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On syzygy modules for polynomial matrices

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Abstract

In this paper, we apply the theory of multivariate polynomial matrices to the study of syzygy modules for a system of homogeneous linear equations with multivariate polynomial coefficients. Several interesting structural properties of syzygy modules are presented and illustrated with examples. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

A classical and important subject in commutative algebra is to obtain the set of all polynomial solutions, called the syzygy module, for a system of homogeneous linear equations with multivariate polynomials coefficients (see, e.g., [1–7] and references therein). This subject has been studied for decades by mathematicians in commutative algebra and various methods have been proposed for computing syzygy modules [1–7]. However, the emphasis has so far mainly been on the computational aspects, such as developing more efficient methods for obtaining syzygy modules using Gröbner bases, and finding polynomial solutions with lower degrees [1–7].

In this paper, we apply the theory of multivariate polynomial matrices developed by researchers in linear multidimensional (nD) systems [8–18] to the study of syzygy

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modules. Using simple polynomial matrix manipulations, we are able to derive several interesting structural properties of syzygy modules. Specifically, we will address the following questions: Can we simplify a given nD polynomial matrix[1] before computing its syzygy module? Does there exist an explicit relationship between a given polynomial matrix and its syzygy module? Can we obtain a globally minimal number of generators for the syzygy module? We show in this paper how to answer these questions by exploiting relevant results in nD system theory.

The organization of the paper is as follows. In the next section, we review some notation, definitions and known results, and also formulate the problems to be discussed mathematically. The main results are presented in Section 3. Three examples are illustrated in Section 4.

2. Preliminaries and problem formulation

In the following, we shall denote $\mathbf{C}(\mathbf{z}) = \mathbf{C}(z_1, \ldots, z_n)$ the set of rational functions in complex variables $z_1, \ldots, z_n$ with coefficients in the field of complex numbers $\mathbf{C}$; $\mathbf{C}[\mathbf{z}]$ the set of polynomials in complex variables $z_1, \ldots, z_n$ with coefficients in $\mathbf{C}$; $\mathbf{C}^{m\times l}[\mathbf{z}]$ the set of $m \times l$ matrices with entries in $\mathbf{C}[\mathbf{z}]$, etc. To be consistent with notation in module theory [2], we simply write $\mathbf{C}^{m\times 1}[\mathbf{z}]$ as $\mathbf{C}^{m}[\mathbf{z}]$. Throughout this paper, the argument $(\mathbf{z})$ is omitted whenever its omission does not cause confusion.

Definition 1 [2]. Let $\mathbf{f}_1, \ldots, \mathbf{f}_l \in \mathbf{C}^m[\mathbf{z}]$. A syzygy of the $m \times l$ matrix $F = [\mathbf{f}_1 \cdots \mathbf{f}_l]$ is a vector $[h_1 \cdots h_l]^{\mathrm{T}} \in \mathbf{C}^l[\mathbf{z}]$, where $(\cdot)^{\mathrm{T}}$ denotes transposition, such that[2]

$$\sum_{i=1}^{l} h_i \mathbf{f}_i = 0_{m,1}. \tag{1}$$

The set of all such syzygies is called the syzygy module of F and is denoted by $\mathrm{Syz}(\mathbf{f}_1, \ldots, \mathbf{f}_l)$ or by $\mathrm{Syz}(F)$.

It is easy to see[3] that $\mathrm{Syz}(F)$ is a submodule of $\mathbf{C}^l[\mathbf{z}]$ and is finitely generated [2]. Let $\mathbf{h}_1, \ldots, \mathbf{h}_s \in \mathbf{C}^l[\mathbf{z}]$ be a *generating set* of $\mathrm{Syz}(F)$, i.e.,

$$F\mathbf{h}_j = 0_{m,1}, \quad j = 1, \ldots, s, \tag{2}$$

and for any $\mathbf{t} \in \mathrm{Syz}(F)$, there exist $w_1, \ldots, w_s \in \mathbf{C}[\mathbf{z}]$ such that

$$\mathbf{t} = w_1\mathbf{h}_1 + \cdots + w_s\mathbf{h}_s. \tag{3}$$

If we let $\mathbf{f}_i = [f_{1i} \cdots f_{mi}]^{\mathrm{T}}$ $(i = 1, \ldots, l)$, $\mathbf{h}_j = [h_{1j} \cdots h_{lj}]^{\mathrm{T}}$ $(j = 1, \ldots, s)$ and $\mathbf{t} = [t_1 \cdots t_l]^{\mathrm{T}}$, (2) and (3) become

[1] With slight abuse of notation, we use the term "nD" to abbreviate "multivariate" or "n-variate". This usage is common among researchers in nD system theory [8,17,18].

[2] Denote $0_{m,l}$ an $m \times l$ zero matrix and I_m an $m \times m$ identity matrix.

