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Opinion diversity and community formation in adaptive networks

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It is interesting and of significant importance to investigate how network structures co-evolve with opinions. In this article, we show that, a simple model integrating consensus formation, link rewiring, and opinion change allows complex system dynamics to emerge, driving the system into a dynamic equilibrium with the co-existence of diversified opinions. Specifically, similar opinion holders may form into communities yet with no strict community consensus; and rather than being separated into disconnected communities, different communities are connected by a non-trivial proportion of inter-community links. More importantly, we show that the complex dynamics may lead to different numbers of communities at the steady state with a given tolerance between different opinion holders. We construct a framework for theoretically analyzing the co-evolution process. Theoretical analysis and extensive simulation results reveal some useful insights into the complex co-evolution process, including the formation of dynamic equilibrium, the transition between different steady states with different numbers of communities, and the dynamics between opinion distribution and network modularity. *Published by AIP Publishing.* <https://doi.org/10.1063/1.4989668>

Individuals with different opinions interact with each other in social systems, leading to complex co-evolution of social opinions and social network structures. Three different actions are believed to be playing important roles in such kind of a co-evolution process. They include (i) consensus formation, where people holding similar opinions agree with each other; (ii) rewiring, where people holding significantly different opinions may tend to remove their links in between; and (iii) opinion changes, while people's opinion may change due to certain other reasons. In this manuscript, we propose a model which integrates all the three factors. It is shown that, with such an integration of actions, rather complex system dynamics emerges, driving the system to a dynamic equilibrium with the coexistence of communities holding diversified opinions. The equilibrium keeps nontrivial connections between different communities rather than forming into disconnected social groups, allowing further interactions and opinion changes to easily happen in the future. More importantly, it is observed that with a given tolerance between different opinion holders, the complex dynamics may lead to different numbers of communities at the steady state, sometimes in a counterintuitive manner. We construct a framework for theoretically analyzing the proposed co-evolution process. There is a good matching between the analytical and simulation results. Our studies have also revealed some other interesting insights, including the detailed process of system evolution into the dynamic equilibrium, the transition between different steady states with different numbers of communities, and

the impacts of a few different factors on the network modularity.

I. INTRODUCTION

Real-life adaptive complex networks co-evolve with the opinions, strategies, and actions of the individuals in the network. For example, upon the outbreak of a dangerous infectious disease, people tend to avoid contact with those who are infected, which changes the contact network topology and consequently changes the dynamics of epidemic spreading.^{1–6} In social networks, generally speaking, people tend to make friends with those who share similar opinions and such friendship may in turn help strengthen their common beliefs.^{1,7} Other examples include the co-evolution of languages and social structures,^{8–11} and the co-evolution of modern cities and urban transportation systems.^{12,13}

Extensive studies have been carried out to investigate the co-evolution of network structure and opinions;^{1,7,14–20} the earliest of which may be the one by Holme and Newman.⁷ Based on the well-known voter model where a randomly chosen node could persuade a random neighbor to adopt the same opinion,^{14,15} they further took into account the evolution of network topology by introducing a simple rewiring mechanism, where nodes have a certain chance to rewire their links with dissenters to similar opinion holders (when the difference between two nodes' opinions is larger than a certain tolerance level, these two nodes are said to be dissenters to each other). It is found that there exists a phase transition in the change of rewiring rate: the final state of the system varies from reaching global consensus to breaking

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into separated communities, each of which reaching its own community consensus. Follow-up studies include deriving analytical solutions for the phase transition,^{16,17} extending the model to directed networks,¹⁸ and introducing opinion noises¹⁹ and different interaction mechanisms such as connecting nodes with different opinions²⁰ and self-interaction,²¹ etc.

Another popular model for studying the interactions and dynamics between different opinions is the Deffuant model,^{22–25} where there is a continuous distribution of different opinions and a randomly chosen node may make consensus with a randomly selected neighbor holding similar opinion; their opinions hence come closer to each other or become the same. Similar to what happened to the voter model, rewiring was later introduced into the Deffuant model.¹ It is shown that the existence of rewiring makes it harder for an adaptive network to reach global consensus; network may finally evolve into a few big opinion communities, each of which holding its community consensus.

Opinion change was introduced in Refs. 26–28, termed as “noises” in these references. It is shown that in the Deffuant model with a continuous opinion distribution and a fixed network topology, different speeds of random opinion change may drive the system either to an ordered state with a set of well-defined opinion groups, or a disordered state where the opinion distribution tends to be uniform.

It can be observed that most of these existing models lead to similar final steady states where the network either reaches a global consensus or breaks into disconnected communities, each of which reaches its own consensus. Such observations however do not reflect common reality: (i) real-world opinions are typically widely diversified with no global consensus;^{29,30} “mainstream opinions” may exist, but they do not become global consensus eliminating other opinions; and mainstream opinions themselves evolve, sometimes rather quickly; and (ii) though people as individuals may cut off links with dissenters, similar opinion holders as a whole seldom get cut off from communities of different opinion holders; some people may keep connections with dissenters for various reasons. While it is understood that many details have to be omitted to allow mathematical modeling focus on the most important factors, the fact that many existing models lead to the steady state essentially different from real-life observations nevertheless calls for closer examination of these models. The questions include what factors are essential for leading to relatively realistic opinion distribution in social systems as discussed above, and how these factors contribute to defining the steady state of the systems, etc.

In this article, we study on a model integrating three different factors, namely (i) consensus formation, where directly connected network nodes may try to reach consensus if their opinions are similar to each other; (ii) link rewiring, where network nodes may rewire some of their links with dissenters to similar opinion holders; and (iii) mutation, where network nodes may change their opinions for various reasons other than consensus making. Examples of opinion mutation include change of mind due to an unexpected critical event, religion conversion,^{31–33} change in opinion due to

abruptly changed environment (e.g., immigration³⁴), etc. In such cases, an individual’s opinion may change quickly and significantly, while still keeping a significant part of his/her original connections.³⁵ These individuals thus may share a similar opinion with one community while being connected to another community. They hence may become the “bridges” between different communities with significantly different opinions. Note that we define an opinion community as a group of opinion holders holding either the same opinion (when there is no mutation) or similar opinions with a bell-curve shape distribution (when mutation exists) and with relatively denser connections in between.

As we will discuss in detail in later parts of this article, with all the three factors being integrated into the same model, complex system dynamics emerge. Specifically, instead of being broken into separate communities each holding its own community consensus, similar opinions would form into communities interconnected by a non-trivial number of inter-community links. In each community, different opinions coexist, typically following a stable, bell-curve-style distribution. As discussed earlier, such a model may arguably better resemble what we may observe in real life, and it allows complex system dynamics to quickly emerge when intruders (e.g., new ideas) come in or when certain internal/external driving forces are applied to the underlying social networks. Interestingly, it is found that the integration of the three factors may also lead to different numbers of communities in the steady-state network with a given tolerance level: the existence of opinion mutation tends to increase the number of opinion communities.

We develop a theoretical framework to describe the co-evolution process. Theoretical analysis and extensive simulation results help reveal some useful insights into the system’s evolution process, including the formation of dynamic equilibrium, the transition between steady states with different number of communities, and the dynamics between opinion distribution and network modularity.

