

Oscillatory and Chaotic Dynamics in Neural Networks Under Varying Operating Conditions

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Abstract—We study the effects of a *time-dependent operating environment* on the dynamics of a neural network. In a previous paper, we studied an exactly solvable model of a higher order neural network [14]. We identified a bifurcation parameter for the system, i.e., the rescaled noise level, which represents the combined effects of incomplete connectivity, interference among stored patterns, and additional stochastic noise. When this bifurcation parameter assumes different but static (time-independent) values, the network shows a spectrum of dynamics ranging from fixed points, to oscillations, to chaos. Here we show that varying operating conditions described by the time-dependence of the rescaled noise level give rise to many more interesting dynamical behaviors, such as disappearances of fixed points and transitions between periodic oscillations and deterministic chaos. These results suggest that a varying environment, such as the one studied in the present model, may be used to facilitate memory retrieval if *dynamic* states are used for information storage in a neural network.

I. INTRODUCTION

RECENT physiological experiments and theoretical studies have suggested [1], [2] that the brain may be using dynamic attractors to store memory, rather than static states as in most artificial neural networks (ANN's) (e.g., [3]–[12]). Investigations of dynamical networks may thus lead to the discovery of powerful algorithms for information processing with ANN's, a prospect that has increased the research interest in the area of dynamic behaviors in ANN's (e.g., [13]–[23]). In particular, several authors have used limit cycles [13], strange attractors [18], [22], [23], and transient behaviors [25] to represent associative memory. Dmitriev and co-workers have effectively applied chaos for information processing in ANN's [15], [24]. Existing work on ANN's focuses on those operating under time-independent conditions, but little effort has been invested in discussing how a varying environment may affect the dynamics of an ANN. In this paper we investigate such effects.

The effects of a dynamical environment have been discussed in areas other than ANN's. May claimed that temporal variations in the environment are a destabilizing influence in ecology [26]. A time-dependent bifurcation parameter in the logistic map was used to study natural populations under varying growth conditions [27]. Many unexpected dynamics were found, among them the noise-induced order, which was discovered earlier in a different system by Matsumoto and Tsuda [28] and which contradicts May's conclusion [26]. Ott

et al. [29] showed theoretically that one can convert a chaotic attractor to any one of a large number of possible attracting time-periodic motions by making small time-dependent perturbations of a system parameter, thereby achieving control of chaos. Their prediction was subsequently observed experimentally in an amorphous magnetoelastic ribbon [30]. The exponential sensitivity of a chaotic system to small perturbations in system parameters was used to develop a method to direct the system to a desired accessible state in a short time [31].

In a previous paper [14], we presented an ANN of McCulloch–Pitts two-state neurons connected by higher order [8] Hebbian-type synapses [5], [32], [33]. Exact solutions are derived for the network dynamics and a variety of dynamical behaviors such as stable retrieving, oscillations, and chaos are revealed. A rescaled noise level that represents the combined effects of the random synaptic dilution, interference between stored patterns, and additional background noise, is found to be an important bifurcation parameter in our system and was assumed to be *independent of time* in the previous work [14].

In the present paper, we study the effects of a *dynamical* environment on an ANN by letting the rescaled noise level, as a system parameter, vary with time. We show that a time-varying environment brings about many interesting changes to the original network dynamics under time-independent conditions. Depending on the fashion in which the bifurcation parameter varies with time, the dynamic environment may stabilize or it may control a chaotic state; e.g., it may turn a chaotic state of the network to an ordered state. It may also destabilize an ordered state of the network and turn it into chaos. The stable fixed points (period-one oscillations) disappear completely under time-varying conditions.

