Controllability of Markovian Jump Boolean Control Networks ⋆

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Abstract

In this paper, controllability of Markovian jump Boolean control networks (MJBCNs) is studied via semi-tensor product of matrices. First, based on the algebraic expression of the considered Boolean control networks, a necessary and sufficient condition for controllability is presented by iteration equations, which however may lead to high-dimensional matrices. To avoid having such matrices, a new matrix is defined and applied to derive another equivalent condition to verify controllability of MJBCNs. Moreover, a maximum principle of MJBCNs is established to further study the minimal controllable time. Finally, two examples are presented to illustrate the obtained results.

Key words: controllability, Markovian jump Boolean networks, minimal controllable time, semi-tensor product.

1 Introduction

Boolean networks, first applied to model genetic networks by Kauffman (Kauffman 1969), have received considerable attention owing to their wide applications in various fields, such as biology (Davidson et al. 2002), game theory (Cheng 2014), and smart home (Kabir et al. 2014), to name just a few. The state variables in these systems are quantized to two values (1 and 0), and the value of each state at time $t+1$ is modified according to logical functions with respect to the values of its neighbors at time t. As many biological systems are affected by extra factors, it is natural to extend Boolean networks to Boolean control networks by adding Boolean inputs. Witnessed by an increasing number of high-quality and landmark results (Akutsu et al. 2007, Cheng, Zhao & Xu 2015, Fornasini & Valcher 2013, Guo et al. 2017, Liang et al. 2017, Meng et al. 2018), Boolean (control) networks have proved to be highly effective in describing numerous practical processes.

It is noteworthy that one gene may update its value according to several possible logical rules at every discrete moment. For example, the bacteriophage lambda, which contains two different models (lysis and lysogeny), may change its state in the light of two different strategies; in an eukaryotic cell, the growth and division of the cell triggered by a set of events usually include four phases. These can be regarded as the switching phenomena in genetic regulatory networks. While reachability, controllability and stability of deterministic Boolean networks have been well investigated (Li et al. 2014), an accurate model cannot be applied when there are uncertainties in switching signals. A novel class of Boolean networks, probabilistic Boolean networks (Liu et al. 2015, Qi et al. 2010, Shmulevich et al. 2003), has therefore been proposed to describe rule-based properties, as well as tackling nondeterminacy in data and model selection. Since in genetic regulatory networks, which are constructed by networks of regulatory interactions among DNA, RNA and proteins, an array of molecular processes are always affected by some intrinsic fluctuations and extrinsic perturbations, stochastic process such as Markov chain has been employed to depict these dynamics (Kim 2002, Sun et al. 2009). With the observation that transitions between states in a gene network occur randomly (Elowitz et al. 2002), Markovian jump Boolean networks (MJBNs) presented in (Meng et al. 2019, 2017) become a favourable option to model some gene expressions with stochastic switching factors.

On the other hand, controllability with the property that the trajectory of a controlled system can be steered from some

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given initial state to the desired one, is one of the core issues in control theory and plays a crucial role in genetic diagnosis and detection. Theoretical research on controllability of Boolean control networks may be helpful for achieving a better understanding of the interaction among genes and designing therapeutic strategies. To date, many results have been reported for controllability of typical, deterministic switched and probabilistic Boolean networks by resorting to the semi-tensor product of matrices (Laschov & Margaliot 2012, Li & Sun 2011, Liang et al. 2017, Liu et al. 2015, Zhao & Cheng 2014). However, it turns out to be a challenge to study on controllability of Markovian jump Boolean control networks (MJBCNs) due to the inherent high complexities of a combination of switching operations and stochastic properties. As the research on MJBCNs as an extension of deterministic switched and probabilistic Boolean control networks can help enlarge the applications in genetic networks, we are motivated to study controllability of MJBCNs in this paper.

In this paper, controllability of MJBCNs is first defined and discussed by using semi-tensor product of matrices. Based on the algebraic expression of the considered network, a necessary and sufficient condition for controllability with probability one is derived by iterations. With the obtained condition, the required control can also be designed. However, the dimensions of involved matrices may be very high when the controllable time and the number of nodes are large, which seriously restricts the applications of the results. To cope with this problem, a new controllability matrix is defined for MJBCNs and based on it, another equivalent condition for verifying controllability is presented. Furthermore, by proposing the maximum principle of MJBCNs, extending the results in Laschov & Margaliot (2011, 2013), a necessary condition for the minimal controllable time is established.

The main contributions of this paper are threefold: i) this is, for the first time, to the best of our knowledge, that a study on controllability of MJBCNs is conducted, which shall lay a theoretical foundation for diagnosing diseases in stochastic genetic regulatory networks; ii) since MJBCNs are more general than typical and probabilistic Boolean networks, the obtained results allow extensions of some previous results (e.g., the results in Li & Sun (2011), Liu, Chen, Lu & Wu (2015)); iii) the newly proposed maximum principle for MJBCNs can be applied to get a necessary condition for the minimal controllable time, which is always difficult to determine.

The rest of this paper is organized as follows. Section 2 introduces some preliminaries of the semi-tensor product and problem formulation. The main results of this paper are presented in Section 3. Section 4 gives two examples to illustrate the effectiveness of the obtained results, which is followed by a brief conclusion in Section 5.

2 Preliminaries and problem formulation

Some primary notations and definitions to be adopted in this study are presented below.

$\mathbb{R}^n$ and $\mathscr{M}_{m\times n}$ denote the sets of n-dimensional column vectors and $m\times n$ real matrices, respectively. The i-th column of the identity matrix I_n is defined as δ_n^i, $i=1,2,\ldots,n$. Denote $\Delta_n := \{\delta_n^i | \ i=1,2,\ldots,n\}$ and $\Delta := \Delta_2$. Denote by Δ^n the Cartesian product $\underbrace{\Delta\times\Delta\times\cdots\times\Delta}_{n}$. A matrix $L\in\mathscr{M}_{n\times r}$, written as $L=[\delta_n^{i_1},\delta_n^{i_2},\ldots,\delta_n^{i_r}]$, is called a logical matrix and can be rewritten as $L=\delta_n[i_1,i_2,\ldots,i_r]$ for simpler notation. $\mathrm{Col}_i(L)$ represents the i-th column of L and $\mathrm{Col}(L)$ is the set of columns of L. Denote by $\mathscr{L}_{n\times r}$ the set of $n\times r$ logical matrices. A column vector $v=(v_1,v_2,\ldots,v_n)^T$ is called a stochastic vector if $\sum_{i=1}^n v_i = 1$ and $v_i\geq 0$. A matrix is called a stochastic matrix if its every column is a stochastic vector. Denote by, respectively, $\mathscr{L}_n^r$ and $\mathscr{L}_{m\times n}^r$ the sets of n-dimensional stochastic vectors and $m\times n$ stochastic matrices. $W_{[m,n]}$ represents an $mn\times mn$ swap matrix defined in Cheng & Qi (2007), Cheng, Qi & Li. (2011). $\mathbf{1}_n$ ($\mathbf{0}_n$) is a column vector in $\mathbb{R}^n$ with all of its elements being 1 (0). $\mathrm{diag}\{M_1,M_2,\ldots,M_n\}$ represents a diagonal matrix with the i-th diagonal as M_i. For two $m\times n$ matrices $A=(a_{ij})$ and $B=(b_{ij})$, matrix disjunction $A\vee B$ and conjunction $A\wedge B$ mean $m\times n$ matrices with the (i,j)-th element being $a_{ij}\vee b_{ij}$ and $a_{ij}\wedge b_{ij}$, respectively, where $\vee$ and $\wedge$ are two traditional logical operators, disjunction and conjunction, respectively.