II. ADAPTIVE MODEL

In this article, we adopt the continuous opinion model where initial opinions are generated following a continuous distribution over the region $[0, 1]$. As discussed earlier, there are three key components integrated into this model, namely consensus formation, link rewiring, and opinion mutation, respectively. Specifically, each node in the network is initialized with a uniformly distributed random opinion value between 0 and 1. At each time step t , a node A is randomly selected together with its random neighbor B . Denote the opinion of a node X at time t as $o(t, X)$. If $|o(t, A) - o(t, B)| > d$, where d is the tolerance threshold, with a probability w , node A would rewire the link to a randomly selected node C sharing similar opinion with node A , i.e., $|o(t, A) - o(t, C)| \leq d$; if $|o(t, A) - o(t, B)| \leq d$, then with a probability $1 - w$, the two nodes would perform consensus making by updating their opinions as follows:

$$o(t+1, A) = o(t, A) - \mu[o(t, A) - o(t, B)], \quad (1)$$

$$o(t+1, B) = o(t, B) + \mu[o(t, A) - o(t, B)], \quad (2)$$

where $\mu \in (0, 1/2]$. Following most of the existing literature,^{22–28} for simplicity we let $\mu = 1/2$ in the rest of the paper. Furthermore, with a probability p , a node may change its opinion to another random value in $[0, 1]$. We term such kind of opinion change as mutation and p as the *mutation rate*.

III. RESULTS

In this section, we show that with the proposed co-evolution model, a network evolves into a dynamic equilibrium with network nodes forming into several communities connected by a non-trivial proportion of inter-community links; and the nodes in each community hold a certain range of opinions with a bell-curve-style distribution. We also show the dynamic process of the co-evolution between the opinions and network topology and explain the formation of the dynamic equilibrium.

Note that our extensive simulation results show that, as long as there exists a positive mutation rate, the initial state of the opinion distribution in the network has virtually no effects on its final steady state, though a lower mutation rate typically leads to a much slower convergence to the final state. While detailed discussions on the effects of the initial state are lengthy and thus have to be left out to another report, in this paper, unless otherwise specified, we assume that the initial state of the system has a uniform opinion distribution.

To start with some relatively simple examples, Fig. 1 shows the snapshots of some networks and their opinion distributions at the steady state. Different values of mutation rate p and tolerance threshold d are adopted. The network starts as an (Erdős–Rényi) random network³⁶ with a size of $N = 10^3$, an average nodal degree of $\langle k \rangle = 10$, and a uniform distribution of different opinions. The simulation lasts for 2×10^6 steps, where at each step a single node is randomly selected, and with a probability $1 - w$, this node may make consensus with one of its neighbors or with a probability w , to rewire one of its links. After that, a node may randomly mutate its own opinion at a probability p . Our simulations

confirm that the number of time steps is large enough for the system to reach the steady state.

Figures 1(a) to 1(d) show the network topologies of 4 different cases with different values of d , w , and p , where nodes form several communities connected by a non-trivial proportion of inter-community links. Figures 1(e) to 1(h) show the corresponding distributions of nodes' opinions in these 4 cases, respectively. We see that in all the 4 cases, several bell-curve-style peaks with approximately equal peak values are formed, and a higher mutation rate leads to denser inter-community connections [comparing Figs. 1(a), 1(b) and 1(c)] and more diversified opinion distributions [comparing Figs. 1(e), 1(f) and 1(g)]. Notice that the system finally reaches a dynamic equilibrium: nodes will keep making consensus, changing their opinions, and constantly rewire their links. Nevertheless, the basic statistical properties such as opinion distribution and the proportion of inter-community links remain stable.

To further examine how nodes with different opinions distribute in different communities, we paint nodes with opinions in different ranges with different colors. Specifically, for a network finally forming into M communities, nodes holding opinions within the interval I_i (we define I_1 as $[0, 1/M]$ and I_i as $[(i-1)/M, i/M]$ for $i = 2, \dots, M$ respectively) are painted in the same color. For example, for networks in Figs. 1(a) to 1(c), where two communities are formed, nodes holding opinions within $[0, 0.5]$ and $(0.5, 1]$ are painted in yellow and red, respectively. Examining Figs. 1(a) to 1(d), we find that nodes with the same color mostly stay in the same community in the final state. This means that nodes in the same community mostly share similar (but not necessarily the same) opinions. More specifically, the opinions of the nodes in the same community typically have a bell curve distribution with a peak value approximately at the center as the “mainstream opinion.” A theoretical framework and detailed analysis will be presented in Sec. IV A.

Note that the existence of opinion mutation plays a critical role in defining the final state: existing studies^{1,7,14–21} have shown that the integration of consensus formation and

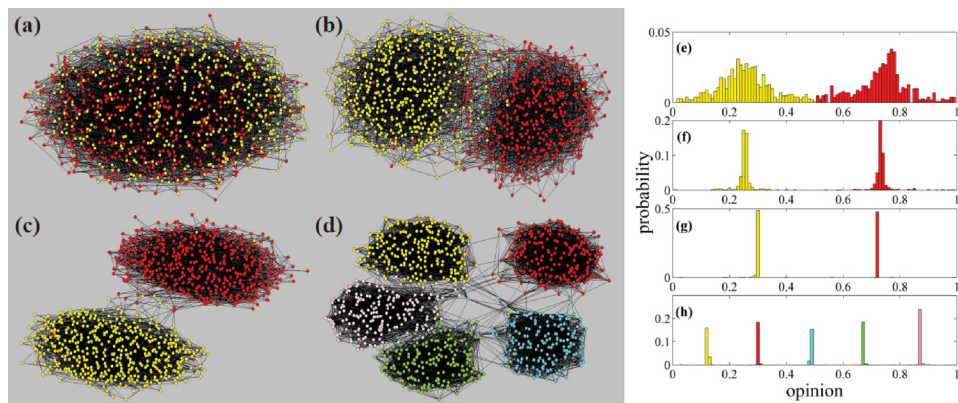


FIG. 1. Network structures at the steady state where (a) $d = 0.25$, $p = 0.1$, and $w = 0.5$; (b) $d = 0.25$, $p = 0.01$, and $w = 0.5$; (c) $d = 0.25$, $p = 0.001$, and $w = 0.5$; and (d) $d = 0.1$, $p = 0.001$, and $w = 0.5$ and their corresponding opinion distribution where (e) $d = 0.25$, $p = 0.1$, and $w = 0.5$; (f) $d = 0.25$, $p = 0.01$, and $w = 0.5$; (g) $d = 0.25$, $p = 0.001$, and $w = 0.5$; and (h) $d = 0.1$, $p = 0.001$, and $w = 0.5$. The network starts as an ER random network with a size of $N = 10^3$ and an average nodal degree of $\langle k \rangle = 10$. For Figs. (a), (b), and (c), the nodes in yellow and red, respectively, hold opinions within the range of $[0, 1/2]$ and $(1/2, 1]$. For Fig. (d), the nodes in yellow, red, blue, green, and pink, respectively, hold opinions within the range of $[0, 1/5]$, $(1/5, 2/5]$, $(2/5, 3/5]$, $(3/5, 4/5]$, and $(4/5, 1]$.

link rewiring leads to a steady state with a few disconnected communities, each holding its single-value community consensus. We now examine the process of the co-evolution of opinion and network topology which finally leads to the formation of the dynamic equilibrium. We examine the case where $d=0.1$, $p=0.001$, and $w=0.5$, which shall finally leads to the formation of 5 different communities. Figures 2(a), 2(b), and 2(c) show the respective opinion distributions at early, middle and final stages of the co-evolution process; while Figs. 2(d) and 2(e) show the corresponding network topologies at the middle and final stages, respectively. We number the 5 communities from 1 to 5 in an increasing order of their respective peak opinion values and paint their nodes in yellow, red, light blue, green, and pink colors, respectively. The whole evolution process can be viewed as an instability propagating from the boundaries towards the center [Figs. 2(a), 2(b) and 2(c)]. Specifically, at the early stage of the evolution, the two peaks of the opinion distribution close to the boundaries (i.e., 0 and 1) form up [Fig. 2(a)] and quickly grow into bell-curve-style peaks [Fig. 2(b)]. Looking at Fig. 2(d), we can see that at this moment, communities 1 and 5, which correspond to these two opinion peaks, have already grown mature and become largely separated from the rest of the network, while the rest part remains as a giant component mainly holding opinions ranging from 1/5 to 4/5 [indicated by the red circles in Figs. 2(b) and 2(d)]. In the middle stage, communities 2 and 4 emerge from this giant component and then largely separate themselves from the other nodes. This process continues until the dynamic equilibrium is finally reached. A similar process has been observed for all the different values of d : formation of communities starts from the two communities holding opinions closest to the boundaries and then propagates inbound towards the central opinion value.