[3] See, e.g., [2] for an introduction to modules and submodules.

$$\underbrace{\begin{bmatrix} f_{11} & \cdots & f_{1l} \\ \vdots & \ddots & \vdots \\ f_{m1} & \cdots & f_{ml} \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} h_{11} & \cdots & h_{1s} \\ \vdots & \ddots & \vdots \\ h_{l1} & \cdots & h_{ls} \end{bmatrix}}_{H} = 0_{m,s}, \tag{4}$$

and

$$\begin{bmatrix} h_{11} & \cdots & h_{1s} \\ \vdots & \ddots & \vdots \\ h_{l1} & \cdots & h_{ls} \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_s \end{bmatrix} = \begin{bmatrix} t_1 \\ \vdots \\ t_l \end{bmatrix}. \tag{5}$$

Since $\mathrm{Syz}(F)$ is generated by the "column space" of the matrix $H = [\mathbf{h}_1, \ldots, \mathbf{h}_s]$, we shall call H a *generating matrix* of $\mathrm{Syz}(F)$ in this paper. If $H_1 = HE_1$ for some nD polynomial matrix E_1, we shall say that H generates H_1. Clearly, if H_1 is a generating matrix of $\mathrm{Syz}(F)$ and H generates H_1, then H is also a generating matrix of $\mathrm{Syz}(F)$.

We can now formulate mathematically the problems to be discussed using the language of matrix theory. Let $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$. Assume that $H \in \mathbf{C}^{l\times s}[\mathbf{z}]$ is a generating matrix of $\mathrm{Syz}(F)$. The following questions arise:

(i) Can we find another nD polynomial matrix F_1 that is simpler than F in some sense,[4] such that $\mathrm{Syz}(F_1) = \mathrm{Syz}(F)$?

(ii) Does there exist an explicit relationship between F (or F_1) and H?

(iii) Can we find an H such that the dimension of H is globally minimal?

We say that the dimension of H is globally minimal if size(H) is equal to or smaller than size(H_0) for any generating matrix H_0 of $\mathrm{Syz}(F)$. Because of the relationship between a generating matrix and a generating set of $\mathrm{Syz}(F)$, obtaining a globally minimal number of generators for $\mathrm{Syz}(F)$ is equivalent to finding a generating matrix whose dimension is globally minimal. It should be pointed out that a minimal generating set defined in [1–7] is in fact only locally minimal since a given generating set is said to be minimal when no proper subset is a generating set of $\mathrm{Syz}(F)$ [1–7]. We shall come back to this in more detail later.

To motivate the discussion, we first consider a simple example.

Example 1. Let

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} = \begin{bmatrix} d_1b_1 & d_1b_2 \\ d_2b_1 & d_2b_2 \end{bmatrix}, \tag{6}$$

where $d_i, b_i \quad (i = 1, 2)$ are all nD polynomials, and $\gcd(d_1, d_2) = 1$ and $\gcd(b_1,\ b_2) = 1$, where $\gcd(d_1, d_2)$ means the greatest common divisor of d_1 and d_2. We first notice that $\operatorname{rank}(F) = 1$ if b_1 and b_2 (d_1 and d_2) are not both identically zero. Let $F_1 = [d_1b_1\ \ d_1b_2]$, $F_2 = [b_1\ b_2]$, and $H = [h_{11}\ h_{21}]^{\mathrm{T}} = [b_2\ \ -b_1]^{\mathrm{T}}$.

[4] By "simpler" we mean that F_1 is a submatrix of F or F_1 is a proper factor of F.

It is easy to see that $\mathrm{Syz}(F) = \mathrm{Syz}(F_1) = \mathrm{Syz}(F_2)$, and H is a generating matrix of $\mathrm{Syz}(F)$. There is also an explicit relationship between F and H, i.e., $h_{11} = f_{12}/d_1$, $h_{21} = -f_{11}/d_1$. Furthermore, for this simple example, it is obvious that the dimension of H is globally minimal.

However, for a general nD polynomial matrix F, it is not straightforward to answer questions (i)–(iii). For example, in general there does not exist an explicit relationship between entries of F and H. Hence, we need to review some useful definitions and known results which have played a central role in nD system theory.

Definition 2 [9]. Let $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$, with $m \leqslant l$. Then F is said to be:

(i) zero left prime (ZLP) if there exists no n-tuple $\mathbf{z}^0 \in \mathbf{C}^n$ which is a zero of *all* the $m \times m$ minors of F;

(ii) minor left prime (MLP) if these $m \times m$ minors of F are relatively prime;

(iii) factor left prime (FLP) if in any polynomial decomposition $F = F_1F_2$ in which F_1 is square, F_1 is a unimodular matrix, i.e., $\det F_1 = k_0 \in \mathbf{C}^*$.[5]

Zero right prime (ZRP), minor right prime (MRP) and factor right prime (FRP) can be similarly defined.

Proposition 1 [9]. *For $n = 1$, the three definitions of zero, minor and factor primeness are equivalent, i.e., ZLP $\equiv$ MLP $\equiv$ FLP; for $n = 2$, ZLP $\not\equiv$ MLP $\equiv$ FLP; for $n \geqslant 3$, ZLP $\not\equiv$ MLP $\not\equiv$ FLP; for all $n \geqslant 1$, ZLP $\Rightarrow$ MLP $\Rightarrow$ FLP.*

Remark 1. Because of the implication of FLP by MLP, we shall use the phrase "strictly FLP" for an nD polynomial matrix that is FLP but not MLP.

