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Applications of Gröbner bases to signal and image processing: a survey

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Abstract

This paper is a tutorial and survey paper on the Gröbner bases method and some of its applications in signal and image processing. Although all the results presented in the paper are available in the literature, we give an elementary treatment here, in the hope that it will further bring awareness of and stimulate interest in Gröbner bases among researchers in signal and image processing. We give first a tutorial on Gröbner bases, and then a survey of the design of multidimensional wavelets and filter banks. Applications of Gröbner bases to other areas of signal and image processing are also briefly reviewed.

Keywords: Gröbner bases; Multidimensional filter banks; Multidimensional wavelets; FIR filter; IIR filter; Multivariate polynomial

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1. Introduction

The theory and algorithm of Gröbner bases were originally developed by Buchberger in the 1960s and later on further enriched with contributions from himself and many other researchers (see [9–13,26] and the references therein). On the one hand, the tool of Gröbner bases is no doubt one of the most powerful methods in mathematics in general, and in algebraic geometry and commutative algebra in particular, as it is evident from the facts that numerous books and research papers on Gröbner bases have been published in recent years and that the Gröbner bases method has been implemented in all major general purpose mathematical software systems like Mathematica, Maple, Derive, etc., see e.g. [53], and in a couple of specialized software systems, notably CoCoA [16], Singular [48], and Macaulay [35]. On the other hand, Gröbner bases have also found wide applications in theoretical physics, applied science and engineering. The main reason for the success of Gröbner bases is that many problems in mathematics, science and engineering can be represented by multivariate polynomials (ideals, modules, matrices etc.), and Gröbner bases are well known to play a similar role in multivariate (nD, $n > 1$) polynomials as Euclidean division algorithm in univariate (1D) polynomials. In signal and image processing, Gröbner bases have been applied to various problems in different areas, such as the design of multidimensional (nD) wavelets and filter banks, robust stability analysis of nD digital filters, balanced multiwavelets and digital filter design, and image processing and computer vision. In particular, for the past decade Gröbner bases have received noticeable attention in the design of nD wavelets and filter banks, which can be represented by nD polynomial and rational matrices. See, for example, the recent special issue on applications of Gröbner bases to multidimensional systems and signal processing, guest edited by the first two co-authors of this paper [33]. However, it is felt that many researchers in signal and image processing are still unaware of the powerful tool of Gröbner bases.

The objectives of this paper are twofold, first to further bring awareness of and stimulate interest in Gröbner bases among researchers in signal and image processing, which is consistent to the purpose of this special issue, and second to survey existing results on applications of Gröbner bases to signal and image processing. Instead of covering the general aspects of the Gröbner bases theory and all of its applications in signal and image processing which would require an extensive discussion beyond the scope of this paper, we will be mainly concerned with the basics of the Gröbner bases method, and its applications to the design of nD wavelets and filter banks. Applications of Gröbner bases to other areas of signal and image processing will then be briefly discussed.

The organization of this paper is as follows. In the next section, we give a tutorial on Gröbner bases. We then present a brief introduction to signal processing in Section 3 to provide some background materials and motivation for readers who are unfamiliar with signal processing. Following that, we review existing results on the design of nD regular nonseparable wavelets in Section 4, the design of nD finite impulse response (FIR), perfect reconstruction (PR) filter banks in Section 5, the design of nD infinite impulse response (IIR), PR filter banks in Section 6, all of which use Gröbner bases. We also show that some existing results on stabilization of nD feedback control systems may be readily applied to the design of nD IIR, PR filter banks. In Section 7, applications of Gröbner bases to other areas of signal and image processing are briefly mentioned. Finally, conclusions are given in Section 8.

2. Gröbner bases

In this section, we give a tutorial on the Gröbner bases approach (see, e.g., [11,13,21,26,36] for details). As several nice tutorial papers have been available in the literature (e.g., [11,13]), we shall only focus on the most fundamental concepts and properties of Gröbner bases of nD polynomial ideals as well as modules over nD polynomial ring.

2.1. Why Gröbner basis?

As it will be shown in later sections, several important problems in filter design and signal processing can be essentially reduced to the nD polynomial equation:

$$f_1(z_1,\dots,z_n)x_1(z_1,\dots,z_n)+\cdots+f_m(z_1,\dots,z_n)x_m(z_1,\dots,z_n) = f(z_1,\dots,z_n), \tag{1}$$

where f and f_i are known while x_i are to be found, $i = 1,\dots,m$. To solve (1), we have to consider whether it is solvable and, if the answer is affirmative, how to construct the solution x_i, $i = 1,\dots,m$ which consists of a particular solution $\hat{x}_i$ of (1) and the solutions $\tilde{x}_i$ to the corresponding homogenous polynomial equation:

$$f_1(z_1,\dots,z_n)\tilde{x}_1(z_1,\dots,z_n)+\cdots+f_m(z_1,\dots,z_n)\tilde{x}_m(z_1,\dots,z_n)=0. \tag{2}$$

Let $K[\mathbf{z}] \triangleq K[z_1,\dots,z_n]$ denote the ring of nD polynomials with coefficients over the field K, and let Ideal(F) stand for the ideal generated by $F = \{f_i \mid f_i \in K[\mathbf{z}],\ i = 1,\dots,m\}$, i.e.,

$$\mathrm{Ideal}(F) = \left\{\sum_{i=1}^{m} h_i f_i \mid h_i \in K[\mathbf{z}], i = 1,\dots,m\right\}. \tag{3}$$

It is then obvious that the solvability problem of (1) is equivalent to the so-called ideal membership problem, i.e., how to verify if a given $f \in K[\mathbf{z}]$ belongs to Ideal(F).

For the 1D ($n = 1$) case, the above problem can be easily solved by utilizing the Euclidean division algorithm as shown in the following simple example.

Example 1. Let $f_1 = z_1^2 - 1,\ f_2 = z_1^3 + 3z_1^2 + 2z_1,\ f_3 = z_1^4 + z_1^3 + 2z_1^2 + 2z_1$ and $f = z_1^2 - 2z_1 - 3$. It is easy to find by the Euclidean algorithm (see, e.g., [17,27]) that

$$\begin{aligned} g \triangleq \mathrm{GCD}(f_1, f_2, f_3) &= \mathrm{GCD}(f_1, \mathrm{GCD}(f_2, f_3)) \\ &= \mathrm{GCD}(z_1^2 - 1, (z_1 + 1)z_1) = z_1 + 1. \end{aligned}$$

Note that, in the 1D case, $\mathrm{Ideal}(F) = \mathrm{Ideal}(g)$ holds for any $F = \{f_i \in K[z_1], i = 1, \ldots, m\}$ and we have $f = qg + r$ where $g = \mathrm{GCD}(f_1, \ldots, f_m\}, q, r \in K[z_1]$ and $r = 0$ or $\deg(r) < \deg(g)$. Therefore, $f \in \mathrm{Ideal}(F)$, or equivalently (1) is solvable, if and only if $r = 0$. For this example, $f = (z_1 - 3)(z_1 + 1)$, i.e., $q = z_1 - 3, r = 0$, thus $f \in \mathrm{Ideal}(f_1, f_2, f_3)$, or equivalently, (1) is solvable.

On the other hand, keeping track of the process of constructing $\mathrm{GCD}(f_2, f_3) = (z_1 + 1)z_1$ and $\mathrm{GCD}(f_1, (z_1 + 1)z_1) = z_1 + 1$, we have

$$\left(\frac{1}{3} - \frac{1}{6}z_1\right) \cdot f_2 + \frac{1}{6} \cdot f_3 = (z_1 + 1)z_1 \tag{4}$$

and

$$-1 \cdot f_1 + 1 \cdot (z_1 + 1)z_1 = z_1 + 1. \tag{5}$$

Substituting (4) into (5) and multiplying the result by $q = z_1 - 3$ yield

$$-(z_1 - 3) \cdot f_1 + (z_1 - 3)\left(\frac{1}{3} - \frac{1}{6}z_1\right) \cdot f_2 + \frac{1}{6}(z_1 - 3) \cdot f_3 = f.$$

The solutions to the corresponding homogenous equation of (2) can also be found similarly based on the Euclidean division algorithm [27].

However, since the Euclidean division algorithm does not apply to nD ($n > 1$) polynomials in general, the above problems are much more difficult and subtle to deal with in the general nD cases. In fact, similar difficulties are encountered for all kinds of nD problems whose 1D counterparts are solved by the Euclidean division algorithm. This has motivated the development of new methods to deal with nD polynomials. Gröbner basis is just such a powerful method as it is essentially an analog of the Euclidean division algorithm for nD polynomials.

2.2. Gröbner basis of polynomial ideal

In order to define Gröbner bases, we have to introduce an *admissible* term ordering for power products or monomials over $K[\mathbf{z}]$. By "admissible" we mean that the defined term ordering $<_T$ should satisfy the following two conditions:

(i) $1 <_T t$ for all $t \in T$ and $t \neq 1$;
(ii) if $s <_T t$ then $s \cdot u <_T t \cdot u, \forall t,\ s,\ u \in T$,

where $T = \{z_1^{i_1} \cdots z_n^{i_n} : z_j \in \mathbf{C}, i_j \in Z_+, j = 1, \ldots, n\}$ and Z_+ denotes the set of nonnegative integers. In fact, these conditions imply that $<_T$ is Noetherian, i.e., there exist no infinitely decreasing chains of the form $h_1 >_T h_2 >_T \cdots$, where $h_1 >_T h_2 \Leftrightarrow h_2 <_T h_1$.

There are infinitely many term orderings that are admissible for Gröbner bases theory. Two most popular term orderings are the lexicographic ordering and the total degree lexicographic ordering, which are in fact sufficient for almost all practical purposes. In this paper, unless otherwise indicated, it is assumed that the variables are ordered in such a way that z_1 has the lowest order while z_n has the highest order. For example, in the 2D case, the lexicographic ordering is given by $1 <_T z_1 <_T z_1^2 <_T \cdots <_T z_2 <_T z_1z_2 <_T z_1^2z_2 <_T \cdots$; and the total degree lexicographic ordering is given by $1 <_T z_1 <_T z_2 <_T z_1^2 <_T z_1z_2 <_T z_2^2 <_T z_1^3 <_T \cdots$.

With respect to (w.r.t.) the chosen term ordering $<_T$, the following notations will be used.

$\mathrm{cf}(f,t)$	the coefficient of power product t in $f \in K[z]$;
$\mathrm{lpp}(f)$	the leading power product, i.e., the maximal power product with nonzero coefficient in $f \in K[z]$ with respect to $<_T$;
$\mathrm{lcf}(f)$	the leading coefficient, i.e., the coefficient of the $\mathrm{lpp}(f)$.

Example 2. Let $f(z_1, z_2) = z_1^3 + 2z_1z_2 + 3z_2^2$. Then we have $\mathrm{cf}(f, z_1^3) = 1$, $\mathrm{cf}(f, z_1z_2) = 2$, $\mathrm{cf}(f, z_2^2) = 3$, and $\mathrm{lpp}(f) = z_2^2$, $\mathrm{lcf}(f) = 3$ for the lexicographic ordering ($z_1^3 <_T z_1z_2 <_T z_2^2$) while $\mathrm{lpp}(f) = z_1^3$, $\mathrm{lcf}(f) = 1$ for the total degree lexicographic ordering ($z_1z_2 <_T z_2^2 <_T z_1^3$).

Having a proper term ordering and the above notation, we can now talk about the division for nD polynomials which are closely related to several basic concepts such as *reduction, normal forms and cofactors*, etc.

Definition 1 (*Reduction* (*division*) [11, 13]). Let $f, g, h \in K[z]$, $g \neq 0$. Then h is called a reduction of f with respect to g, denoted by $f \rightarrow_g h$, if and only if there exist $b \in K$ and a power product u such that $\mathrm{cf}(f,\ u \cdot \mathrm{lpp}(g)) \neq 0$, $b = \mathrm{cf}(f, u \cdot \mathrm{lpp}(g))/\mathrm{lcf}(g)$, and

$$h = f - b \cdot u \cdot g. \tag{6}$$

Let $F \subseteq K[z]$. Then h is called a reduction of f modulo F, denoted by $f \rightarrow_F h$, if and only if there exists $g \in F$ such that $f \rightarrow_g h$.

