MDS problem of multiplex network since 1) the nodes of multiplex network have different neighbors in different layers which should be considered simultaneously; 2) the nodes of multiplex network should be either occupied or observed in every layers rather than just in one layer; 3) the nodes of multiplex network may have different states in different network layers. In this paper, we extend the spin glass and message passing theory to multiplex network for solving the MDS problem of multiplex network. A novel spin glass model of the MDS problem of multiplex network is built and solved through the derived belief-propagation (BP) equation based on the replica symmetry mean field theory. As a consequence, we can predict the relative size of the MDS of multiplex network theoretically and we also propose a message passing algorithm and a leaf removal strategy to construct approximate optimal dominating set in practice.

A. Spin glass model and message passing theory for MDS problem of multiplex network

In the construction of dominating set, each node $i \in V$ should be either occupied (i.e., $i \in D$, denoted by c_i=1) or empty ($i \in V \backslash D$, c_i=0). Therefore, the total number of the possible occupation configurations is 2^N. Therefore, we can built a spin glass model for the MDS problem of multiplex network following the partition function

$$Z = \sum_{\underline{c}} \prod_{i \in V} \left\{ e^{-x c_i} \prod_{\alpha} \left[1 - (1 - c_i) \prod_{j \in \partial^\alpha i} (1 - c_j) \right] \right\} \tag{3}$$

where $x > 0$ is the re-weighting parameter which emphasizes the dominating sets of smaller cardinality; $\underline{c} = \{c_1, c_2, ..., c_N\}$ denotes an occupation configuration, $\partial^\alpha i$ indicates the set of neighboring nodes of i in layer G^α. $1 - c_i = 0$ or $\prod_{j \in \partial^\alpha i} (1 - c_j) = 0$ means node i is occupied or observed in layer G^α. Therefore, the term $\prod_{\alpha} \left[1 - (1 - c_i) \prod_{j \in \partial^\alpha i} (1 - c_j) \right]$ being equals to 1 means node i is observed in every layer; otherwise it equals 0. In this sense, only the dominating set of the multiplex network contributes to Z. Particularly in the limit of $x \to \infty$, the partition function will be exclusively contributed by configurations of the MDS.

The spin glass model Eq. (3) can be solved using the replica-symmetric mean field theory, which can be derived from the expansions of the partition function or understood from the perspective of Bethe-Peierls approximation [67]. Therefore,

the marginal probability $q_i^{c_i}$ that node i is in the state $c_i \sim (0, 1)$ can be expressed as

$$q_i^{c_i} = \frac{\tilde{q}_i^{c_i}}{z_i} \tag{4}$$

where

$$\tilde{q}_i^1 = e^{-x} \prod_{j\in\partial i} \sum_{c_j} q_{j\to i}^{(c_j,1)}, \tag{5}$$

$$\tilde{q}_i^0 = \sum_{\underline{c}_{\partial i}} \prod_{j\in\partial i} q_{j\to i}^{(c_j,0)} \prod_{\alpha} H(\sum_{j\in\partial^\alpha i} c_j - 1), \tag{6}$$

and

$$z_i = \sum_{c_i} \tilde{q}_i^{c_i}. \tag{7}$$

∂i is the union of all neighbor sets of node i in every layer, i.e., $\partial i = \bigcup_\alpha \partial^\alpha i$. $\underline{c}_{\partial i}$ indicates an occupation configuration of all neighbors of node i. $H(a)$ is the Heaviside step function such that $H(a) = 1$ if $a \geq 0$, and $H(a) = 0$ if $a < 0$. $\sum_{j\in\partial^\alpha i} c_j$ denotes the number of neighbors in occupied state of node i in layer G^α. Therefore, the term $\prod_\alpha H(\sum_{j\in\partial^\alpha i} c_j - 1)$ being equals to 1 means node i has at least one occupied neighbor in every layer, that is, i is observed in every layer; otherwise it equals 0. $q_{j\to i}^{(c_j,c_i)}$ is the joint probability of i being in c_i state and its neighbor j being c_j state under the assumption that the constraint of node i is not considered. One simple way of explaining Eq. (4) is that if node i is occupied, i.e., c_i=1, it is observed in every layer thus its neighboring nodes in each layer could be in any state (see Eq. (5)); however if i is empty, i.e., c_i=0, it should be observed by its occupied neighboring nodes in each layer thus it should have at least one neighboring node being state occupied in each layer (see Eq. (6)).

Similarly, under the Bethe-Peierls approximation, we present the expression of $q_{j\to i}^{(c_j,c_i)}$ which is also called the belief propagation equation of the spin glass model

$$q_{j\to i}^{(c_j,c_i)} = \frac{\tilde{q}_{j\to i}^{(c_j,c_i)}}{z_{j\to i}} \tag{8}$$

where

$$\tilde{q}_{j\to i}^{(1,0)} = e^{-x} \prod_{k\in\partial j\setminus i} \sum_{c_k} q_{k\to j}^{(c_k,1)}, \tag{9}$$

$$\tilde{q}_{j\to i}^{(1,1)} = \tilde{q}_{j\to i}^{(1,0)}, \tag{10}$$

$$\tilde{q}_{j\to i}^{(0,1)} = \sum_{\underline{c}_{\partial j\setminus i}} \prod_{k\in\partial j\setminus i} q_{k\to j}^{(c_k,0)} \prod_{\alpha: j\notin\partial^\alpha i} H(\sum_{k\in\partial^\alpha j} c_k - 1), \tag{11}$$