Existing studies show that when a system does not reach a single global consensus but evolves to have multiple communities instead, the number of communities M and the tolerance d roughly follow a relationship of $M \sim 1/2d$.^{1,23} While such observations reveal the relationship between M

and d , to the best of our knowledge, none of them has investigated the relation between M and mutation rate p . We show that, with other parameters given and fixed, different values of p may lead to different numbers of communities at the steady state.

To start, we first illustrate the relationship between w and M . In Fig. 3, we show the results in an ER random network with $N = 10^5$, $\langle k \rangle = 20$, $d=0.1$, and $p=0.01$. We can see that when there is no rewiring operation, i.e., $w=0$, 4 communities will be formed up in the steady-state network; when $w=0.9$, however, 5 communities will be formed up in the steady-state network. In fact, it is a common conclusion applying to all the cases we have tested that a higher rewiring probability tends to lead to having more communities in the steady-state network.

Such an observation can be understood. To illustrate the dynamic process leading to the equilibrium, we show in Fig. 3 the opinion distribution in the two networks when the numbers of consensus formation operations (denoted as NCF in the figure) are 10^4 , 8×10^4 and 1.5×10^5 respectively, and at the final steady state, respectively. As discussed earlier, in both networks, there is an instability propagating inbound starting from the boundary. Specifically, two peaks form up close to the boundaries and gradually shift inbound. The inbound shifting however may be interrupted by rewiring operations since such operations, if fast enough, may make the two communities holding the outmost opinions quickly get largely separated from the rest part of the network. The opinion distribution of two communities shall then get largely stabilized, leaving a relatively larger opinion range in between, and consequently may allow more communities to be formed up in the further system evolution.

An example can be observed in Fig. 3. In Fig. 3(a), peaks 1 and 5 shift inbound until they reach 0.130 and 0.870, respectively, when NCF roughly equals 1.5×10^5 . In Fig. 3(b), the inbound propagation is interrupted: the two communities holding the outmost opinions are largely stabilized when NCF roughly equals 8×10^4 , with respective opinion peaks at 0.110 and 0.890. Such interruptions change the

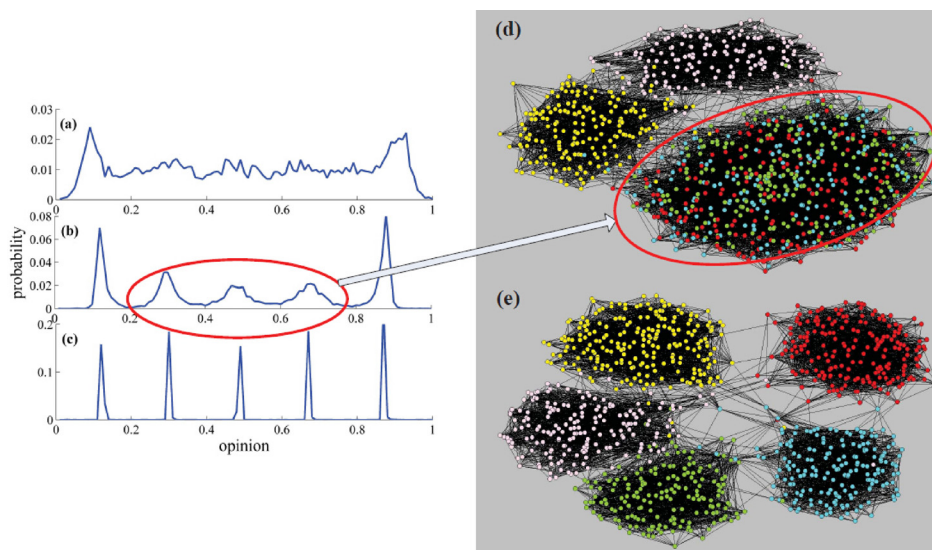


FIG. 2. Opinion distribution at different time steps of the evolution process: (a) $t=500$, (b) $t=3000$, and (c) $t=2 \times 10^6$; and the network topology at (d) $t=3000$ and (e) $t=2 \times 10^6$. $d=0.1$, $p=0.001$, and $w=0.5$. The network starts as an ER random network with a size of $N=10^3$ and an average nodal degree of $\langle k \rangle = 10$. In each time step, a single pair of nodes is chosen.

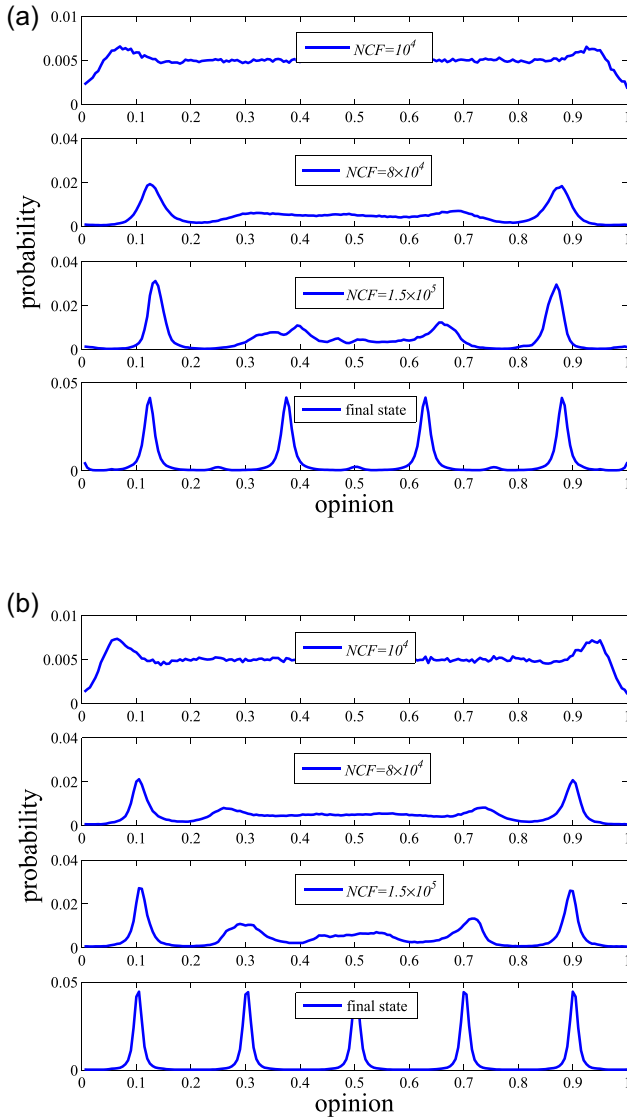


FIG. 3. Evolution of opinions in the ER network with different rewiring probabilities at (a) $w=0$ and (b) $w=0.9$. Let $N=10^5$, $\langle k \rangle=20$, $d=0.1$, and $p=0.01$. The results show the average in 100 independent networks.

further co-evolution of opinion and network topology: the peaks of communities 2 and 3, after their emergence, propagate inbound and stop approximately at opinions $o=0.375$ and $o=0.625$, respectively, in Fig. 3(a), which does not leave enough space for any further community to grow mature. For the network in Fig. 3(b), however, peaks 2 and 4 shift inbound and stop approximately at $o=0.305$ and $o=0.695$, respectively. Enough space is left for the central peak 3 to grow to the full size. Note that such phenomena can be more easily observed for small values of d . For example, when $d=0.05$, by increasing w , we may observe two transitions in community number: first from 8 communities to 9 communities, and then from 9 to 10.