Definition 2 (*Normal form* (*remainder*) *and cofactors* [11, 13]). $h \in K[z]$ is called in normal form (or reduced form, or remainder) modulo $F = \{f_1, \ldots, f_m\}$ if and only if there is no $h' \in K[z]$ such that $h \rightarrow_F h'$. Further, h is called a normal form of $f \in K[z]$ modulo F, denoted by $\mathrm{NF}(F, f)$, if and only if there is a sequence of reductions

$$f = k_0 \rightarrow_F k_1 \rightarrow_F k_2 \rightarrow_F \cdots \rightarrow_F k_q = h \tag{7}$$

and h is in normal form modulo F. As a result of the above reduction sequence, h can be finally expressed in the form

$$h = f - \sum_{i=1}^{m} c_i f_i, \tag{8}$$

where $c_i \in K[z]$, $i = 1, \ldots, m$, are called the cofactors of the representation of h from f modulo F.

Example 3. Let $F = \{f_1, f_2\}$, $f_1 = 2 + 3z_1z_2 + z_1^2z_2$, $f_2 = 2z_1z_2 + z_2^2$ and $f = -z_1z_2 + 3z_1^2z_2^2$ with the lexicographic ordering. As $\mathrm{lpp}(f) = z_1^2z_2^2$ is a multiple of $\mathrm{lpp}(f_2) = z_2^2$, we may choose $u = z_1^2$ such that $\mathrm{lpp}(f) = u \cdot \mathrm{lpp}(f_2)$ and $\mathrm{cf}(f, u \cdot \mathrm{lpp}(f_2)) = 3 \neq 0$. Then,

$$f \to_{f_2} -z_1z_2 - 6z_1^3z_2 \triangleq h_1 = f - b_1 \cdot u \cdot f_2 \tag{9}$$

is a (one step) reduction of f modulo F where $b_1 = \mathrm{cf}(f, u \cdot \mathrm{lpp}(f_2))/\mathrm{lcf}(f_2) = 3$. In this way, the leading term of f, $\mathrm{cf}(f, \mathrm{lpp}(f)) \cdot \mathrm{lpp}(f) = 3z_1^2z_2^2$, is replaced by monomials whose power products are lower in the lexicographic ordering. In the same way, it is ready to have the following reductions

$$\begin{aligned} h_1 &\to_{f_1} 12z_1 - z_1z_2 + 18z_1^2z_2 \triangleq h_2 \quad (h_2 = h_1 - (-6)z_1f_1) \\ &\to_{f_1} -36 + 12z_1 - 55z_1z_2 \triangleq h_3 \quad (h_3 = h_2 - 18f_1). \end{aligned} \tag{10}$$

Now, we see that no more reduction modulo F is possible as $\mathrm{lpp}(h_3) = z_1z_2$ is neither a multiple of $\mathrm{lpp}(f_1) = z_1^2z_2$ nor that of $\mathrm{lpp}(f_2) = z_2^2$. Thus, h_3 is a normal form (remainder) of f modulo F, i.e., $\mathrm{NF}(F, f) = h_3$. It is also easy to see from (9) and (10) that

$$h_3 = f - c_1f_1 - c_2f_2, \tag{11}$$

where $c_1 = 18 - 6z_1$, $c_2 = 3z_1^2$ are the cofactors of h_3 modulo F.

Note that it is also possible to have another normal form h_4 of f modulo F as follows.

$$\begin{aligned} f &\to_{f_1} -6z_2 - z_1z_2 - 9z_1z_2^2 \to_{f_2} -6z_2 - z_1z_2 + 18z_1^2z_2 \\ &\to_{f_1} -36 - 6z_2 - 55z_1z_2 \triangleq h_4 = f - (18 + 3z_2)f_1 + 9z_1f_2. \end{aligned} \tag{12}$$

The above example shows that the normal forms for arbitrarily given f and F are not unique in general. That is, if we choose a different sequence of reduction operations, we may have a different normal form and thus different cofactors. This is in fact substantially different to the 1D case. See [13] for more details and examples on these points.

Here, let us just have a look on the difficulties arising from the property of non-uniqueness of normal forms through a simple example given in [17].

Example 4. Let $f_1 = 1 + z_1z_2$, $f_2 = -1 + z_2^2$ with the lexicographic ordering. It is easy to find the following two different normal forms h_1 and h_2 of $f = -z_1 + z_1z_2^2$ modulo $F = \{f_1, f_2\}$.

$$h_1 = f - z_2 \cdot f_1 - 0 \cdot f_2 = -z_1 - z_2,$$
$$h_2 = f - 0 \cdot f_1 - z_1 \cdot f_2 = 0.$$

The result of $h_2 = 0$ shows that $f \in \mathrm{Ideal}(f_1, f_2)$ while the result of $h_1 \neq 0$ shows that f may have nonzero normal form even if $f \in \mathrm{Ideal}(f_1, f_2)$.

Therefore, we see from this example that, different to the 1D case, $\mathrm{NF}(F, f) = 0$ is only a sufficient condition but not a necessary one for $f \in \mathrm{Ideal}(F)$ in the nD ($n \geqslant 2$) case. It is then desirable to have a set G of nD polynomials such that $\mathrm{Ideal}(G) = \mathrm{Ideal}(F)$ and $\mathrm{NF}(G, f)$ can be uniquely determined. A set G of nD polynomials with such properties is just the Gröbner basis introduced by Buchberger [9–11,13].

Definition 3 (*Gröbner basis* [11, 13]). A subset G of $K[z]$ is called a Gröbner basis of $\mathrm{Ideal}(G)$ (w.r.t. the term ordering considered) if and only if any $f \in K[z]$ has a unique normal form modulo G, or equivalently, for any $f \in \mathrm{Ideal}(G)$, $\mathrm{NF}(G, f) = 0$.

The notion of Gröbner bases can be further standardized to the notion of *completely reduced Gröbner bases* [9–11].

Definition 4 (*Completely reduced Gröbner basis*). A Gröbner basis G is further called a completely reduced Gröbner basis (w.r.t. the term ordering considered) if and only if for all $g \in G$, g is monic, i.e. $\mathrm{lcf}(g) = 1$, and is in normal form modulo $G - \{g\}$.

The problems we have now are how to verify if a given set $F \subset K[z]$ is a Gröbner basis and, in the case that F is not a Gröbner basis, how to transform F into a Gröbner basis G with $\mathrm{Ideal}(G) = \mathrm{Ideal}(F)$. To this end, we need the important concept of S-polynomial and some of its nice properties [9–11,13].

Definition 5 (*S-polynomial*). The S-polynomial corresponding to $f_1, f_2 \in K[z]$, denoted by $\mathrm{Sp}(f_1, f_2)$, is defined by

$$\mathrm{Sp}(f_1, f_2) = u_1 \cdot f_1 - \frac{\mathrm{lcf}(f_1)}{\mathrm{lcf}(f_2)} \cdot u_2 \cdot f_2, \tag{13}$$

where u_1, u_2 are such that $\mathrm{lcm}(\mathrm{lpp}(f_1), \mathrm{lpp}(f_2)) = u_1 \cdot \mathrm{lpp}(f_1) = u_2 \cdot \mathrm{lpp}(f_2)$, with lcm being the least common multiple.

Theorem 1 [11]. *A set G of polynomials is a Gröbner basis if and only if, for all $g_1, g_2 \in G$, $\mathrm{NF}(G, \mathrm{Sp}(g_1, g_2)) = 0$.*

Let us now go through an example to see how the Gröbnerianity can be tested and a Gröbner basis can be constructed algorithmically based on the above property of S-polynomials. More details and examples on the algorithms and further applications of Gröbner basis can be found in [11,13,26].

Example 5. Let $F = \{f_1, f_2\}$, $f_1 = 2 + 3z_1z_2 + z_1^2z_2$, $f_2 = 2z_1z_2 + z_2^2$ with the lexicographic ordering, as given in Example 3.

It is ready to have

$$\mathrm{Sp}(f_1, f_2) = z_2 \cdot f_1 - z_1^2 \cdot f_2 = 2z_1 - 2z_1^3z_2 + 3z_1z_2^2, \tag{14}$$

$$\begin{aligned}\mathrm{Sp}(f_1, f_2) &\to_{f_2} 2z_2 - 6z_1^2z_2 - 2z_1^3z_2 \triangleq h \quad (h = \mathrm{Sp}(f_1, f_2) - 3z_1f_2)\\ &\to_{f_1} 4z_1 + 2z_2 = \mathrm{NF}(F, \mathrm{Sp}(f_1, f_2))\\ &\qquad (\mathrm{NF}(F, \mathrm{Sp}(f_1, f_2)) = h - (-2)z_1f_1).\end{aligned} \tag{15}$$

Thus we know that F is not a Gröbner basis by Theorem 1. In fact, we have seen in Example 3 that the normal forms of $f = -z_1z_2 + 3z_1^2z_2^2$ modulo F are not unique and thus F is not a Gröbner basis by the definition.

To transform F into a Gröbner basis (see, e.g., [11,13]), adjoin (the monic version of) $\mathrm{NF}(F, \mathrm{Sp}(f_1, f_2))$ $f_3 \triangleq 2z_1 + z_2$ to F, and get a new set $\hat{F} = \{f_1, f_2, f_3\}$. It is easy to see that $\mathrm{Ideal}(F) = \mathrm{Ideal}(\hat{F})$ as f_3 can be presented, due to (14) and (15), in the form $f_3 = c_1f_1 + c_2f_2$ with $c_1 = z_1 + z_2/2$ and $c_2 = -(3z_1 + z_1^2)/2$. Obviously, $\mathrm{NF}(\hat{F}, \mathrm{Sp}(f_1, f_2)) = 0$ since $\mathrm{Sp}(f_1, f_2)$ can be first reduced to $\mathrm{NF}(F, \mathrm{Sp}(f_1, f_2)) = 4z_1 + 2z_2$ as shown above and then to 0 by just subtracting $2f_3$.

To see if $\hat{F}$ is a Gröbner basis, we still need to check at least if $\mathrm{NF}(\hat{F}, \mathrm{Sp}(f_1, f_3)) = 0$. By similar calculations as above, we have

$$\mathrm{Sp}(f_1, f_3) = f_1 - z_1^2f_3 = 2 - 2z_1^3 + 3z_1z_2, \tag{16}$$

$$\mathrm{NF}(\hat{F}, \mathrm{Sp}(f_1, f_3)) = \mathrm{Sp}(f_1, f_3) - 3z_1f_3 = 2 - 6z_1^2 - 2z_1^3 \neq 0, \tag{17}$$

which shows that $\hat{F}$ is still not yet a Gröbner basis. Again, we can construct $f_4 \triangleq -1 + 3z_1^2 + z_1^3$ from $\mathrm{NF}(\hat{F}, \mathrm{Sp}(f_1, f_3))$ and adjoin f_4 to $\hat{F}$ to get $\tilde{F} = \{f_1, f_2, f_3, f_4\}$ which satisfies $\mathrm{Ideal}(\tilde{F}) = \mathrm{Ideal}(\hat{F}) = \mathrm{Ideal}(F)$ and $\mathrm{NF}(\tilde{F}, \mathrm{Sp}(f_1, f_3)) = 0$.

It is ready to see that $\mathrm{NF}(\tilde{F}, \mathrm{Sp}(f_2, f_3)) = 0$, $\mathrm{NF}(\tilde{F}, \mathrm{Sp}(f_1, f_4)) = 0$, $\mathrm{NF}(\tilde{F}, \mathrm{Sp}(f_2, f_4)) = 0$ and $\mathrm{NF}(\tilde{F}, \mathrm{Sp}(f_3, f_4)) = 0$. Thus, we have obtained a Gröbner basis $\tilde{F}$ equivalent to F. Moreover, since $f_1 - (3z_1 + z_1^2)f_3 + 2f_4 = 0$, $f_2 - z_2f_3 = 0$, that is, the first two polynomials f_1 and f_2 in $\tilde{F}$ can be reduced to zero with respect to f_3 and f_4, $\tilde{F}$ is not a completely reduced Gröbner basis for F. By removing f_1 and f_2, we then have the completely reduced Gröbner basis $G \triangleq \{f_3, f_4\}$ equivalent to F.

After having a Gröbner basis $G = \{g_1, \ldots, g_s\}$ equivalent to $F = \{f_1, \ldots, f_m\}$, the solvability problem of (1), or the corresponding ideal membership problem, can be solved by just testing whether or not $\mathrm{NF}(G, f) = 0$. Also, by keeping track of the reduction processes for constructing $g_i \in G$ from $f_j \in F$ in the above construction algorithm of Gröbner bases, we can obtain the cofactors of $g_i \in G$ modulo F [11,13] in the form

$$g_i = \sum_{j=1}^{m} c_{ij} f_j, \quad c_{ij} \in K[z], \quad i = 1, \ldots, s \tag{18}$$

or in a slightly sloppy notation,

$$G = UF, \quad U \triangleq \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{s1} & \cdots & c_{sm} \end{bmatrix}. \tag{19}$$

It is then easy to obtain a particular solution to (1) and the solutions to (2) by the reduction of f modulo G and the above relation between G and F. For more detailed tutorial examples, see, e.g., [13].

