BIBO Stability of Inverse 2-D Digital Filters in the Presence of Nonessential Singularities of the Second Kind

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***Abstract*—This paper considers the open problem concerning the BIBO stability of inverse 2-D digital filters in the presence of nonessential singularities of the second kind. It is shown that there exist BIBO stable 2-D filter transfer functions having simple nonessential singularities of the second kind on T^2, that also admit BIBO stable inverses. A class of such functions is obtained. Another class of BIBO stable 2-D functions that cannot have BIBO stable inverses is also characterized.**

I. Introduction

INVERSE multidimensional (n-D) digital filters have applications in areas such as digital image processing [1], [2]. For example, inverse 2-D digital filters may be used as tools to restore degraded images [1]. The invertibility of 2-D digital filter (or 2-D system) transfer functions has been investigated by a number of researchers (see, e.g., [1], [3]). In this paper, we study the problem regarding the bounded-input bounded-output (BIBO) stability of inverse 2-D digital filters in the presence of nonessential singularities of the second kind.

Consider the class of n-D ($n \geqslant 2$) linear shift-invariant (LSI) quarter-plane causal digital filters whose region of support is the first quarter-plane and which is described by a real rational transfer function

$$G(z_1,\cdots,z_n)=\frac{P(z_1,\cdots,z_n)}{Q(z_1,\cdots,z_n)},\qquad Q(0,\cdots,0)\neq 0$$

where P and Q are relatively prime polynomials in n complex variables. If $P(0,\cdots,0)\neq 0$, then $G^{-1}(z_1,\cdots,z_n)$, the inverse of $G(z_1,\cdots,z_n)$, also represents an n-D LSI quarter-plane causal digital filter. It is well known [5] that $G^{-1}(z_1,\cdots,z_n)$ is BIBO stable if $P(z_1,\cdots,z_n)$ has no zeros in the closed unit polydisk

$$\overline{U}^n=\{(z_1,\cdots,z_n)\colon |z_1|\leqslant 1,\cdots,|z_n|\leqslant 1\}.$$

However, as established in [4], [8], $G^{-1}(z_1,\cdots,z_n)$ may still be BIBO stable even when it has some nonessential singularities of the second kind on the distinguished boundary of the unit polydisk

$$T^n=\{(z_1,\cdots,z_n)\colon |z_1|=1,\cdots,|z_n|=1\}.$$

Manuscript received November 23, 1987; revised July 20, 1988. This paper was recommended by Associate Editor D. M. Goodman.
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IEEE Log Number 8825004.

The problem concerning the BIBO stability of n-D ($n \geqslant 2$) digital filters in the presence of nonessential singularities of the second kind on T^n has been the subject of intensive research for a decade (see, e.g., [4]–[15]). In particular, exploiting certain resultants of polynomials in two variables, Roytman *et al.* [12] have recently developed a useful method for testing the BIBO stability of 2-D digital filter transfer functions having simple nonessential singularities of the second kind on T^2. This method can also be used for checking the BIBO stability of inverse 2-D digital filter transfer functions with such singularities on T^2.

An interesting question arises [9]: Does there exist BIBO stable n-D ($n \geqslant 2$) digital filter transfer functions having nonessential singularities of the second kind on T^n that also admit BIBO stable inverses? Bose [9, p. 245] conjectured that such a BIBO stable n-D transfer function cannot have a BIBO stable inverse. In this contribution, we show that this conjecture does not hold in general. In fact, as will be seen in Section III, there are an infinite number of BIBO stable 2-D rational functions having a simple nonessential singularity of the second kind on T^2 that also admit BIBO stable inverses. A class of such 2-D stable functions is parametrized. In Section IV, we give a characterization of another class of BIBO stable 2-D digital filter transfer functions which cannot have BIBO stable inverses.