$$\tilde{q}_{j\to i}^{(0,0)} = \sum_{\underline{c}_{\partial j\setminus i}} \prod_{k\in\partial j\setminus i} q_{k\to j}^{(c_k,0)} \prod_{\alpha} H(\sum_{k\in\partial^\alpha j} c_k - 1) \tag{12}$$

and

$$z_{j\to i} = \sum_{(c_j,c_i)} \tilde{q}_{j\to i}^{(c_j,c_i)} \tag{13}$$

where $\partial j \setminus i$ is the set of all neighbors of node j except node i. The BP equation can be explained as $q_i^{c_i}$ but it is defined on the cavity graph where node i is absent.

For a given multiplex network, the BP equation can be calculated by iterating the equation starting from random initial values until convergence. After the BP equation is solved, some thermal dynamical quantities of the spin glass model can be obtained. First, the energy density ϵ of the spin glass model is given by

$$\epsilon = \frac{1}{N} \sum_i q_i^1. \tag{14}$$

In addition, the total free energy related to the partition function can be expressed as

$$F = \sum_i f_i - \sum_{(i,j)} f_{ij}, \tag{15}$$

where f_i and f_{ij} are the contributions of the free energy of node i and edge (i, j) respectively,

$$f_i = -\frac{1}{x} \ln z_i \tag{16}$$

$$f_{ij} = -\frac{1}{x} \ln \sum_{c_i,c_j} q_{i\to j}^{(c_i,c_j)} q_{j\to i}^{(c_j,c_i)}. \tag{17}$$

Then we can compute the free energy density through

$$f = \frac{F}{N}. \tag{18}$$

At last, the entropy density of the system is evaluated as

$$s = x(\epsilon - f). \tag{19}$$

In Fig. 5, we display the energy density ϵ, free energy density f and entropy density s of the corresponding spin glass system for a given MDS problem of multiplex network. One sees that s decreases as x grows. In addition, as shown in panel (a) with the growth of x, the energy density ϵ decreases, that is the partition function Eq. (3) will be dominated more by the dominating sets of smaller cardinality. Therefore, since the entropy density of real system can't be negative, we can obtain the lower bound of the theoretical size of the MDS, denoted by ϵ_c, at the $x = x^*$ where $s(x^*) = 0$ (see panel (a) and (c)). In Table 1, we present the theoretical size of the MDS of many multiplex network instances. It should be noted that the BP iteration may be unable to converge to a fixed point or jump between several possible optimal solutions in some cases. A possible way is to select the optimal solution from several computations using different x. Some other discussions on the convergence of the solution of the spin glass model and auxiliary methods can be found in [16], [67], [71]–[73].

B. Belief-propagation-guided decimation algorithm

Under the spin glass model and message passing theory, we can also propose heuristic algorithm to construct approximate optimal dominating set in practice.

From Eq. (4), we can derive the probability q_i^1 that node i is occupied which naturally prompts us to develop a belief-propagation-guided decimation (BPD) algorithm to construct the MDS for a given multiplex network. Since such BPD algorithm will be improved by embedding a leaf-removal strategy in the next section, we here call this BPD algorithm

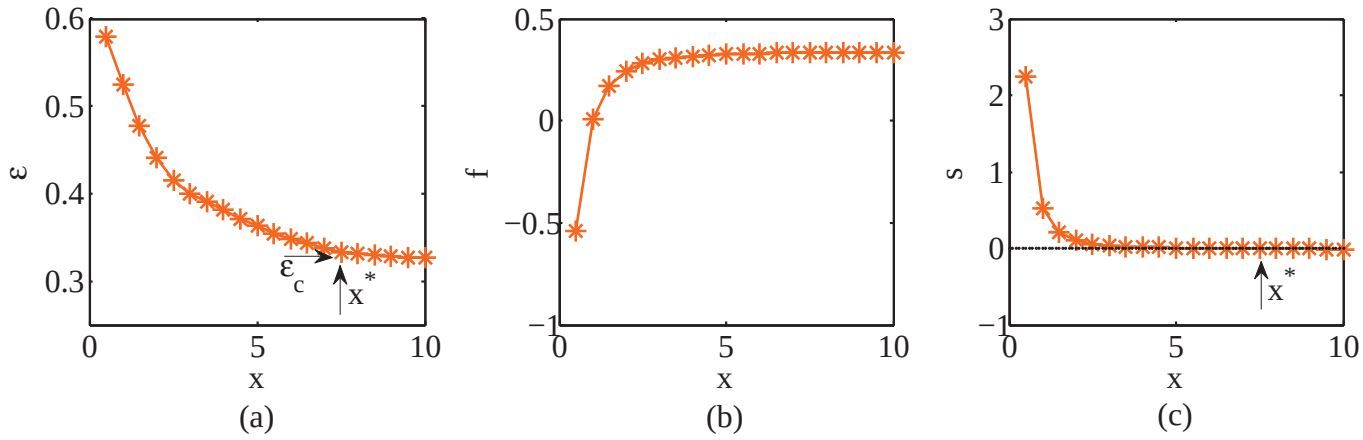

Fig. 5. The energy density ϵ, free energy density f and the entropy density s versus x. The multiplex networks used are the 6th network shown in Table 1.

the pure BPD algorithm. The core idea of the pure BPD is to iteratively choose the node with the highest occupation probability to be occupied until a dominating set is obtained. The procedure of the pure BPD is described as follows.