With the understanding that a large value of w may lead to a larger number of opinion communities in the final-state networks, it would be interesting to study on the threshold value of w , denoted as w_c hereafter, leading to the transition in the number of communities in different networks. Figure 4(a) illustrates the threshold values in ER random networks

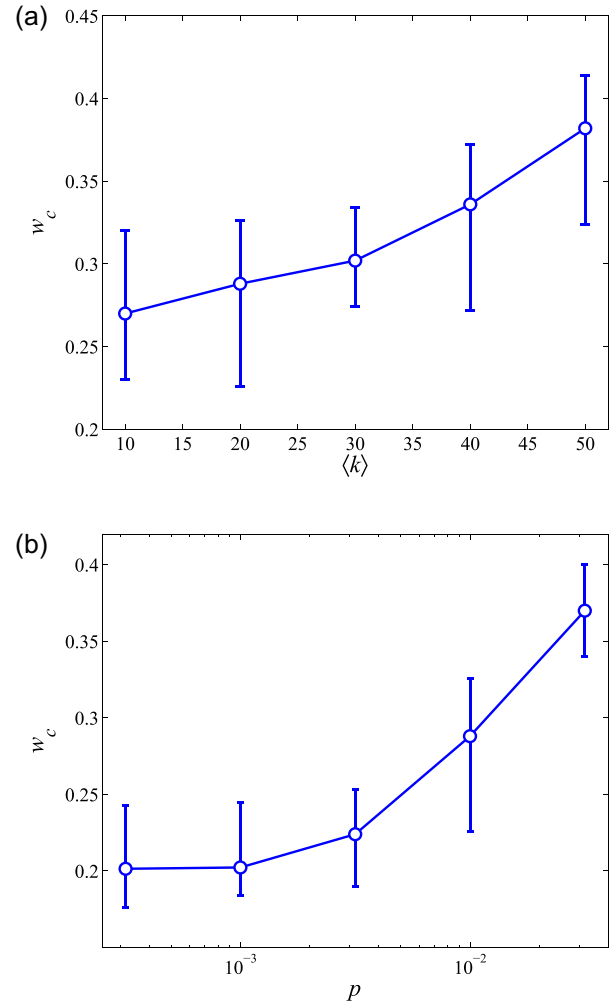


FIG. 4. Threshold value w_c vs. (a) average degree $\langle k \rangle$, for $d=0.1$ and $p=0.01$; (b) mutation probability p , for $d=0.1$ and $\langle k \rangle=20$ in the ER random network. When $w > w_c$, 5 communities shall coexist in the steady-state network; while for $w < w_c$, only 4 communities would coexist. The 95% confidence intervals are given for the average in 10 independent networks.

with different average nodal degrees $\langle k \rangle$. For each $\langle k \rangle$ value, 10 random networks are generated and in each network, the threshold value w_c is the average of 10 rounds of independent simulations. In each round of simulation, a trial-and-error approach is adopted to decide the value of w_c with a step length of 0.01. The figure shows the average in all the 10 networks with 95% confidence range. It can be observed that in all these networks, a higher value of w averagely leads to a steady state with more communities in the network; and the threshold value of the transition in the community number increases with the average nodal degree. This can be understood since a higher average nodal degree slows down the separation between different communities during the inbound instability propagation. Consequently it requests a higher rewiring rate to cause an increase in the number of communities at the steady state.

Figure 4(b) shows the relationship between the mutation probability p and threshold value w_c . Again, 10 random networks are generated for each p value, and in each network, the threshold value w_c is the average of 10 rounds of independent simulations. The results show that a small value of p

(e.g., when p increases from 0 to 1.0×10^{-3}) may not make any significant difference to the value of w_c . When the value of p is large enough, however, a larger p basically leads to a larger w_c . Such an observation is understandable: for a too small value of p , mutation influences on the network evolution are trivial compared to those of consensus formation and rewiring, and thus make hardly any visible changes to w_c ; while for a large enough value of p , random mutations essentially compromise the effects of rewiring in separating different communities, and hence increase the threshold value w_c . Equivalently we may draw the conclusion that for a given value of w , a larger value of p tends to decrease the number of communities M in the steady state. It is interesting, and somewhat counterintuitive, that a higher mutation rate, which we may expect to observe in a more tolerant or less stable society, actually decreases the number of opinion communities at the steady state.

Note that the above observations are valid only when p is large but not too large. A further increased p value may further change the opinion distribution in the steady-state network, as we shall discuss in detail in Sec. IV A.

IV. THEORETICAL ANALYSIS

In this section, we present theoretical analysis on the co-evolution of opinion and network structures. Specifically, a theoretical framework for describing the co-evolution process is proposed in Sec. IV A, which matches well with the simulation results. In Sec. IV B, we study in further details the impacts of a few factors on the modularity of the steady-state network.

A. Analysis of the co-evolution process

The co-evolution process and the final steady state can be mathematically described by mean-field analysis. The basic idea is to reveal the correlation between opinions on two end nodes of a randomly selected link. Due to consensus formation and rewiring operations, the two end nodes of each link tend to have similar opinions. Assume that time is continuous and nodes are selected at a rate ϵ to make decisions. Denote the probability density that a randomly selected node has an opinion x_1 and its random neighbor has an opinion x_2 at time t as $P(t, x_1, x_2)$. We have $P(t, x_1, x_2) = P(t, x_2, x_1)$. Further define a conditional probability density function $c(t, x_1, x_2)$, which denotes that at time t , given that a node has an opinion x_1 , the probability that its random neighbor holds an opinion x_2 . We have $c(t, x_1, x_2) = P(t, x_1, x_2) / \rho(t, x_1)$, where $\rho(t, x_1)$ is the probability density of opinion x_1 at time t , i.e., $\rho(t, x_1) = \int_0^1 P(t, x_1, x) dx = \int_0^1 P(t, x, x_1) dx$. Assume that at each time step, a randomly selected node may mutate to another opinion x with a probability density $b(x)$. For the simple random uniform mutation being considered in this article, $b(x) = 1$ for any $x \in [0, 1]$. The whole process can be described by the simple mean field analysis as follows.

Considering how mutation may help increase $P(t, x_1, x_2)$, we have

$$\begin{aligned} P_{\text{mutation}}^+(t, x_1, x_2) &= \langle k \rangle \epsilon p \int_0^1 \rho(t, x) c(t, x, x_2) b(x_1) dx \\ &\quad + \langle k \rangle \epsilon p \int_0^1 \rho(t, x) c(t, x, x_1) b(x_2) dx \\ &= \langle k \rangle \epsilon p \int_0^1 P(t, x, x_2) b(x_1) dx + \langle k \rangle \epsilon p \int_0^1 P(t, x, x_1) b(x_2) dx \\ &= \langle k \rangle \epsilon p \cdot \rho(t, x_2) + \epsilon \langle k \rangle p \cdot \rho(t, x_1). \end{aligned} \quad (3)$$

Meanwhile, mutation may decrease $P(t, x_1, x_2)$ at a rate of

$$\begin{aligned} P_{\text{mutation}}^-(t, x_1, x_2) &= \langle k \rangle \epsilon p \cdot \rho(t, x_1) c(t, x_1, x_2) + \langle k \rangle \epsilon p \cdot \rho(t, x_2) c(t, x_2, x_1) \\ &= \langle k \rangle \epsilon p \cdot P(t, x_1, x_2) + \langle k \rangle \epsilon p \cdot P(t, x_2, x_1). \end{aligned} \quad (4)$$

For each pair of nodes with an opinion difference less than d , they make consensus at a probability of $1 - w$. Hence for $x_1 = x_2$, consensus formation increases $P(t, x_1, x_2)$ at a rate of

$$\begin{aligned} P_{\text{consensus1}}^+(t, x_1, x_2) &= \epsilon(1 - w) \cdot \int_{-d/2}^{d/2} \rho(t, x_1 - x) c(t, x_1 - x, x_2 + x) dx \\ &= \epsilon(1 - w) \int_{-d/2}^{d/2} P(t, x_1 - x, x_2 + x) dx. \end{aligned} \quad (5)$$