II. Preliminaries

Consider the class of 2-D LSI quarter-plane causal digital filters whose region of support is the first quarter-plane and which is described by a real rational transfer function

$$G(z_1,z_2)=\frac{P(z_1,z_2)}{Q(z_1,z_2)},\qquad Q(0,0)\neq 0 \tag{1}$$

where P and Q are relatively prime. A 2-tuple z_a such that $P(z_a)\neq 0$ and $Q(z_a)=0$ is called a pole or a nonessential singularity of the first kind of $G(z_1,z_2)$; z_a is called a nonessential singularity of the second kind of $G(z_1,z_2)$ if $P(z_a)=Q(z_a)=0$. Because any singularities of a rational function in n variables are necessarily nonessential, we shall, in this paper, abbreviate the term "nonessential singularity of the second kind" as "second kind singularity," for the sake of simplicity.

Since $Q(z_1, z_2)$ in (1) is nonzero in some neighborhood around the origin (0,0), $G(z_1, z_2)$ is analytic and has a power series expansion in this neighborhood

$$G(z_1, z_2) = \sum_{m,n=0} g_{mn} z_1^m z_2^n$$

where g_{mn} is the impulse response of $G(z_1, z_2)$. The filter is BIBO stable if and only if

$$\sum_{m,n=0} |g_{mn}| < \infty.$$

In practice, however, it is often more convenient to determine the stability of a filter from its transfer function $G(z_1, z_2)$. A simple criterion giving a necessary and sufficient condition for the BIBO stability of 2-D digital filter transfer functions having simple second kind singularities on T^2 has recently been presented by Roytman *et al.* [12]. Since this important criterion will be frequently used in this paper, we reproduce it here for convenience of exposition. As in [12], we denote by $\tilde{A}(z_1, z_2)$ the discrete paraconjugate of a 2-D polynomial $A(z_1, z_2)$ and by $R_{z_2}[A(z_1, z_2), B(z_1, z_2)]$ the z_2-resultant of the polynomials $A(z_1, z_2)$ and $B(z_1, z_2)$.

Theorem 1 [12]:[1] Let

$$G(z_1, z_2) = \frac{P(z_1, z_2)}{[Q(z_1, z_2)]^t}$$

where P and Q are relatively prime and t is a positive integer. Assume that P/Q has no poles in $\bar{U}^2$, nor any second kind singularities in $\bar{U}^2$ except for the simple ones[2] at (α, β) and $(1/\bar{\alpha}, 1/\bar{\beta})$ on T^2. We further assume that P/Q has no simple or multiple second kind singularities of the form (α, γ) outside $\bar{U}^2$. Then, $G(z_1, z_2)$ is BIBO stable if and only if

$$t m_\alpha\left(R_{z_2}[Q, \tilde{Q}]\right) < m_\alpha\left(R_{z_2}[P, \tilde{Q}]\right) \tag{2}$$

where $m_\alpha(f(z_1))$ denotes the multiplicity of the factor $(z_1 - \alpha)$ in $f(z_1)$.

III. Existence of Stable Inverses for a Class of BIBO Stable 2-D Functions

This section provides an answer to the question of whether or not a BIBO stable rational function $G(z_1, z_2)$ with second kind singularities on T^2 can admit a BIBO stable inverse [12]. Let us begin our discussion with an example.

Example 1: Consider the 2-D rational function

$$G_1(z_1, z_2) = \frac{P_1(z_1, z_2)}{Q_1(z_1, z_2)} = \frac{4 - 3z_1 - 3z_1 z_2 + 2z_1^2 z_2}{3 - 2z_1 - z_2}.$$

It can be easily checked that $P_1(z_1, z_2)$ and $Q_1(z_1, z_2)$ are relatively prime, and that $G_1(z_1, z_2)$ has no poles in $\bar{U}^2$, nor second kind singularities anywhere except for a simple one at (1,1) on T^2.