Step-B0. All nodes of the given multiplex network are initially set as empty and unobserved in all layers. The joint probability $q_{j\to i}^{(c_j,c_i)}$ should be set as arbitrary value, and the re-weighting parameter x should be a relatively large value.

Step-B1. Repeat calculating the belief propagation equation Eq. (8) for a fixed round T_0 of iterations and then calculate the occupation probability q_i^1 of each node i based on Eq. (4).

Step-B2. Occupy r unoccupied nodes which have the largest occupation probabilities.

Step-B3. Delete the links between the observed nodes which are in the same layer, and then delete the isolated nodes that have been observed in all layers.

Step-B4. If the resulting multiplex network still contains nodes that have not been observed in one or more layers, we repeat calculating the belief propagation equation Eq. (8) for a fixed round T_1 of iterations, and then calculate the occupation probability q_i^1 of each node i based on Eq. (4).

Note: In this step, if node j is not observed in all layers, $\tilde{q}_{j\to i}^{\{0,1\}}$ and $\tilde{q}_{j\to i}^{\{0,0\}}$ could be derived from Eqs. (11) and (12) respectively. While if node j has been observed in one or more layers, it presents no restriction to the occupation states of all its neighbors in the corresponding layers. In this sense, we could simplify the expressions of $\tilde{q}_{j\to i}^{\{0,1\}}$ and $\tilde{q}_{j\to i}^{\{0,0\}}$ which are given by

$$\tilde{q}_{j\to i}^{\{0,1\}} = \sum_{\underline{c}_{\partial j\backslash i}} \prod_{k\in\partial j\backslash i} q_{k\to j}^{(c_k,0)} \prod_{\alpha: j\notin\partial^\alpha i, j\notin O^\alpha} H(\sum_{k\in\partial^\alpha j} c_k - 1), \tag{20}$$

$$\tilde{q}_{j\to i}^{(0,0)} = \sum_{\underline{c}_{\partial j\backslash i}} \prod_{k\in\partial j\backslash i} q_{k\to j}^{(c_k,0)} \prod_{\alpha: j\notin O^\alpha} H(\sum_{k\in\partial^\alpha j} c_k - 1) \tag{21}$$

where O^α denotes the set of nodes that have been observed in layer G^α. Similarly, if node i is empty but observed in one or more layers, we have

$$\tilde{q}_i^0 = \sum_{\underline{c}_{\partial i}} \prod_{j\in\partial i} q_{j\to i}^{(c_j,0)} \prod_{\alpha: i\notin O^\alpha} H(\sum_{j\in\partial^\alpha i} c_j - 1). \tag{22}$$

Step-B5. Repeat operations B2-B4 until all nodes are observed in all layers.

Algorithm 1 also shows the procedure of the pure BPD algorithm. For the pure BPD, T_0, r and T_1 are all tunable parameters. The impacts of their values are discussed in Section V.

Algorithm 1 The pure BPD algorithm.

Input: A multiplex network G;
Output: A dominating set D of G;
Initialize: r, T_0, T_1;
Perform operations B0, B1 in turn;
do
    Perform operations B2, B3, B4 in turn;
while ($\exists$ i and α, such that $i \notin O^\alpha$);
return D;

C. Leaf-removal algorithm of multiplex network

The leaf-removal (LR) algorithm is a well-known strategy for combinatorial optimization problems which is very fast and very easy to implement. It has been proved that it can produce exact solutions for the vertex cover problem and the MDS problem in single-layer tree networks. Even if the LR algorithm cannot make all nodes be observed, the LR can also be used to identify part of the nodes of an exact MDS and simplify the network by deleting occupied nodes and observed nodes.

To the best of our knowledge, there has no studies on the LR for multiplex network. Therefore, we here present the LR algorithm for multiplex network. The procedure of LR performed on a given multiplex network G is described as follows.

Step-L0. If there is an isolated unobserved node i in one layer, occupy it (Fig. 6(a)) and all its unobserved neighboring nodes in each layer thus become observed. Then delete i from G.

Step-L1. If i is an unobserved leaf node linked with j in one or more layers, we occupy node j if one of the following conditions is met: 1) i is an isolated observed node in all the other layers (see Fig. 6(b)); 2) i is also an unobserved leaf