It should be emphasized that the uniqueness property of normal form entails also a lot of other important properties of Gröbner bases which play a crucial role in obtaining algorithmic solutions to numerous fundamental algebraic problems (see, e.g., [11,13]). One of the most important properties of Gröbner bases is the so-called elimination property with respect to lexicographic ordering, which provides a basis for solutions to many problems in commutative algebra such as the solution of systems of algebraic equations, the implicitization problem for algebraic manifolds, etc (see, e.g., [11–13]). This property is stated in the following theorem.

Theorem 2 [11, 49]. *Let G be a Gröbner basis with respect to the lexicographic ordering of power products. Assume without loss of generality that $z_1 <_T \cdots <_T z_n$. Then*

$$\mathrm{Ideal}(G) \cap K[z_1, \ldots, z_i] = \mathrm{Ideal}(G \cap K[z_1, \ldots, z_i]), \quad i = 1, \ldots, n, \tag{20}$$

where the ideal on the right-hand side is formed in $K[z_1, \ldots, z_i]$.

This result means that the ith *elimination ideal* of G is generated by just those polynomials in G that depend only on the variables $z_1, \ldots, z_i$. For simplicity, we consider only the case where the ideal generated by $F = \{f_1(z), \ldots, f_m(z) \in K[z]\}$ is of zero-dimension, i.e., the system of algebraic equations defined by $f_i(z) = 0$, $i = 1, \ldots, m$, has a finite number of solutions. By Theorem 2, the Gröbner basis of F with respect to the term ordering $z_1 <_T \ldots <_T z_n$ will be in the form $G = \{g_1(z_1), g_2(z_1, z_2), \ldots, g_{n-1}(z_1, \ldots, z_{n-1}), g_n(z_1, \ldots, z_n)\}$. Thus, due to the elimination property, it is possible to solve the algebraic system $f_i(z) = 0$, $i = 1, \ldots, m$ through solving the corresponding algebraic system $g_1(z_1) = 0$, $g_2(z_1, z_2) = 0, \ldots,$ $g_n(z_1, \ldots, z_n) = 0$ in a "variable by variable" way. For more details and various illustrative examples, see [11,13] and the references therein.

2.3. Gröbner basis of modules over polynomial rings

In system theory and signal processing, it is often required to solve, instead of the polynomial equation of (1), the following polynomial matrix equation:

$$A(z)X(z) = C(z), \tag{21}$$

where $A(z) \in K^{r\times m}[z]$ and $C(z) \in K^{r\times r}[z]$ are given, and $X(z) \in K^{m\times r}[z]$ is to be found. Let $A = [\boldsymbol{a}_1 \cdots \boldsymbol{a}_m]$, $X = [x_{ij}]$, $C = [\boldsymbol{c}_1 \cdots \boldsymbol{c}_r]$ with $\boldsymbol{a}_i, \boldsymbol{c}_j \in K^r[z]$, $i = 1, \dots, m$ and $j = 1, \dots, r$. It is then obvious that (21) can be equivalently expressed as

$$x_{1j}\boldsymbol{a}_1 + \cdots + x_{mj}\boldsymbol{a}_m = \boldsymbol{c}_j, \quad j = 1, \dots, r. \tag{22}$$

Though it is possible to reduce (21) to (1) (see, e.g., [22]), it is in fact more natural and efficient to solve (21), or equivalently (22), by applying the Gröbner basis of modules over an nD polynomial ring to be reviewed in this section [21,55].

Furukawa et al. [21] and Mora and Möller [36] have independently generalized the concept of Gröbner basis for polynomial ideal to Gröbner basis of modules over polynomial ring, which is briefly summarized as follows [51].

Let $F = \{\boldsymbol{f}_1, \dots, \boldsymbol{f}_m\}$ be a subset of $K^r[z]$. By Module(F) we mean the module generated by F, i.e.,

$$\text{Module}(F) = \{h_1\boldsymbol{f}_1 + \cdots + h_m\boldsymbol{f}_m \mid h_i \in K[z], i = 1, \dots, m\}. \tag{23}$$

To generalize the notion of *reduction*, we need first to fix an ordering on the r-tuples of power products under certain admissible conditions. In fact we can do this by only fixing the ordering on a subset P of r-tuples of power products which consist of tuples with only one nonzero component, i.e.,

$$P \triangleq \{(0, \dots, 0, z_1^{i_1} \cdots z_n^{i_n}, 0, \dots, 0)^{\mathrm{T}} \mid i_1, \dots, i_n \in Z_+\}. \tag{24}$$

The elements of P are called *power product tuples*. Then a partial ordering $<_M$ on P is defined by

$$(\forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in P)\ [\boldsymbol{p}_1 <_M \boldsymbol{p}_2 \Leftrightarrow ((\exists q \neq 1, q \text{ power product})\ \boldsymbol{p}_2 = q \cdot \boldsymbol{p}_1)]. \tag{25}$$

By an admissible ordering $<_{M(T)}$ on P, we mean any total ordering which satisfies the following properties:

(i) $(\forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in P)\ [\boldsymbol{p}_1 <_M \boldsymbol{p}_2 \Rightarrow \boldsymbol{p}_1 <_{M(T)} \boldsymbol{p}_2]$.
(ii) $(\forall \boldsymbol{p}_1, \boldsymbol{p}_2 \in P)\ [\boldsymbol{p}_1 <_{M(T)} \boldsymbol{p}_2 \Rightarrow ((\forall q, q \text{ power product}) q \cdot \boldsymbol{p}_1 <_{M(T)} q \cdot \boldsymbol{p}_2)]$.

It can be shown that every admissible ordering on P is Noetherian [51].

Further, the notations $\leqslant_M$ and $\leqslant_{M(T)}$ are defined as $(\boldsymbol{p}_1 \leqslant_M \boldsymbol{p}_2) \Leftrightarrow [\boldsymbol{p}_1 <_M \boldsymbol{p}_2$ or $\boldsymbol{p}_1 = \boldsymbol{p}_2]$ and $(\boldsymbol{p}_1 \leqslant_{M(T)} \boldsymbol{p}_2) \Leftrightarrow [\boldsymbol{p}_1 <_{M(T)} \boldsymbol{p}_2$ or $\boldsymbol{p}_1 = \boldsymbol{p}_2]$ for all $\boldsymbol{p}_1, \boldsymbol{p}_2 \in P$.

Let $<_T$ be an admissible ordering on the power products of $K[z]$, for example the lexicographic ordering or the total degree (lexicographic) ordering. Let $\boldsymbol{p} = (0, \dots, 0, p_i, 0, \dots, 0)^{\mathrm{T}}$ and $\boldsymbol{q} = (0, \dots, 0, q_j, 0, \dots, 0)^{\mathrm{T}} \in P$, where $p_i \neq 0$ occurs at the ith position of $\boldsymbol{p}$ and $q_j \neq 0$ at the jth position of $\boldsymbol{q}$. The *term first ordering based on* $<_T$ [51], or called *highest-order smallest-suffix ordering* [21], is such an example which determines the ordering $<_{M(T)}$ on P by comparing first p_i and q_j with respect to $<_T$, i.e.,

$$\boldsymbol{p} <_{M(T)} \boldsymbol{q} \Leftrightarrow [p_i <_T q_j \text{ or } (p_i = q_j \text{ and } i > j)]. \tag{26}$$

Another admissible ordering is the *index first ordering based on* $<_T$ which defines $<_{M(T)}$ on P by comparing first the indices i and j, i.e.,

$$\boldsymbol{p} <_{M(T)} \boldsymbol{q} \Leftrightarrow [i > j \text{ or } (i = j \text{ and } p_i <_T q_j)]. \tag{27}$$

For example, consider the following elements of $\boldsymbol{Z}^2[z_1, z_2]$ where $\boldsymbol{Z}$ is the domain of integers, and choose the total degree lexicographic ordering $(z_1 <_T z_2)$ as a term ordering on $\boldsymbol{Z}[z_1, z_2]$. Then by the term first ordering based on $<_T$ we have

$$\begin{aligned}
&\begin{bmatrix}0\\1\end{bmatrix} <_{M(T)} \begin{bmatrix}1\\0\end{bmatrix} <_{M(T)} \begin{bmatrix}0\\z_1\end{bmatrix} <_{M(T)} \begin{bmatrix}z_1\\0\end{bmatrix} <_{M(T)} \begin{bmatrix}0\\z_2\end{bmatrix} <_{M(T)} \begin{bmatrix}z_2\\0\end{bmatrix}\\
&\quad <_{M(T)} \begin{bmatrix}0\\z_1^2\end{bmatrix} <_{M(T)} \begin{bmatrix}z_1^2\\0\end{bmatrix} <_{M(T)} \begin{bmatrix}0\\z_1z_2\end{bmatrix} <_{M(T)} \begin{bmatrix}z_1z_2\\0\end{bmatrix} <_{M(T)} \cdots
\end{aligned}$$

while according to the index first ordering based on $<_T$ we get

$$\begin{aligned}
&\begin{bmatrix}0\\1\end{bmatrix} <_{M(T)} \begin{bmatrix}0\\z_1\end{bmatrix} <_{M(T)} \begin{bmatrix}0\\z_2\end{bmatrix} <_{M(T)} \begin{bmatrix}0\\z_1^2\end{bmatrix} <_{M(T)} \begin{bmatrix}0\\z_1z_2\end{bmatrix} <_{M(T)} \cdots\\
&\quad <_{M(T)} \begin{bmatrix}1\\0\end{bmatrix} <_{M(T)} \begin{bmatrix}z_1\\0\end{bmatrix} <_{M(T)} \begin{bmatrix}z_2\\0\end{bmatrix} <_{M(T)} \begin{bmatrix}z_1^2\\0\end{bmatrix} <_{M(T)} \begin{bmatrix}z_1z_2\\0\end{bmatrix}\\
&\quad <_{M(T)} \cdots
\end{aligned}$$

For a chosen admissible ordering $<_{M(T)}$, we can uniquely represent any nonzero r-tuple of polynomial $\boldsymbol{f}$ as

$$\begin{aligned}
&\boldsymbol{f} = \sum_{i=1}^{\sigma} \mathrm{cf}(\boldsymbol{f}, \boldsymbol{p}_i) \cdot \boldsymbol{p}_i, \quad \mathrm{cf}(\boldsymbol{f}, \boldsymbol{p}_i) \in K \setminus \{0\}, \quad \boldsymbol{p}_i \in P,\\
&\qquad \boldsymbol{p}_1 <_{M(T)} \boldsymbol{p}_2 <_{M(T)} \cdots <_{M(T)} \boldsymbol{p}_\sigma
\end{aligned} \tag{28}$$

where $\mathrm{cf}(\boldsymbol{f}, \boldsymbol{p}_i)$ is the *coefficient of* $\boldsymbol{p}_i$ in $\boldsymbol{f}$.

Further, the following notations with respect to the chosen ordering are defined.

- $\mathrm{lppt}(\boldsymbol{f})$ the *leading power product tuple of* $\boldsymbol{f}$, i.e., $\boldsymbol{p}_\sigma$;
- $\mathrm{lpp}(\boldsymbol{f})$ the *leading power product of* $\boldsymbol{f}$, i.e., the nonzero component of $\boldsymbol{p}_\sigma$;
- $\mathrm{lcf}(\boldsymbol{f})$ the *leading coefficient of* $\boldsymbol{f}$, i.e., $\mathrm{cf}(\boldsymbol{f}, \boldsymbol{p}_\sigma)$;
- $\mathrm{lt}(\boldsymbol{f})$ the *leading term of* $\boldsymbol{f}$, i.e., $\mathrm{lcf}(\boldsymbol{f}) \cdot \mathrm{lpp}(\boldsymbol{f})$;
- $\mathrm{hp}(\boldsymbol{f})$ the *head position of* $\boldsymbol{f}$, i.e., if the nonzero component of $\boldsymbol{p}_\sigma$ occurs at the kth position, then $\mathrm{hp}(\boldsymbol{f}) = k$.

Similarly as for the polynomial case, the notions of reduction, normal form and Gröbner basis can be defined for elements of a module over $K[z]$ (see, e.g., [21,36,51]).