II. THE NETWORK MODEL AND ITS ANALYSIS

A. Under Time-Independent Operating Conditions

The artificial neurons used in the present network model are McCulloch–Pitts two-state neurons. They are connected by both first-order and second-order Hebbian-type rules. Explicitly, the state of the i th neuron is

$$S_i(t + \Delta t) = \text{sign}[h_i(t)] \quad (1)$$

where $\text{sign}(x) = -1$ for negative x and $\text{sign}(x) = +1$ otherwise. The total input for the i th neuron is

$$h_i(t) = \gamma_1 \sum_{j=1}^N T_{ij} S_j(t) + \gamma_2 \sum_{j,k=1}^N T_{ijk} S_j(t) S_k(t) + \eta_i \quad (2)$$

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where

$$T_{ij} = C_{ij} \sum_{\mu=1}^p S_i^\mu S_j^\mu, \quad T_{ijk} = C_{ijk} \sum_{\mu=1}^p S_i^\mu S_j^\mu S_k^\mu \quad (3)$$

are the modified Hebbian synaptic efficacies, \vec{S}^μ is the μ th stored pattern, and p is the number of patterns stored. The coefficients γ_1 and γ_2 measure the relative strengths of first-order and second-order interactions. We have introduced random asymmetric dilution in the efficacies T_{ij} and T_{ijk} by choosing random variables C_{ij} and C_{ijk} as follows: C_{ij} takes the value one with a probability (C/N) and the value zero with a probability $(1 - C/N)$, and C_{ijk} takes the value one with a probability $(2C/N^2)$ and the value zero with a probability $(1 - 2C/N^2)$. Synaptic dilution is essential in both modeling the observed incomplete connectivity in real neurophysiological systems and assuring an *exact* solution [14], [33]. Although the synaptic disruptions are randomly carried out, information (patterns $\vec{S}^\mu, \mu = 1 \cdots p$) can still be stored with the remaining Hebbian synapses. We also include in (2) a background random Gaussian noise η_i with an average zero and a standard deviation σ_o in order to take into account the presence of noise (temperature).

For *parallel* network dynamics where all neurons are updated simultaneously (for effects of updating synchronicity, see [17]), it can be shown [14] that the network obeys the following dynamical equation:

$$m(t+1) = \text{erf}\left\{\frac{\gamma_1 m(t) + \gamma_2 [m(t)]^2}{\sqrt{2}\sigma}\right\} \equiv F[m(t), \sigma]. \quad (4)$$

Here $m(t) = \langle 1/N \vec{S}^1 \cdot \vec{S}(t) \rangle$ is the average overlap between the state of the system at time t and \vec{S}^1 , the memory state that is closest to the initial condition of the system

$$\text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-x^2} dx \quad (5)$$

is the standard error function and σ is the rescaled noise level deviation

$$\sigma \equiv \sqrt{(\gamma_1^2 + \gamma_2^2)[(p-1)/C] + (\sigma_o/C)^2}. \quad (6)$$

If $\gamma_1 = 1$ and $\gamma_2 = -1$, the positive attractors m of the network can be obtained by iterating (4) for many time steps (see [14] for discussions of other γ_1 and γ_2 values). As shown in Fig. 1, when $\sigma > 0.8$ and for any positive initial overlap $m(0) > 0$, the only nonnegative fixed points are zeros. For $0.8 > \sigma > 0.2$, the system converges to a single branch of stable positive fixed points with any positive initial overlap. This corresponds to static retrievals. As σ decreases below 0.2, oscillations start to appear and the network dynamics becomes chaotic through a sequence of period-doubling bifurcations as σ is further reduced. Amidst the largely chaotic behaviors, there are small “windows” of σ values in which the network dynamics exhibits periodic oscillations, for example, a period-3 oscillation [34]. We shall discuss this period-3 oscillatory region further below.

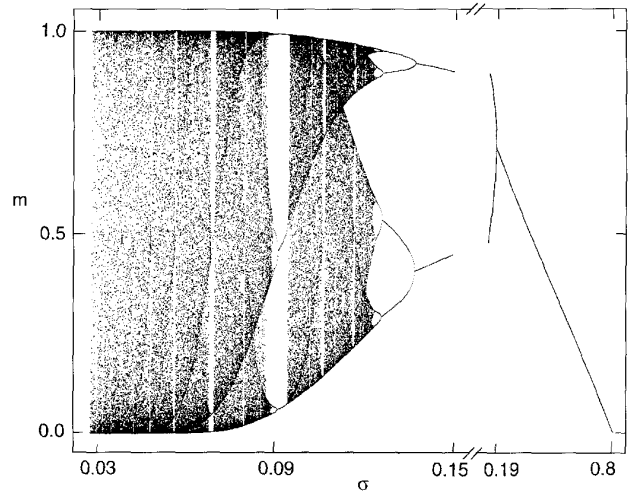


Fig. 1. Fixed points of m , the overlap between the network state and the initial attracting memory, as a function of the rescaled noise level σ under time-independent operating conditions.

