An Efficient Constrained Predictive Control Algorithm for 2-input Systems

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Abstract. A multivariable constrained predictive controller based on the generalised predictive control approach is formulated. A special case of 2-input/2-output system is treated in detail and a simple solution which avoids the quadratic programming solution is obtained. In addition to its ability to handle directional non-linearity and constraints systematically, the algorithm also gives an interpretation to the sequencing and range-splitting control strategies often found in industrial controllers.

Keywords: predictive control; constrained optimisation

1 Introduction

Processes are normally affected by physical or operational limitations. Input constraints arise, for example, when valves are used: the valve position can only vary between 'fully closed' and 'fully open'. Speed and torque limits also form the input constraints when motors are used. Output constraints occur in high precision applications when limits need to be placed on the acceptable variations of the output. Traditionally, the control calculations are made under the assumption that the resulting control action will be implemented. The inability to implement the calculated control action due to physical constraints usually leads to a degraded process response. In predictive control, since future control moves are also calculated by the algorithm, violations of constraints on the manipulated variables can be dealt with before they occur. This may result in a better performance and a lesser risk of instability. Using a process model, the constrained predictive controller can also determine when a process output is going to exceed some acceptable limit. Corrective action can then be taken before the limit is exceeded.

The main motivation for developing control algorithm that can handle process constraints comes from the chemical process industry in which constraints constitute an important part of most chemical process control problems (Prett and Garcia, 1988). The minimisation of a performance index subject to inequality constraints does not yield a closed-form solution; some form of search technique must be employed. In the case of a quadratic minimisation with linear inequality constraints, least squares (Lawson and Hanson, 1974) or quadratic programming (QP) (Fletcher, 1987) techniques are often used. Garcia (Garcia and Morshedi, 1986)

proposed to optimise the constrained dynamic matrix control (DMC) cost function using a gradient based QP technique. Dion (Dion et al., 1991) applied a similar idea to the generalised predictive control (GPC) cost function, while Wilkinson (Wilkinson et al., 1990) used singular-value decomposition together with least-distance-programming to find an admissible solution to multivariable constrained GPC. Tsang and Clarke (Tsang and Clarke, 1988) considered input-constrained GPC for high speed applications and devised an efficient search algorithm for the case when the control horizon is two.

In this paper, a multivariable constrained predictive controller based on the GPC (Clarke et al., 1987) approach is formulated. A special case of 2-input/2-output system is treated in detail and a simple solution which avoids the quadratic programming is obtained. It is shown that the algorithm can handle directional non-linearity and constraints systematically and it also gives an interpretation to the split-range or sequencing control strategies often found in industrial controllers.

2 Constrained predictive control algorithm

The formulation of a multivariable constrained predictive control algorithm is briefly stated to provide readers with sufficient background information for the rest of the paper. The details can be found in (Ling, 1992).

By minimising a moving cost of the form

$$J = \sum_{i=1}^{p} \sum_{j=N_1}^{N_2} (w_i(t+j) - y_i(t+j))^2$$

$$+ \sum_{i=1}^{m} \lambda_i \sum_{j=1}^{N_u} (u_i(t+j-1))^2$$

$$+ \sum_{i=1}^{m} \gamma_i \sum_{j=1}^{N_u} (\Delta u_i(t+j-1))^2 \qquad (1)$$

subject to the constraints

$$v_i(t+j) \le d_{v_i},$$

where λ_i and γ_i are weightings on the control signals and their increments respectively; v_i and d_{v_i} represent the constrained variables and their respective limits and w_i, y_i and u_i are the set-points, outputs and inputs of a m-input/p-output plant, the problem of constrained predictive control can be cast into a standard optimisation problem:

Minimise the cost function

$$J = \mathbf{u}^T A \mathbf{u} - 2B^T \mathbf{u} + D \qquad (2)$$

subject to the constraints

$$\bar{C}\mathbf{u} \le \bar{\mathbf{d}},$$

If the set of constraints on which the optimum lies is known, the solution is given by

$$\mathbf{u} = A^{-1}B - A^{-1}C^T(CA^{-1}C^T)^{-1}(CA^{-1}B - \mathbf{d}) \quad (3)$$

where C is the active constraints matrix, and $\mathbf{d}$ is the active constraints vector.