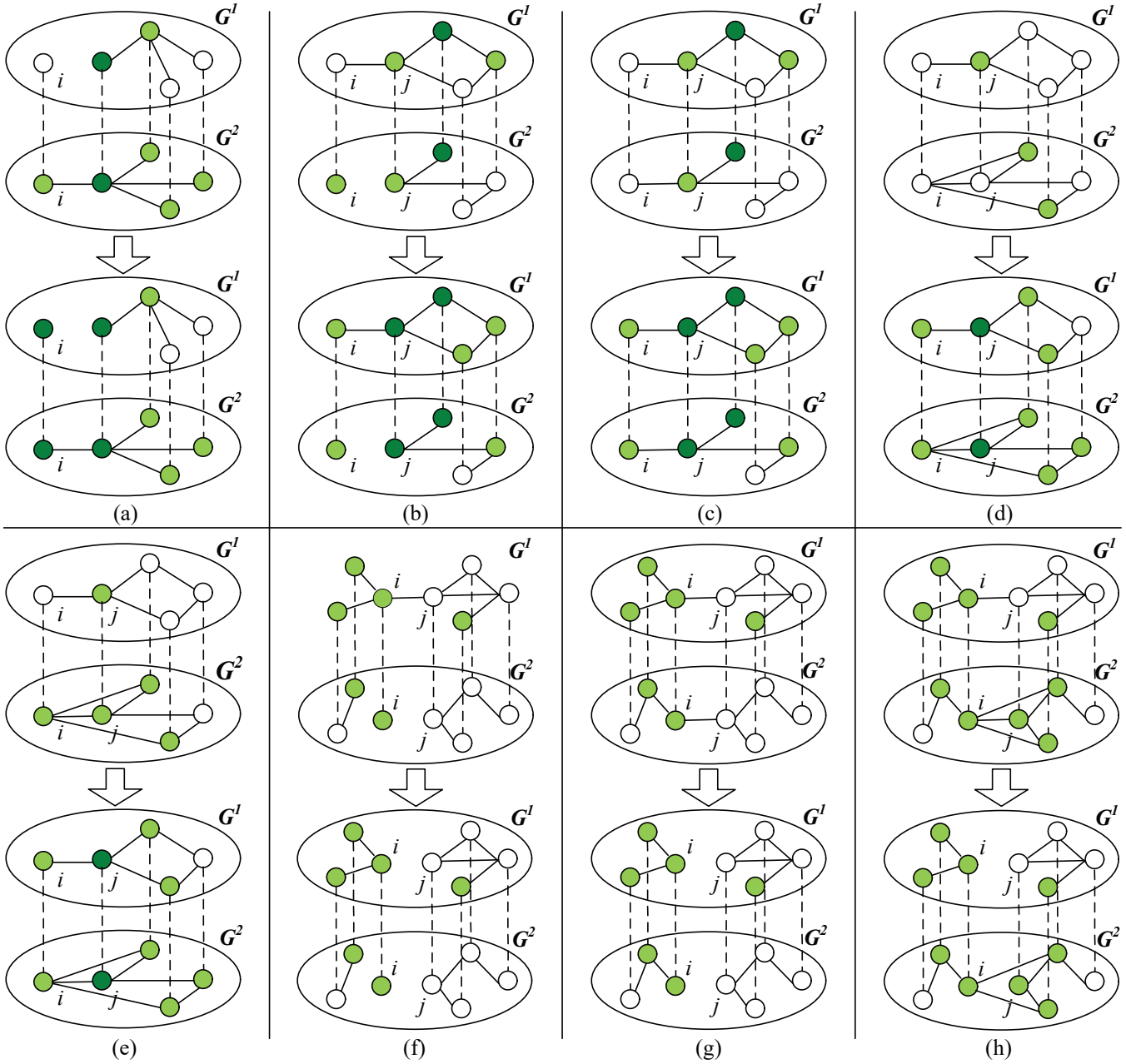

Fig. 6. Basic operations of LR in multiplex networks. The dark green node is occupied, the light green node is empty but observed.

node and adjacent to the same unique neighbor j in all the other layers (Fig. 6(c)); 3) i has only a same single unobserved neighbor j in all the other layers (Fig. 6(d)); 4) i is observed and has no unobserved neighbors in all the other layers (Fig. 6(e)). If j is occupied, all its unobserved neighboring nodes in each layer thus become observed. We then delete j from G.

Step-L2. If i is an observed node and has only one unobserved neighboring node j in one or more layers, we can 1) remove the edge (i,j) from these layers if i is an isolated observed node in all the other layers (Fig. 6(f)); 2) remove the edge (i,j) from all layers if i is also an observed node and has the same single unobserved neighbor j in all the other layers (Fig. 6(g)); 3) remove the edge (i,j) from all layers if i is observed and has no unobserved neighbors in all the other layers (Fig. 6(h)).

Step-L3. If i is an observed node and has no unobserved neighbors in all layers, we delete i from G.

Step-L4. Repeat operations L0-L3 until no nodes can be further observed.

We now present our ultimate BPD algorithm for solving the MDS problem of multiplex network which combines the pure BPD algorithm and the LR process: before each calculations of the occupation probabilities of network nodes in pure BPD algorithm, we first carry out the LR algorithm one time to identify some of the occupied nodes and simplify the network as much as possible. The BPD algorithm is also described in Algorithm 2.

The advantages of the BPD algorithm can be summarized

as follows, 1) The LR is very fast and very easy to implement; 2) The chosen occupied nodes by the LR process are optimal (occupying the only neighbor of the leaf node must be an optimal strategy) for constructing an MDS of multiplex network; 3) LR simplifies the network and consequently reduces the running time of the pure BPD process; 4) When an LR process is finished and some nodes are still unobserved, the pure BPD could continue to find new nodes to occupy and then introduce another LR implement; 5) The BPD could slightly reduce the size of the constructed dominating set compared with that constructed by the pure BPD.