The following rather lengthy definition is necessary to establish an explicit relationship between an nD polynomial matrix and its syzygy module.

Definition 3 [13,16]. Let $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$ and $H \in \mathbf{C}^{l\times r}[\mathbf{z}]$ be of full rank with $l = m + r$ and $FH = 0_{m,r}$. Consider first all the $m \times m$ submatrices of F. If a submatrix F_i ($1 \leqslant i \leqslant \beta = \binom{l}{m}$) is formed by selecting columns $1 \leqslant i_1 < \cdots < i_m \leqslant l$ from F, we associate F_i with an m-tuple $(i_1, \ldots, i_m)$. Clearly, there exists a one to one correspondence between all the $m \times m$ submatrices of F and the collection of all strictly increasing m-tuple $(i_1, \ldots, i_m)$, where $1 \leqslant i_1 < \cdots < i_m \leqslant l$. By enumerating the above m-tuple $(i_1, \ldots, i_m)$ in the lexicographic order, the $m \times m$ submatrices of F are ordered accordingly. Let $\det F_i = \tilde{b}_i\,\tilde{d},\ i = 1, \ldots, \beta$, where $\tilde{d}$ is $\gcd(\det F_1, \ldots, \det F_\beta)$. We call $\{\tilde{b}_1, \ldots, \tilde{b}_\beta\}$ the reduced minors of F.

Now consider the matrix H. If a submatrix H_i is formed by selecting rows $1 \leqslant j_1 < \cdots < j_r \leqslant l$ from H, we delete $j_1, \ldots, j_r$ from the finite set of integers $\{1, 2, \ldots, l\}$ and keep the remaining m integers, denoted by $i_1, \ldots, i_m$. Associating the index i with the ordered m-tuple $(i_1, \ldots, i_m)$ as we do for F, we can establish the

[5] $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$, the set of nonzero complex numbers.

order for all the $r \times r$ submatrices of H. Let $\det H_i = b_i\, d,\ i = 1, \ldots, \beta$, where d is $\gcd(\det H_1, \ldots, \det H_\beta)$. We call $\{b_1, \ldots, b_\beta\}$ the complementary reduced minors of H.

Proposition 2 [13,16]. *Let $P = \tilde{D}^{-1}\tilde{N} = ND^{-1} \in \mathbf{C}^{m\times r}(\mathbf{z})$, where $\tilde{D} \in \mathbf{C}^{m\times m}[\mathbf{z}]$, $D \in \mathbf{C}^{r\times r}[\mathbf{z}],\ \tilde{N}, N \in \mathbf{C}^{m\times r}[\mathbf{z}]$, and let $b_1, \ldots, b_\beta$ denote the complementary reduced minors of $[D^{\mathrm{T}}\ N^{\mathrm{T}}]^{\mathrm{T}}$, $\tilde{b}_1, \ldots, \tilde{b}_\beta$ the reduced minors of $[-\tilde{N}\ \tilde{D}]$. Then*

$$b_i = \pm\tilde{b}_i, \quad i = 1, \ldots, \beta, \tag{7}$$

where the sign depends on the index i.

3. Some properties of syzygy modules

In this section, we answer the questions (i)–(iii) raised in the previous section one by one. We shall begin with question (i) on how to obtain an nD polynomial matrix F_1, simpler than a given matrix F, such that $\mathrm{Syz}(F_1) = \mathrm{Syz}(F)$.

Proposition 3. *Let $F \in \mathbf{C}^{q\times l}[\mathbf{z}]$ be of rank m, where $m < \min\{q, l\}$. Let F_1 be a full row rank $m \times l$ submatrix of F. Then* $\mathrm{Syz}(F) = \mathrm{Syz}(F_1)$.

Proof. Since F_1 is a submatrix of F, it is obvious that $\mathrm{Syz}(F_1) \supset \mathrm{Syz}(F)$. We next show that $\mathrm{Syz}(F) \supset \mathrm{Syz}(F_1)$. Let $H_1 \in \mathbf{C}^{l\times s}[\mathbf{z}]$ be a generating matrix of $\mathrm{Syz}(F_1)$. We have $F_1H_1 = 0_{m,s}$. Since both F and F_1 are of rank m and F_1 is a submatrix of F, all rows of F can be generated by linear combinations of the m rows of F_1 over $C(\mathbf{z})$, i.e., there exists $W \in \mathbf{C}^{q\times m}(\mathbf{z})$ such that $F = WF_1$. We then have $FH_1 = WF_1H_1 = 0_{q,s}$, i.e., $\mathrm{Syz}(F) \supset \mathrm{Syz}(F_1)$. Therefore, $\mathrm{Syz}(F) = \mathrm{Syz}(F_1)$. $\square$

Since F_1 in the above proposition is a submatrix of F, it will be computationally more efficient to compute $\mathrm{Syz}(F_1)$ than $\mathrm{Syz}(F)$. The next result shows that this kind of simplification can also be achieved if a given matrix F admits certain factorizations.