For $x_1 \neq x_2$

$$\begin{aligned} P_{\text{consensus2}}^+(t, x_1, x_2) &= (\langle k \rangle - 1) \epsilon(1 - w) \cdot \int_{-d/2}^{d/2} P(t, x_1 + x, x_1 - x) c(t, x_1 + x, x_2) dx \\ &\quad + (\langle k \rangle - 1) \epsilon(1 - w) \cdot \int_{-d/2}^{d/2} P(t, x_2 + x, x_2 - x) c(t, x_2 + x, x_1) dx. \end{aligned} \quad (6)$$

Combining the above two different cases, we have

$$\begin{aligned} P_{\text{consensus}}^+(t, x_1, x_2) &= \epsilon(1 - w) \int_{-d/2}^{d/2} P(t, x_1 - x, x_2 + x) dx \cdot \delta_1(x_1, x_2) \\ &\quad + (\langle k \rangle - 1) \epsilon(1 - w) \\ &\quad \cdot \int_{-d/2}^{d/2} P(t, x_1 + x, x_1 - x) c(t, x_1 + x, x_2) dx \\ &\quad + (\langle k \rangle - 1) \epsilon(1 - w) \\ &\quad \cdot \int_{-d/2}^{d/2} P(t, x_2 + x, x_2 - x) c(t, x_2 + x, x_1) dx, \end{aligned} \quad (7)$$

where $\delta_1(x_1, x_2) = 1$ if $x_1 = x_2$; otherwise $\delta_1(x_1, x_2) = 0$.

Consensus formation decreases $P(t, x_1, x_2)$ at

$$\begin{aligned} P_{\text{consensus}}^-(t, x_1, x_2) &= \langle k \rangle \epsilon(1 - w) \int_{-d}^d P(t, x_1, x_1 - x) c(t, x_1, x_2) dx \\ &\quad + \langle k \rangle \epsilon(1 - w) \int_{-d}^d P(t, x_2, x_2 - x) c(t, x_2, x_1) dx. \end{aligned} \quad (8)$$

For each pair of nodes with an opinion difference greater than d , rewiring happens at a probability of w . The link between the two nodes will be removed; and one of the end nodes of the link will be connected to a random node holding a similar opinion (with a difference less than d). Thus link rewiring increases $P(t, x_1, x_2)$ at a rate of

$$\begin{aligned} P_{\text{rewiring}}^+(t, x_1, x_2) &= \epsilon w \int_d^1 P(t, x_1, x_1 \pm x) \rho(t, x_2) / \phi(t, x_1, d) dx \cdot \delta_2(x_1, x_2) \\ &+ \epsilon w \int_d^1 P(t, x_2, x_2 \pm x) \rho(t, x_1) / \phi(t, x_2, d) dx \cdot \delta_2(x_1, x_2), \end{aligned} \quad (9)$$

where $\delta_2(x_1, x_2) = 1$ for $|x_1 - x_2| \leq d$; and $\delta_2(x_1, x_2) = 0$ otherwise. Here we also define $\phi(t, x_1, d) = \int_{-d}^d \rho(t, x_1 + x) dx$. Note that $\rho(t, x_1) / \phi(t, x_1, d)$ denotes the probability that at time t , a node with an opinion x_2 is chosen as the rewiring target and gets connected with a node with opinion x_1 .

Similarly, it can be shown that link rewiring decreases $P(t, x_1, x_2)$ at a rate of

$$\begin{aligned} P_{\text{rewiring}}^-(t, x_1, x_2) &= \epsilon w P(t, x_1, x_2) (1 - \delta_2(x_1, x_2)) \\ &+ \epsilon w P(t, x_2, x_1) (1 - \delta_2(x_1, x_2)). \end{aligned} \quad (10)$$

Combining all the terms above, we have that the probability density function $P(t, x_1, x_2)$ changes at a rate of

$$\begin{aligned} \Delta P(t, x_1, x_2) &= P_{\text{mutation}}^+(t, x_1, x_2) + P_{\text{consensus}}^+(t, x_1, x_2) \\ &+ P_{\text{rewiring}}^+(t, x_1, x_2) - P_{\text{mutation}}^-(t, x_1, x_2) \\ &- P_{\text{consensus}}^-(t, x_1, x_2) - P_{\text{rewiring}}^-(t, x_1, x_2). \end{aligned} \quad (11)$$

The whole co-evolution process can be numerically calculated using Eqs. (2)–(10). Extensive simulation results

show that, for systems with a mutation probability $p > 0$, with or without link rewiring, the initial opinion distribution in most cases has no significant influence on the final opinion distribution. Similar observations have been made on Deffuant model with noises (what we call mutation).²⁸

We first compare the theoretical results of the final-state opinion distribution with the simulation results. For each set of parameters, 100 ER random networks are generated, each with a size of $N = 10^5$ and an average nodal degree of $\langle k \rangle = 20$. Assume that one pair of nodes is selected at each time step. We record and average the opinion distribution every 2000 time steps from $t = 8 \times 10^6$ to $t = 10^7$ as the final state opinion distribution. It has been confirmed that the time is long enough for the network to reach the steady state. The average results in the 100 networks are considered as the steady-state opinion distribution corresponding to the given set of parameters.

Figure 5 compares the simulation and theoretical results of opinion distribution in the steady-state network. The theoretical results are numerically calculated using Eq. (10). We find that the theoretical results match well with the simulation results. The opinion distribution has a few bell curves with approximately equal peak values. Note that the only exception is the case in Fig. 5(c) when $d = 0.1$ and $p = 0.1$. The relatively small opinion tolerance combined with the high mutation rate lead to a high density of inter-community links, which account for about 65% of all the links. The high mutation rate weakens the community separation and growth to the level that communities 2, 3, and 4 never get a chance to be fully formed up. Even in the steady state, these three communities have significantly lower peak values than those of communities 1 and 5.

To achieve further insights into the co-evolution process, we examine the evolution of $P(t, x_1, x_2)$ in detail. Let

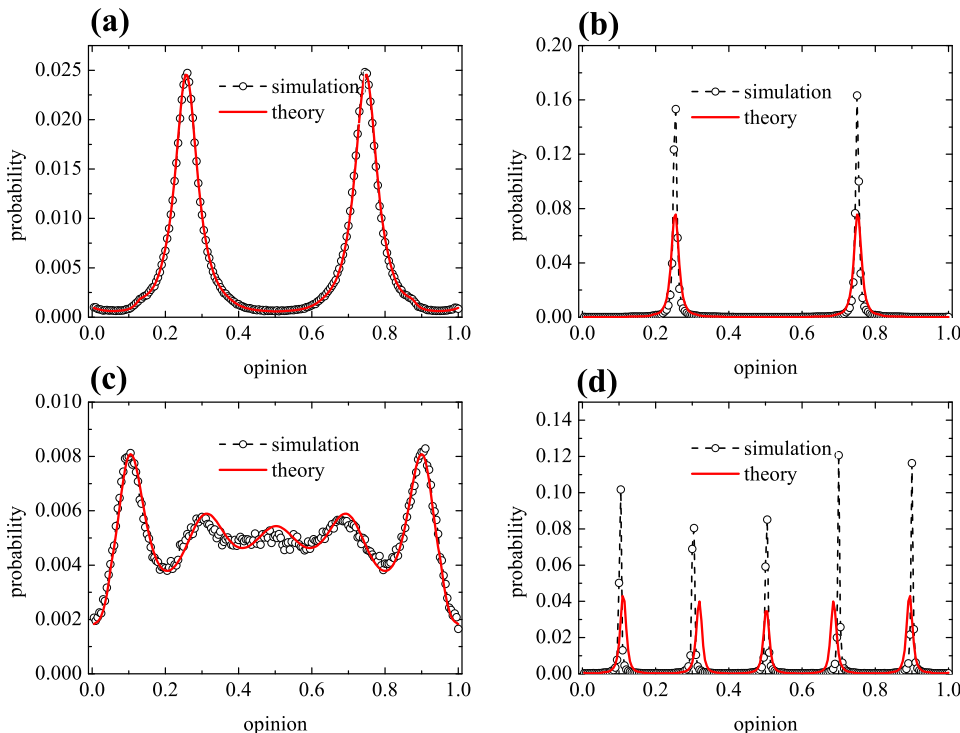


FIG. 5. Comparisons between the simulation and theoretical results of steady-state opinion distribution in the ER random network with different sets of parameters: (a) $d = 0.25$, $p = 0.1$, and $w = 0.5$; (b) $d = 0.25$, $p = 0.01$, and $w = 0.5$; (c) $d = 0.1$, $p = 0.1$, and $w = 0.5$; and (d) $d = 0.1$, $p = 0.01$, and $w = 0.5$. We assume that the network size is $N = 10^5$ and the average nodal degree is $\langle k \rangle = 20$.