Definition 6 (*Reduction* (*division*)). Let $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in K^r[z]$, $\boldsymbol{f} \neq 0 \triangleq (0, \ldots, 0)^{\mathrm{T}}$. Then the reduction relation $\boldsymbol{g} \rightarrow_{\boldsymbol{f}}$ is defined as

$$\boldsymbol{g} \rightarrow_{\boldsymbol{f}} \boldsymbol{h} \Leftrightarrow (\exists v, v \text{ power product})\Big[\mathrm{cf}(\boldsymbol{g}, v \cdot \mathrm{lppt}(\boldsymbol{f})) \neq 0$$
$$\text{and} \quad \boldsymbol{h} = \boldsymbol{g} - \frac{\mathrm{cf}(\boldsymbol{g}, v \cdot \mathrm{lppt}(\boldsymbol{f}))}{\mathrm{lcf}(\boldsymbol{f})} \cdot v \cdot \boldsymbol{f}\Big]. \tag{29}$$

Let $F \subseteq K^r[z]$. Then $\boldsymbol{h}$ is a reduction of $\boldsymbol{g}$ modulo F, denoted by $\boldsymbol{g} \rightarrow_F \boldsymbol{h}$, if and only if there exists $\boldsymbol{f} \in F$ such that $\boldsymbol{g} \rightarrow_{\boldsymbol{f}} \boldsymbol{h}$.

Definition 7 (*Normal form* (*remainder*)). Let $\boldsymbol{h} \in K^r[z]$ and F be a finite subset of $K^r[z]$. $\boldsymbol{h}$ is in normal form (or reduced form, or remainder) modulo F if and only if there is no $\boldsymbol{h}' \in K^r[z]$ such that $\boldsymbol{h} \rightarrow_F \boldsymbol{h}'$. Then $\boldsymbol{h}$ is a normal form of $\boldsymbol{f}$ modulo F, denoted by $\mathrm{NF}(F, \boldsymbol{f})$, if and only if there is a sequence of reductions such that

$$\boldsymbol{f} = \boldsymbol{k}_0 \rightarrow_F \boldsymbol{k}_1 \rightarrow_F \boldsymbol{k}_2 \rightarrow_F \cdots \rightarrow_F \boldsymbol{k}_q = \boldsymbol{h} \tag{30}$$

and $\boldsymbol{h}$ is in normal form modulo F.

Definition 8 (*Gröbner basis*). A finite subset G of $K^r[z]$ is a Gröbner basis of Module(G) (w.r.t. the ordering considered) if and only if any $\boldsymbol{f} \in K^r[z]$ has a unique normal form modulo G, or equivalently, for any $\boldsymbol{f} \in \mathrm{Module}(G)$, $\mathrm{NF}(G, \boldsymbol{f}) = 0$.

Definition 9 (*Completely reduced Gröbner basis*). A Gröbner basis $G \subseteq K^r[z]$ is further called a completely reduced Gröbner basis (w.r.t. the ordering considered) if and only if for all $\boldsymbol{g} \in G$, $\mathrm{lcf}(\boldsymbol{g}) = 1$ and $\boldsymbol{g}$ is in normal form modulo $G - \{\boldsymbol{g}\}$.

Similar to the polynomial case, constructive algorithms can be established to test the Gröbnerianity and to calculate the Gröbner basis G for a given module based on the generalization of the notion of S-polynomial and the related properties [21,36,51].

Definition 10 (S-*polynomial*). Let $\boldsymbol{f}_1, \boldsymbol{f}_2 \in K^r[z]$. The S-polynomial of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$, denoted by $\mathrm{Sp}(\boldsymbol{f}_1, \boldsymbol{f}_2)$, is defined by

$$\mathrm{Sp}(\boldsymbol{f}_1, \boldsymbol{f}_2) = \begin{cases} u_1 \cdot \boldsymbol{f}_1 - \frac{\mathrm{lcf}(\boldsymbol{f}_1)}{\mathrm{lcf}(\boldsymbol{f}_2)} \cdot u_2 \cdot \boldsymbol{f}_2 & \text{if } \mathrm{hp}(\boldsymbol{f}_1) = \mathrm{hp}(\boldsymbol{f}_2), \\ 0 & \text{otherwise,} \end{cases} \tag{31}$$

where u_1, u_2 are such that $\mathrm{lcm}(\mathrm{lpp}(\boldsymbol{f}_1), \mathrm{lpp}(\boldsymbol{f}_2)) = u_1 \cdot \mathrm{lpp}(\boldsymbol{f}_1) = u_2 \cdot \mathrm{lpp}(\boldsymbol{f}_2)$.

Theorem 3 [21, 36, 51]. *A finite subset G of $K^r[\boldsymbol{z}]$ is a Gröbner basis if and only if, for all $\boldsymbol{g}_1, \boldsymbol{g}_2 \in G$,* $\mathrm{NF}(G, \mathrm{Sp}(\boldsymbol{g}_1, \boldsymbol{g}_2)) = 0$.

Example 6. Let $F = \{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ where $\boldsymbol{f}_1 = \begin{bmatrix} -z_1 + 3z_2^2 \\ 1 + 2z_1z_2 \end{bmatrix}, \boldsymbol{f}_2 = \begin{bmatrix} 2z_1 - z_2 \\ z_1z_2 \end{bmatrix} \in \boldsymbol{Q}^2[z_1, z_2]$ with $\boldsymbol{Q}$ being the field of rational numbers, and choose the term first ordering based on the total degree lexicographic ordering on $\boldsymbol{Q}[z_1, z_2]$. It follows that $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ can be expressed in the form

$$\boldsymbol{f}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} z_1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ z_1z_2 \end{bmatrix} + 3\begin{bmatrix} z_2^2 \\ 0 \end{bmatrix},$$

$$\boldsymbol{f}_2 = 2\begin{bmatrix} z_1 \\ 0 \end{bmatrix} - \begin{bmatrix} z_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ z_1z_2 \end{bmatrix}$$

and $\mathrm{lppt}(\boldsymbol{f}_1) = \begin{bmatrix} z_2^2 \\ 0 \end{bmatrix}$, $\mathrm{lppt}(\boldsymbol{f}_2) = \begin{bmatrix} 0 \\ z_1z_2 \end{bmatrix}$, $\mathrm{lpp}(\boldsymbol{f}_1) = z_2^2$, $\mathrm{lpp}(\boldsymbol{f}_2) = z_1z_2$, $\mathrm{lcf}(\boldsymbol{f}_1) = 3$, $\mathrm{lcf}(\boldsymbol{f}_2) = 1$, $\mathrm{lt}(\boldsymbol{f}_1) = 3z_2^2$, $\mathrm{lt}(\boldsymbol{f}_2) = z_1z_2$, $\mathrm{hp}(\boldsymbol{f}_1) = 1$, $\mathrm{hp}(\boldsymbol{f}_2) = 2$.

Let us now obtain the reduction of a given $\boldsymbol{f} \in \boldsymbol{Q}[z_1, z_2]$ modulo F.

$$\begin{aligned} \boldsymbol{f} \triangleq \begin{bmatrix} 1 + z_1z_2^2 \\ z_1^2z_2 \end{bmatrix} &\rightarrow_{\boldsymbol{f}_1} \begin{bmatrix} 1 + \frac{1}{3}z_1^2 \\ -\frac{1}{3}z_1 + \frac{1}{3}z_1^2z_2 \end{bmatrix} \triangleq \boldsymbol{f}' \quad \left(\boldsymbol{f}' = \boldsymbol{f} = -\tfrac{1}{3}z_1\boldsymbol{f}_1\right) \\ &\rightarrow_{\boldsymbol{f}_2} \begin{bmatrix} 1 - \frac{1}{3}z_1^2 + \frac{1}{3}z_1z_2 \\ -\frac{1}{3}z_1 \end{bmatrix} \triangleq \boldsymbol{h} \quad \left(\boldsymbol{h} = \boldsymbol{f}' - \tfrac{1}{3}z_1\boldsymbol{f}_2\right). \end{aligned}$$

It is easy to see that $\boldsymbol{h}$ is a normal form of $\boldsymbol{f}$ modulo F and

$$\boldsymbol{h} = \boldsymbol{f} - c_1\boldsymbol{f}_1 - c_2\boldsymbol{f}_2$$

with $c_1 = c_2 = (1/3)z_1$.

As $\mathrm{hp}(\boldsymbol{f}_1) \neq \mathrm{hp}(\boldsymbol{f}_2)$, we see that $\mathrm{Sp}(\boldsymbol{f}_1, \boldsymbol{f}_2) = 0$, which means that F is already a Gröbner basis by Theorem 3. In fact, we can reach the same normal form $\boldsymbol{h}$ of $\boldsymbol{f}$ modulo F by any different sequence of reductions such as

$$\begin{aligned} \boldsymbol{f} &\rightarrow_{\boldsymbol{f}_2} \begin{bmatrix} 1 - 2z_1^2 + z_1z_2 + z_1z_2^2 \\ 0 \end{bmatrix} \rightarrow_{\boldsymbol{f}_1} \begin{bmatrix} 1 - \frac{5}{3}z_1^2 + z_1z_2 \\ -\frac{1}{3}z_1 - \frac{2}{3}z_1^2z_2 \end{bmatrix} \\ &\rightarrow_{\boldsymbol{f}_2} \begin{bmatrix} 1 - \frac{1}{3}z_1^2 + \frac{1}{3}z_1z_2 \\ -\frac{1}{3}z_1 \end{bmatrix} = \boldsymbol{h}. \end{aligned}$$

In the case that the given F is not a Gröbner basis, it is possible to transform it to a Gröbner basis G equivalent to F in a similar way as for the polynomial case. For more details on the construction algorithm and various applications of Gröbner basis of modules over $K[\boldsymbol{z}]$, see, e.g., [21,36,51,55].

3. Introduction to signal processing

In this section, we give a brief introduction to some fundamental concepts in signal processing. This is to prepare the background for the review of applications of Gröbner bases to multidimensional wavelets and filter banks to be presented in the subsequent sections. Our presentation follows [50] closely. For simplicity of exposition, we consider only one-dimensional (1D) signal processing in this section. Most concepts could be easily generalized to the multidimensional (nD) case.

Discrete signals are typically denoted as $u(m)$, $x(m)$, and so on, where m is an integer called the time index. It is often convenient to work with z-transform which is defined as follows:

$$X(z) = \sum_{m=-\infty}^{\infty} x(m) z^m.$$

The z-transform exists only when the summation converges for some region in the z-plane. For a finite length sequence the z-transform converges everywhere except possibly at $z = 0$ and/or $z = \infty$.

Note that in the literature, it is a common practice to represent the delay operator by z^{-1} in the 1D signal and system community but by z in the nD signal and system community. As this paper is mainly on nD signals and systems, we will adopt z instead of z^{-1} throughout the paper, for both the 1D and nD cases.

A discrete system operates on an input sequence $u(m)$ to produce an output sequence $y(m)$. The simplest and yet most important class of discrete systems is the class of linear shift-invariant (LSI) systems. An LSI system can be completely characterized by the impulse response sequence $h(m)$ which is the output $y(m)$ in response to a unit-pulse input $\delta(m)$ defined by

$$\delta(m) = \begin{cases} 1 & m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For LSI systems, the input–output relation is given by

$$y(m) = \sum_{i=-\infty}^{\infty} h(i) u(m-i),$$

which can also be expressed in the z-transform domain as

$$Y(z) = H(z) U(z),$$

where $Y(z)$, $H(z)$, $U(z)$ are the z-transform of $y(m)$, $h(m)$, $u(m)$, respectively. $H(z)$ is called the transfer function of the LSI system. In most practical applications, transfer functions are rational functions of the form

$$H(z) = \frac{A(z)}{B(z)}, \tag{32}$$

where $A(z)$, $B(z)$ are relatively prime (or coprime) polynomials.

A discrete system is said to be causal if the output $y(m)$ at time m does not depend on the future values of the input sequence, i.e., does not depend on $u(i)$, $i > m$. An LSI system is causal if and only if the impulse response $h(m) = 0$ for $m < 0$. An LSI system is stable if and only if $\sum_{m=-\infty}^{\infty} |h(m)| < \infty$. The stability condition can also be conveniently expressed in terms of $H(z)$, that is, an LSI system is stable if and only if $H(z)$ has no poles in the closed unit disc $\overline{U} \triangleq \{z \in \mathbf{C} : |z| \leqslant 1\}$, where $\mathbf{C}$ is the field of complex numbers. This condition is equivalent to that $B(z)$ has no zeros in $\overline{U}$. In such a case, we also call $B(z)$ a stable polynomial.

A finite impulse response (FIR) system is one for which $B(z) = 1$ in (32). A causal Nth order FIR filter can be represented as

$$H(z) = \sum_{m=0}^{N} h(m) z^m, \quad h(N) \neq 0. \tag{33}$$

Obviously, FIR systems are inherently stable. An LSI system which is not FIR is said to be an infinite impulse response (IIR) system.