Proposition 4. *Let $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of rank m. If $F = E_1F_1$ for some $E_1 \in \mathbf{C}^{m\times m}[\mathbf{z}]$, $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, then* $\mathrm{Syz}(F) = \mathrm{Syz}(F_1)$.

Proof. Since F is of rank m, E_1 must also be of rank m. Hence, E_1 is nonsingular. Let $H \in \mathbf{C}^{l\times s}[\mathbf{z}]$ be a generating matrix of $\mathrm{Syz}(F)$. We have $FH = 0_{m,s}$, or $E_1F_1H = 0_{m,s}$, or $F_1H = 0_{m,s}$ since E_1 is nonsingular. Thus, $\mathrm{Syz}(F_1) \supset \mathrm{Syz}(F)$. On the other hand, let $H_1 \in \mathbf{C}^{l\times s}[\mathbf{z}]$ be a generating matrix of $\mathrm{Syz}(F_1)$. We have $F_1H_1 = 0_{m,s}$. It then follows that $FH_1 = E_1F_1H_1 = 0_{m,s}$, i.e., $\mathrm{Syz}(F) \supset \mathrm{Syz}(F_1)$. Therefore, $\mathrm{Syz}(F) = \mathrm{Syz}(F_1)$. $\square$

Proposition 4 is very useful for obtaining a syzygy module for a given matrix admitting certain prime factorizations, as will be demonstrated by an example in the next section.

Because of Proposition 3, it suffices to consider a full row rank matrix $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$. This requires $l \geqslant m$. If $l = m$, then $\mathrm{Syz}(F) = [0 \dots 0]^{\mathrm{T}}$. Therefore, we shall only consider the nontrivial case where $l > m$. In order to make good use of relevant results in nD system theory, we shall assume, without loss of generality, that $F = [-\tilde{N}\ \tilde{D}]$ with D being nonsingular. For convenience of exposition, we state the following assumption which will be adopted in the remainder of the paper:

Assumption 1. Let $F = [-\tilde{N}\ \tilde{D}] \in \mathbf{C}^{m\times l}[\mathbf{z}]$ be of rank m, with $l > m$ and $\tilde{D} \in \mathbf{C}^{m\times m}[\mathbf{z}]$ being nonsingular. We also let $r = l - m > 0$.

Before we answer question (ii), the following two lemmas are required.

Lemma 1. *Let F be given as in Assumption* 1 *and suppose*

$$FH = [-\tilde{N}\ \tilde{D}]\begin{bmatrix} D \\ N \end{bmatrix} = 0_{m,r}. \tag{8}$$

If $H \in \mathbf{C}^{l\times r}[\mathbf{z}]$ *is of full rank and* $D \in \mathbf{C}^{r\times r}[\mathbf{z}]$, *then* $\det D \not\equiv 0$.

Proof. We can view both F and H as rational matrices. Since F is of full row rank and H is of full column rank, there exist $B \in \mathbf{C}^{l\times m}(\mathbf{z})$ and $G' \in \mathbf{C}^{r\times l}(\mathbf{z})$ such that

$$FB = I_m \tag{9}$$

and

$$G'H = I_r. \tag{10}$$

Thus,

$$\begin{bmatrix} G' \\ F \end{bmatrix}[H\ \ B] = \begin{bmatrix} I_r & W \\ 0_{m,r} & I_m \end{bmatrix}, \tag{11}$$

where $W = G'B \in \mathbf{C}^{r\times m}(\mathbf{z})$. Let $G = G' - WF$. Simple algebra on (11) gives

$$\begin{bmatrix} G \\ F \end{bmatrix}[H\ \ B] = \begin{bmatrix} I_r & 0_{r,m} \\ 0_{m,r} & I_m \end{bmatrix} \tag{12}$$

or

$$\underbrace{\begin{bmatrix} X & Y \\ -\tilde{N} & \tilde{D} \end{bmatrix}}_{V}\underbrace{\begin{bmatrix} D & \tilde{Y} \\ N & \tilde{X} \end{bmatrix}}_{U} = \begin{bmatrix} I_r & 0_{r,m} \\ 0_{m,r} & I_m \end{bmatrix}, \tag{13}$$

where X, Y and $\tilde{X}, \tilde{Y}$ are submatrices of G and B, respectively, with appropriate dimension. According to a well-known result on matrix theory [19, p. 29], the fact that V and $\tilde{D}$ are nonsingular implies that D is also nonsingular, i.e., $\det D \not\equiv 0$. □

Lemma 2. *Let F be given as in Assumption* 1. *Then there exists a generating matrix* $H \in \mathbf{C}^{l\times s}[\mathbf{z}]$ *of* $\mathrm{Syz}(F)$, *with* $r \leqslant s < \infty$. *Moreoever, every generating matrix of* $\mathrm{Syz}(F)$ *is of rank r.*

Proof. Obvious. □

We are now ready to establish an explicit relationship between a given matrix and its syzygy module.