Algorithm 2 The BPD algorithm.

Input: A multiplex network G;
Output: A dominating set D of G;
Initialize: r, T_0, T_1, step←0;
do
  Perform operations L1, L2, L3, L4 in turn;
  If (step=0)
    Perform operations B0, B1, B2, B3 in turn;
    step++;
  Else If (step>0)
    Perform operations B4, B2, B3 in turn;
    step++;
  End If
while ($\exists$ i and α, such that $i \notin O^{\alpha}$);
return D;

V. Analysis

In this section, we discuss the appropriate values of parameters of the BPD algorithm and verify the effectiveness of BPD by comparing with other algorithms.

A. Parameter evaluation

In the BPD algorithm, intuitively, the smaller the parameter r is, the smaller the constructed dominating set through the BPD and consequently the longer the running time of the algorithm. In present work, we define

$$r = \rho \sum_{\alpha} |O^{\alpha}|/M, \tag{23}$$

which is updated in each step of the algorithm according to the number of unobserved nodes. It get smaller as the number of unobserved nodes decreases. ρ is a tunable parameter in (0,1). The smaller the ρ is, the smaller the r. ρN is the maximum value of the number of occupied nodes in one step of the BPD algorithm. When $\sum_{\alpha} |O^{\alpha}|$ is greater than 1 and r is smaller than 1, we set r as equal to 1.

By setting $\rho = 0.01$, in Fig. 7 we shown the impacts of T_0 and T_1. On can see that a relatively large T_0 is necessary for constructing a small dominating set, while T_1 has no obvious impact on the performance of the BPD. Therefore, we set $T_0 = 100$ and $T_1 = 10$ for all BPD in present work.

Figures 8 and 9 show the impact of r on the performance of BPD algorithm. The *size* refers to the size of the constructed dominating set. The *step* indicates the number of iterations of the BPD algorithm which can be regarded as the time costs

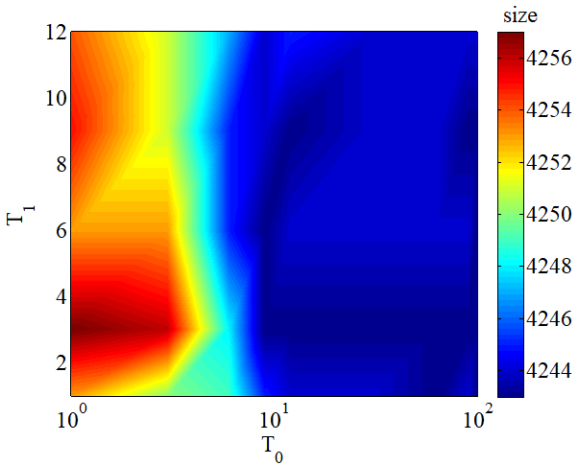

Fig. 7. The size of the constructed dominating set through BPD versus T_0, T_1. We set $x = 10$, $\rho = 0.01$. The multiplex networks used are the 9th network shown in Table 1.

we calculate the occupation probabilities of the unoccupied nodes. From Fig. 8, one sees that the sizes of the constructed dominating sets via the BPD and pure BPD decrease and become closer to each other with a decreasing value of ρ (r). They have the almost same performances on constructing the dominating sets when a relatively small ρ is used. However, there is a growing value gap between the steps of the BPD and the pure BPD with the decrease of ρ as shown in Fig. 9. The number of steps of the BPD is much smaller than that of the pure BPD when ρ is small. In this sense, we safely draw the conclusion that the BPD algorithm has a better performance in both accuracy and efficiency than the pure BPD. In the following experiments, we set ρ to be 0.0001 in all BPD algorithms.

B. Performance comparisons

Though the MDS problem of multiplex network is first proposed in present work and no related solution algorithms have been developed, some general algorithms for the combinatorial optimization problem could be easily extended for solving the MDS problem of multiplex network. The most well-known ones include the centrality-based algorithm, the greedy strategy and the simulated annealing algorithm. In this part, we verify the effectiveness of the BPD algorithm by comparing it with these general algorithms in various multiplex network examples.

1) Centrality-based algorithm: Centrality has been widely used to identify the vital nodes of network which characterizes the importance of network node according to the network topology. The most commonly adopted centrality indexes include degree, betweenness, pagerank, closeness, k-shell, eigenvector, collective influence etc. In a multiplex network, the importance of nodes depend on the connectivity patterns within and across different network layers. In recent years, a number of centrality indexes of multiplex network have been developed [34]. Among these measures, here we consider the degree centrality and use it to construct the dominating set of a multiplex network.