B. Under Time-Dependent Operating Conditions

In the above discussions [14], the bifurcation parameter σ has been assumed to be a constant of time. This is analogous to the brain working in a time-invariant environment. However, the environment often changes with time. For instance, when one is listening to a piece of music, the temporal tones of the music give rise to varying conditions in the brain. As an initial effort to explore the effects of such varying conditions, we let σ vary with time in our neural-network model, which resembles the study of varying growth conditions with a time-dependent bifurcation parameter in a logistic map [27] and work on chaos control with time-dependent perturbations to various system parameters [29]–[31].

A simple choice of the time-dependence of σ is that σ alternates periodically between two values, for example, $\sigma = A$ at odd time steps and $\sigma = B$ at even time steps. Fig. 2(a) and (b) show the bifurcation diagrams for $B = 0.61A$ and $B = 0.25A$, respectively. Compared with the constant σ case where $A = B$ (Fig. 1), we find that positive (period-1) fixed points disappear completely and the most ordered cases are period-2 oscillations. The noise threshold at which the average final overlap $m = 0$, the onset threshold at which chaos first appears, and the amplitudes of period-2 oscillations all increase as the difference between A and B increases.

Lyapunov exponents, which measure the randomness of the network dynamics, are calculated as a function of (A, B)

$$\lambda = \frac{1}{N} \sum_{t=t_r}^{N+t_r} \log_2 \left| \frac{dm(t+1)}{dm(t)} \right| \quad (7)$$

where $t_r = 10^3$ is a transient interval before the network dynamics stabilizes and $N = 10^6$ is a time-window in which the Lyapunov exponent is calculated.

In addition to the simple periodic variation of the rescaled noise level σ between two parameters (A, B) as discussed above, we consider a more general form of variation described