Proposition 5. *Let F be given as in Assumption* 1 *and* $H \in \mathbf{C}^{l\times s}[\mathbf{z}]$ *be a generating matrix of* $\mathrm{Syz}(F)$. *Let* H_1 *be an arbitrary but fixed full rank* $l \times r$ *submatrix of H. Let* $b_1, \ldots, b_\beta$ *denote the complementary reduced minors of* H_1, *and* $\tilde{b}_1, \ldots, \tilde{b}_\beta$ *the reduced minors of F. Then*

$$b_i = \pm\tilde{b}_i, \quad i = 1, \ldots, \beta, \tag{14}$$

where the sign depends on the index i.

Proof. By Lemma 2, $s \geqslant r$. Hence, it is meaningful to talk about a full rank $l \times r$ submatrix of H. Let $H_1 = [D_1^{\mathrm{T}}\ N_1^{\mathrm{T}}]^{\mathrm{T}}$ with $D_1 \in \mathbf{C}^{r\times r}[\mathbf{z}]$ and $N_1 \in \mathbf{C}^{m\times r}[\mathbf{z}]$. Since H_1 is of full column rank by assumption, $\det D_1 \not\equiv 0$ by Lemma 1. We then have

$$FH = [-\tilde{N}\ \tilde{D}]\begin{bmatrix} D_1 \\ N_1 \end{bmatrix} = 0_{m,r} \tag{15}$$

or

$$-\tilde{N}D_1 + \tilde{D}N_1 = 0_{m,r} \tag{16}$$

or

$$P = \tilde{D}^{-1}\tilde{N} = N_1 D_1^{-1}, \tag{17}$$

where $P \in \mathbf{C}^{m\times r}(\mathbf{z})$. By Proposition 2, we have

$$b_i = \pm\tilde{b}_i, \quad i = 1, \ldots, \beta, \tag{18}$$

where the sign depends on the index i. □

A by-product of the above proposition is that the complementary reduced minors of all full rank $l \times r$ submatrices of H are identical. For this reason, we can simply call $b_1, \ldots, b_\beta$, as defined in Proposition 5, the complementary reduced minors of H. Proposition 5 shows that for a general full rank nD polynomial matrix F, although there does not exist an explicit relationship between entries of F and of H that is a generating matrix of $\mathrm{Syz}(F)$, there does exist a simple relationship between the reduced minors of F and the complementary reduced minors of H. This relationship is also useful for answering question (iii), as will be discussed in detail in the following.

Consider again an F as given in Assumption 1. We know that if F is over a field, such as $\mathbf{C}(\mathbf{z})$, then we can always find an $l \times r$ generating matrix over $\mathbf{C}(\mathbf{z})$

for $\mathrm{Syz}(F)$. However, this is not the case when F is over a ring, and the following proposition gives a condition for the existence of such a generating matrix.

Proposition 6. *Let F be given as in Assumption* 1. *Then* $\mathrm{Syz}(F)$ *has a generating matrix of dimension* $l \times r$ *if and only if there exists an MRP matrix* $H \in \mathbf{C}^{l\times r}[\mathbf{z}]$ *such that* $FH = 0_{m,r}$.

Proof. *Sufficiency*: Assume that there exists an MRP matrix $H \in \mathbf{C}^{l\times r}[\mathbf{z}]$ such that $FH = 0_{m,r}$. By Lemma 2, there exists a generating matrix $H_1 \in \mathbf{C}^{l\times s}[\mathbf{z}]$ of Syz(F). It suffices to show that H generates H_1. Let $H_0 = [H\ \ H_1] \in \mathbf{C}^{l\times(r+s)}[\mathbf{z}]$. It is obvious that $FH_0 = 0_{m,(r+s)}$. Hence, H_0 is also a generating matrix of Syz(F) and is of rank r by Lemma 2. Since H is MRP, by a result due to Youla and Gnavi [9], we have $H_0 = [H\ \ H_1] = H[I_r\ \ E_1]$ for some $E_1 \in \mathbf{C}^{r\times s}[\mathbf{z}]$, or $H_1 = HE_1$. That is, H_1 is generated by H. It follows that H is a generating matrix of Syz(F).

Necessity: Let $H \in \mathbf{C}^{l\times r}[\mathbf{z}]$ be a generating matrix of Syz(F), i.e.

$$FH = 0_{m,r}. \tag{19}$$

By Lemma 2, H is of rank r. We first show that H cannot have a nontrivial right factor. Suppose that H has a nontrivial right factor $E_2 \in \mathbf{C}^{r\times r}[\mathbf{z}]$, i.e.,

$$H = H_2E_2 \tag{20}$$

for some $H_2 \in \mathbf{C}^{l\times r}[\mathbf{z}]$, with $\det E_2 \not\equiv 0$ and E_2 not a unimodular matrix. Combining (19) and (20) gives

$$FH = FH_2E_2 = 0_{m,r}. \tag{21}$$

Since $\det E_2 \not\equiv 0$, (21) leads to

$$FH_2 = 0_{m,r}, \tag{22}$$

implying that each column of H_2 belongs to Syz(F). Clearly, from (20) H cannot generate H_2 since E_2 is not a unimodular matrix. Thus, H is not a generating matrix of Syz(F), a contradiction. Therefore, H cannot have a nontrivial right factor. By Definition 2 and Remark 1, H is either strictly FRP or MRP.