Definition 1 (Cheng, Qi & Li. 2011) *The semi-tensor product of matrices $M\in\mathscr{M}_{a\times b}$ and $N\in\mathscr{M}_{c\times d}$, denoted by $M\ltimes N$, is defined as $M\ltimes N=(M\otimes I_{l/b})(N\otimes I_{l/c})$, where l is the least common multiple of b and c, and $\otimes$ is the Kronecker product (Liu & Trenkler 2008).*

The semi-tensor product of matrices in Definition 1 generalizes the traditional matrix product: $M\ltimes N=MN$ when $b=c$. Therefore, in this paper the symbol "$\ltimes$" is omitted if no confusion arises. Further discussions on properties and applications of the semi-tensor product can be found in Cheng & Qi (2007), Cheng, Qi & Li. (2011).

Consider a Boolean control network with n nodes and s updated rules for every node as follows:

$$\begin{cases} x_1(t+1)=f_1^{\sigma(t)}(X(t),U(t)),\\ x_2(t+1)=f_2^{\sigma(t)}(X(t),U(t)),\\ \qquad\vdots\\ x_n(t+1)=f_n^{\sigma(t)}(X(t),U(t)), \end{cases} \tag{1}$$

where $x_i(t)\in\Delta$ is the state, $u_j(t)\in\Delta$ is the control input, $f_i^{\sigma(t)}:\Delta^{n+m}\to\Delta$ is a Boolean function, $i=1,2,\ldots,n$, $j=1,2,\ldots,m$, and $\sigma(t)$ is a switching signal. $X(t)=(x_1(t),x_2(t),\ldots,x_n(t))$, $U(t)=(u_1(t),u_2(t),\ldots,u_m(t))$. In this paper, the studied control inputs $u_j(t) \in \Delta$, $j = 1, 2, \dots, m$ are deterministic and free Boolean sequences, as the first kind of control considered in Li & Sun (2011). $\{\sigma(t)|t \geq 0\}$ is modeled as a discrete time homogeneous Markov chain with the finite mode set $\mathscr{S} = \{1, 2 \dots, s\}$ and its transition probability matrix $\Lambda = (\lambda_{ij})_{s\times s}$ is given by

$$\lambda_{ij} = \mathbf{Pr}\{\sigma(t+1) = j|\sigma(t) = i\}, \tag{2}$$

where $\lambda_{ij} \geq 0$ for $i, j \in \mathscr{S}$ and $\sum_{j=1}^{s} \lambda_{ij} = 1$ for any $i \in \mathscr{S}$. In this setting, switching-based Boolean function $f_i^{\sigma(t)}$ is in the possible update rule set $\{f_i^1, f_i^2, \dots, f_i^s\}$ and $f_i^{\sigma(t)} = f_i^j$ when $\sigma(t) = j$ for $j \in \mathscr{S}$. Network (1) is called an MJBCN.

Via semi-tensor product, the following lemma is presented, which is applied to convert Boolean networks into an algebraic form similar to that of linear systems.

Lemma 1 (Cheng & Qi 2007) *Let $f(x_1, x_2, \dots, x_n) \in \Delta$ be a Boolean function with $x_i \in \Delta$, $i = 1, 2, \dots, n$. There exists a unique matrix $M_f \in \mathscr{L}_{2\times 2^n}$, called the structure matrix of f, which satisfies $f(x_1, x_2, \dots, x_n) = M_f \ltimes_{i=1}^{n} x_i := x_1 \ltimes x_2 \ltimes \cdots \ltimes x_n$.*

Subsequently, by Lemma 1, MJBCN (1) can be equivalently transformed into an algebraic form as follows (Cheng & Qi 2007, Cheng et al. 2011):

$$x(t+1) = F_{\sigma(t)} u(t) x(t), \tag{3}$$

where $x(t) = \ltimes_{i=1}^{n} x_i(t) \in \Delta_{2^n}$, $u(t) = \ltimes_{i=1}^{m} u_i(t) \in \Delta_{2^m}$. $F_{\sigma(t)} \in \mathscr{L}_{2^n\times 2^{n+m}}$ is called the transition matrix of network (1). The definition of controllability with probability one is given based on the algebraic form (3) in the following.

Definition 2 (Li & Sun 2011, Liu et al. 2015) *Consider MJBCN (3). Given the initial state $x_0 \in \Delta_{2^n}$ and the destination state $x_d \in \Delta_{2^n}$, x_d is controllable from x_0 at time $N \geq 1$ with probability one if there exists a sequence of deterministic controls $u(0)$, $u(1)$, $\dots, u(N-1)$ such that under any initial distribution of $\sigma(t)$,*

$$\mathbf{Pr}\{x(N) = x_d|x(0) = x_0\} = 1. \tag{4}$$

The aim of this paper is to derive some equivalent conditions to verify whether there exists a sequence of controls steering a given initial state x_0 to a required state x_d at time N with probability one, and to determine the minimal controllable time, i.e., the minimal $N > 0$ satisfying (4).

3 Main results

In this section, controllability of MJBCNs is first discussed and then a new maximum principle is proposed to establish a necessary condition for the minimal controllable time.