Corresponding to the transfer function $H(z)$, the quantity $H(\mathrm{e}^{\mathrm{j}\omega})$ is called frequency response where the real variable ω stands for frequency. The frequency response, which in general is a complex quantity, can be expressed as

$$H(\mathrm{e}^{\mathrm{j}\omega}) = |H(\mathrm{e}^{\mathrm{j}\omega})| \mathrm{e}^{\mathrm{j}\phi(\omega)}. \tag{34}$$

The real-valued quantities $|H(\mathrm{e}^{\mathrm{j}\omega})|$ and $\phi(\omega)$ are called the magnitude response and the phase response of the filter, respectively.

A digital filter is said to have linear phase (LP) if the phase response $\phi(\omega)$ is linear in ω. However, in the signal and image processing community, a less stringent definition for LP is often adopted, which is given as

$$H(\mathrm{e}^{\mathrm{j}\omega}) = c\,\mathrm{e}^{-\mathrm{j}K\omega} H_{\mathrm{R}}(\omega), \tag{35}$$

where c is a possibly complex constant, $\mathrm{j} = \sqrt{-1}$, K is real, and $H_{\mathrm{R}}(\omega)$ is a real valued function of ω. According to this definition, a real coefficient FIR filter $H(z) = \sum_{m=0}^{N} h(m) z^m$ is LP if and only if $h(m) = h(N - m)$ or $h(m) = -h(N - m)$, for $m = 0, 1, \ldots, N$. In some applications such as image and video signal processing, the LP property is very important.

A digital filter bank is a collection of digital filters, with a common input or a common output. Both of these cases are shown in Fig. 1. The system in Fig. 1(a) is called an analysis bank while the system in Fig. 1(b) is called a synthesis bank. Suppose that the filter bank consists of Q filters, we say that the filter bank is a Q-channel (or Q-band) filter bank and the individual filters are called subband filters. Usually each filter in a filter bank covers only a certain band of frequencies in the spectrum and hence the usage of “subband filter” is justified. A discrete system consisting of the cascade of an analysis filter bank and a synthesis filter bank is said to have the perfect reconstruction (PR) property if its output and input signals are identical except possibly for delay. In such a case, we have a PR analysis filter bank and a PR synthesis filter bank. Filter banks are very useful in a variety of applications

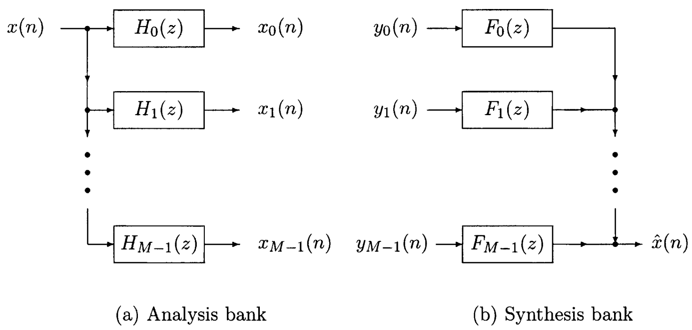

Fig. 1. Digital filter banks.

as they can be used to decompose signals into several subband signals, which can be processed more effectively.

Another powerful tool in signal and image processing is the wavelet transform. Unlike the traditional Fourier transform which operates on the whole duration of a given signal and detects global features of the given signal, wavelet transform operates on a short segment of a given signal at a time and hence can effectively identify local features of the given signal, such as transience. Wavelet functions and the associated scaling functions are closely related to filter banks. We now review briefly some fundamentals of 1D wavelets.

If $\psi(x)$ is a real-valued function whose Fourier transform $\Psi(u)$ satisfies the following admissibility criterion

$$\int_{-\infty}^{\infty} \frac{|\Psi(u)|^2}{|u|} du < \infty,$$

then $\psi(x)$ is called a (basic) wavelet function.

A wavelet function $\psi(x)$ and its associated scaling function $\phi(x)$ may be generated from a two-channel PR analysis filter bank consisting of a subband lowpass filter $H_0(z)$ and a subband highpass filter $H_1(z)$ as follows:

$$\phi(x) = \sqrt{2} \sum_i H_0(i)\phi(2x - i)$$

and

$$\psi(x) = \sqrt{2} \sum_i H_1(i)\phi(2x - i).$$

To ensure the regularity of the designed wavelet $\psi(x)$, it is often required that the filter bank has maximal flatness. A filter $H(z)$ is said to have flatness of order one at $z = 1$ (or $z = -1$) if $H(z)$ has a zero at $z = 1$ (or $z = -1$). $H(z)$ is said to have maximal flatness if all its zeros are either at $z = 1$ or at $z = -1$.

With this background, we are now ready to review the applications of Gröbner bases to nD wavelets and filter banks in the following sections, starting with nD wavelets.

4. Regular nonseparable multidimensional wavelets

Although 1D wavelets have been extensively investigated in the past decades, much less attention has been directed to nD wavelets due to the increasing complexity in dealing with the latter. On the other hand, image and video signals are nD in nature and the common practice in dealing with these nD signals is to exploit separable nD wavelets, i.e., wavelets which are products of 1D wavelets. It is known that separable wavelets are in general only suboptimal and hence it is desirable to have nD nonseparable wavelets.

In this section, we review a method for the design of regular nonseparable two-dimensional (2D) wavelets using Gröbner bases techniques presented in [20]. Consider a four-channel 2D PR analysis filter bank consisting of four 2D FIR filters $H_0(z_1, z_2), \ldots, H_3(z_1, z_2)$. Wavelets may be generated from the above filter bank using the following equations:

$$\phi(x, y) = \sum_{i,j} \phi(2x - i, 2y - j) H_0(i, j), \tag{36}$$

$$\psi_1(x, y) = \sum_{i,j} \phi(2x - i, 2y - j) H_1(i, j), \tag{37}$$

$$\psi_2(x, y) = \sum_{i,j} \phi(2x - i, 2y - j) H_2(i, j), \tag{38}$$

$$\psi_3(x, y) = \sum_{i,j} \phi(2x - i, 2y - j) H_3(i, j), \tag{39}$$

where ϕ is the scaling function and ψ_1, ψ_2, ψ_3 are the wavelets. To ensure maximal flatness (regularity) of the designed wavelets, some additional conditions have to be imposed. There are several ways to design regular nonseparable 2D PR filter banks, such as numerical optimization, cascade form [25] and state-space representation. It turns out that the cascade form approach directly leads to the application of Gröbner bases, as presented in [20] and reviewed in the following. Note that an additional advantage of the cascade form is that linear phase is guaranteed.

Let

$$R_i = \begin{bmatrix} \cos\alpha_i & -\sin\alpha_i & 0 & 0 \\ \sin\alpha_i & \cos\alpha_i & 0 & 0 \\ 0 & 0 & \cos\beta_i & -\sin\beta_i \\ 0 & 0 & \sin\beta_i & \cos\beta_i \end{bmatrix}, \quad W = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad D(z_1, z_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1 & 0 & 0 \\ 0 & 0 & z_2 & 0 \\ 0 & 0 & 0 & z_1 z_2 \end{bmatrix},$$

$$\mathscr{H}(z_1, z_2) = R_1 W P \prod_{i=2}^{K} D(z_1, z_2) P W R_i W P, \tag{40}$$

where $2K \times 2K$ is the size of support for the filters. The cascade form for the family of nonseparable 2D PR linear phase filter banks is then given by (with the sampling matrix equal to $2I$)

$$\begin{aligned} H_i(z_1, z_2) = {} & \mathscr{H}_{i,0}(z_1^2, z_2^2) + \mathscr{H}_{i,1}(z_1^2, z_2^2) z_1 + \mathscr{H}_{i,2}(z_1^2, z_2^2) z_2 \\ & + \mathscr{H}_{i,3}(z_1^2, z_2^2) z_1 z_2, \quad i = 0, \ldots, 3 \end{aligned} \tag{41}$$

where $\mathscr{H}_{i,j}$ denotes the i, j component of the matrix $\mathscr{H}$. To ensure that the filters $H_0, \ldots, H_3$ are maximally flat, i.e., with flatness of order N at given points, the following conditions have to be satisfied for all $k_1, k_2, k_1 + k_2 < N$ (Note: $<$ was mistaken as $\leqslant$ in [20])

$(\partial^{k_1+k_2} H_0)/\partial z_1^{k_1} \partial z_2^{k_2}$ vanishes at $(1, -1), (-1, -1), (-1, 1)$;

$(\partial^{k_1+k_2} H_1)/\partial z_1^{k_1} \partial z_2^{k_2}$ vanishes at $(1, -1), (1, 1), (-1, 1)$;

$(\partial^{k_1+k_2} H_2)/\partial z_1^{k_1} \partial z_2^{k_2}$ vanishes at $(1, 1), (-1, -1), (-1, 1)$;

$(\partial^{k_1+k_2} H_3)/\partial z_1^{k_1} \partial z_2^{k_2}$ vanishes at $(1, -1), (-1, -1), (1, 1)$.

Note that the above flatness equations are polynomial equations with respect to $\cos\alpha_i, \sin\alpha_i, \cos\beta_i, \sin\beta_i$, and hence, together with the extra equations of $\sin^2\gamma + \cos^2\gamma = 1$ ($\gamma = \alpha_i, \beta_i$), can be solved by using the Gröbner basis algorithm. For example, when $N = 2$ and $K = 3$, there are six angles and hence 12 variables in the polynomial system. For each H_i $(i = 0, \ldots, 3)$, $H_i = 0$, $\partial H_i/\partial z_1 = 0$, $\partial H_i/\partial z_2 = 0$ at three points will make 36 equations. Considering the six extra equations imposing $\sin^2 + \cos^2 = 1$, there will be 40 equations in the polynomial system altogether. It turns out that this polynomial system is a zero-dimensional one and hence there is a unique solution to the system. Although the same wavelet coefficients were produced in [20,25], the Gröbner basis algorithm adopted in [20] is much simpler computationally compared with the direct method reported in [25].

When N and/or K increase, the number of equations in the polynomial system increases drastically and it is hence very important to develop symbolic computation software package that is able to solve polynomial system with a large number of equations. A novel method was proposed in [20] which was able to design 2D nonseparable PR linear phase FIR filter banks for N up to 5 and K up to 8 by combining the method of substitution of variables with Gröbner bases tools. The detailed discussion on this method is rather involved and the reader is referred to [20] for more details. The above method could also be extended to the design of regular nonseparable higher-dimensional wavelets, although the number of polynomial equations would increase significantly.

5. Multidimensional FIR filter banks

One of the main applications of Gröbner bases to signal and image processing is the design of multidimensional FIR perfect reconstruction filter banks. We require some notation and background adopted from [14,50].

For a vector $\boldsymbol{p} = (p_1, \ldots, p_n)^{\mathrm{T}}$, and a matrix $M = [\boldsymbol{m}_1 \cdots \boldsymbol{m}_n]$ composed of vectors $\boldsymbol{m}_i$, $i = 1, \ldots, n$, define $z^{\boldsymbol{p}} \triangleq \prod_{i=1}^{n} z_i^{p_i}$, and $z^M \triangleq (z^{\boldsymbol{m}_1}, \ldots, z^{\boldsymbol{m}_n})$.

In an nD multirate system, the decimation (sampling) matrix is an $n \times n$ nonsingular integer matrix and the number of bands m is equal to $|\det M|$. There are exactly m distinct cosets for M. Select one vector from one coset, and these m vectors are called coset vectors. The sampled version of an nD signal $x_a(\boldsymbol{k})$ is given by

$$x(\boldsymbol{k}) = x_a(M\boldsymbol{k}),$$

where $\boldsymbol{k}$ is an integer vector. The set of all sample points, $\boldsymbol{t} = M\boldsymbol{k}$, is called the lattice generated by the matrix M, denoted by $\mathrm{LAT}(M)$.

In filter bank analysis and design, it is often convenient to use polyphase decomposition.

Definition 11 [14]. A polynomial $a(z)$ is self-(anti)symmetric with index $\boldsymbol{n}$, denoted by $\mathrm{Ind}(a) = \boldsymbol{n}$, if it satisfies $a(z) = \pm z^{\boldsymbol{n}} a(z^{-1})$; a pair of polynomials $a(z)$ and $b(z)$ are cross-(anti)symmetric with index $\boldsymbol{m}$, if they satisfy $a(z) = \pm z^{\boldsymbol{m}} b(z^{-1})$.

Definition 12 [14]. The kth subband filter $H_k(z)$ of the m-channel filter bank is of type $\boldsymbol{k}_i$ if its index $\boldsymbol{n}_k$ can be represented as $\boldsymbol{n}_k = M\boldsymbol{m}_k + \boldsymbol{k}_i$, where $\boldsymbol{k}_i$ is an integer vector in $LAT(M)$.