Let us compute the discrete paraconjugate of $Q_1(z_1, z_2)$

$$\tilde{Q}_1(z_1, z_2) = z_1 z_2 Q_1(z_1^{-1}, z_2^{-1}) = z_1 z_2 (3 - 2z_1^{-1} - z_2^{-1}) = 3z_1 z_2 - 2z_2 - z_1.$$

The z_2-resultant of Q_1 and $\tilde{Q}_1$, and that of P_1 and $\tilde{Q}_1$ are

$$R_{z_2}[Q_1, \tilde{Q}_1] = \begin{vmatrix} 3z_1 - 2 & -z_1 \\ -1 & 3 - 2z_1 \end{vmatrix} = -6(z_1 - 1)^2$$

and

$$R_{z_2}[P_1, \tilde{Q}_1] = \begin{vmatrix} 2z_1^2 - 3z_1 & 4 - 3z_1 \\ -1 & 3 - 2z_1 \end{vmatrix} = -4(z_1 - 1)^3$$

from which it follows:

$$m_1\left(R_{z_2}[Q_1, \tilde{Q}_1]\right) = 2 \quad \text{and} \quad m_1\left(R_{z_2}[P_1, \tilde{Q}_1]\right) = 3.$$

Therefore, by Theorem 1, $G_1(z_1, z_2)$ is BIBO stable [12].

Now consider the BIBO stability of the inverse of $G_1(z_1, z_2)$ given by

$$G_1^{-1}(z_1, z_2) = \frac{Q_1(z_1, z_2)}{P_1(z_1, z_2)}.$$

Since $Q_1(z_1, z_2)$ has no zeros in the region $\{\bar{U}^2 - (1,1)\}$, it is necessary for BIBO stability of $G^{-1}(z_1, z_2)$ that $P_1(z_1, z_2)$ has no zeros in $\{\bar{U}^2 - (1,1)\}$. This is shown as follows.

Rewriting $P_1(z_1, z_2)$ as

$$P_1(z_1, z_2) = (4 - 3z_1)\left\{1 - \frac{1}{3}\left(2 + \frac{1}{4 - 3z_1}\right) z_1 z_2\right\}$$

and because

$$\left|\frac{1}{4 - 3z_1}\right| < 1 \qquad \forall\ |z_1| \le 1,\ z_1 \ne 1$$

it follows that

$$\left|-\frac{1}{3}\left(2 + \frac{1}{4 - 3z_1}\right)\right| < 1 \qquad \forall\ |z_1| \le 1;\ z_1 \ne 1$$

from which it is easily established that

$$P_1(z_1, z_2) \ne 0 \qquad \forall\ |z_1| \le 1,\ z_1 \ne 1;\ |z_2| \le 1. \tag{3}$$

On the other hand, $P_1(1, z_2) = 1 - z_2$ and, therefore,

$$P_1(1, z_2) \ne 0 \qquad \forall\ z_2 \ne 1. \tag{4}$$

It follows from (3) and (4) that $P_1(z_1, z_2)$ has no zeros in $\{\bar{U}^2 - (1,1)\}$ and, therefore, $G^{-1}(z_1, z_2)$ has no poles in $\bar{U}^2$.

It is easy to show that $G^{-1}(z_1, z_2)$ has no second kind singularities anywhere except for a simple one at (1,1) on T^2, and therefore Theorem 1 can be applied to check the

[1] This theorem is sufficient for the purpose of this paper, although it can be extended to a wider class of 2-D functions as shown in [12], [14].

[2] As in [12], a second kind singularity (α, β) of the function P/Q is said to be simple if $(\partial Q/\partial z_1)_{(\alpha,\beta)} \ne 0$ and $(\partial Q/\partial z_2)_{(\alpha,\beta)} \ne 0$.