$d=0.1$, $w=0.5$, $p=0.01$, and $\langle k \rangle = 20$ where the network would finally form into 5 communities. Figure 6 shows both the theoretical and simulation results for the evolution of $P(t, x_1, x_2)$. The theoretical results are obtained from iterative numerical calculations of Eq. (10) by setting $\epsilon = 0.02$, which means that in each iteration, 2% of the nodes make decisions. For the corresponding numerical simulations, we let each iteration contain 2000 time steps, which also allows about 2% of the nodes make decision in each iteration. Denote the iteration number as T . We only show the results of a single round of simulation as the evolution speeds for different rounds of simulations vary due to noises, making it difficult to generate a clear image of average results. However, massive rounds of simulations have revealed that the system co-evolution always goes through nearly the same process, though not necessarily at the same speed; and there is a good match between theoretical and simulation results at different time as far as the pattern of $P(t, x_1, x_2)$ is concerned.

As we could observe in Fig. 6, at the early stage of the co-evolution, nodes holding similar opinions gradually connect to each other at relatively higher probabilities. This is indicated by the high probability stripe along the diagonal of opinion distribution map as can be seen in Figs. 6(a) and 6(e). Then the boundary-opinion holders (holding opinions close to 0 or 1) would make consensus first and gather together to form up the first two communities indicated by the two bright dots at the corner of Figs. 6(a) and 6(e). The formation of communities then propagates inbound, forming up two more communities, indicated by the two bright dots closer to the centers in Figs. 6(b) and 6(f). Finally, an opinion peak emerges around the central opinion value, as indicated by the bright dots in the center of Figs. 6(c) and 6(g). When the system reaches the final dynamic equilibrium, five communities are formed up, each of which mostly containing

similar opinions around a peak value. Similar opinion holders connect to each other with a relatively high density of connections in these communities while they still preserve some connections with dissenters.

B. Modularity of the steady-state networks

In this subsection, we further examine the impacts of a few factors, namely the rewiring speed, the mutation rate and the average nodal degree, on the modularity of the steady-state networks. From the discussions in Sec. III, intuitively we may expect that since a higher mutation rate and a large average nodal degree tend to weaken the community structures, they may lead to a lower modularity level of the steady-state network. As to the rewiring speed, a higher rewiring speed tends to enhance community structures and consequently increase the network modularity at the steady state. While such intuitive results can be confirmed by simulation as later we would see, it is of interest to study in detail how the network modularity is changed with the changes in these parameters, which is the main goal of this subsection.

We start by proposing an approximate analysis on the proportion of intercommunity links at the steady state. To simplify the discussions, we assume that in the final state there are M equal-size communities; that is, the i th community mainly contains those nodes holding opinions within the interval I_i {defined in Sec. III that I_1 is $[0, 1/M]$ and I_i is $[(i-1)/M, i/M]$ for $i=2, \dots, M$ respectively}. This assumption is supported by the extensive simulation results reported in the Sec. III, with the only exception for the case when the mutation rate is quite high. We argue that in most real-life systems, the mutation rate is of a low value, especially in those systems that can sustain a steady state. With such an assumption, the simple analysis can be derived as below.

Suppose that at the steady state, a randomly selected node averagely has q proportion of its links being

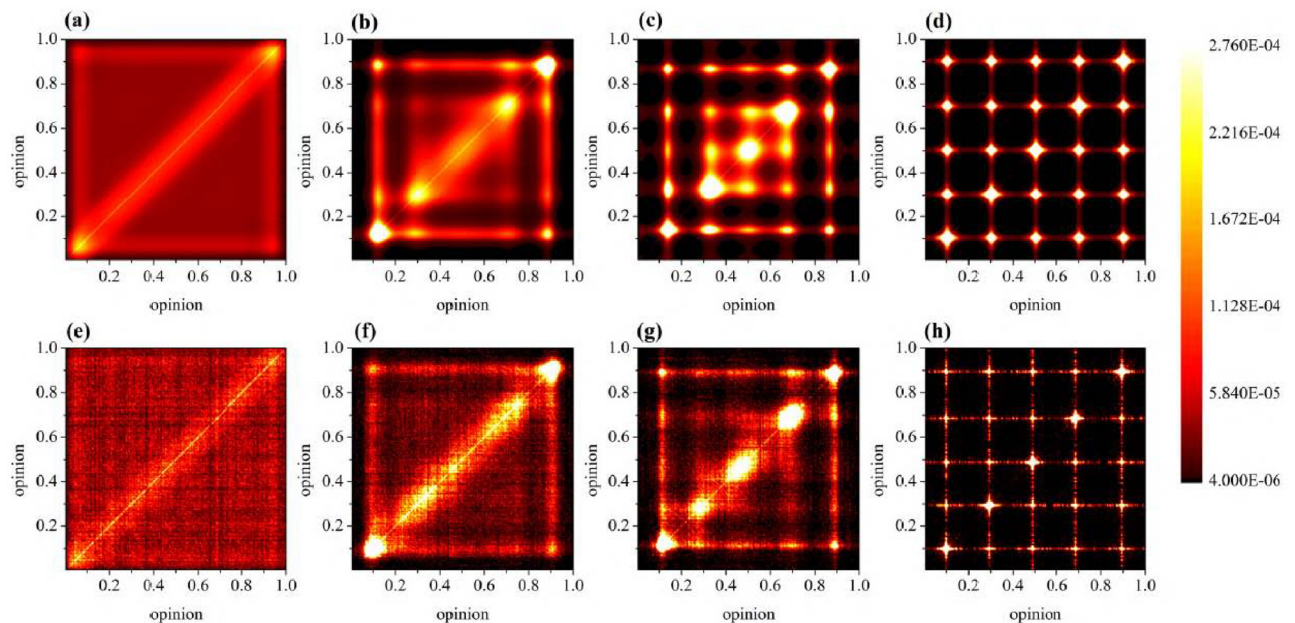


FIG. 6. Evolution of $P(t, x_1, x_2)$ for theoretical results at (a) $T=25$, (b) $T=100$, (c) $T=150$, and (d) $T=5000$; and a single simulation case at time step (e) $T=25$, (f) $T=100$, (g) $T=150$, and (h) $T=5000$. Note that for the number of time steps denoted as t , we have $t=2000T$.

inter-community links. Let node pairs still be randomly selected at a rate of ϵ . Due to mutation, the proportion of inter-community links is changed at a rate of

$$q^+ = \epsilon \left(p\langle k \rangle (1 - q) - \frac{p\langle k \rangle q}{1 - M} \right) \frac{M - 1}{M}. \quad (12)$$

To explain Eq. (12), we see that when a node with q proportion of inter-community links and $(1 - q)$ proportion of intra-community links is randomly selected and mutate to an opinion belonging to the range of a different community, all its intra-community links become inter-community links while averagely $1/(M - 1)$ of its inter-community links become intra-community links. The absolute number of inter-community links therefore is changed by $\langle k \rangle p(1 - q) - \langle k \rangle pq/(1 - M)$. The probability that a mutation does change a randomly selected node's opinion to be within the opinion range of another community is $(M - 1)/M$.