Let A represent an analysis polyphase matrix whose (k, l)-element H_{kl} is obtained from

$$H_k(z) = \sum_{l=0}^{m-1} z^{\boldsymbol{k}_l} H_{kl}(z^M), \tag{42}$$

where $H_k(z)$ is the kth analysis subband filter.

Similarly, let B represent a synthesis polyphase matrix whose (l, k)-element F_{lk} is obtained from

$$F_k(z) = \sum_{l=0}^{m-1} z^{\boldsymbol{k}_l} F_{lk}(z^M), \tag{43}$$

where $F_k(z)$ is the kth synthesis subband filter.

Perfect reconstruction (PR) is achieved if $B(z)A(z) = z^{\boldsymbol{s}} I$ for some integer vector $\boldsymbol{s}$. A special case is the delay free case when $B(z)A(z) = I$. For an nD PR FIR system consisting of an analysis filter bank and a synthesis filter bank, the output signal of this PR FIR system will be the same as the input system except for some possible delays. In the case of delay free, the output signal will be identical to the input signal.

Note that the polyphase matrices $A(z)$ and $B(z)$ are in general Laurent polynomial matrices whose elements are Laurent polynomials, i.e., polynomials in the variables $z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}$. Using a technique proposed by Park et al. [40] for converting a Laurent polynomial into a polynomial, we can assume that $A(z)$ and $B(z)$ are just polynomial matrices for convenience of exposition.

Consider a Q-channel nD FIR filter bank. When $Q = m$ where $m = |\det M|$, with M being the sampling matrix, it becomes the maximally decimated (or critically sampled) filter bank. When $Q < m$, it is clear that PR cannot be achieved. On the other hand, if $Q > m$, the so-called nonmaximally decimated (or oversampled) filter bank case, there are infinite number of synthesis polyphase matrices $B(z)$ satisfying the PR condition for a given analysis polyphase matrix $A(z)$ that is zero right prime (see [32,37,56] for more details regarding factorizations and primeness for nD polynomial matrices), i.e., $A(z)$ is of full rank for all $z \in \mathbf{C}^n$.

With this background, we are now ready to give a survey of results on FIR, PR filter bank design using Gröbner bases.

5.1. Analysis filters in nD FIR, PR filter banks

One of the important questions in the design of analysis filters in 1D and nD filter banks is that given one or more analysis subband filters, how to construct the remaining analysis subband filters such that the resultant filter bank satisfies the PR condition, or mathematically, the resultant polyphase matrix is unimodular, i.e., matrix whose determinant is a nonzero constant. This problem has been well studied in the 1D context [50] because the classical Euclidean division algorithm can be readily applied here. In the nD setting, the situation is more complicated. Although a zero right prime nD polynomial matrix can always be completed into a unimodular matrix, current construction algorithms for this purpose are still fairly complicated and inefficient [29,39]. A heuristic yet more efficient algorithm for nD unimodular matrix completion was proposed in [38] and its improved version has recently been proposed in [14], based on new results on nD polynomial matrix factorizations [8,31] and Gröbner bases.

The problem of constructing an nD FIR filter bank with both the PR and LP properties becomes more difficult and remains open except for some special cases. In the following, we review a recent result on the construction of two-channel nD FIR, LP, PR filter banks [14].

Fact 1 [14]. Consider the two-channel ($m = 2$) case, i.e., the polyphase matrix $H(z)$ is given by

$$H(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{bmatrix}, \tag{44}$$

where $H_{00}(z)$ and $H_{01}(z)$ are the polyphase components of the specified FIR, LP, PR subband filter $H_0(z)$ while $H_{10}(z)$ and $H_{11}(z)$ will be the polyphase components of the other FIR subband filter $H_1(z)$. Assume that $H_{00}(z)$ and $H_{01}(z)$ do not have any nontrivial common zeros, then it is possible to construct $H_{10}(z)$ and $H_{11}(z)$ such that $H_1(z)$ is LP and the resultant filter bank is PR, i.e., $\det H(z) = 1$. The construction of $H_{10}(z)$ and $H_{11}(z)$ can be carried out via the following four steps.

1. Compute the Gröbner basis of the ideal generated by $H_{00}(z)$ and $H_{01}(z)$.
2. Trace back the Gröbner basis computation to obtain $H'_{10}(z)$ and $H'_{11}(z)$ such that

$$H_{00}(z)H'_{10}(z) - H_{01}(z)H'_{11}(z) = z^{\boldsymbol{s}}, \tag{45}$$

   where $\boldsymbol{s} = (\boldsymbol{m}_{01} + \boldsymbol{m}_{10})/2$ and $\boldsymbol{m}_{ij}$ is an index of the self-(anti)symmetric filter $H_{ij}(z)$. $\boldsymbol{m}_{00}$ and $\boldsymbol{m}_{11}$ are chosen such that $\boldsymbol{m}_{00} + \boldsymbol{m}_{11} = \boldsymbol{m}_{01} + \boldsymbol{m}_{10}$ and $\boldsymbol{s}$ is an integer vector.
3. Let $H''_{10}(z) \triangleq z^{\boldsymbol{m}_{10}} H'_{10}(z^{-1})$, $H''_{11}(z) \triangleq z^{\boldsymbol{m}_{11}} H'_{11}(z^{-1})$.
4. Let

$$H_{10}(z) = \frac{1}{2}\left[H'_{10}(z) + H''_{10}(z)\right], \quad H_{11}(z) = \frac{1}{2}\left[H'_{11}(z) + H''_{11}(z)\right]. \tag{46}$$

   $H_{10}(z)$ and $H_{11}(z)$ will then be the polyphase components of the desired FIR, LP, PR filter corresponding to $H_{00}(z)$ and $H_{01}(z)$.

A nontrivial design example is also provided in [14] for a two-channel 2D LP, PR filter bank, where the 2D low-pass filter is generated using McClellan transformation. The reader is referred to [14] for more details.

While the design of two-channel nD FIR, LP, PR analysis filter banks is solved, the design for the general m-channel ($m > 2$) nD FIR LP, PR filter banks remains as an open problem at present [14].

5.2. Synthesis filters in nD FIR, PR filter banks

The design of synthesis filters in nD PR filter banks arises from both the maximally decimated [43] and the nonmaximally decimated [41] filter bank cases. In either case, the objective is to design the corresponding synthesis filter bank given an analysis nD filter bank subject to the PR condition. Mathematically, the design of synthesis nD filter banks can be formulated as the following problem:

Consider an nD polynomial matrix $A(z) \in \mathbf{R}^{Q\times J}[z]$, with $Q \geqslant J$, find another nD polynomial matrix $B(z) \in \mathbf{R}^{J\times Q}[z]$ such that

$$B(z)A(z) = I. \tag{47}$$

$B(z)$ is also called the left inverse of $A(z)$. A necessary and sufficient condition for the existence of $B(z)$ is that $A(z)$ is zero right prime. The case of $Q = J$ is trivial here since $A(z)$ being zero right prime means that $\det A(z)$ is a nonzero constant. Hence $B(z) = A^{-1}(z) \in \mathbf{R}^{J\times Q}[z]$. Consider now the case of $Q < J$.

There are several methods for solving (47) (see [2,40,44,55,56]). As Gröbner basis algorithm is constructive and efficient, we review here two methods that use Gröbner bases to solve (47) in the design of synthesis nD FIR, PR filter banks.

Fact 2 [44]. Let $A(z) \in \mathbf{R}^{Q\times J}[z]$, with $Q > J$. Assume that $A(z)$ is zero right prime. Then the left inverse $B(z)$ of $A(z)$ can be obtained constructively by the following five steps:

1. Compute all the $J \times J$ minors of A, denoted by $e_1, \ldots, e_\beta$.
2. Compute the reduced Gröbner basis G for the ideal generated by $e_1, \ldots, e_\beta$. Since $A(z)$ is zero right prime by assumption, $G = \{1\}$.
3. Trace back the Gröbner basis computation to obtain $\lambda_1, \ldots, \lambda_\beta$, such that $\sum_{i=1}^{\beta} \lambda_i e_i = 1$.
4. For $i = 1, \ldots, \beta$, construct polynomial matrix B_i from A such that $B_i A = e_i I$.
5. Let $B = \sum_{i=1}^{\beta} \lambda_i B_i$. It is then easy to verify that $BA = \sum_{i=1}^{\beta} \lambda_i B_i A = \sum_{i=1}^{\beta} \lambda_i e_i I = I$. Therefore, $B(z)$ is the required left inverse of $A(z)$.

Fact 3 [40]. Let $A(z) \in \mathbf{R}^{Q\times J}[z]$, with $Q > J$. Assume that $A(z)$ is zero right prime. Then the left inverse $B(z)$ of $A(z)$ can be obtained constructively by the following three steps:

1. Compute the reduced Gröbner G basis for the module generated by all the rows of A, i.e., $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_Q$. Since $A(z)$ is zero right prime by assumption, $G = \{\mathbf{e}_1, \ldots, \mathbf{e}_J\}$, where $\mathbf{e}_1, \ldots, \mathbf{e}_J$ are the standard basis row vectors.
2. Trace back the Gröbner basis computation to obtain

$$\begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_J \end{bmatrix} = \begin{bmatrix} b_{11} & \ldots & b_{1Q} \\ \vdots & \ddots & \vdots \\ b_{J1} & \ldots & b_{JQ} \end{bmatrix} \begin{bmatrix} \boldsymbol{a}_1 \\ \vdots \\ \boldsymbol{a}_Q \end{bmatrix}$$

   or

$$I = BA.$$

   Therefore, $B(z)$ is the required left inverse of $A(z)$.

Once the polyphase matrix $B(z)$ is obtained, the synthesis filter bank can be calculated according to Eq. (43).

Remark 1. When $J = 1$, Facts 2 and 3 are the same. When $J > 1$, Fact 3 is in general more efficient to implement than Fact 2, as the number of minors of a rectangular matrix increases drastically when the size of the matrix increases moderately [30].

Remark 2. Facts 2 and 3 were actually proposed by researchers in systems and control much earlier (see [7,55] respectively). The relationship between signal and image processing and systems and control will be further explored when we discuss nD IIR filter bank design.

So far we have only discussed the computation of one left inverse of $A(z)$, or the design of one synthesis filter bank. Clearly, when $Q > J$, there are infinite number of such synthesis filter banks for the given analysis filter bank represented by $A(z)$. The parameterization of the class of all such synthesis filter banks was in fact done by Park [41,43] using Gröbner bases again and reviewed as follows.

Fact 4 [41, 43]. Consider $A(z) \in \mathbf{R}^{Q\times J}[z]$, with $Q > J$. Assume that $A(z)$ is zero right prime. Then the set of all left inverses $B(z)$ of $A(z)$ can be parameterized as follows:

1. Compute a particular left inverse B_{part} using Fact 2 or Fact 3.
2. Compute the reduced Gröbner basis $G = \{\boldsymbol{g}_1, \ldots, \boldsymbol{g}_l\}$ for the syzygy module of A, i.e. $\operatorname{Syz}(A)$, where $\boldsymbol{g}_i$ $(i = 1, \ldots, l)$ are row vectors. Let
$$G_{\text{mat}} \triangleq \begin{bmatrix} \boldsymbol{g}_1 \\ \vdots \\ \boldsymbol{g}_l \end{bmatrix} \in \mathbf{R}^{l\times Q}[z].$$
3. The parameterization of the class of all such synthesis filter banks is then given by $B = B_{\text{part}} + UG_{\text{mat}}$, where $U \in \mathbf{R}^{J\times l}[z]$ is arbitrary.

The parameterization of all the synthesis filter banks is very useful in the design of optimal synthesis nD FIR, PR filter bank as we are now able to optimize the filter coefficients of U freely while satisfying the PR condition. Details on the optimal design of nD synthesis filter banks and also a design example can be found in [43].

In image and video processing, other than the PR condition, linear phase (LP) for the filter banks is often required as well. While it is not easy to give a closed-form expression for the parameterization of all the synthesis filter banks satisfying both PR and LP conditions, it is possible to impose linear phase or zero phase condition during the design stage [43].

5.3. Two-dimensional FIR lossless systems

A square Laurent polynomial matrix $H(z)$ is called paraunitary if it satisfies $\hat{H}H = H\hat{H} = I$, where $\hat{H}(z_1, \ldots, z_n) \triangleq H(z_1^{-1}, \ldots, z_n^{-1})^{\mathrm{T}}$. Two-dimensional (2D) FIR lossless systems are characterized by 2D paraunitary matrices, that is, a 2D FIR system is lossless if its transfer matrix is a 2D paraunitary matrix. Note that the filter banks associated with paraunitary matrices are both FIR and have the PR property. However, the converse is in general not true.