In general, the above minimisation constitute a quadratic program (QP) which no closed-form solution is available; some form of search technique must be employed. However, for a special case of a 2-input/2-output process, and with the control horizon in the predictive algorithm selected to be unity ($N_u = 1$), the geometry of the problem can be exploited to give a simple solution. Note that if no constraint is active at the optimum, i.e., C and $\mathbf{d}$ are null matrix and vector respectively, eqn. 3 reduces to the familiar least squares solution:

$$\mathbf{u}^* = A^{-1}B. \qquad (4)$$

3 Two-input/two-output plant

The motivation for studying this particular case is that temperature controllers often have to handle process non-linearity due to differences in heating and cooling dynamics, and gain-scheduling PIDs are commonly used. Another example can be found from the heating, ventilating and air-conditioning applications where a single PI controller is often used to control different components in the air-handling plant. In addition to the differences in

the dynamic characteristics of these components, there are also operational constraints that need to be respected while operating the plant. For example, a minimum amount of fresh air intake must be maintained while operating the air-handling plant. Constrained predictive control approach can be applied to these situations. In this paper, an efficient algorithm for the solution to this particular case is developed. The application of this controller to an environmental control system and the experimental results are given in (Ling and Dexter, 1991).

For a two-input/two-output plant, if the constrained predictive control algorithm is formulated for the case of unit control horizon ($N_u = 1$) and for simplicity, only input amplitude constraints are considered, then some of the quantities involved in eqn. 2 reduces to

$$\bar{C} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}^T,$$

$$\bar{\mathbf{d}} = [\overline{u_1} \; \overline{u_2} \; \underline{u_1} \; \underline{u_2}]^T, \qquad \text{and}$$

$$\mathbf{u} = [u_1(t) \; u_2(t)]^T$$

We shall also denote the matrix A in eqn. 2 by

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Geometrically, the problem is depicted in figure 1. On the $u_1(t)$–$u_2(t)$ plane, the constraints form a rectangular feasible region. The contours of $J = $ constant are a family of ellipses. The centre of the ellipse family is given by the unconstrained optimum u^*. If u^* is in the feasibility region, the solution is simply $u^\dagger = u^*$ where $u^\dagger$ denotes the constrained optimum. Otherwise, the objective of the constrained optimisation is to find the smallest ellipse that touches the feasibility boundary. This problem has been considered by Tsang and Clarke (1988); however, the solution given only ensures that the first control signal is calculated correctly. This is sufficient for a single-input/single-output problem because of the receding horizon implementation. In the present situation, it is required that both the control signals are calculated correctly. Such a solution can be obtained by considering the following cases:

3.1 Only one constraint is violated

In this case, $u^\dagger$ is given by minimising J along the line PQ or RS as shown in Fig. 2a and 2b, depending on which constraint is being violated, and clipping $u_1(t)$ or $u_2(t)$ if necessary.

Minimising J along a line can be interpreted geometrically as finding a tangential point of an ellipse that touches the line PQ or RS. The solution can be

found be selecting an appropriate C and d in equation 3. For example, to minimise along line RS, we have $C = [1 \; 0]$, $d = \overline{u_1}$. From eqn. 3, yields

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \overline{u_1} \\ u_2^* - \frac{a_{21}}{a_{11}}(u_1^* - \overline{u_1}) \end{bmatrix}$$

and after clipping

$$\begin{bmatrix} u_1^\dagger \\ u_2^\dagger \end{bmatrix} = \begin{bmatrix} \overline{u_1} \\ \text{sat}(u_2, \underline{u_2}, \overline{u_2}) \end{bmatrix}, \qquad (5)$$

where

$$\text{sat}(x, \underline{x}, \overline{x}) = \begin{cases} \underline{x}, & \text{if } x < \underline{x} \\ \overline{x}, & \text{if } x > \overline{x} \\ x, & \text{otherwise} \end{cases}$$

The remaining three cases can be arrived at in a similar manner and it can be shown (Ling, 1992) that the solutions to the various cases of constraint violations are all similar to equation 5.