Consider the algebraic expression (3) of (1). Then, reviewing some basic results of probability theory and Markov jump systems (Borovkov 2013, Costa et al. 2006) and denoting $\xi_i(t) = \mathbf{E}\{x(t)\mathbf{1}_{\{\sigma(t)=i\}}\}$ where $\mathbf{1}_{\{\sigma(t)=i\}}$ stands for the indicator function of the set $\{\sigma(t) = i\}$ and $i \in \mathscr{S}$, it can be obtained that $\mathbf{E}\{x(t)\} = \mathbf{E}\left\{\sum_{i=1}^{s} x(t)\mathbf{1}_{\{\sigma(t)=i\}}\right\} = \sum_{i=1}^{s} \xi_i(t)$, and

$$\begin{aligned}\xi_j(t+1) &= \sum_{i=1}^{s} \mathbf{E}\{x(t+1)\mathbf{1}_{\{\sigma(t+1)=j\}}\mathbf{1}_{\{\sigma(t)=i\}}\} \\ &= \sum_{i=1}^{s} \lambda_{ij} F_i W_{[2^n,2^m]} \mathbf{E}\{x(t)\mathbf{1}_{\{\sigma(t)=i\}}\} u(t) \\ &= \sum_{i=1}^{s} \lambda_{ij} \tilde{F}_i \xi_i(t) u(t),\end{aligned}$$

where $\tilde{F}_i = F_i W_{[2^n,2^m]}$. Let $\xi(t) = (\xi_1^T(t), \dots, \xi_s^T(t))^T$, then

$$\begin{aligned}\xi(t+1) &= (\Lambda^T \otimes I_{2^n})\mathrm{diag}\{\tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_s\}\xi(t)u(t) \\ &= \tilde{F}\xi(t)u(t),\end{aligned} \tag{5}$$

where $\tilde{F} = (\Lambda^T \otimes I_{2^n})\mathrm{diag}\{\tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_s\} \in \mathscr{M}_{s2^n\times s2^{n+m}}$.

Remark 1 *The algebraic expression (5) cannot be obtained by simply taking expectation on both sides of (3) since the probability distribution of $\sigma(t)$ is not easy to get. Besides, probabilistic Boolean networks studied in Li & Sun (2011), Liu, Chen, Lu & Wu (2015) can be viewed as a special case of MJBCNs. The reason is as follows. Consider an arbitrary probabilistic Boolean control network, in which there are s subnetworks admitting a probability distribution as $\pi = (\pi_1, \pi_2, \dots, \pi_s)$. If one associates the probabilistic Boolean control network with a Markov chain $\{\sigma(t)|t \geq 0\}$ with its transition probability matrix being $\Lambda = \mathbf{1}_s \otimes \pi$, then this probabilistic Boolean control network is an MJBCN.*

Remark 2 *By now, MJBCN (1) is equivalently converted into an algebraic form (5) similar to that of typical Boolean control networks. However, it should be pointed out that the state space of (5) is no longer Δ_{s2^n}, which is different from that of typical Boolean control networks. The original state variable $x(t)$ is in Δ_{2^n}, that is, the state variable $x(t)$ is a 2^n-dimensional vector with only one element equaling 1 and the others 0, which implies that $\mathbf{1}_{2^n}^T x(t) = 1$. Hence, one obtains $\mathbf{1}_{2^n}^T \mathbf{E}\{x(t)\} = 1$, and $\mathbf{1}_{s2^n}^T \xi(t) = \sum_{i=1}^{s} \mathbf{1}_{2^n}^T \xi_i(t) = 1$. That is, $\xi(t) \in \mathscr{L}_{s2^n}^r$. Moreover, based on the construction of $\tilde{F}$, it can be obtained that*

$$\begin{aligned}\mathbf{1}_{s2^n}^T \tilde{F} &= \mathbf{1}_{s2^n}^T (\Lambda^T \otimes I_{2^n})\mathrm{diag}\{\tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_s\} \\ &= \mathbf{1}_{s2^n}^T \mathrm{diag}\{\tilde{F}_1, \tilde{F}_2, \dots, \tilde{F}_s\} \\ &= \mathbf{1}_{s2^{n+m}}^T, \\ \mathbf{1}_{s2^n}^T \tilde{F}^N &= \mathbf{1}_{s2^{n+m}}^T \tilde{F}^{N-1} \\ &= (\mathbf{1}_{s2^n}^T \otimes \mathbf{1}_{2^m}^T)(\tilde{F} \otimes I_{2^m})\tilde{F}^{N-2} \\ &= \mathbf{1}_{s2^{n+2m}}^T \tilde{F}^{N-2} = \cdots \\ &= \mathbf{1}_{s2^{n+Nm}}^T.\end{aligned}$$

Thus the transition matrix $\tilde{F}$, as well as its any power $\tilde{F}^N$, is a stochastic matrix. Therefore, the state variable and transition matrix in the algebraic expression (5) of (1) remain the property of column sum being equal to 1, although their elements do not attain 1 or 0 any more.

To discuss controllability, first we assume that the initial state is $x_0 = \delta_{2^n}^{\alpha}$ and the destination state is $x_d = \delta_{2^n}^{\beta}$ for given $\alpha, \beta \in \{1, 2, \ldots, 2^n\}$. The following result is presented based on the algebraic expression (5).

Lemma 2 *Consider MJBCN (3). x_d is controllable from x_0 at time N with probability one if and only if under any initial distribution of $\sigma(t)$, $(\mathbf{1}_s^T \otimes I_{2^n})\xi(N) = x_d$.*

Proof. Combining Definition 2 and the definition of expectation, condition (4) is equivalent to $\mathbf{E}\{x(N)|x(0) = x_0\} = x_d$. Taking the notation of $\xi(t)$ into consideration, one has for the given x_0, $\mathbf{E}\{x(N)\} = \sum_{i=1}^{s} \xi_i(N) = (\mathbf{1}_s^T \otimes I_{2^n})\xi(N)$. The proof is thus completed. ■

Theorem 1 *Consider MJBCN (3). x_d is controllable from $x_0 = \delta_{2^n}^{\alpha}$ at time N with probability one if and only if $x_d \in \mathrm{Col}(\Phi_N)$, where*

$$\Phi_N = \left((\mathbf{1}_s^T \otimes I_{2^n})\tilde{F}^N \delta_{s2^n}^{\alpha}\right) \bigwedge \left((\mathbf{1}_s^T \otimes I_{2^n})\tilde{F}^N \delta_{s2^n}^{2^n+\alpha}\right) \bigwedge \cdots \bigwedge \left((\mathbf{1}_s^T \otimes I_{2^n})\tilde{F}^N \delta_{s2^n}^{(s-1)2^n+\alpha}\right).$$