BIBO stability of $G^{-1}(z_1, z_2)$. Direct calculation gives

$$P_1(z_1, z_2) = 4z_1^2 z_2 - 3z_1 z_2 - 3z_1 + 2$$

$$R_{z_2}[P_1, P_1] = \begin{vmatrix} 4z_1^2 - 3z_1 & -3z_1 + 2 \\ 2z_1^2 - 3z_1 & 4 - 3z_1 \end{vmatrix} = -6z_1(z_1 - 1)^2$$

$$R_{z_2}[Q_1, P_1] = -R_{z_2}[P_1, Q_1] = 4(z_1 - 1)^3.$$

Hence,

$$m_1(R_{z_2}[P_1, P_1]) = 2 \quad \text{and} \quad m_1(R_{z_2}[Q_1, P_1]) = 3.$$

By Theorem 1, $G_1^{-1}(z_1, z_2)$ is BIBO stable. Therefore, it is concluded that both $G_1(z_1, z_2)$ and $G_1^{-1}(z_1, z_2)$ are BIBO stable.

Remark 1: It is interesting to notice that the inverses of all the BIBO stable 2-D transfer functions having second kind singularities on T^2 discussed in the examples of [4], [7], [12]–[14] are either quarter-plane noncausal, or BIBO unstable. In fact, most of the above-mentioned 2-D functions have separable numerators, and consequently have an infinite number of zeros on T^2.

A natural question arises at this point: Does there exist other BIBO stable 2-D transfer functions having second kind singularities on T^2 whose inverses are also BIBO stable? This question is important since it is sometimes desirable to obtain BIBO stable 2-D transfer functions with stable inverses (the so-called minimum phase stable 2-D transfer function) [9]. In the remainder of this section, we provide an affirmative answer to this question by deriving a necessary and sufficient condition for the class of 2-D real rational functions of the form

$$G_2(z_1, z_2) = k\frac{1 + a_1 z_1 + e_1 z_1^2 + b_1 z_2 + c_1 z_1 z_2 + d_1 z_1^2 z_2}{1 - a_2 z_1 - b_2 z_2} = k\frac{P_2(z_1, z_2)}{Q_2(z_1, z_2)} \tag{5}$$

to be BIBO stable as well as inverse stable. It is assumed that P_2 and Q_2 are relatively prime, and that $a_2 > 0$, $b_2 > 0$, $a_2 + b_2 = 1$, $k \neq 0$.

Following Roytman *et al.* [12], a necessary and sufficient condition for $G_2(z_1, z_2)$ to be BIBO stable is obtained in the following lemma.

Lemma 1: $G_2(z_1, z_2)$ in (5) is BIBO stable if and only if $d_1 \neq 0$ and

$$1 + a_1 + e_1 + b_1 + c_1 + d_1 = 0 \tag{6a}$$

$$\frac{d_1 - a_2 c_1 + b_2 e_1}{a_2 d_1} = 3 \tag{6b}$$

$$\frac{a_2 b_1 - b_2 a_1 - c_1}{a_2 d_1} = 3 \tag{6c}$$

$$\frac{b_2 + b_1}{a_2 d_1} = 1. \tag{6d}$$

Proof: Since $a_2 + b_2 = 1$ by assumption, $Q_2(1,1) = 0$. A necessary condition for $G_2(z_1, z_2)$ to be BIBO stable is that $P_2(1,1) = 0$, i.e., (6a) must hold. Suppose in the following that (6a) is satisfied.

It is easily seen that $(1,1)$ is the only simple second kind singularity of $G_2(z_1, z_2)$ in $\overline{U}^2$, and that $G_2(z_1, z_2)$ has no simple or multiple second kind singularities of the form $(1, \gamma)$ outside $\overline{U}^2$. Thus by Theorem 1, $G_2(z_1, z_2)$ is BIBO stable if and only if [12]

$$m_1(R_{z_2}[Q_2, Q_2]) < m_1(R_{z_2}[P_2, Q_2]). \tag{7}$$

Direct calculation gives

$$R_{z_2}[Q_2, Q_2] = \begin{vmatrix} z_1 - a_2 & -b_2 z_1 \\ -b_2 & 1 - a_2 z_1 \end{vmatrix} = -a_2(z_1 - 1)^2$$

$$R_{z_2}[P_2, Q_2] = \begin{vmatrix} b_1 + c_1 z_1 + d_1 z_1^2 & 1 + a_1 z_1 + e_1 z_1^2 \\ -b_2 & 1 - a_2 z_1 \end{vmatrix}$$
$$= -a_2 d_1 z_1^3 + (d_1 - a_2 c_1 + b_2 e_1) z_1^2 - (a_2 b_1 - b_2 a_1 - c_1) z_1 + (b_2 + b_1). \tag{8a}$$