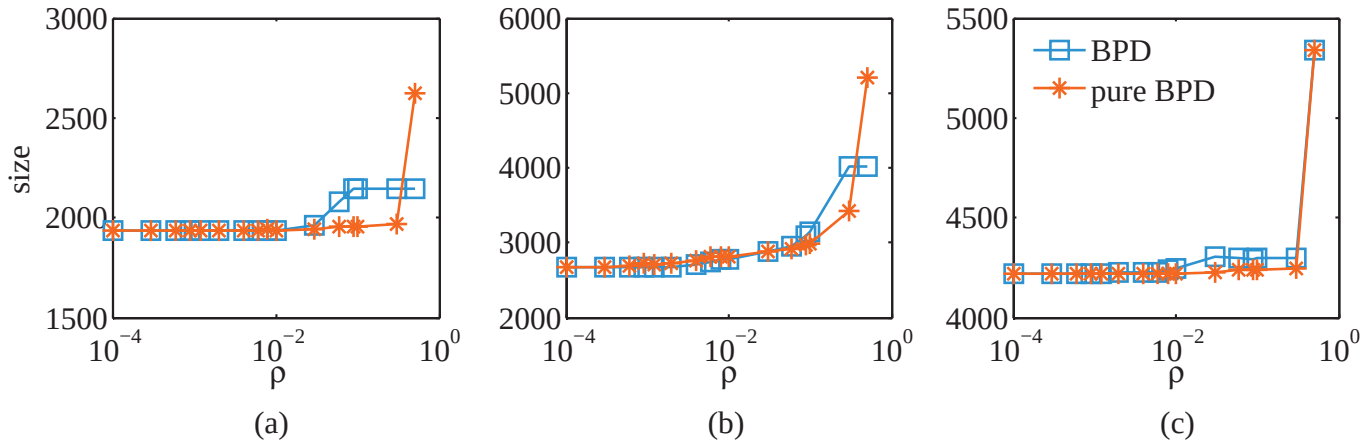

Fig. 8. The *size* of BPD and pure BPD versus the parameter ρ. The multiplex networks used are the 7th, 8th and 9th networks shown in Table 1.

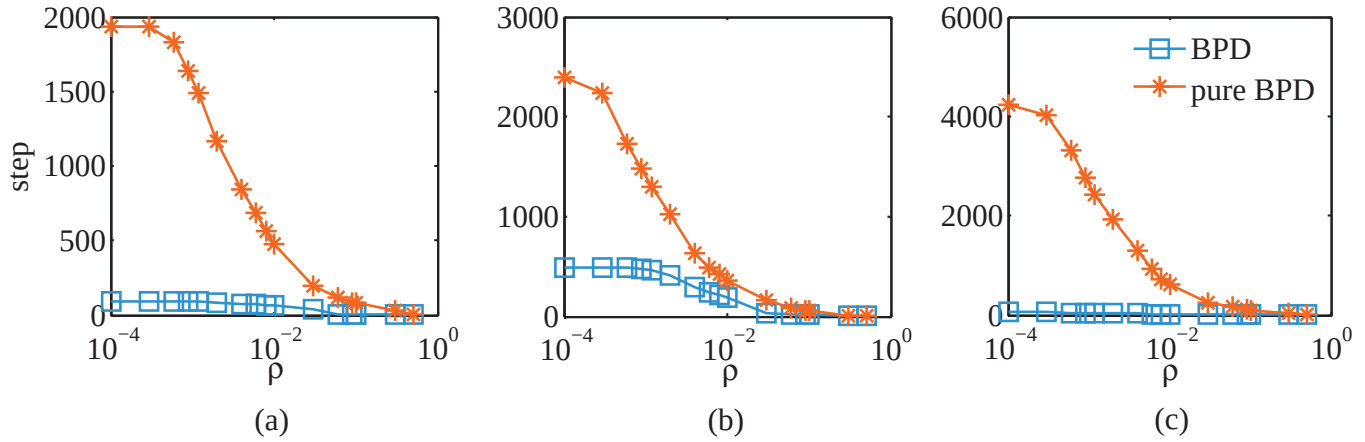

Fig. 9. The *step* of BPD and pure BPD versus the parameter ρ. The multiplex networks used are the 7th, 8th and 9th networks shown in Table 1.

In a multiplex network, the degree of node can be defined in various ways. We consider a simple case that the degree of a node in multiplex network is the sum of its degrees across all the layers, that is

$$d_i = \sum_{\alpha} d_i^{\alpha}. \tag{24}$$

Therefore, for the degree algorithm to the MDS problem of multiplex network, we select the node with the largest degree d_i as an occupied node and then remove it from the network. We repeatedly select the node with the largest d_i by recalculating d_i for the remaining network at each step, until a dominating set is constructed.

2) Greedy algorithm: The greedy algorithm is very simple and widely used for optimization problems. For the greedy algorithm to the MDS problem of multiplex network, we repeatedly select node that has the largest number of edges connecting with unobserved nodes in all layers. That is, at each time step, the node with the highest scores is selected to be occupied, and the score for the remaining unoccupied nodes are then updated. The algorithm continues until the occupied nodes form a dominating set. Here, the score of node i is defined as

$$\sigma_i = \sum_{\alpha} |O_i^{\alpha}| \tag{25}$$

where O_i^{α} denotes the set of new observed nodes in layer G^{α} introduced due to the occupation of node i. Particularly, node i also belongs to O_i^{α} if it is unobserved in layer G^{α} before its occupation. This score takes into account the influence of the node's occupation for each network layer.