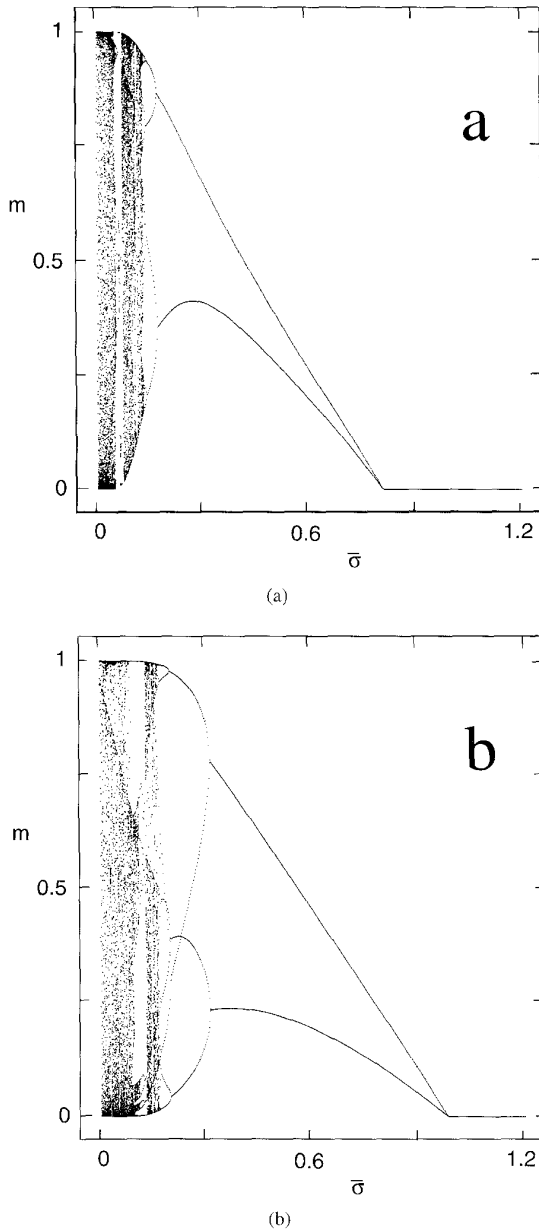


Fig. 2. Fixed points of m , the overlap between the network state and the initial attracting memory, as a function of the average rescaled noise level $\bar{\sigma} = (A + B)/2$, when the rescaled noise level σ alternates periodically between A and B . (a) $B = 0.61A$. (b) $B = 0.25A$.

by the Cowley parameter [35], [27]

$$\gamma = 1 - \frac{P_{B|A}}{P_A} \quad (8)$$

where P_A is the probability for $\sigma = A$ and $P_{B|A}$ is the conditional probability for $\sigma = B$ under the condition that $\sigma = A$ in the previous time step. Since $P_A P_{A|B} = P_B P_{B|A}$ and $P_A + P_B = P_{A|B} + P_{A|A} = P_{B|A} + P_{B|B} = 1$, we have

$$P_{A|A} = P_{B|B} = (1 + \gamma)/2 \quad (9)$$

$$P_{A|B} = P_{B|A} = (1 - \gamma)/2 \quad (10)$$

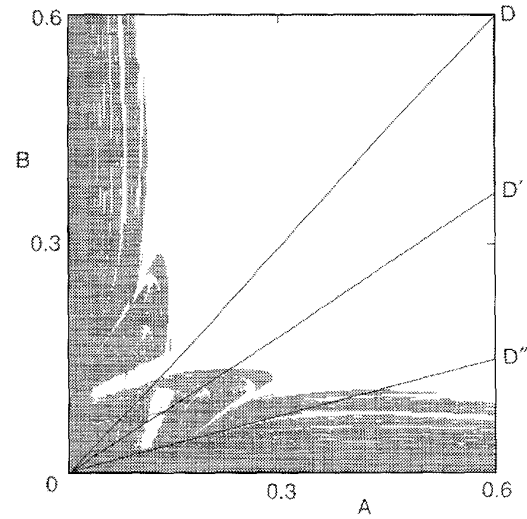


Fig. 3. The signs of the Lyapunov exponents λ given by (7) when the rescaled noise level σ alternates periodically between A and B , i.e., when the Cowley parameter $\gamma = -1$. Black dots represent positive Lyapunov exponents (chaos) and white dots represent negative Lyapunov exponents (order). The case where $A = B$ or σ does not vary with time (Fig. 1) is represented by the diagonal line OD . OD' represents $B = 0.61A$ [Fig. 2(a)] and OD'' represents $B = 0.25A$ [Fig. 2(b)].

if we let $P_A = P_B = 0.5$, which we shall assume in our subsequent discussion. Equations (9) and (10) indicate that the Cowley parameter γ serves as a measure of randomness for σ to switch between values (A, B) in any two adjacent time steps. For example, when γ is close to but less than one, (9) and (10) show that switching between A and B is unlikely, hence the sequence $\{\dots \sigma(t-1)\sigma(t)\sigma(t+1)\dots\}$ looks like $\{\dots AAAAAABBBBBB\dots\}$. Conversely, when γ is close to but greater than -1 , (9) and (10) show that switching between A and B is extremely likely, which corresponds to the discussed case where σ alternates periodically between A and B . The case where $\gamma = 0$ corresponds to the completely random situation where $P_{A|A} = P_{B|B} = P_{A|B} = P_{B|A} = 0.5$.

The results for the Lyapunov exponent are presented in Figs. 3 and 4.