Proof. Necessity: From Lemma 2, under any initial distribution of $\sigma(t)$, $(\mathbf{1}_s^T \otimes I_{2^n})\xi(N) = x_d$. Assume that the initial distribution of $\sigma(t)$ is $\mathbf{Pr}\{\sigma(0) = i\} = p_i$ with any nonnegative numbers p_i's satisfying $\sum_{i=1}^{s} p_i = 1$. Note that $\xi_i(0) = \mathbf{E}\{x(0)\mathbf{1}_{\{\sigma(0)=i\}}\} = p_i\delta_{2^n}^{\alpha}$ and $\xi(0) = (\xi_1^T(0), \xi_2^T(0), \ldots, \xi_s^T(0))^T$, then

$$\begin{aligned}\xi(0) &= (p_1(\delta_{2^n}^{\alpha})^T, p_2(\delta_{2^n}^{\alpha})^T, \ldots, p_s(\delta_{2^n}^{\alpha})^T)^T \\ &= p_1\delta_s^1\delta_{2^n}^{\alpha} + p_2\delta_s^2\delta_{2^n}^{\alpha} + \cdots + p_s\delta_s^s\delta_{2^n}^{\alpha} \\ &= \sum_{i=1}^{s} p_i \delta_{s2^n}^{(i-1)2^n+\alpha}. \end{aligned} \tag{6}$$

Consider the dynamics of $\xi(t)$ in (5). By iteration, it leads to

$$\begin{aligned}\xi(N) &= \tilde{F}\xi(N-1)u(N-1) = \cdots \\ &= \tilde{F}^N\xi(0)u(0)\cdots u(N-2)u(N-1) \\ &= \sum_{i=1}^{s} p_i\tilde{F}^N\delta_{s2^n}^{(i-1)2^n+\alpha}u(0)\cdots u(N-2)u(N-1).\end{aligned}$$

Accordingly, for any nonnegative numbers p_i's satisfying $\sum_{i=1}^{s} p_i = 1$, one gets

$$\begin{aligned}&(\mathbf{1}_s^T \otimes I_{2^n})\sum_{i=1}^{s} p_i\tilde{F}^N\delta_{s2^n}^{(i-1)2^n+\alpha}u(0)\cdots u(N-2)u(N-1) \\ &= (\mathbf{1}_s^T \otimes I_{2^n})\xi(N) \\ &= x_d.\end{aligned} \tag{7}$$

That is, for any required p_i's, an integer $h \in \{1, 2, \ldots, 2^{Nm}\}$ exists such that

$$x_d = \mathrm{Col}_h\left(\sum_{i=1}^{s} p_i(\mathbf{1}_s^T \otimes I_{2^n})\tilde{F}^N\delta_{s2^n}^{(i-1)2^n+\alpha}\right). \tag{8}$$

Recalling Remark 2, we note that $\tilde{F}^N \in \mathscr{L}^r_{s2^n \times s2^{n+Nm}}$, then $(\mathbf{1}_s^T \otimes I_{2^n})\tilde{F}^N\delta_{s2^n}^{(i-1)2^n+\alpha} \in \mathscr{L}^r_{2^n \times 2^{Nm}}$. Thereby, x_d equals $\mathrm{Col}_h((\mathbf{1}_s^T \otimes I_{2^n})\tilde{F}^N\delta_{s2^n}^{(i-1)2^n+\alpha})$ for any $i \in \mathscr{S}$, which is equivalent to $x_d \in \mathrm{Col}(\Phi_N)$.

Sufficiency: If $x_d \in \mathrm{Col}(\Phi_N)$, then an integer h exists such that for any $i \in \mathscr{S}$, $x_d = \mathrm{Col}_h\left((\mathbf{1}_s^T \otimes I_{2^n})\tilde{F}^N\delta_{s2^n}^{(i-1)2^n+\alpha}\right)$. Immediately, (8) holds for any nonnegative p_i's satisfying $\sum_{i=1}^{s} p_i = 1$. From the analysis on necessity, x_d is controllable from $x_0 = \delta_{2^n}^{\alpha}$ at time N with probability one. ■

Remark 3 *From the proof of Theorem 1, x_d is controllable from x_0 at time N with probability one, and the required control sequence can be designed as $u(0) = \delta_{2^m}^{i_1}$, $u(1) = \delta_{2^m}^{i_2}, \ldots, u(N-1) = \delta_{2^m}^{i_N}$, where $i_1, i_2, \ldots, i_N$ are uniformly determined by $\delta_{2^m}^{i_1} \ltimes \cdots \ltimes \delta_{2^m}^{i_{N-1}} \ltimes \delta_{2^m}^{i_N} = \delta_{2^{Nm}}^{h}$, i.e., $(i_1-1)2^{(N-1)m} + \cdots + (i_{N-2}-1)2^{2m} + (i_{N-1}-1)2^m + i_N = h$ with h satisfying (8). In addition, the result in Theorem 1 looks similar to Theorem 3.1 in Li & Sun (2011), however, the derivations are rather different. The results in Li & Sun (2011) are not applicable for dealing with the network studied in this paper.*

From Theorem 1, to check whether x_d is controllable from $x(0) = \delta_{2^n}^{\alpha}$ at time N with probability one, we need to compute Φ_N and check whether the condition $x_d \in \mathrm{Col}(\Phi_N)$ is satisfied. Moreover, if $x_d = \mathrm{Col}_{\gamma}(\Phi_N)$, it can be seen from the proof of Theorem 1 that the product $u(0)u(1)\cdots u(N-1) = \delta_{2^{Nm}}^{\gamma}$. By the results in (Cheng & Qi 2007), the control sequence $u(0), u(1), \ldots, u(N-1)$ can be uniquely determined. Despite of this, the dimensions of matrices in Theorem 1 may be very large, restricting applicability of the method when n, m and N are large. To have a more practical solution for the controllability problem, next we shall tackle this problem from a different angle by constructing a new controllable matrix.

Note that the dynamics in (5) can be rewritten as

$$\xi(t+1) = Lu(t)\xi(t), \tag{9}$$

where $L = \tilde{F}W_{[2^m, s2^n]} \in \mathscr{L}^r_{s2^n \times s2^{n+m}}$. Denote

$$\Theta_1 = \{(\mathbf{1}_s^T \otimes I_{2^n})L_1, (\mathbf{1}_s^T \otimes I_{2^n})L_2, \ldots, (\mathbf{1}_s^T \otimes I_{2^n})L_{2^m}\}, \quad (10)$$

where $[L_1, L_2, \ldots, L_{2^m}] = L$ and $L_i \in \mathscr{L}^r_{s2^n \times s2^n}$, $i = 1, 2, \ldots, 2^m$. For a positive integer k, define

$$\begin{aligned}\Theta_k = \{&(\mathbf{1}_s^T \otimes I_{2^n})L_{i_k}L_{i_{k-1}} \cdots L_{i_1} |\ i_j = 1, 2, \ldots, 2^m, \\ &j = 1, 2, \ldots, k\}, \quad (11)\end{aligned}$$