As for 1D systems, 2D paraunitary matrices play an important role in 2D nonseparable FIR filter bank design, lossless FIR filter bank realization and other related areas. It is straightforward to characterize 2D FIR paraunitary matrices which are factorable into rotations and delays. A challenging problem was whether there existed 2D paraunitary matrices that were not factorable into rotations and delays, i.e., 2D nonfactorable paraunitary matrices. Recently, using Gröbner bases, Park has shown [42] that the lowest *total degree* of a 2D FIR nonfactorable paraunitary matrix is 4 and *of type* (2, 2) (the *total degree* and *type* of a 2D FIR paraunitary matrix will be defined shortly). Furthermore, he also gave a closed-form expression for the class of 2D 2×2 nonfactorable paraunitary matrices. We now give a review of the results on 2D FIR paraunitary matrices presented in [42], with an emphasis on the application of Gröbner bases.

For simplicity of discussion, only 2D paraunitary polynomial matrices are considered since 2D paraunitary Laurent matrices can be easily converted to the former. A 2D polynomial matrix $H(z_1, z_2) = \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} H_{ij} z_1^i z_2^j$ is said to be *of type* (k_1, k_2) and of *total degree* $k = k_1 + k_2$, where H_{ij} are constant matrices with $H_{k_1 k_2} \neq 0$ [42].

Let $H(z_1, z_2)$ be a 2×2 paraunitary polynomial matrix, and let $\boldsymbol{v}(z_1, z_2)$ be its first column vector. It was shown in [42] that the factorability of H is equivalent to that of $\boldsymbol{v}$. When $\boldsymbol{v}$ is of type (2, 0) and of type (0, 2), $\boldsymbol{v}$ is always factorable since these are just the 1D cases. If $\boldsymbol{v}$ is of type (1, 1), it can be expressed as

$$\boldsymbol{v} = \boldsymbol{v}_{00} + \boldsymbol{v}_{10} z_1 + \boldsymbol{v}_{01} z_2 + \boldsymbol{v}_{11} z_1 z_2 \tag{48}$$

for some constant real vectors $\boldsymbol{v}_{00}, \boldsymbol{v}_{10}, \boldsymbol{v}_{01}, \boldsymbol{v}_{11}$. Let $\boldsymbol{v}_{ij} = \begin{bmatrix} a_{ij} \\ b_{ij} \end{bmatrix}$, where a_{ij}, b_{ij} are real numbers to be determined. Define the five polynomials h_i $(i = 1, \ldots, 5)$ in the polynomial ring

$$\mathbf{R}[a_{00}, a_{10}, a_{01}, a_{11}, b_{00}, b_{10}, b_{01}, b_{11}]$$

by

$$h_1 = a_{01} a_{10} + b_{01} b_{10}, \tag{49}$$

$$h_2 = a_{00} a_{11} + b_{00} b_{11}, \tag{50}$$

$$h_3 = a_{00} a_{10} + a_{01} a_{11} + b_{00} b_{10} + b_{01} b_{11}, \tag{51}$$

$$h_4 = a_{00} a_{01} + a_{10} a_{11} + b_{00} b_{01} + b_{10} b_{11}, \tag{52}$$

$$h_5 = a_{00}^2 + a_{01}^2 + a_{10}^2 + a_{11}^2 + b_{00}^2 + b_{01}^2 + b_{10}^2 + b_{11}^2 - 1. \tag{53}$$

It can be easily shown [42] by setting $\boldsymbol{v}\hat{\boldsymbol{v}} = 1$ that the set of common real zeros of the polynomials h_i's, or equivalently the variety of the ideal generated by h_i's denoted by $V(h_1, \ldots, h_5)$, completely characterizes $\boldsymbol{v}$ of type (1, 1). Next, it was shown in [42] that $\boldsymbol{v}$, and hence H_{ij}, of type (1, 1) is factorable for all the a_{ij}, b_{ij} that are the real zeros of the polynomial $h \triangleq (a_{00} b_{01} - b_{00} a_{01})(a_{00} b_{10} - b_{00} a_{10})$, or equivalently are in $V(h)$. Therefore, the problem of determining the factorability of H_{ij} of type (1, 1) is equivalent to testing whether

$$V(h_1, \ldots, h_5) \subset V(h)? \tag{54}$$

This is in fact the determination of the radical membership problem. Using Gröbner bases, Park showed [42] that the above question indeed has a positive answer and hence, all 2×2 paraunitary polynomial matrices of type (1, 1) are factorable. Therefore, all 2×2 2D FIR paraunitary polynomial matrices of total degree 2 are factorable. Using Gröbner bases, Park then went on to show that all 2×2 2D FIR paraunitary polynomial matrices of total degree 3 are also factorable. For 2×2 2D FIR paraunitary polynomial matrices of total degree 4, those of types (3, 1) and (1, 3) are factorable, while there exist 2×2 nonfactorable paraunitary polynomial matrices of type (2, 2). In fact, although the nonfactorability conditions for this case were formulated into a system of nD polynomial equations, which could be solved in principle using Gröbner bases, the solution became too complicated for the currently available software that implements Gröbner bases computation. The problem was then resolved by using a convex geometric approach instead. This reinforces the fact that while Gröbner bases are very powerful in solving problems involving nD polynomials, much is still needed to improve software packages implementing the computation of Gröbner bases. For more details on the parameterization of 2D FIR lossless systems, see [42].

Remark 3. Before ending this section, it is worth emphasizing the advantages of using Gröbner bases for the analysis and design of nD FIR filter banks. Although there exist other methods for this topic (see, e.g., [1,50]), one of the advantages of adopting Gröbner bases over other methods is that the Gröbner bases method is a systematic and effective approach in dealing with nD filter banks which can be represented by nD polynomial matrices. Another advantage of using the Gröbner bases method is that it can often give a parameterization of all the solutions when they exist [41,43].

6. Multidimensional IIR filter banks

In comparison with nD FIR, PR filter banks, relatively less attention has been paid to nD IIR, PR filter banks, both in the 1D and nD cases. This is not because IIR filter banks are not as useful as FIR filters. In fact, IIR filter banks are well known to have superior frequency behavior at low computational cost [3]. However, due to the inherent stability problem and structure complexity associated with rational functions (matrices), IIR, PR filter banks are more difficult to analyze and design than FIR filter banks. This is particularly so for the nD case. Despite of these difficulties, some progress has been made in the design of nD causal, stable PR IIR filter banks in recent years. In the following, we review results on the design of IIR filter banks using Gröbner bases, and also present some new results by exploiting existing results in nD feedback control system design developed by us in recent years. We will be mainly concerned with this problem: Given a specific analysis 2D stable IIR filter, design the remaining analysis filters such that the resultant filter bank is PR. As the general nD ($n > 2$) case is more difficult and several problems are still outstanding, we begin our discussion with 2D IIR filter banks.

6.1. Two-dimensional IIR, PR filter banks

Similar to the discussion on FIR filter banks, we first review some properties and results on stable nD/2D rational matrices, which are associated with nD/2D IIR filter banks in the same way as nD/2D polynomials matrices with nD/2D FIR filter banks.

Let $f(z) \in \mathbf{R}[z]$. $f(z)$ is said to be a stable polynomial if $f(z)$ has no zeros in the closed unit polydisc $\overline{U}^n \triangleq \{z \in \mathbf{C}^n : |z_1| \leqslant 1, |z_2| \leqslant 1, \ldots, |z_n| \leqslant 1\}$. A rational function $p(z)/d(z)$, with $p(z)$, $d(z)$ being factor coprime polynomials, i.e., they do not have any nontrivial common factors [4–6,19], is said to be stable if $d(z)$ is a stable polynomial. We denote the ring of all stable rational functions by $\mathbf{R}_s(z)$.

Fact 5. Let $\tilde{f}_1(z_1, z_2), \ldots, \tilde{f}_Q(z_1, z_2) \in \mathbf{R}_s(z_1, z_2)$ be factor coprime. If $\tilde{f}_1(z_1, z_2), \ldots, \tilde{f}_Q(z_1, z_2)$ have no common zeros in $\overline{U}^2$, then there exist $\tilde{x}_1(z_1, z_2), \ldots, \tilde{x}_Q(z_1, z_2) \in \mathbf{R}_s(z_1, z_2)$ such that

$$\tilde{f}_1(z_1, z_2)\tilde{x}_1(z_1, z_2) + \cdots + \tilde{f}_Q(z_1, z_2)\tilde{x}_Q(z_1, z_2) = 1, \tag{55}$$

which can be constructively obtained by using the Gröbner bases approach.

Several methods have been proposed for solving Eq. (55). It was shown in [23,55] that Gröbner bases are an attractive and efficient method. The key step in all the methods for the solution to Eq. (55) is the construction of a 2D stable polynomial $s(z_1, z_2)$ that vanishes in the variety of the ideal generated by $\tilde{f}_i(z_1, z_2)$'s. This is always possible since for factor coprime rational functions $\tilde{f}_i(z_1, z_2)$'s, the variety of the ideal generated by $\tilde{f}_i(z_1, z_2)$'s is of zero-dimension. The result has only been generalized to nD ($n > 2$) in some special cases. We will discuss this in the next section.

Now we review a result on the design of analysis 2D IIR, PR filter banks. For convenience of exposition we consider the two-channel case resulting from the quincunx sampling, i.e., the sampling matrix being $M = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Fact 6 [15]. Consider a 2D stable IIR filter $H_0(z_1, z_2)$, and assume that its two polyphase components $H_{00}(z_1, z_2)$ and $H_{01}(z_1, z_2)$ have no common zeros in the closed unit bi-disc $\overline{U}^2$. Then a 2D stable IIR filter $H_1(z_1, z_2)$ can be constructed such that the resultant 2D IIR filter bank is stable and PR. The construction of $H_1(z_1, z_2)$ can be carried out in the following three steps:

1. Decompose $H_0(z_1, z_2)$ into polyphase components as:

$$H_0(z_1, z_2) = H_{00}(z_1 z_2, z_1 z_2^{-1}) + z_1 H_{01}(z_1 z_2, z_1 z_2^{-1}), \tag{56}$$

where $H_{00}(z_1, z_2)$ and $H_{01}(z_1, z_2)$ are stable 2D rational functions.

2. By assumption, $H_{00}(z_1, z_2)$ and $H_{01}(z_1, z_2)$ have no common zeros in $\overline{U}^2$. By Fact 5, using Gröbner bases, we can construct stable 2D rational functions $H_{10}(z_1, z_2)$ and $H_{11}(z_1, z_2)$ such that

$$H_{00}(z_1, z_2)H_{11}(z_1, z_2) - H_{01}(z_1, z_2)H_{10}(z_1, z_2) = 1. \tag{57}$$

3. Let

$$H_1(z_1, z_2) = H_{10}(z_1 z_2, z_1 z_2^{-1}) + z_1 H_{11}(z_1 z_2, z_1 z_2^{-1}) \tag{58}$$

and

$$H(z_1, z_2) = \begin{bmatrix} H_{00}(z_1, z_2) & H_{01}(z_1, z_2) \\ H_{10}(z_1, z_2) & H_{11}(z_1, z_2) \end{bmatrix}. \tag{59}$$

Obviously, $\det H(z_1, z_2) = 1$, and hence $H_1(z_1, z_2)$ is the required stable 2D IIR filter such that the resultant filter bank is PR.

The above result can in fact be generalized to the m-channel ($m > 2$) 2D IIR filter bank case, using existing results on stabilization of 2D feedback control systems. The details are omitted here. For a similar result using methods other than Gröbner bases, see [3].

6.2. Multidimensional IIR, PR filter banks

Now consider the two-channel nD ($n > 2$) case.

Fact 7 [15]. Consider an nD ($n > 2$) stable IIR filter $H_0(z)$, and assume that its two polyphase components $H_{00}(z)$ and $H_{01}(z)$ have no common zeros in the closed unit polydisc $\overline{U}^n$. Then an nD stable IIR filter $H_1(z)$ can be constructed such that the resultant nD IIR filter bank is stable and PR.

The construction step for $H_1(z)$ is similar to Fact 6 and is omitted here. For more details, see [15]. However, it should be pointed out here that Charoenlarpnopparut and Bose did not provide in [15] a constructive method for solving Eq. (55). As we mentioned earlier, a crucial step in the solution to Eq. (55) is the construction of a stable polynomial that vanishes on all the common zeros of $H_{00}(z)$ and $H_{01}(z)$. To the best of our knowledge, up to now, for the nD ($n > 2$) case, a stable polynomial that vanishes on all the common zeros of $H_{00}(z)$ and $H_{01}(z)$ can be obtained constructively only for the following two special cases.