In Fig. 3, the black and white dots represent positive (chaotic) and negative (ordered) Lyapunov exponents at the corresponding (A, B) , respectively, for the case where $\gamma = -1$, i.e., a periodic σ . The diagonal line OD represents the case where $A = B$ or σ does not change with time. Fig. 2(a) is represented by OD' , where $B = 0.61A$. OD'' represents $B = 0.25A$ and passes through a larger black region, which supports the fact that chaos is more dominant in Fig. 2(b) compared to Fig. 2(a). In these situations, the time-dependence of σ promotes chaos or disorder, as concluded by May [26].

Fig. 3 shows only the signs of the Lyapunov exponents. The actual values of the Lyapunov exponents are important in studying trends of changes between order and chaos. Due to the complexity of chaotic structures, a three-dimensional plot is not meaningful. We represent in Fig. 4(a)–(h) the values of the Lyapunov exponents by vivid colors. We are able to illustrate the enormous information content of the chaotic

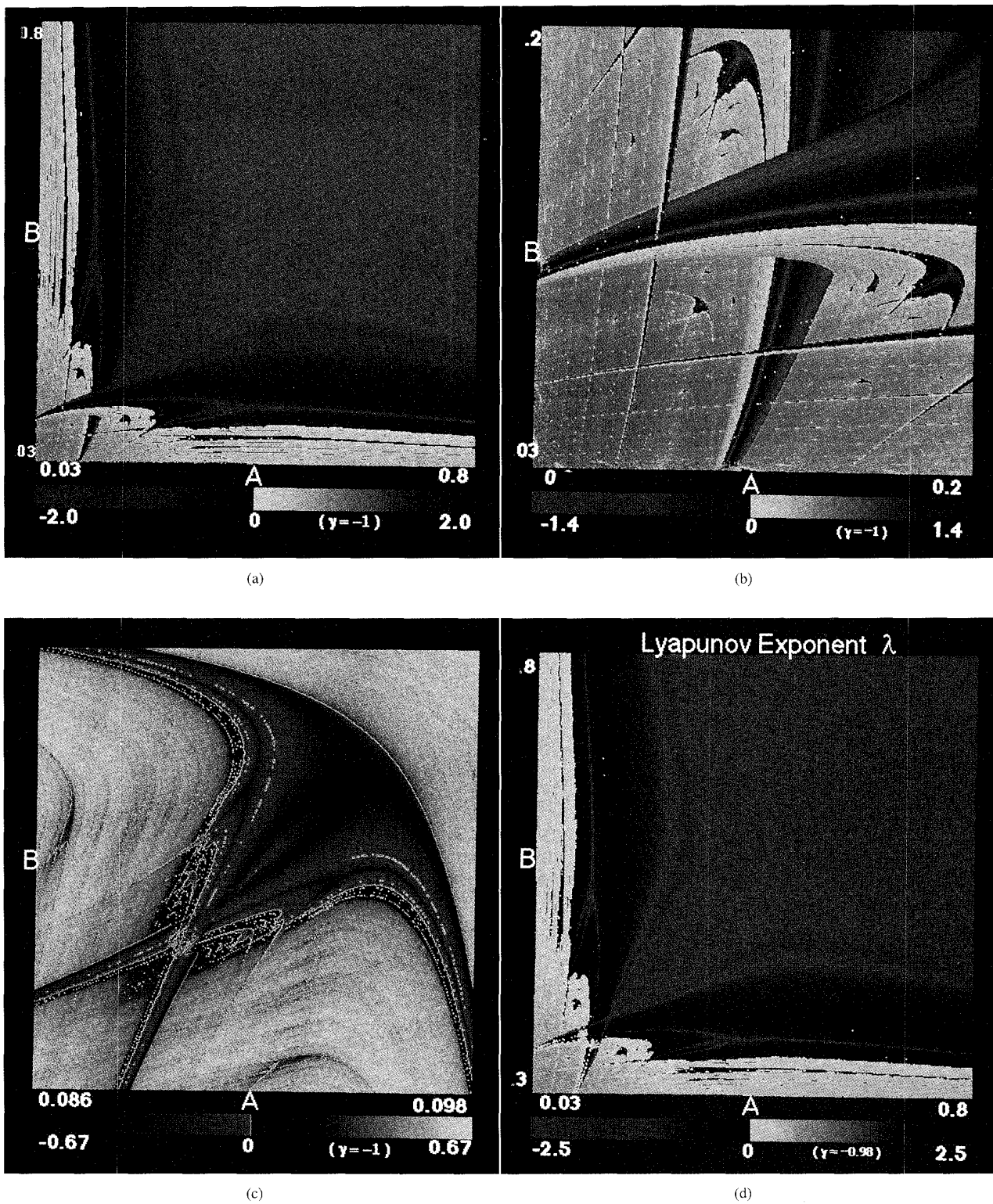


Fig. 4. Lyapunov exponents λ given by (7), when the rescaled noise level σ alternates between A and B according to (8) with various choices of the Cowley parameter γ . Positive Lyapunov exponents indicate chaos and negative Lyapunov exponents indicate order. (a) $\gamma = -1$ (σ alternates periodically), $0.03 \leq A, B \leq 0.8$. The figure is asymmetric with respect to the diagonal line. (b) the region with $0.03 \leq A, B \leq 0.2$ in (a) enlarged. (c) the region with $0.086 \leq A, B \leq 0.098$ in (b) enlarged (the period-3 island). (d) $\gamma = -0.98$ (σ alternates slightly randomly), $0.03 \leq A, B \leq 0.8$. The figure is symmetric with respect to the diagonal line, in contrast to (a).