$$\lfloor \Theta_k \rfloor = \left\{ (\lfloor \Delta \rfloor \delta_s^1) \wedge (\lfloor \Delta \rfloor \delta_s^2) \wedge \cdots \wedge (\lfloor \Delta \rfloor \delta_s^s) | \Delta \in \Theta_k \right\} \quad (12)$$

where for a matrix $A = (a_{ij})$, $\lfloor A \rfloor$ represents that every element a_{ij} of matrix A is mapped to its corresponding floor function $\lfloor a_{ij} \rfloor$, i.e., the largest integer less than or equal to a_{ij}. Let $\mathscr{R}_k = \langle \lfloor \Theta_k \rfloor \rangle$ denote the disjunction of all the matrices in the set $\lfloor \Theta_k \rfloor$, then $\mathscr{R}_k \in \mathscr{M}_{2^n \times 2^n}$ is a 0-1 matrix since every element in $\lfloor \Theta_k \rfloor$ is not only in $\mathscr{M}_{2^n \times 2^n}$ but also a 0-1 matrix. Another equivalent condition for controllability with probability one can be derived as follows.

Theorem 2 *Consider MJBCN (3). $x_d = \delta_{2^n}^\beta$ is controllable from $x_0 = \delta_{2^n}^\alpha$ at time N with probability one if and only if the (β, α)-th element of $\mathscr{R}_N$ is*

$$(\mathscr{R}_N)_{\beta, \alpha} = 1. \quad (13)$$

Proof. Necessity: If $x_d = \delta_{2^n}^\beta$ is controllable from $x_0 = \delta_{2^n}^\alpha$ at time N with probability one, then by Lemma 2 and (9), there exists a control sequence $u(0) = \delta_{2^m}^{i_1}$, $u(1) = \delta_{2^m}^{i_2}$, ..., $u(N-1) = \delta_{2^m}^{i_N}$, such that

$$\begin{aligned}\delta_{2^n}^\beta &= (\mathbf{1}_s^T \otimes I_{2^n})\xi(N) \\ &= (\mathbf{1}_s^T \otimes I_{2^n})Lu(N-1)\xi(N-1) \\ &= (\mathbf{1}_s^T \otimes I_{2^n})L_{i_N}\xi(N-1) = \cdots \\ &= (\mathbf{1}_s^T \otimes I_{2^n})L_{i_N}L_{i_{N-1}} \cdots L_{i_1}\xi(0).\end{aligned}$$

Let the distribution of $\sigma(0)$ be $\mathbf{Pr}\{\sigma(0) = i\} = p_i$ for any nonnegative numbers p_i's satisfying $\sum_{i=1}^s p_i = 1$. By the proof of Theorem 1, we have $\xi(0) = \sum_{i=1}^s p_i \delta_{s2^n}^{(i-1)2^n + \alpha}$. Thus

$$\delta_{2^n}^\beta = \sum_{i=1}^s p_i (\mathbf{1}_s^T \otimes I_{2^n})L_{i_N}L_{i_{N-1}} \cdots L_{i_1}\delta_{s2^n}^{(i-1)2^n + \alpha}, \quad (14)$$

which means that for any $i \in \mathscr{S}$, the $(\beta, (i-1)2^n + \alpha)$-th element of $(\mathbf{1}_s^T \otimes I_{2^n})L_{i_N}L_{i_{N-1}} \cdots L_{i_1}$ is 1. That is, for any $i \in \mathscr{S}$, the (β, α)-th element of $\lfloor (\mathbf{1}_s^T \otimes I_{2^n})L_{i_N}L_{i_{N-1}} \cdots L_{i_1} \rfloor \delta_s^i$ is 1. From the definition of $\mathscr{R}_N$ based on Θ_N and $\lfloor \Theta_N \rfloor$ in (11), (12), one can get (13).

Sufficiency: If condition (13) holds, then there exists a sequence of integers $i_1, i_2, \ldots, i_N$ such that for any $i \in \mathscr{S}$, the (β, α)-th element of $\lfloor (\mathbf{1}_s^T \otimes I_{2^n})L_{i_N}L_{i_{N-1}} \cdots L_{i_1} \rfloor \delta_s^i$ is 1, which implies that $(\mathbf{1}_s^T \otimes I_{2^n})L_{i_N}L_{i_{N-1}} \cdots L_{i_1} \delta_s^i \delta_{2^n}^\alpha = \delta_{2^n}^\beta$, since $(\mathbf{1}_s^T \otimes I_{2^n})L_{i_N}L_{i_{N-1}} \cdots L_{i_1}$ is a stochastic matrix. Therefore, under any initial distribution of $\sigma(t)$ given as $\mathbf{Pr}\{\sigma(0) = i\} = p_i$ for any nonnegative numbers p_i's satisfying $\sum_{i=1}^s p_i = 1$, one can get

$$\begin{aligned}(\mathbf{1}_s^T \otimes I_{2^n})\xi(N) &= (\mathbf{1}_s^T \otimes I_{2^n})L_{i_N}L_{i_{N-1}} \cdots L_{i_1} \sum_{i=1}^s p_i \delta_{s2^n}^{(i-1)2^n + \alpha} \\ &= \delta_{2^n}^\beta.\end{aligned}$$

The proof is thus completed. ∎

Remark 4 *The Markovian switching signal $\sigma(t)$ is not assumed to be ergodic; that is, $\sigma(t)$ may not admit a stationary distribution $(\pi_1, \pi_2, \ldots, \pi_s)$ with $\pi_i > 0$ for any $i \in \mathscr{S}$. Thus, at any time t, $\mathbf{Pr}\{\sigma(t) = i\}$ may be zero for some $i \in \mathscr{S}$. From this, one cannot claim that the β-th element of $\xi_i(N)$ is 1 when $x_d = \delta_{2^n}^\beta$ is controllable from x_0 at time N with probability one. On the other hand, as explained in Remark 1, probabilistic Boolean control networks are a special class of MJBCNs. Therefore, our results obtained in this paper are applicable for studying controllability of probabilistic Boolean control networks (Li & Sun 2011, Liu et al. 2015), while the results in Li & Sun (2011), Liu, Chen, Lu & Wu (2015) cannot be utilized to deal with the problem for MJBCNs studied in this paper.*