3.2 Both constraints are violated

The method described for the one constraint violation case will still give the correct optimum if $u^\dagger$ is on the line PQ or RS (Fig. 2c). If $u^\dagger$ is on one of the corners of the feasibility region, then any of the above minimisation will result in both $u_1(t)$ and $u_2(t)$ violating the constraints and clipping them will bring the solution to the correct corner (Fig. 2d).

3.3 The total solution

Combining all the possible cases of constraint violations, the total solution for this special case of 2-input systems can be described by Fig. 3.

Remark 1 *Although only input amplitude constraints has been considered in arriving at the solution, rate constraints can also be handled in a similar manner. In addition, for the case of unity control horizon ($N_u = 1$), both rate and amplitude constraints can be handled simultaneously. Note that the constraints are now time-varying, depending on the past inputs to the plant.*

4 Interpretations

The constrained predictive controller described earlier can also be considered as a scheduling controller, where the operating condition is divided into two regions and the scheduling strategy is implicitly prescribed by the model. This allows, for example, handling separately different dynamics of the cooling and heating elements in the loop.

4.1 Split-range or sequencing controller

A split-range or sequencing controller arises when a two-input/one-output plant

$$\mathcal{A}(q^{-1})y(t) = \mathcal{B}_1(q^{-1})u_1(t) + \mathcal{B}_2(q^{-1})u_2(t)$$

with $\mathcal{B}_2 = k\mathcal{B}_1$ and no weighting is placed on the control increments is considered. Here, we need at least either one of the weightings on the control signals, λ_1 or λ_2, to be nonzero in order to have an unique solution.

Let H_1 and H_2 be vectors of step response coefficients of $\frac{\mathcal{B}_1}{\mathcal{A}}$ and $\frac{\mathcal{B}_2}{\mathcal{A}}$ respectively. Then from eqn. 4, for the case of $N_u = 1$, the unconstrained optimum is

$$\mathbf{u}^* = \frac{1}{\det} \begin{bmatrix} \lambda_2 H_1^T \\ \lambda_1 k H_1^T \end{bmatrix} (\mathbf{w} - \mathbf{f} + G\tilde{\mathbf{u}}). \qquad (6)$$

where $\det = k^2\lambda_1 H_1^T H_1 + \lambda_2 H_1^T H_1 + \lambda_1\lambda_2$.

Suppose we choose $\lambda_1 = 0$, $\lambda_2 \neq 0$, i.e. assuming that the cost of u_1 is negligible compare to that of u_2, then

$$u_1^* = \frac{\lambda_2 H_1^T}{\lambda_2 H_1^T H_1}(\mathbf{w} - \mathbf{f} + G\tilde{\mathbf{u}})$$

$$u_2^* = 0$$

$$\frac{a_{21}}{a_{11}} = \frac{-k H_1^T H_1}{k^2 H_1^T H_1 + \lambda_2} \approx -\frac{1}{k},$$

$$\text{if } \lambda_2 \ll k^2 H_1^T H_1$$

Hence, Fig. 3 simplifies to Fig. 4 and the resulting controller structure can be interpreted as a split-range or sequencing strategy commonly found in industrial controllers. The derivation also shows that u_1^* is independent of λ_2 and the value of λ_2 should be chosen small so that the approximation $\lambda_2 \ll k^2 H_1^T H_1$ holds.