We next show that H cannot be strictly FRP. Suppose, on the contrast, that H is strictly FRP, i.e., the $r \times r$ minors of H have a nontrivial gcd $d(\mathbf{z})$. Partition F and H conformably as $F = [-\tilde{N}\ \ \tilde{D}]$ and $H = [D^{\mathrm{T}}\ \ N^{\mathrm{T}}]^{\mathrm{T}}$. The assumption $\det \tilde{D} \not\equiv 0$ implies $\det D \not\equiv 0$ by Lemma 1. We then have

$$FH = [-\tilde{N}\ \ \tilde{D}]\begin{bmatrix} D \\ N \end{bmatrix} = 0_{m,r} \tag{23}$$

or

$$-\tilde{N}D + \tilde{D}N = 0_{m,r} \tag{24}$$

or

$$P = \tilde{D}^{-1}\tilde{N} = ND^{-1}, \tag{25}$$

where $P \in \mathbf{C}^{m\times r}(\mathbf{z})$. Since $H = [D^{\mathrm{T}}\ N^{\mathrm{T}}]^{\mathrm{T}}$ is strictly FRP, P has another right matrix fraction description (MFD) $P = N_3 D_3^{-1}$ such that $D_3 \neq DW_3$ and $N_3 \neq NW_3$ for any $W_3 \in \mathbf{C}^{r\times r}[\mathbf{z}]$ (see [13]). This is equivalent to

$$\begin{bmatrix} D_3 \\ N_3 \end{bmatrix} \neq \begin{bmatrix} D \\ N \end{bmatrix} W_3 \tag{26}$$

for any $W_3 \in \mathbf{C}^{r\times r}[\mathbf{z}]$. This means that H cannot generate $H_3 = [D_3^{\mathrm{T}}\ N_3^{\mathrm{T}}]^{\mathrm{T}}$. On the other hand, from $P = \tilde{D}^{-1}\tilde{N} = N_3 D_3^{-1}$, we have

$$[-\tilde{N}\ \tilde{D}] \begin{bmatrix} D_3 \\ N_3 \end{bmatrix} = 0_{m,r}, \tag{27}$$

implying that each column of H_3 belongs to Syz(F). Combining (26) and (27) leads to a conclusion that H cannot generate Syz(F), another contradiction. Therefore, H cannot be strictly FRP either, and must be MRP. □

The above proposition gives a characterization of an $l \times r$ generating matrix of Syz(F) when such a generating matrix exists. An interesting question arises at this point. Given an arbitrary F as in Assumption 1, can we always find an $l \times r$ generating matrix for Syz(F)? The answer is positive for $n \leqslant 2$, but negative for $n > 2$.

Proposition 7. *Let F be given as in Assumption* 1 *except that* $F \in \mathbf{C}^{m\times l}[z_1, z_2]$. *Then, there exists a generating matrix* $H \in \mathbf{C}^{l\times r}[z_1, z_2]$ *of* Syz(F).

Proof. By Assumption 1, $F = [-\tilde{N}\ \tilde{D}]$ and det $\tilde{D} \not\equiv 0$. Associate F with a 2D rational matrix $P = \tilde{D}^{-1}\tilde{N}$. By a well-known result in 2D polynomial matrix theory [8,11], P has a right MFD, $P = ND^{-1}$ such that $H = [D^{\mathrm{T}}\ N^{\mathrm{T}}]^{\mathrm{T}} \in \mathbf{C}^{l\times r}[z_1, z_2]$ is MRP. Clearly, $P = \tilde{D}^{-1}\tilde{N} = ND^{-1}$ gives rise to

$$FH = [-\tilde{N}\ \tilde{D}] \begin{bmatrix} D \\ N \end{bmatrix} = 0_{m,r}. \tag{28}$$

By Proposition 6, H is a generating matrix of Syz(F). □

The existence of an $l \times r$ generating matrix of Syz(F) is due to the equivalence of factor and minor primeness for 2D (including 1D) polynomial matrices [9] and the availability of computational methods for the extraction of any nontrivial right (or left) factors from a given 1D or 2D polynomial matrix [10,8,11]. Unfortunately, factor primeness is no longer equivalent to minor primeness for nD ($n > 2$) polynomial matrices [9]. Moreover, it is still an open problem to extract a nontrivial right (or left) factor from a given nD ($n > 2$) polynomial matrix [9,17], although some partial results in this direction are now available [14,15,18].

On the other hand, researchers in commutative algebra have developed methods for the construction of generating matricesof Syz(F) [1,7]. However, these generating matrices are not necessarily of size $l \times r$. In fact, generating matrices for a given nD polynomial matrix F may even be different in size, depending on F, the method adopted and the ordering of terms and positions [1,7]. Another interesting question then arises. Given an $l \times s$ $(s > r)$ generating matrix H_1 of Syz(F), can we decide from H_1 whether or not there exists an $l \times r$ generating matrix of Syz(F)? The following proposition gives an answer to this question.