The dimension of $\tilde{F}$ involved in Theorem 1 is $s2^n \times s2^{n+m}$ and hence the dimension of $\tilde{F}^N$ is $s2^n \times s2^{n+Nm}$. By the properties of semi-tensor product, the dimension of Φ_N in Theorem 1 is obtained as $2^n \times 2^{Nm}$. For Theorem 2, considering the notation of Θ_N, there may be 2^{Nm} matrices in Θ_N. However, since $L_i \in \mathscr{M}_{s2^n \times s2^n}$ for $i = 1, 2, \ldots, 2^m$, every matrix in Θ_N is of dimension $2^n \times s2^n$. From the above analysis, we can see that the large value of N may lead to high-dimensional matrices involved in Theorem 1 and makes the computation infeasible. In contrast, although the number of matrices in Θ_N is large, the dimensions of matrices in Θ_N remain unchanged. The result of Theorem 2 therefore has a winning margin from this perspective. Moreover, the computational time complexities of Theorems 1 and 2 are, respectively, $O((s2^n)^3 \cdot 2^{(2N-1)m})$ and $O((s2^n)^3 \cdot N \cdot 2^{Nm})$ in the worst case. It can be seen that when $N = 1$ or $N = 2$, $m = 1$, the computational time complexities of the results in Theorems 1 and 2 are exactly the same; otherwise the computational time complexity of the result in Theorem 2 is lower than that of Theorem 1.

To study the minimal controllable time, consider the equivalent algebraic form (3) of (1). Since a payoff function in a game can be described by a pseudo-Boolean function (Cheng, He, Qi & Xu 2015), we define a cost function associated with (3) as

$$J(x(t_f); u) = \mathbf{E}\{q_{\sigma(t_f)}^T x(t_f)\}, \quad (15)$$

where the terminal time t_f and $q_i \in \mathbb{R}^{2^n}$, $i \in \mathscr{S}$, are fixed.

The aim is to find a control sequence $u(0), u(1), \dots, u(t_f - 1)$ steering the state trajectory of (3) from $x(0) = x_0$ to a terminal state $x(t_f)$ such that $J(x(t_f); u)$ is maximal. This can be viewed as an optimal control problem formulated as

$$\max_u J(x(t_f); u) \tag{16}$$

where the state $x(t)$ and the control input $u(t)$ are with respect to equation (3). Here, if a control sequence $u^* = \{u^*(0), u^*(1), \dots, u^*(t_f - 1)\}$ exists such that $J(x^*(t_f), u^*)$, where $x^*(t)$ is the corresponding state trajectory, is maximal, then u^* is called an optimal control sequence. Reviewing the maximum principle in Laschov & Margaliot (2011) dealing with optimal control problem for deterministic Boolean control networks, a new maximum principle for solving the optimal control problem in (16) for MJBCNs can be derived.

Lemma 3 *Assume that $u^* = \{u^*(0), u^*(1), \dots, u^*(t_f - 1)\}$ is an optimal control sequence and $x^*(t)$ denotes the corresponding state trajectory. Let $q = (q_1^T, q_2^T, \dots, q_s^T)^T$. Define a mapping $a : \{1, 2, \dots, t_f\} \to \mathbb{R}^{s2^n}$ as*

$$a(t) = (Lu^*(t))^T a(t+1), \quad a(t_f) = q \tag{17}$$

and functions $b_h : \{0, 1, \dots, t_f - 1\} \to \mathbb{R}$, $h = 1, 2, \dots, 2^m$, as

$$b_h(t) = a^T(t+1) L_h \xi^*(t), \tag{18}$$

where $\xi^*(t)$ corresponds to $x^*(t)$. For any $t \in \{0, 1, \dots, t_f - 1\}$, if there exists an integer $h \in \{1, 2, \dots, 2^m\}$ such that $b_h(\hat{t}) \geq b_j(\hat{t})$ for any $j \neq h$, then $u^*(t) = \delta_{2^m}^h$.

Proof. With the description of $\xi(t)$, one has

$$J(x(t_f); u) = \sum_{i=1}^{s} q_i^T \mathbf{E}\{x(t_f) \mathbf{1}_{\{\sigma(t_f) = i\}}\} = q^T \xi(t_f). \tag{19}$$

To develop an analytical characterization of the optimal controls, based on (9), denote

$$\Phi(t_1, t_2; u) = Lu(t_1 - 1) Lu(t_1 - 2) \cdots Lu(t_2), \ t_1 \geq t_2. \tag{20}$$

Then $\xi(t_1) = \Phi(t_1, t_2; u) \xi(t_2)$. Fix any time $\hat{t} \in \{0, 1, \dots, t_f - 1\}$ and define a new control input as

$$u(t) = \begin{cases} \omega, & t = \hat{t}, \\ u^*(t), & \text{otherwise}, \end{cases} \tag{21}$$

where $\omega \in \Delta_{2^m}$ and $\omega \neq u^*(\hat{t})$. Based on the dynamics of $\xi(t)$ in (9), one gets that

$$\begin{aligned}
&J(x^*(t_f); u^*) - J(x(t_f); u) \\
&= q^T (\xi^*(t_f) - \xi(t_f)) \\
&= q^T (\Phi(t_f, \hat{t}+1; u^*) L u^*(\hat{t}) \xi^*(\hat{t}) - \Phi(t_f, \hat{t}+1; u^*) L \omega \xi^*(\hat{t})) \\
&= q^T \Phi(t_f, \hat{t}+1; u^*) L (u^*(\hat{t}) - \omega) \xi^*(\hat{t}).
\end{aligned}$$

By utilizing $a(t)$ and $b_h(t)$ defined in (17) and (18), one has

$$\begin{aligned}
a^T(t) &= a^T(t+1) L u^*(t) \\
&= a^T(t+2) L u^*(t+1) L u^*(t) = \cdots \\
&= a^T(t_f) L u^*(t_f - 1) \cdots L u^*(t+1) L u^*(t) \\
&= q^T \Phi(t_f, t; u^*),
\end{aligned}$$

which derives that $J(x^*(t_f); u^*) - J(x(t_f); u) = a^T(\hat{t} + 1) L(u^*(\hat{t}) - \omega) \xi^*(\hat{t})$. If there exists an integer $h \in \{1, 2, \dots, 2^m\}$ such that $b_h(\hat{t}) \geq b_j(\hat{t})$ for any $j \neq h$, then $u^*(\hat{t}) = \delta_{2^m}^h$, since for any $\omega = \delta_{2^m}^j$ with $j \neq h$, $J(x^*(t_f); u^*) - J(x(t_f); u) = a^T(\hat{t} + 1) L(\delta_{2^m}^h - \omega) \xi^*(\hat{t}) = b_h(\hat{t}) - b_j(\hat{t}) \geq 0$. ■

Consider the controllability problem again. Let $q_i = \delta_{2^n}^\beta - \mathbf{1}_{2^n}$, $i \in \mathscr{S}$, then $J(x(N); u) = q^T \xi(N) \leq 0$ since every element of $-q = -(q_1^T, q_2^T, \dots, q_s^T)^T$ and $\xi(N)$ is nonnegative. If $x_d = \delta_{2^n}^\beta$ is controllable from x_0 at time N with probability one, then the corresponding u maximizes $J(x(N); u) = q_i^T \delta_{2^n}^\beta = 0$. Based on this, a necessary condition for the minimal controllable time is obtained.