3) Simulated annealing algorithm: The simulated annealing (SA) algorithm has been widely used for solving different kinds of optimization problems. It aims to find the optimal configuration under the definition of energy function. We here present an SA algorithm for solving the MDS problem of multiplex network. The energy function is defined as

$$E(\underline{c}) = \mu \sum_i c_i + \sum_{\alpha} (N - O^{\alpha}) \tag{26}$$

where $\mu < 1$ is a tunable parameter. $\sum_i c_i$ is the number of occupied nodes and $N - O^{\alpha}$ the number of unobserved nodes in layer α. At the beginning of the SA algorithm, we construct a $\underline{c}$ which could form a dominating set of the multiplex network by randomly selecting occupied nodes. Then at each time step of SA, we perform the following operation:

TABLE I
SOLUTIONS OF THE MDS PROBLEM OF MULTIPLEX NETWORK BASED ON DIFFERENT ALGORITHMS.

Number	Multiplex network	N	M	Degree	Greedy	SA	pure BPD	BPD	Theoretical size
1	G^1-Dolphins $G^2 - \mathrm{SF}_1$	62 62	159 181	40	18	17	17	17	16.48
2	G^1-Email $G^2 - \mathrm{SF}_2$	1133 1133	5451 2266	715	326	304	303	302	292.78
3	G^1-RoadEU $G^2 - \mathrm{SF}_3$	1174 1174	1417 2343	703	460	419	414	412	400.41
4	G^1-PPI $G^2 - \mathrm{SF}_4$	2361 2361	6646 4721	1793	752	716	711	710	704.26
5	G^1-Protein $G^2 - \mathrm{SF}_5$	3133 3133	6726 6261	3116	1041	1001	994	992	990.44
6	G^1-Grid $G^2 - \mathrm{SF}_6$	4941 4941	6594 9879	2938	1842	1722	1676	1674	1664.37
7	G^1-GrQc $G^2 - \mathrm{SF}_7$	5242 5242	14496 5242	3083	2045	1956	1936	1936	1935.89
8	G^1-HepTh $G^2 - \mathrm{SF}_8$	9877 9877	25998 25998	9856	2718	2767	2664	2664	2657.57
9	G^1-PGP $G^2 - \mathrm{SF}_9$	10680 10680	24316 10696	5453	4385	4325	4221	4221	4218.62
10	G^1-AstroPh $G^2 - \mathrm{SF}_{10}$	18772 18772	198110 37550	13238	4667	4785	4286	4286	4272.49
11	$G^1 - \mathrm{SF}_{11}$ $G^2 - \mathrm{ER}_{12}$	58228 58228	58245 84578	58221	26211	28273	24371	24355	24351.02

Operation-S0. If the current occupation configuration $\underline{c}$ forms a dominating set of the multiplex network, we select an occupied node randomly and changed its state to unoccupied.

Operation-S1. If the current occupation configuration $\underline{c}$ cannot form a dominating set of the multiplex network, we select an unoccupied node randomly and changed its state to occupied.

The operation S0 or S1 is accepted with probability $e^{-\beta\Delta E}$; otherwise, the new occupation should be omitted and the old one be preserved. ΔE is the energy difference between the new configuration and the old configuration. β represents the inverse of the temperature. In this paper, we set $\mu = 0.6$; the initial value of β is set to be 0.5 and it is increased by 10^{-6} at each time step. The SA algorithm terminates when β equals β_{max}.

4) Results: We perform experiments on several different kinds of multiplex networks, each of which is formed by randomly connecting an scale-free (SF) network [74] with an Erdős-Rényi (ER) random network [75] or an SF network with a real-world network (including dolphins network (Dolphins [76]), e-mail communication network (Email [77]), European express road network (RoadEU [78]), protein-protein interaction network (PPI [79] and Protein [80]), US power grid (Grid [81]), collaboration network of high-energy physics authors (HepTh [82]), interaction network of users of Pretty Good Privacy (PGP [83]), Arxiv ASTRO-PH collaboration network (Author [82]).

The experiment results in Table 1 show that the degree centrality-based algorithm has the weakest performance in all networks. One major reason is that the degree of node in multiplex network does not accurately reflect the importance of the node in different layers. A node with large d_i may have a small degree in a particular layer and thus cannot let more nodes be observed in the layer. The greedy algorithm has high time complexity and its accuracy is significantly weaker than that of the BPD algorithms. In addition, the SA algorithm performs similarly to the BPD algorithm in small networks, while is worse than BPD in large networks at the same setting of $\beta_{max} = 20$. In Fig. 10, we test the influence of β_{max} on SA. It can be found that large β_{max} is more capable of finding smaller dominating set since more occupation configurations are evaluated. However, large β_{max} also means high time complexity which makes the SA algorithm unfeasible in large-scale networks.

Purely from the the size of the constructed dominating set, the BPD and pure BPD have the almost same best performance where the BPD is slightly better than the pure BPD. They are quite close to the theoretical size of the MDS. However, the complexity of the pure BPD is much higher than that of the BPD, as illustrated in the above subsection (see Fig. 9). Therefore, considering both accuracy and efficiency, the BPD algorithm is optimal for solving the MDS problem of multiplex network.