Since $m_1(R_{z_2}[Q_2, Q_2]) = 2$, a necessary and sufficient condition for (7) to hold is that $d_1 \neq 0$ and

$$R_{z_2}[P_2, Q_2] = x_1(z_1 - 1)^3 \tag{8b}$$

for some nonzero constant x_1. Equations (6b)–(d) now follow as a consequence of comparing the coefficients of (8a) and (8b), thereby proving Lemma 1.

Assuming that $G_2(z_1, z_2)$ is BIBO stable, i.e. (6a–d) are satisfied, let us now consider the BIBO stability of its inverse, namely

$$G_2^{-1}(z_1, z_2) = \frac{1}{k}\frac{Q_2(z_1, z_2)}{P_2(z_1, z_2)}. \tag{9}$$

Since $Q_2(z_1, z_2) \neq 0$ in the region $\hat{U}^2 \triangleq \{\overline{U}^2 - (1,1)\}$, it is necessary for BIBO stability of $G_2^{-1}(z_1, z_2)$ that $P_2(z_1, z_2) \neq 0$ in $\hat{U}^2$.

Lemma 2: $P_2(z_1, z_2) \neq 0$ in $\hat{U}^2$ if and only if

$$|e_1| < 1 \tag{10a}$$

$$|a_1| - e_1 < 1 \tag{10b}$$

$$(1 - a_1 + e_1)^2 - (b_1 + d_1 - c_1)^2 > 0 \tag{10c}$$

$$1 + a_1^2 + e_1^2 - b_1^2 - c_1^2 - d_1^2 - 6e_1 + 6b_1 d_1 \geqslant 0. \tag{10d}$$

Proof: That $P_2(z_1, z_2) \neq 0$ in $\hat{U}^2$ implies $P_1(z_1, 0) \neq 0$ for $|z_1| \leqslant 1$, which is equivalent to the conditions (10a) and (10b) [16]. We assume hereafter that (10a) and (10b) are already satisfied.

By Lemma 1, the assumption that $G_2(z_1, z_2)$ is BIBO stable implies

$$b_1 + c_1 + d_1 = -(1 + a_1 + e_1) \neq 0 \tag{11}$$

since $P_2(1,0) = 1 + a_1 + e_1 \neq 0$, as implied by (10a) and (10b) [16]. Thus

$$P_2(1, z_2) = (1 + a_1 + e_1)(1 - z_2) \neq 0 \quad \forall\, z_2 \neq 1. \tag{12}$$

Hence, the condition $P_2(z_1, z_2) \neq 0$ in $\hat{U}^2$ is equivalent to

$$P_2(z_1, z_2) \neq 0 \qquad \forall\, (z_1, z_2) \in \overline{U}^2;\ z_1 \neq 1. \tag{13}$$

Rewriting $P_2(z_1, z_2)$ as

$$P_2(z_1, z_2) = (1 + a_1 z_1 + e_1 z_1^2)[1 + f_2(z_1) z_2] \quad (14)$$

where $f_2(z_1) \triangleq (b_1 + c_1 z_1 + d_1 z_1^2)/(1 + a_1 z_1 + e_1 z_1^2)$, (13) is equivalent to

$$|f_2(z_1)| < 1 \qquad \forall\, |z_1| \leqslant 1,\ z_1 \neq 1. \quad (15)$$

Recalling (11) gives $|f_2(1)| = 1$. By the well known Maximum Modulus Principle (see, e.g., [17, p. 229]), (15) holds if and only if

$$|f_2(z_1)| < 1 \qquad \forall\, |z_1| = 1;\ z_1 \neq 1. \quad (16)$$

By means of the bilinear transformation $z_1 = (s_1 - 1)/(s_1 + 1)$, (16) is equivalent to

$$\left| f_2\left(\frac{j\omega_1 - 1}{j\omega_1 + 1} \right) \right|^2 < 1, \qquad -\infty < \omega_1 < +\infty. \quad (17)$$