An intuitive explanation for the logic of a split-range or sequencing controller is as follows: Whenever a single controlled variable responds to two manipulated variables, there is opportunity for optimisation. If one of the manipulated variables were free, then it can be maximised, which will in turn minimise the other. Therefore, only when the maximum usage of the free variable is still insufficient to meet the control demand then will the second variable be used; hence the sequencing logic. The derivation also showed how the differences in gains can be accounted for in the scheduling logic. In many plant situations, neither of the input variables is free; there exist some combinations of the two which will provide the necessary output at a minimum cost. This optimum can be found by the general procedure described where neither λ_1 nor λ_2 is zero.

5 Conclusions

In this paper, a constrained predictive control approach to the design of a controller is presented. For the special case of 2-input/2-output systems, a simple solution is obtained which is suitable for implementation on a low-cost dedicated hardware. In addition to the ability to handle non-linearity and constraints systematically, it also gives an interpretation of the split-range or sequencing control strategy often found in industrial controllers. This controller has been applied to environment control systems and experimental results suggest that significant cost savings on operating the air-handling plant can be achieved.

Figure 1: *The geometry of the problem of the case of* $N_u = 1$.

Figure 2: *Various cases of constraint violations* $(N_u = 1)$.
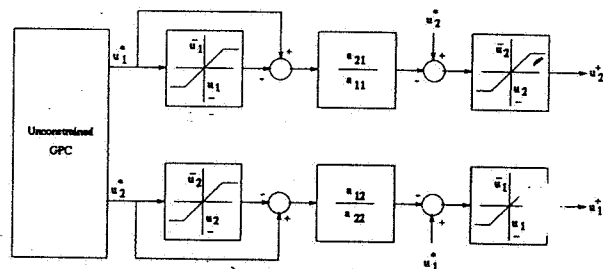
Figure 3: *A total solution to two-input amplitude constrained optimisation,* $N_u = 1$.

Figure 4: *Input constrained GPC as a split-range controller*

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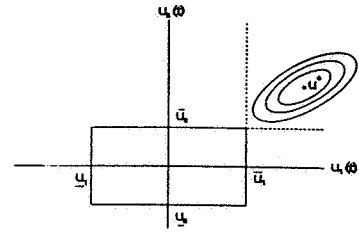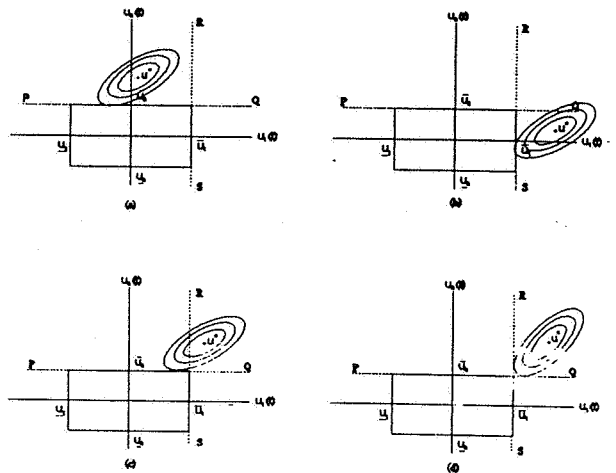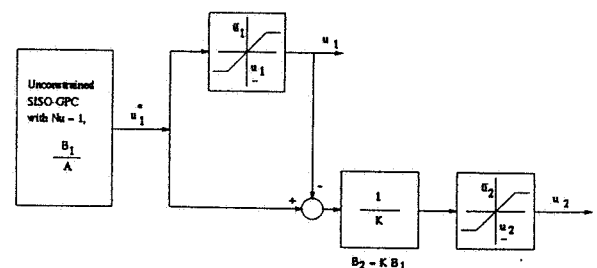Wilkinson, D.J., A.J. Morris and M.T. Tham (1990). Multivariable constrained generalised predictive control, a comparison with QDMC. ACC 1990.