C. Application verification

In Section III, we discussed three applications of the MDS of multiplex network. The effectiveness of the proposed BPD algorithm on monitoring epidemic and constructing extractive text summarization is straightforward. According to the experiments in Table 1, the size of the observer nodes in monitoring

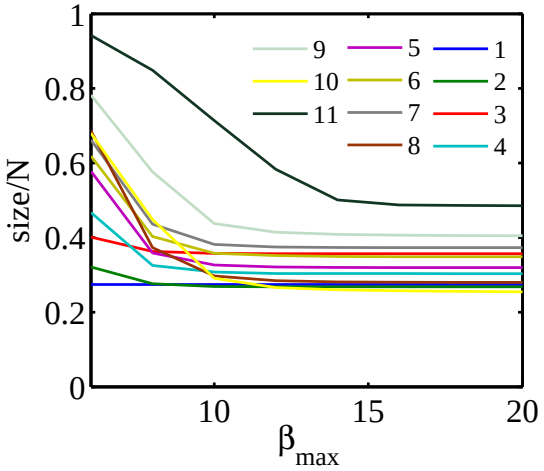

Fig. 10. The influence of β_{max} on SA. The multiplex networks used are the networks shown in Table 1.

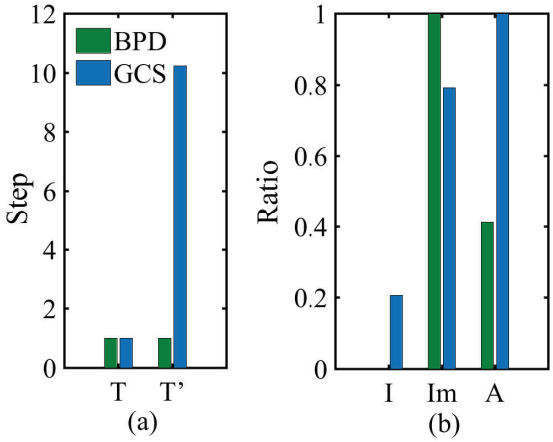

Fig. 11. Comparisons of the BPD-based MDS method and the GCS method on detection and controlling of epidemic spreading in multiplex network. The multiplex networks used contain two SF network layers with average degree 4. The dynamical parameters of SIR are λ_e=δ_e=λ_a=δ_a=1.

epidemic or the summary in extractive text summarization located based on the BPD is smaller than that obtained through other methods. Therefore, in this section, we just verify the effectiveness of the BPD on the early detection and controlling of epidemic spreading in multiplex network.

We assume an epidemic spreads on physical network layer following the susceptible-infected-recovered (SIR) model. Each node in SIR can be in three different states: susceptible, infected, or recovered. At each time step, the susceptible node could be infected by its infected neighbor with probability λ_e and the infected node could become recovered with probability δ_e. As discussed in Section III, to suppress the epidemic spreading, parts of network nodes, called observers, could be selected to distribute the awareness information or patch to other nodes through the virtual network layer to immunize them. When nodes belonging to the MDS of the multiplex network are selected to be observers, the epidemic could be detected and stopped immediately. In this sense, according to Table 1, the number of observers selected based on the BPD algorithm is the least compared with other algorithms. In addition, we compare the MDS method with the general competing spreading (GCS) method where no observers are specified [54], [59], [60]. In GCS, when a node is infected in physical network, it will become aware in the virtual network and spread the awareness information to its neighbors. If the susceptible nodes acquire the awareness information from their neighbors, they will become immunized against the epidemic. We assume the propagation of awareness information in virtual network also following the SIR model which is the same as the epidemic spreading on physical network and differs only in the infection and recovery rates, λ_a and δ_a, respectively.

Figure 11 presents the comparisons of the BPD-based MDS method and the GCS method. T and T' are the times that the epidemic is detected and eliminated respectively. One sees that BPD could find and stop the epidemic by using only one time step, since it can distribute the awareness information or patch to all network nodes as soon as the epidemic appears. Though the GCS method could immediately detect the epidemic, it requires relatively more time to stop the epidemic. Panel (b) shows the ratios of infected nodes (I), immunized nodes (Im) and nodes participating in distributing awareness information (A). Except the initial infected node, all the other nodes are immunized, thus no nodes are infected in the BPD algorithm. However, sine the distribution of the awareness information requires a certain amount of time, part of the nodes are infected and not all nodes can be immunized. In addition, the GCS needs almost all of the network nodes participating in distributing awareness information, while only the nodes of the constructed dominating set are needed in the BPD algorithm.

VI. Conclusion

In present work, we proposed the definition of the MDS of multiplex network and discussed several of its potential applications on epidemic monitor in multiplex network, early detection and control of epidemic in multiplex network and the construction of extractive text summarization. We built a spin glass model for the MDS problem of multiplex network which was then solved through the message passing theory. In addition, we proposed a BPD algorithm and a leaf removal strategy. Based on the proposed theories and algorithms, we can either predict the theoretical size of the MDS of multiplex network or construct approximate optimal MDS in practice.

In the future, we will look for more possible applications of the MDS of multiplex network, such as clustering or cross-view retrieval and recognition [84]–[88]. Another goal is to find better tools or methods to solve the MDS problem of multiplex network, such as machine learning and neural networks [89]–[91]. In addition, we could investigate the variants of the MDS of multiplex network, including the minimum-weight dominating set or minimum-connected dominating set of the multiplex networks. Several other classical optimization problems in graph can be also extended to multiplex networks, such as the independent set problem, the vertex cover problem, the hitting set problem, the graph coloring problem and the feedback vertex set problem, etc [16], [68]–[70].

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