Proposition 8. *Let F be given as in Assumption* 1 *and* $H_1 \in \mathbf{C}^{l\times s}[\mathbf{z}]$ *be a generating matrix of* $\mathrm{Syz}(F)$, *with* $s > r$. *Then* $\mathrm{Syz}(F)$ *has a generating matrix of dimension* $l \times r$ *if and only if* H_1 *can be factorized as* $H_1 = HE$ *for some* $H \in \mathbf{C}^{l\times r}[\mathbf{z}]$, $E \in \mathbf{C}^{r\times s}[\mathbf{z}]$ *with H being MRP.*

Proof. *Sufficiency*: Suppose that H_1 can be factorized as $H_1 = HE$ for some $H \in \mathbf{C}^{l\times r}[\mathbf{z}]$, $E \in \mathbf{C}^{r\times s}[\mathbf{z}]$ with H being MRP. Since H_1 is a generating matrix for Syz(F), H_1 is of rank r by Lemma 2. It follows immediately that E must also be of rank r. Let $T = FH \in \mathbf{C}^{m\times r}[\mathbf{z}]$. From $FH_1 = 0_{m,s}$, we have $FHE = 0_{m,s}$, or $TE = 0_{m,s}$, or $TE_r = 0_{m,r}$ where E_r is a nonsingular $r \times r$ submatrix of E. Since E_r is nonsingular, it is obvious that $T = 0_{m,r}$. Thus, $FH = 0_{m,r}$. Since H is MRP by assumption, H is an $l \times r$ generating matrix of Syz(F) by Proposition 6.

Necessity: Suppose that $H \in \mathbf{C}^{l\times r}[\mathbf{z}]$ is a generating matrix of Syz(F). By Proposition 6, H is MRP and $FH = 0_{m,r}$. Since $FH_1 = 0_{m,s}$, arguing similarly as in the proof procedure for the sufficiency of Proposition 6, we have $H_1 = HE$ for some $E \in \mathbf{C}^{r\times s}[\mathbf{z}]$. $\square$

Unfortunately, to the best knowledge of this author, in the case of $n > 2$, there still does not exist an algebraic method for testing whether or not an arbitrary nD $l \times s$ $(s > r)$ polynomial matrix of rank r can be factorized as $H_1 = HE$ for some $H \in \mathbf{C}^{l\times r}[\mathbf{z}]$, $E \in \mathbf{C}^{r\times s}[\mathbf{z}]$ [9,17]. Nevertheless, there do exist several methods for testing the factorizability and carrying out factorizations for some special nD polynomial matrices [14,15,18]. Therefore, it is sometimes possible to derive an $l \times r$ generating matrix of Syz(F) from an $l \times s$ $(s > r)$ generating matrix. This will be demonstrated by an example in the following section.

4. Examples

In this section, we present three examples to illustrate the new results derived in the previous section. The examples are all taken from the literature and are chosen in such a way that each example corresponds mainly to each question raised in Section 2. For consistency with the notation adopted in this paper, we use z_1, z_2, z_3 for the complex variables instead of the usual x, y, z commonly adopted in commutative algebra.

Example 2 [2, p. 165]. Let

$$F = \begin{bmatrix} z_2 + 2z_1^2 + z_1 & z_1 - z_2 & z_1^2 + z_1 \\ z_2 & z_2 & z_2 \end{bmatrix}. \tag{29}$$

Instead of directly applying the Gröbner basis approach to obtaining $\mathrm{Syz}(F)$, as was done in [2], we first check whether F is FLP. The 2×2 minors of F are:

$$2z_2(z_1 + z_2), \quad z_2(z_1 + z_2), \quad -z_2(z_1 + z_2), \tag{30}$$

and the reduced minors of F are just $2, 1, -1$. Clearly, F is not FLP. Applying the factorization methods proposed in [8,11], we can factorize F as

$$F = E_1 F_1 = \begin{bmatrix} z_1 - z_2 & z_1^2 + z_1 \\ z_2 & z_2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}. \tag{31}$$

It is straightforward to see that $H = [1\ 1\ -2]^{\mathrm{T}}$ is a generating matrix of $\mathrm{Syz}(F_1)$. By Proposition 4, $\mathrm{Syz}(F) = \mathrm{Syz}(F_1)$. Hence, we have obtained the same result as the one in [2] without even applying Gröbner bases.

The above example shows that the potential advantege of applying nD polynomial matrix factorization techniques have not yet been fully realized by researchers in algebra.