Theorem 3 *Assume that for network (3), $x_d = \delta_{2^n}^\beta$ is controllable from x_0 at minimal time $t_f = N^*$ with probability one, and denote the corresponding control and state trajectory by u^* and x^*, respectively. Mapping $a(t)$ is defined as that in (17) with $q = \mathbf{1}_s \otimes q_i$ and $q_i = \delta_{2^n}^\beta - \mathbf{1}_{2^n}$. Then for any $t_1 > t_2$,*

$$a^T(t_1) \xi^*(t_1) = 0, \tag{22}$$

$$a^T(t_1) \xi^*(t_2) < 0, \tag{23}$$

where $\xi^*(t)$ corresponds to $x^*(t)$ with the dynamics in (9).

Proof. Since $J(x^*(N^*); u^*) = q^T \xi^*(N^*) = 0$, one has

$$0 = q^T \xi^*(N^*) = q^T \Phi(N^*, t_1; u^*) \xi^*(t_1) = a^T(t_1) \xi^*(t_1).$$

To prove the condition (23), suppose that there exists $t_2 < t_1$ such that $a^T(t_1) \xi^*(t_2) \geq 0$. Owing to the nonnegativity of $\xi^*(t)$ and $-a(t)$ according to the dynamics in (17), it can only hold that $a^T(t_1) \xi^*(t_2) = 0$, i.e.,

$$q^T \Phi(N^*, t_1; u^*) \xi^*(t_2) = 0. \tag{24}$$

Since $q = \mathbf{1}_s \otimes q_i$ with $q_i = \delta_{2^n}^\beta - \mathbf{1}_{2^n}$, from (24), we have $(\mathbf{1}_s^T \otimes (\delta_{2^n}^\beta)^T) \Phi(N^*, t_1; u^*) \xi^*(t_2) = \mathbf{1}_{s2^n}^T \Phi(N^*, t_1; u^*) \xi^*(t_2)$ implying $(\delta_{2^n}^\beta)^T (\mathbf{1}_s^T \otimes I_{2^n}) \Phi(N^*, t_1; u^*) \xi^*(t_2) = 1$, i.e., $(\mathbf{1}_s^T \otimes I_{2^n}) \Phi(N^*, t_1; u^*) \xi^*(t_2) = \delta_{2^n}^\beta$. Therefore, $x_d = \delta_{2^n}^\beta$ is controllable at time $N^* - t_1 + t_2 < N^*$ with probability one. This is contrary to the minimal controllable time N^*. Therefore, the proof is completed. ■

Based on Theorem 3, if N^* is the minimal controllable time, then $a^T(N^*)x^*(N^*) = q^T x^*(N^*) = 0$ and $a^T(N^*)x^*(t) = q^T x^*(t) < 0$ for any $t < N^*$.

Remark 5 *Unlike that in deterministic Boolean control networks with finite and deterministic states, calculating the controllable time is a challenging issue for stochastic Boolean control networks. Although the studied MJBCN in this paper can be equivalently converted into algebraic expressions (5) and (9), which are similar to that of Boolean control networks, the state space and the transition matrices are no longer Δ_{2^n} and logical matrices, respectively. This leads to major difficulties in determining an upper bound of the controllable time for MJBCNs. However, the minimal controllable time can be verified by the conditions in Theorem 3.*

4 Examples

Example 1 Motivated by the fact that Boolean networks have been widely applied in genetic regulatory networks (Kobayashi & Hiraishi 2011), we consider a Markovian jump network:

$$\begin{cases} x_1(t+1) = f_1^{\sigma(t)}(x_1(t), x_2(t), u(t)), \\ x_2(t+1) = f_2^{\sigma(t)}(x_1(t), x_2(t), u(t)), \end{cases} \tag{25}$$

where $x_1(t), x_2(t)$ denote some gene states, $u(t)$ represents the factor determined by the external environment, and $\sigma(t) \in \mathscr{S} = \{1,2\}$ is modeled as a Markov chain with its transition probability matrix being $\Lambda = \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix}$. The possible update rules are: $f_1 \in \{u \vee (x_1 \vee x_2), u \vee (x_1 \wedge x_2)\}$, $f_2 \in \{x_1 \leftrightarrow x_2, x_1 \vee x_2\}$. By a simple computation, the transition matrices are obtained as $F_1 = \delta_4[1,4,4,3,3,4,4,3]$, $F_2 = \delta_4[1,2,2,4,3,4,4,4]$. Network (25) switches stochastically between two networks according to the Markov chain $\{\sigma(t) | t \geq 0\}$. Assume that the initial state $x_0 = \delta_4^1$ represents a sick state. The purpose is to find a proper control such that state variable is steered to $(x_1, x_2) = (\delta_2^2, \delta_2^2)$ or $x_d = \delta_4^4$ corresponding to a desirable target, namely cell survival.

First, based on Theorem 1, after a computation by mathematical software (MATLAB in this paper), one has

$$x_d = \delta_4^4 \in \mathrm{Col}\left((\mathbf{1}_2^T \otimes I_4)\tilde{F}^2 \delta_8^1 \wedge (\mathbf{1}_2^T \otimes I_4)\tilde{F}^2 \delta_8^5 \right),$$

which means that for network (25), $x_d = \delta_4^4$ is controllable from $x_0 = \delta_4^1$ at $N = 2$ with probability one.

The dimension of $\tilde{F}^N$ increases with N, which can be very high. To avoid having an extra-large matrix, we use the result in Theorem 2 to discuss the controllability. With the obtained $\mathscr{R}_1$ and $\mathscr{R}_2$ where

$$\mathscr{R}_1 = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 0\,1\,1\,0 \end{bmatrix}, \quad \mathscr{R}_2 = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 1\,0\,0\,0 \end{bmatrix},$$

one finds that $(\mathscr{R}_2)_{4,1} = 1$. Then by Theorem 2, $x_d = \delta_4^4$ is controllable from $x_0 = \delta_4^1$ at $N = 2$ with probability one.

For network (25), $n = 2$, $m = 1$, $s = 2$ and the controllable time is $N = 2$. Thus the computational times for verifying the conditions in both Theorems 1 and 2 are $O(2^{12})$ in the worst case. The actual computational times for calculating the conditions in Theorems 1 and 2 are, respectively, about 0.001009 and 0.000589 seconds by MATLAB running on i7-7500U CPU@2.7 GHz, which is reasonably short.