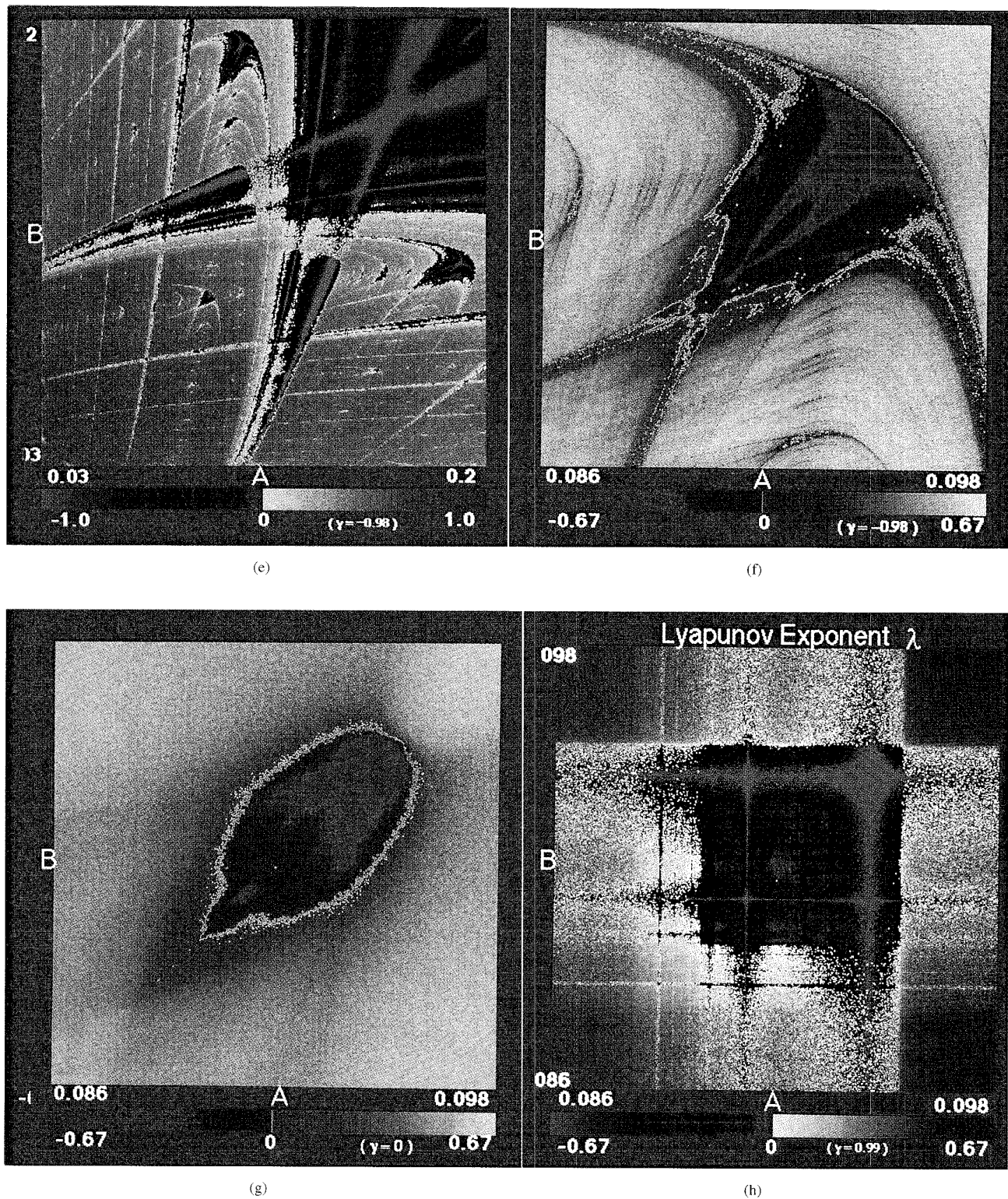


Fig. 4. (Continued.) Lyapunov exponents λ given by (7), when the rescaled noise level σ alternates between A and B according to (8) with various choices of the Cowley parameter γ . Positive Lyapunov exponents indicate chaos and negative Lyapunov exponents indicate order. (e) The region with $0.03 \leq A, B \leq 0.2$ in (d) enlarged. (f) The region with $0.086 \leq A, B \leq 0.098$ in (e) enlarged (the period-3 island). (g) Period-3 island with $\gamma = 0$ (σ varies completely randomly between A and B), $0.086 \leq A, B \leq 0.098$. (h) Period-3 island with $\gamma = 0.99$ (σ hardly varies between A and B), $0.086 \leq A, B \leq 0.098$.

structures in the present model. This suggests that chaos in neural networks may be used to store even more information than conventional static storage prescriptions [3]–[12].

Fig. 4(a)–(c) shows the values of Lyapunov exponents λ when the rescaled noise level σ alternates periodically between A and B , i.e., when the Cowley parameter $\gamma = -1$. The asymmetry of these figures with respect to the diagonals shows the sensitivity of the network dynamics to the order in which σ assumes the values A and B . In contrast, Fig. 4(d)–(h) are symmetric with respect to the diagonals when σ assumes the values A and B with certain randomness.