At the same time, due to the rewiring operation, inter-community links are broken at a rate of

$$q^- = \epsilon q w. \quad (13)$$

This equation shows that for a randomly chosen node and its random neighbor, the link connecting them has a probability q to be an inter-community link; and if it is an inter-community link, it is broken at a probability w . Note that here we neglect the effects of consensus formation as we assume that in the final steady state, consensus formation almost only happens between nodes in the same community. The dynamics of the inter-community links at the steady state therefore can be expressed as

$$\epsilon q w = \epsilon \left(p\langle k \rangle (1 - q) - \frac{p\langle k \rangle q}{1 - M} \right) \frac{M - 1}{M}. \quad (14)$$

And the inter-community link, proportion q can be expressed as

$$q = \frac{p\langle k \rangle}{w + p\langle k \rangle} \frac{M - 1}{M}. \quad (15)$$

With Eq. (15), we can then proceed to calculate the modularity of the network. We adopt the definition in Ref. 37 that the network modularity can be calculated as

$$Q = \frac{1}{4m} \sum_{ij} \frac{A_{ij} - (k_i k_j)/2m}{2m} s_i s_j, \quad (16)$$

where A is the adjacency matrix, m is the number of edges, and k_i is the nodal degree of node i . For all the possible combinations of i and j , if nodes i and j are in different communities, $s_i s_j + 1 = 0$; and $s_i s_j + 1 = 2$ otherwise. Since a randomly selected node averagely has q proportion of inter-community links, for a given node i and a randomly selected node j , the probability that they are connected and in the same community is $k_i(1 - q)/N$, while with a probability $1/M$, the two nodes are in the same community. Therefore there exists simple relationship between Q and q that

$$\begin{aligned} Q &= \frac{1}{4m} \sum_i \left(k_i - \frac{k_i 2m/M}{2m} \right) (1 - q) \\ &\quad - \frac{1}{4m} \sum_i \left(k_i - \frac{k_i 2m(M - 1)/M}{2m} \right) q \\ &= \frac{1}{2m} \sum_i \left(k_i \frac{M - 1}{M} - k_i q \right) \\ &= \frac{M - 1}{M} - q. \end{aligned} \quad (17)$$

To verify the analysis, we carry out simulations on the ER random networks with $N = 10^5$ and $\langle k \rangle = 10, 20, 40$, respectively. We adopt parameter values $d = 0.25$, $w = 0.1$, and 0.5 where the networks will finally have 2 communities, and $d = 0.1$, $w = 0.5$ where the networks will finally have 5 communities. Different opinions are uniformly distributed at the beginning. We examine different cases with different mutation rates of $p = 10^{(-4+n/4)}$ for n from 0 to 16. For each parameter set, we generate 100 random networks. And in each time step, we let a single pair of nodes be randomly selected. We record the proportion of inter-community links and calculate the network modularity every 2000 time steps from $t = 8 \times 10^6$ to $t = 10^7$; and finally average the results of these 1000 calculations.

Figure 7(a) shows the change of inter-community link proportion q with the mutation rate p . We also evaluate the effects of rewiring speed w and the average degree $\langle k \rangle$ on q . We see that in all the different cases, theoretical analyses match well with simulation results. Notice that when the mutation rate p is very low, we have that $\langle k \rangle p \ll w$; from Eq. (14) we have $w + \langle k \rangle p \approx w$ and hence $q \sim p$. The proportion of inter-community links therefore increases approximately linearly with the mutation rate. This matches the observation in Fig. 7(a) that in a log-log scale plot, the relation between p and q approximates a linear function with a slop value 1 when p is small enough. The increase of q becomes slower when the mutation rate p is high, which can be derived from Eq. (14) and also can be easily understood: as aforementioned, a higher mutation rate leads to a larger proportion of inter-community links and a smaller proportion of intra-community links. When a mutation operation changes the opinion of a node to be within the range of another community, all the node's intra-community links become inter-community links while a part of their inter-community links become intra-community links. Such mutations become less effective in increasing the number of inter-community links when there is already a large proportion of inter-community links. The increasing speed of q thus becomes slower than a linear function of p .

Rewiring rate w also has its impacts on the inter-community link proportion q . A larger value of w basically leads to a smaller value of q when other parameters are fixed. The reason is simple: a higher rewiring speed leads to a smaller number of inter-community links. While such an observation is obvious, it is however interesting to observe from (14), and Fig. 7(a) as well, that when p is small enough, for given p and $\langle k \rangle$, q increases approximately linearly proportional to $1/w$.

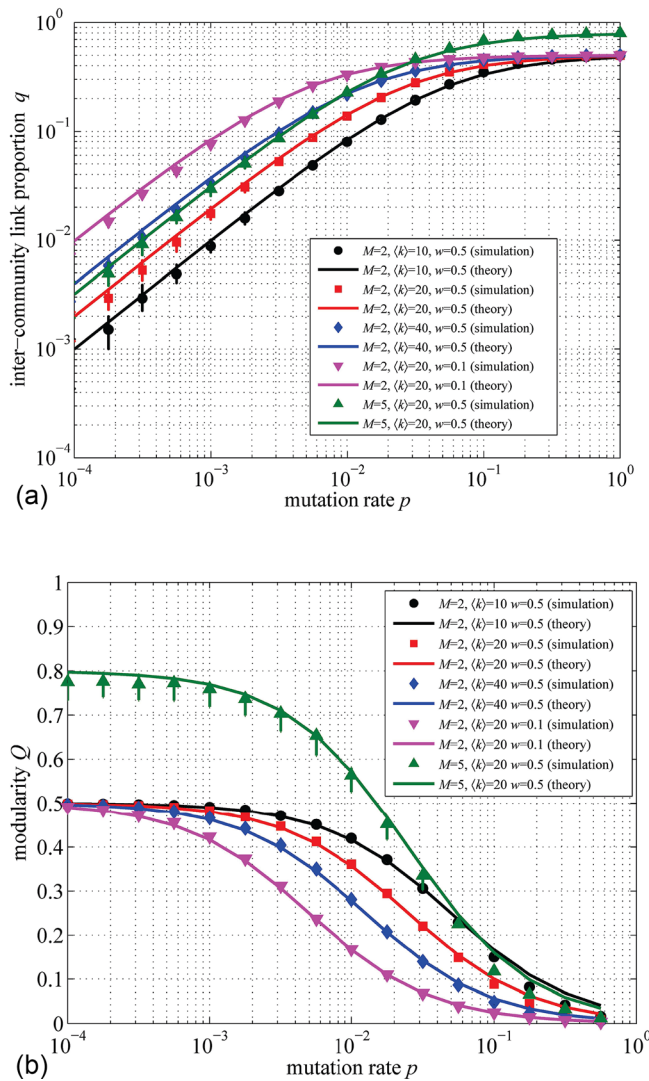


FIG. 7. (a) Proportion of inter-community links vs. mutation rate and (b) modularity vs. mutation rate. The 95% confidence interval is given for the average results in 100 random networks.

As to the effects of $\langle k \rangle$ on q , it can be easily derived from Eq. (14) that when $\langle k \rangle p \ll w$, q increases almost linearly with $\langle k \rangle$. Note that the steady state is a dynamic equilibrium where the speed of increasing inter-community links by opinion mutation statistically speaking equals the speed of removing inter-community links by rewiring. A high average nodal degree allows the number of inter-community links to be increased faster by a given number of mutation operations (i.e., a given p). It thus takes a higher proportion of inter-community links at the steady state to allow a larger number of inter-community link removals such that the dynamic balance can be achieved. The effects again become less significant when p is large, due to the same reason as we discussed earlier that mutation becomes less effective in increasing the number of intercommunity links under such a case.