On the other hand, let $q_1 = q_2 = \delta_4^4 - \mathbf{1}_4$ in (15). As analyzed above, at time $t_f = 2$, the cost function $J_{\max} = 0$. In what follows, the maximum principle of MJBCNs in Lemma 3 and Theorem 3 is applied to validate whether $t_f = 2$ is the minimal controllable time and meanwhile determine the optimal control. By a simple computation, since $a(2) = q$ and

$$b_1(1) = a^T(2)L_1\xi^*(1) = -[1\ 0\ 0\ 1\ 1\ 1\ 1\ 0]\xi^*(1),$$
$$b_2(1) = a^T(2)L_2\xi^*(1) = -[1\ 0\ 0\ 1\ 1\ 0\ 0\ 0]\xi^*(1),$$

one has that $b_2(1) \geq b_1(1)$. Thus $u^*(1) = \delta_2^2$. Then $a(1) = (Lu^*(1))^T a(2) = -[1\ 0\ 0\ 1\ 1\ 0\ 0\ 0]^T$. Assume that the initial distribution of $\sigma(t)$ is $\mathbf{Pr}\{\sigma(0) = i\} = p_i$, $i = 1,2$, with $p_1 + p_2 = 1$. Then $\xi^*(0) = p_1\delta_8^1 + p_2\delta_8^5$. From $b_1(0) = a^T(1)L_1\xi^*(0) = -1$ and $b_2(0) = a^T(1)L_2\xi^*(0) = 0$, one can also get that $u^*(0) = \delta_2^2$. Moreover, $\xi^*(1) = L_2\xi^*(0)$ and $a(0) = -[0\ 0.3\ 0.3\ 0\ 0\ 0.6\ 0.6\ 0.6]^T$. It follows that $q^T\xi^*(2) = 0$, $q^T\xi^*(1) < 0$ and $q^T\xi^*(0) < 0$. That is, by Theorem 3, $t_f = 2$ is the minimal controllable time from $x_0 = \delta_4^1$ to $x_d = \delta_4^4$ with probability one.

Example 2 The *lac* operon in *Escherichia coli* has been studied extensively and used as a model system in gene regulation (Jacob & Monod 1961). There are many mathematical models, most of which are differential equations, describing the behavior and interactions of the *lac* gene. It has been proved that information of network topology and interaction type (activation/inhibition) is sufficient for capturing dynamics of gene networks (Albert & Othmer 2003). Therefore, Boolean networks provide an effective tool for conducting qualitative analysis on genetic networks (Jacob & Monod 1961). A reduced Boolean network whose dynamics is equivalent to that of the original model of the *lac* operon is considered as follows (Jacob & Monod 1961):

$$\begin{cases} M(t+1) = \neg G_e(t) \wedge (L(t) \vee L_m(t)), \\ L(t+1) = M(t) \wedge L_e(t) \wedge \neg G_e(t), \\ L_m(t+1) = ((L_{em}(t) \wedge M(t)) \vee L_e(t)) \wedge \neg G_e(t), \end{cases} \tag{26}$$

where M, L, L_m, G_e, L_e, and L_{em} represent *lac* mRNA, lactose, medium concentration of lactose, extracellular glucose, low concentration of extracellular lactose, and medium concentration of extracellular lactose, respectively. As in Jacob & Monod (1961), (L_e, L_{em}) attains three values, i.e., $(L_e, L_{em}) \in \{(\delta_2^2, \delta_2^2), (\delta_2^2, \delta_2^1), (\delta_2^1, \delta_2^1)\}$ (here, δ_2^1 and δ_2^2 are equivalent to 1 and 0, respectively). If the extracellular lactose is in a stochastic circumstance modeled by a Markov chain $\sigma(t)$, then (26) becomes an MJBCN as in (1) with G_e acting as a control input. This MJBCN switches among the following three subnetworks:

$$\begin{cases} M(t+1) = \neg G_e(t) \wedge (L(t) \vee L_m(t)), \\ L(t+1) = \delta_2^2, \\ L_m(t+1) = \delta_2^2, \end{cases} \tag{27}$$

$$\begin{cases} M(t+1) = \neg G_e(t) \wedge (L(t) \vee L_m(t)), \\ L(t+1) = \delta_2^2, \\ L_m(t+1) = M(t) \wedge \neg G_e(t), \end{cases} \tag{28}$$

$$\begin{cases} M(t+1) = \neg G_e(t) \wedge (L(t) \vee L_m(t)), \\ L(t+1) = M(t) \wedge \neg G_e(t), \\ L_m(t+1) = \neg G_e(t), \end{cases} \tag{29}$$

and the transition probability matrix of $\sigma(t)$ is given as $\Lambda = \begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.4 & 0.3 & 0.3 \end{bmatrix}$. In these settings, the operon is ON when $M = \delta_2^1$ and OFF when $M = \delta_2^2$. The research problem is whether the operon can be streered from ON to OFF by a sequence of extracellular glucose G_e.

With this aim, let $x(t) = M(t)L(t)L_m(t)$ and $u(t) = G_e(t)$, then the MJBCN model in this example can be converted into the form as (3) and the studied problem is changed to the controllability problem from $\delta_8^4, \delta_8^3, \delta_8^2, \delta_8^1$ to $\delta_8^8, \delta_8^7, \delta_8^6, \delta_8^5$ with probability one. Here, we just consider whether the state δ_8^8 is controllability from δ_8^4 with probability one. By Theorem 1 and some computation, the condition $\delta_8^8 \in \mathbf{Col}(\Phi_N)$ for $N = 2$ holds; that is, the state δ_8^8 is controllability from δ_8^4 with probability one.

5 Conclusion

In this paper, controllability of MJBCNs was investigated by the semi-tensor product of matrices. By converting MJBCNs into a deterministic network with the dynamics similar to that of typical Boolean control networks, one necessary and sufficient condition was obtained by iteration. To lower the dimensions of involved matrices, another equivalent condition was derived based on a newly defined controllable matrix. Moreover, to deal with the minimal controllable time, a necessary condition was derived by utilizing the maximum principle of MJBCNs. Future research of interest is to study the minimal controllable time via a more general condition that is independent of the optimal control and optimal trajectory. Besides, observability is one of the fundamentally important topics in Boolean networks. To our best knowledge, observability of deterministic Boolean networks has four different definitions as stated in Zhang & Zhang (2016). As pointed out in Zhang & Zhang (2016), the methods and techniques dealing with observability are different from those tackling controllability. Observability of MJBCNs is another topic which shall be of our future research interest.

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