By comparing Fig. 4(c), (f), (g), and (h), which illustrate the period-3 “island” (green in color) for different choices of the Cowley parameter γ , we observe that the period-3 island has the smallest area for $\gamma = 0$, when σ switches between A and B completely randomly. Hence the randomness in sequence $\sigma(t)$ seems to promote chaotic behaviors in the network.

In Fig. 4(b) and (c), for example, there are off-diagonal green (ordered) regions with the corresponding portion of the diagonal line (time-independent σ) passing through a yellow-red (chaotic) region. This implies the following. Suppose the network dynamics is chaotic if σ is kept constant at one or both of two values, e.g., (A_1, B_1) . For some (A_1, B_1) , the network dynamics can become ordered when σ alternates between (A_1, B_1) . In these cases, the time-dependence of σ controls or suppresses chaos and promotes order [27]–[31].

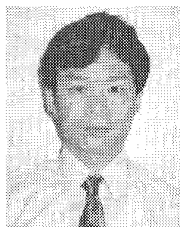
The results presented in this paper are primarily theoretical; however, we have concluded that a varying environment can induce transitions from ordered states to chaotic states and vice versa. If dynamical behaviors are used to efficiently store memory, as suggested by recent physiological and theoretical studies (e.g., [1], [2], [13], [15], and [22]–[25]), a varying environment can be used to facilitate information processing, such as memory retrieval, by inducing transitions between various dynamic network states, which will be a subject of future investigations.

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