With the understanding of the relation between q and a few factors, the effects of these factors on the system modularity value Q can be easily derived from Eq. (16): when p is very small, Q decreases approximately linearly with p , $\langle k \rangle$ and $1/w$. The decreasing speed becomes slower when p gets

larger. To allow better observation of modularity values within the range that we are interested, Fig. 7(b) plots the results in the log-linear scale. Note that it is confirmed that Q decreases approximately linearly with p when p is small though it does not appear to be an obvious observation in the figure. Further, it is interesting to observe that when w is of a high value, e.g., at 0.5, it takes a relatively high value of p (higher than 10^{-3}) to let Q significantly shift away from $(M - 1)/M$. For a smaller value of w , e.g., at 0.1, a low value of p can already lead to nontrivial changes to Q . This matches the real-life experiences that in a more tolerant social system (with a lower value of w), opinion changes are more effective in increasing the number of interconnections between different opinion communities.

V. CONCLUSIONS

In this work, we evaluated the complex co-evolution process of opinions and system structures where three different factors, namely opinion formation, link rewiring, and opinion mutation are integrated together into the same model. It was observed that such a model would allow the system to evolve into a dynamic equilibrium with multiple communities, each of which holds a range of opinions with a bell-curve-style distribution and with non-trivial intercommunity links in between. Such a system, as we claim, better resembles the observations in the real life and would allow the system to easily evolve further when the internal/external conditions change. It is also observed that a few different factors, rather than tolerance between different opinions alone, may contribute to deciding the number of communities in the final-state networks. An analytical framework was proposed to describe the co-evolution process with satisfactory precision. Further, the relation between system co-evolution and system modularity was carefully studied. It was revealed that there exists a linear relationship between system modularity and a few factors when the mutation rate is low. Our study shall help better understand how different factors work together leading to the complex co-evolution dynamics that we may observe in many complex social systems.

It is revealed in this article that different mutation rates may lead to different final steady states. In our preliminary studies,^{38,39} it has been further observed that the effects of different mutation patterns may be equally, if not more, significant. Further studies are needed to fully understand the effects of mutation in opinion formation on complex social networks.

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¹B. Kozma and A. Barrat, *Phys. Rev. E* **77**, 016102 (2008).

²T. Gross, C. J. D. D'Lima, and B. Blasius, *Phys. Rev. Lett.* **96**, 208701 (2006).

³L. B. Shaw and I. B. Schwartz, *Phys. Rev. E* **77**, 066101 (2008).

⁴S. Bansal, J. Read, B. Pourbohloul, and L. A. Meyers, *J. Biol. Dyn.* **4**, 478 (2010).

- ⁵V. Marceau, P. A. Noel, L. Hebert-Dufresne, A. Allard, and L. J. Dube, *Phys. Rev. E* **82**, 036116 (2010).
- ⁶J. Zhou, G. Xiao, S. A. Cheong, X. Fu, L. Wong, S. Ma, and T. H. Cheng, *Phys. Rev. E* **85**, 036107 (2012).
- ⁷P. Holme and M. E. J. Newman, *Phys. Rev. E* **74**, 056108 (2006).
- ⁸R. I. M. Dunbar, *Behav. Brain Sci.* **16**, 681 (1993).
- ⁹D. M. Abrams and S. H. Strogatz, *Nature* **424**, 900 (2003).
- ¹⁰E. Lieberman, J.-B. Michel, J. Jackson, T. Tang, and M. A. Nowak, *Nature* **449**, 713 (2007).
- ¹¹C. Beckner, R. Blythe, J. Bybee, M. H. Christiansen, W. Croft, N. C. Ellis, J. Holland, J. Ke, D. Larsen-Freeman, and T. Schoenemann, *Lang. Learn.* **59**, 1 (2009).
- ¹²F. Xie and D. Levinson, *Networks Spat. Econ.* **9**, 291 (2009).
- ¹³M. Barthélemy and A. Flammini, *Networks Spat. Econ.* **9**, 401 (2009).
- ¹⁴C. Castellano, D. Vilone, and A. Vespignani, *Europhys. Lett.* **63**, 153 (2003).
- ¹⁵V. Sood and S. Redner, *Phys. Rev. Lett.* **94**, 178701 (2005).
- ¹⁶G. A. Bohme and T. Gross, *Phys. Rev. E* **83**, 035101 (2011).
- ¹⁷R. Durrett, J. P. Gleeson, A. L. Lloyd, P. J. Mucha, F. Shi, D. Sivakoff, J. E. S. Socolar, and C. Varghese, *Proc. Natl. Acad. Sci. U.S.A.* **109**, 3682 (2012).
- ¹⁸G. Zschaler, G. A. Bohme, M. Seisinger, C. Huepe, and T. Gross, *Phys. Rev. E* **85**, 046107 (2012).
- ¹⁹A. Carro, R. Torra, and M. San Miguel, *Sci. Rep.* **6**, 24775 (2016).
- ²⁰D. Kimura and Y. Hayakawa, *Phys. Rev. E* **78**, 016103 (2008).
- ²¹N. Crokidakis and C. Anteneodo, *Phys. Rev. E* **86**, 061127 (2012).
- ²²G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, *Adv. Complex Syst.* **03**, 87 (2000).
- ²³R. Hegselmann and U. Krause, *J. Artif. Soc. Simul.* **5**, 2 (2002), available at <http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.454.3597&rep=rep1&type=pdf>.
- ²⁴E. Ben-Naim, P. L. Krapivsky, and S. Redner, *Physica D* **183**, 190 (2003).
- ²⁵C. Jin, Y. Li, and X. Jin, *Physica A* **467**, 267 (2017).
- ²⁶M. Pineda, R. Toral, and E. Hernandez-Garcia, *J. Stat. Mech.: Theory Exp.* **7**, P08001 (2009), available at <http://iopscience.iop.org/article/10.1088/1742-5468/2009/08/P08001/meta>.
- ²⁷M. Pineda, R. Toral, and E. Hernández-García, *Eur. Phys. J. D* **62**, 109 (2011).
- ²⁸A. Carro, R. Toral, and M. San Miguel, *J. Stat. Phys.* **151**, 131 (2013).
- ²⁹L. R. Olson, W. Cadge, and J. T. Harrison, *Soc. Sci. Q.* **87**, 340 (2006).
- ³⁰D. L. Lasorsa, *J. Mass Comm. Q.* **68**, 38 (1991), available at <http://journals.sagepub.com/doi/abs/10.1177/107769909106800105>.
- ³¹D. A. Snow and R. Machalek, *Annu. Rev. Sociol.* **10**, 167 (1984).
- ³²R. F. Paloutzian, J. T. Richardson, and L. R. Rambo, *J. Pers.* **67**, 1047 (1999).
- ³³L. R. Rambo, *Soc. Compass* **46**, 259 (1999).
- ³⁴J. S. Phinney, G. Horenczyk, K. Liebkind, and P. Vedder, *J. Soc. Issues* **57**, 493 (2001).
- ³⁵S. Phithakkitnukoon, F. Calabrese, Z. Smoreda, and C. Ratti, in *Privacy, Security, Risk and Trust (PASSAT) and 2011 IEEE Third International Conference on Social Computing (SocialCom)* (2011), p. 515.
- ³⁶P. Erdos and A. Renyi, *Publ. Math. Debrecen* **6**, 290 (1959), available at <https://pdfs.semanticscholar.org/4201/73e087bca0bdb31985e28ff69c60a129c8ef.pdf>.
- ³⁷M. E. J. Newman, *Proc. Natl. Acad. Sci. U.S.A.* **103**, 8577 (2006).
- ³⁸Y. Yu and G. Xiao, in 2015 First Conference on Human and Social Analytics (HUSO) (2015).
- ³⁹Y. Yu and G. Xiao, in 2015 IEEE International Conference on Digital Signal Processing (DSP) (2015).