Fact 8 [54]. Let $\tilde{f}_1(z), \ldots, \tilde{f}_Q(z) \in \mathbf{R}_s(z)$. If $\tilde{f}_i(z)$, $i = 1, \ldots, Q$, have only finitely many common zeros outside $\overline{U}^n$, then there exist $\tilde{x}_1(z), \ldots, \tilde{x}_Q(z) \in \mathbf{R}_s(z)$ such that

$$\tilde{f}_1(z)\tilde{x}_1(z) + \cdots + \tilde{f}_Q(z)\tilde{x}_Q(z) = 1, \tag{60}$$

which can be constructively obtained by using the Gröbner bases approach as follows.

Let $l(z)$ be the least common multiple of the denominators of $\tilde{f}_1(z), \ldots, \tilde{f}_Q(z)$, and let $f_i(z) = l(z)\tilde{f}_i(z) \in \mathbf{R}[z]$, $i = 1, \ldots, Q$. Then, it is easy to see that (60) holds if and only if

$$f_1(z)x_1(z) + \cdots + f_Q(z)x_Q(z) = s(z) \tag{61}$$

holds for some $x_1(z), \ldots, x_Q(z), s(z) \in \mathbf{R}[z]$ and $s(z) \neq 0$ in $\overline{U}^n$.

Let $\mathscr{I}$ denote the ideal generated by $f_1(z), \ldots, f_Q(z)$, and $\mathscr{V}(\mathscr{I})$ the algebraic variety of $\mathscr{I}$, i.e., $\mathscr{V}(\mathscr{I}) = \{z \in \mathbf{C}^n : f_i(z) = 0, i = 1, \ldots, Q\}$. By the assumption, we have that $\mathscr{V}(\mathscr{I}) \cap \overline{U}^n = \emptyset$ and $\mathscr{I}$ is of zero-dimension. Then, using the Gröbner bases approach (see, e.g, Method 6.10 of [11]), one can obtain n 1D polynomials $g_i(z_i)$, $i = 1, \ldots, n$, in $\mathscr{I}$. By the well-known 1D methods, we can factorize $g_i(z_i)$ as

$$g_i(z_i) = g_{i\mathrm{s}}(z_i)g_{i\mathrm{u}}(z_i), \quad i = 1, \ldots, n \tag{62}$$

such that $g_{is}(z_i)$ is stable while $g_{i\mathrm{u}}(z_i)$ is completely unstable (i.e., all the zeros of $g_{i\mathrm{u}}(z_i)$ lie in $\overline{U}$) [55]. Now, construct the stable polynomial

$$\tilde{s}(z) = \prod_{i=1}^{n} g_{is}(z_i), \tag{63}$$

which vanishes on $\mathscr{V}(\mathscr{I})$ [54,55].

Introduce a new indeterminate t. It is then obvious that the polynomials $(1 - t\tilde{s}(z))$ and $f_1(z), \ldots, f_Q(z)$ share no common zeros. According to Hilbert's Nullstellensatz, there exist $\hat{x}(z,t)$, $\hat{x}_1(z,t), \ldots, \hat{x}_Q(z,t)$, which can be constructively obtained by the Gröbner bases approach, such that

$$f_1(z)\hat{x}_1(z,t) + \cdots + f_Q(z)\hat{x}_Q(z,t) + (1 - t\tilde{s}(z))\hat{x}(z,t) = 1. \tag{64}$$

Substituting $1/s(z)$ for t and clearing out the denominators yield

$$f_1(z)x_1(z) + \cdots + f_Q(z)x_Q(z) = \tilde{s}^r(z) \tag{65}$$

with r being a certain positive integer. Now, we get the solutions $x_1(z), \ldots, x_Q(z) \in \mathbf{R}[z]$, $s(z) = \tilde{s}^r(z) \neq 0$ in $\overline{U}^n$ to (61), and further the solutions $\tilde{x}_i(z) = l(z)x_i(z)/s(z) \in \mathbf{R}_s(z)$, $i = 1, \ldots, Q$, to (60).

Fact 9 [34]. Let $\tilde{f}_i(z)$, $f_i(z)$, $i = 1 \ldots, Q$, $\mathscr{I}$ and $\mathscr{V}(\mathscr{I})$ be defined as in Fact 8. Assume that $\mathscr{V}(\mathscr{I})$ is finite w.r.t. the variables $z_1, \ldots, z_{n-2}$, then it can be shown that the reduced Gröbner basis $\{g_1, \ldots, g_{n-2}, g_{n-1}, \ldots, g_r\}$ of $\mathscr{I}$ is such that the ideal generated by $\{g_1, \ldots, g_{n-2}\}$ is finite w.r.t. the variables $z_1, \ldots, z_{n-2}$ (see [34] for the details).

Further, denote $\mathscr{V}_k = \mathscr{V}(\{g_1, \ldots, g_{n-2}\}) \subset \mathbf{C}^{n-2}$, and assume that $\mathscr{V}_k$ contains only p points, $(z_{11}, \ldots, z_{n-2,1}), \ldots, (z_{1p}, \ldots, z_{n-2,p})$. Order these p points such that the first v points are in $\overline{U}^{n-2} = \{(z_1, \ldots, z_{n-2}) \in \mathbf{C}^{n-2} : |z_1| \leqslant 1, \ldots, |z_{n-2}| \leqslant 1\}$, and the last $p - v$ points are not in $\overline{U}^{n-2}$. Then for every point in $\mathscr{V}_k$, i.e.,

$(z_{1i}, \dots, z_{n-2,i})$, $i = 1, \dots, v$, the corresponding set of 2D polynomials $g_{n-1}(z_{1i}, \dots, z_{n-2,i}, z_{n-1}, z_n), \dots, g_r(z_{1i}, \dots, z_{n-2,i}, z_{n-1}, z_n)$ can be reordered and rewritten as

$$d_i(z_{n-1}, z_n)b_{i1}(z_{n-1}, z_n), \dots, d_i(z_{n-1}, z_n)b_{iq}(z_{n-1}, z_n), 0, \dots, 0, \tag{66}$$

where $d_i(z_{n-1}, z_n) \not\equiv 0 \in \mathbf{R}[z_{n-1}, z_n]$, and $b_{ij}(z_{n-1}, z_n) \not\equiv 0 \in \mathbf{R}[z_{n-1}, z_n]$, $j = 1, \dots, q$ are factor coprime.

Let $\mathscr{I}_i$ denote the ideal generated by $d_i(z_{n-1}, z_n)b_{ij}(z_{n-1}, z_n)$, $j = 1, \dots, q$ and $\mathscr{V}(\mathscr{I}_i) \subset \mathbf{C}^2$ the variety of $\mathscr{I}_i$. If $\mathscr{V}(\mathscr{I}_i) \cap \overline{U}^2 = \{(z_{n-1}, z_n) \in \mathbf{C}^2 \| z_{n-1}| \leqslant 1, |z_n| \leqslant 1\} = \emptyset$, then $d_i(z_{n-1}, z_n)$ is stable and a stable 2D polynomial $s_{b_i}(z_{n-1}, z_n)$ can be constructed such that $s_{b_i}(z_{n-1}, z_n)$ vanishes in $\mathscr{V}(\{b_{i1}(z_{n-1}, z_n), \dots, b_{iq}(z_{n-1}, z_n)\})$. Thus, a stable nD polynomial $\tilde{s}(z)$ vanishing on $\mathscr{V}(\mathscr{I})$ can be constructed as follows:

$$\begin{aligned}\tilde{s}(z) = &\prod_{k=v+1}^{p} (z_1 - z_{1k})^{w_{1k}} (z_2 - z_{2k})^{w_{2k}} \cdots (z_{n-2} - z_{n-2,k})^{w_{n-2,k}} \\ &\times \prod_{i=1}^{v} d_i(z_{n-1}, z_n) s_{b_i}(z_{n-1}, z_n),\end{aligned} \tag{67}$$

where

$$w_{mk} = \begin{cases} 0 & \text{if } |z_{mk}| \leqslant 1, \\ 1 & \text{otherwise,} \end{cases} \quad m = 1, \dots, n-2; k = v+1, \dots, p.$$

Utilizing $\tilde{s}(z)$ and the method shown in Fact 8, the solutions to (61) and further to (60) can be constructively obtained.

Therefore, the design of general nD ($n > 2$) stable IIR PR filter banks remains to be a challenging open problem.

Remark 4. As far as we are aware, results on nD IIR filter bank analysis and design are only reported in [3,15]. While Basu presented a state-space approach in [3], he indicated in the conclusion section that a computationally more efficient approach would be the Gröbner bases method. The pioneering work reported in [15] and the results adopted from the literature in nD control system design and presented in this section have clearly shown the advantages of using Gröbner bases in the analysis and design of nD IIR filter banks and we expect more results would appear in this direction in the future.

7. Other applications

In this section, we give a brief review of applications of Gröbner bases to other areas in signal and image processing.

Stability test and stability margin computation: Stability is undoubtedly the most important requirement for an nD filter. To ensure satisfactory performance of a stable nD filter, it is often necessary to know the stability margin which indicates how far away from being unstable the considered system is. It has been shown that the stability test and stability margin computation problems can be formulated in a unified way as a system of algebraic equations characterized by $n+1$ polynomials in $n+1$ variables, which can be solved using the Gröbner bases approach [15,18].

Balanced multiwavelet bases: It is well known that except for the Haar wavelet, it is not possible to design symmetric and orthogonal wavelet bases based on a single scaling-wavelet function pair. Hence, much attention has been directed to the design of multiwavelet bases in recent years. Selesnick has recently shown [46,47] that the design of K-balanced symmetric and orthogonal multiwavelet bases would lead to a system of polynomial equations, which is difficult to solve using conventional methods. However, using Gröbner bases, Selesnick has successfully designed minimal-length K-balanced orthogonal multiwavelet bases for $K = 1, 2, 3$, and further shown that multiwavelet bases based on even-length symmetric FIR filters were smoother than those based on odd-length symmetric FIR filters. The limitation of the implementation of Gröbner bases for a large number of system of polynomial equations was also observed.

Design of nonsymmetric FIR filters: As indicated in [28,45], the exact linear phase and minimum phase solutions only provide the extreme solutions, i.e., the former unnecessarily constrains the linear phase response in the full frequency domain while the latter drops the phase approximation altogether, therefore it is sometimes desired to design FIR filters whose properties are between those of exact linear and minimum phase filters. In [45] Selesnick and Burrus investigated the design of a new class of nonsymmetric maximally flat low-pass FIR filters, which can achieve a smaller group delay than symmetric filters while maintaining relatively constant group delay around $\omega = 0$, with no degradation of the frequency response magnitude. It has been shown that the design problem of such nonsymmetric FIR filters for several different cases of specifications, can be reduced to the problem of solving a system of multivariate polynomial equations and thus can be solved constructively by utilizing the Gröbner bases approach [45].

Image processing and computer vision: Many problems in image processing and computer vision naturally lead to simultaneous polynomial equations. These equations were traditionally solved using numerical methods as they were too complicated to render analytical solutions in the past. Holt et al. recently have applied Gröbner bases to some of these problems and obtained analytical solutions [24]. The problems addressed in [24] were mainly on rigid motion estimation, which is fundamental in image processing and computer vision. Specifically, it was shown that both the problem of one rigid link moving in a plane with one endpoint known and the problem of optical flow all led to the mathematical problem of a system of polynomial equations, which were then solved using Gröbner bases. Potential applications of Gröbner bases to other related areas, such as surface intersection in computer-aided design, and inverse position problems in kinematics/robotics, were also pointed out in [24], although not fully addressed. Another problem in computer vision is the study of 3D-from-2D using elimination, which was addressed in [52] by Werman and Shashua. Most of the results on 3D geometric invariants from point correspondences across multiple 2D views have been unified in [52] by exploiting the elimination theory using Gröbner bases. The topics discussed in [52] include reconstruction from two and three views, epipolar geometry from seven points, trilinearity of three views, the use of a priori 3D information such as bilateral symmetry, shading and color constancy etc. The results presented in [52] have reinforced the promise and versatile of Gröbner bases for problems which could be formulated as simultaneous polynomial equations.

8. Conclusions

In this paper, we have given a tutorial of the basics of Gröbner bases, and a review of its application to signal and image processing, with emphasis on multidimensional wavelets and filter banks. Both FIR and IIR, PR filter banks are considered. Applications of Gröbner bases to other areas in signal and image processing have also been reviewed briefly. It has been demonstrated in the paper that in a wide variety of problems arising from signal and image processing, the Gröbner bases approach either solves problems which would be difficult to solve by other methods, or provides a computationally more efficient method than traditional methods. The wide applications obtained so far show the promise of Gröbner bases as an attractive and efficient algebraic method in signal and image processing and we are positive that many more applications of Gröbner bases to signal and image processing will appear in the future.

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