Example 3 [5, p. 140]. Let $F = [z_1\ z_2\ z_3]$. A generating matrix $H \in \mathbf{C}^{3\times 3}[z_1, z_2, z_3]$ has been given in [5]:

$$H = \begin{bmatrix} z_2 & z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & -z_1 & -z_2 \end{bmatrix}. \tag{32}$$

Although H is of rank 2, it cannot be factorized as a product of two 3D polynomial matrices of smaller size. Therefore, by Proposition 8, there does not exist any 3×2 generating matrix of $\mathrm{Syz}(F)$. Now let H_1 be a 3×2 submatrix formed from selecting columns 1 and 2 of H, i.e.,

$$H_1 = \begin{bmatrix} z_2 & z_3 \\ -z_1 & 0 \\ 0 & -z_1 \end{bmatrix}. \tag{33}$$

It is obvious that the reduced minors of F are z_1, z_2, z_3, and the complementary reduced minors of H_1 are $z_1, -z_2, z_3$. Proposition 5 is therefore verified.

Finally, we present a nontrivial example which demonstrates the validity of Propositions 6 and 8.

Example 4 [1, p. 151]. Let

$$F = [f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5], \tag{34}$$

where

$$f_1 = z_3^2 - z_1 z_2 + z_3,$$

$$f_2 = z_2^4 - z_1z_2z_3 + z_1z_2 - z_3,$$

$$f_3 = z_2^3z_3 + z_2^3 - z_1^2z_2 + z_1z_3,$$

$$f_4 = z_1z_2^3 + z_2^2z_3 - z_1^2z_3 + z_2^2,$$

$$f_5 = z_1z_2^2z_3 + 2z_1z_2^2 - z_1^3 + z_2z_3 + z_2.$$

Using Gröbner bases, a generating matrix was obtained as follows [1]:

$$H = [\mathbf{h}_1\ \mathbf{h}_2\ \mathbf{h}_3\ \mathbf{h}_4\ \mathbf{h}_5\ \mathbf{h}_6]$$
$$= \begin{bmatrix} z_1z_2+1 & 0 & 0 & z_2^2 - z_1^2 & z_1z_2^2 + z_2 & z_2^3 + z_1 \\ z_3+1 & z_1 & 0 & 0 & 0 & z_1 \\ -z_2 & 1 & z_1 & z_1 & 0 & -z_3 \\ 0 & -z_2 & 1 & -z_3 & z_1 & 0 \\ 0 & 0 & -z_2 & 0 & -z_3 & 0 \end{bmatrix}.$$

It was claimed that the set of $\mathbf{h}_1, \ldots, \mathbf{h}_6$ is already a (locally) minimal generating set for $\mathrm{Syz}(F)$ with respect to the T-representation introduced in [1]. Since our main interest is to obtain a generating matrix whose dimension is globally minimal, i.e., to obtain a globally minimal generating set, we want to know whether H can be further reduced.

We first observe $\mathbf{h}_6 = z_1\mathbf{h}_1 - z_3\mathbf{h}_2 + z_2\mathbf{h}_4$. Hence, $H_1 = [\mathbf{h}_1 \cdots \mathbf{h}_5]$ is also a generating matrix of $\mathrm{Syz}(F)$, which is of smaller dimension than that of H. Direct computation shows that none of the 5×4 submatrices of H_1 is MRP, and hence, it is not possible to pick any 4 columns from H_1 as a globally minimal generating set of $\mathrm{Syz}(F)$. (We omit the details for this argument to save space.) However, applying the primitive factorization algorithm proposed previously by the author [14,15] to the submatrix H_2 formed from the first 4 columns of H_1, we are able to carry out a primitive factorization for H_2 as $H_2 = H_3E_3$, where

$$H_3 = \begin{bmatrix} z_1z_2+1 & 0 & 0 & z_1^3 + z_2 \\ z_3+1 & z_1 & 0 & 0 \\ -z_2 & 1 & z_1 & -z_1^2 \\ 0 & -z_2 & 1 & z_1z_3 + z_1 \\ 0 & 0 & -z_2 & -z_3 \end{bmatrix}$$

and

$$E_3 = \begin{bmatrix} 1 & 0 & 0 & -z_1^2 \\ 0 & 1 & 0 & z_1z_3 + z_1 \\ 0 & 0 & 1 & -z_3 \\ 0 & 0 & 0 & z_2 \end{bmatrix}.$$

It is straightforward to test that H_3 is MRP and $FH_3 = 0_{1,4}$. By Proposition 6, H_3 is a generating matrix of $\mathrm{Syz}(F)$ and the dimension of H_3 is now globally minimal.

To convince the reader that H_3 is indeed a generating matrix of $\mathrm{Syz}(F)$, we give E_4 explicitly in the following:

$$E_4 = \begin{bmatrix} 1 & 0 & 0 & -z_1^2 & -z_1^3 \\ 0 & 1 & 0 & z_1 z_3 + z_1 & z_1^2 z_3 + z_1^2 \\ 0 & 0 & 1 & -z_3 & -z_1 z_3 \\ 0 & 0 & 0 & z_2 & z_1 z_2 + 1 \end{bmatrix}.$$

It can then be easily verified that $H_1 = H_3 E_4$.

Finally, although the entries of F and of H_3 look very different from each other, it is straightforward to test that there does exist a simple relationship between the reduced minors of F and the complementary reduced minors of H_3 as stated in Proposition 5.

It is hoped that this paper will motivate more research in the investigation of nD polynomial matrices and related open problems.

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