Direct calculation gives

$$\left| f_2\left(\frac{j\omega_1 - 1}{j\omega_1 + 1} \right) \right|^2 = \frac{[(b_1 + c_1 + d_1)\omega_1^2 + (c_1 - b_1 - d_1)]^2 + 4(b_1 - d_1)^2 \omega_1^2}{[(1 + a_1 + e_1)\omega_1^2 + (a_1 - 1 - e_1)]^2 + 4(1 - e_1)^2 \omega_1^2}.$$

Recalling (11) and simplifying yields

$$\left| f_2\left(\frac{j\omega_1 - 1}{j\omega_1 + 1} \right) \right|^2 = 1 - \frac{2[1 + a_1^2 + e_1^2 - b_1^2 - c_1^2 - d_1^2 - 6e_1 + 6b_1 d_1]}{[(1 + a_1 + e_1)\omega_1^2 + (a_1 - 1 - e_1)]^2 + 4(1 - e_1)^2 \omega_1^2} \omega_1^2 - \frac{(1 - a_1 + e_1)^2 - (b_1 + d_1 - c_1)^2}{[(1 + a_1 + e_1)\omega_1^2 + (a_1 - 1 - e_1)]^2 + 4(1 - e_1)^2 \omega_1^2}.$$

It is now easy to see that (17) holds if and only if (10c) and (10d) are satisfied, thereby proving Lemma 2.

Remark 2: Several methods for testing a 2-D polynomial $P(z_1, z_2) \neq 0$ in $\overline{U}^2$ are available in the literature (see, e.g., [5], [16]). However, these tests may not be applicable to the case where $P(z_1, z_2)$ has some zeros on T^2, as discussed in the above lemma.

We are now in a position to state the main result of this section.

Theorem 2: Let

$$G_2(z_1, z_2) = k \frac{P_2(z_1, z_2)}{Q_2(z_1, z_2)}$$

where $P_2(z_1, z_2)$ and $Q_2(z_1, z_2)$ are defined as in (5). Then, $G_2(z_1, z_2)$ and $G_2^{-1}(z_1, z_2)$ are both BIBO stable if and only if a_1, b_1, c_1, d_1, and e_1 satisfy

$$a_1 = -\frac{1}{a_2}[a_2^2 + (1 - a_2)^2 d_1 + e_1] \quad (18a)$$

$$b_1 = a_2 d_1 + (a_2 - 1) \quad (18b)$$

$$c_1 = \frac{1}{a_2}[(1 - 3a_2) d_1 + (1 - a_2) e_1] \quad (18c)$$

where

$$\begin{cases} 2a_2 - 1 < e_1 < 1, & \text{if } 0 < a_2 < \frac{1}{3} \\ -a_2 < e_1 < 1, & \text{if } \frac{1}{3} \leqslant a_2 < 1 \end{cases} \quad (18d)$$

and

$$\begin{cases} \frac{a_2 + e_1}{3a_2 - 1} < d_1 < \frac{a_2 - e_1}{1 - a_2}, & \text{if } 2a_2 - 1 < e_1 < -a_2 \\ -\frac{a_2 + e_1}{1 + a_2} < d_1 < \frac{a_2 - e_1}{1 - a_2}, & \text{if } e_1 \geqslant -a_2 \text{ and } e_1 > 2a_2 - 1 \\ -\frac{a_2 + e_1}{1 + a_2} < d_1 < \frac{a_2 + e_1}{3a_2 - 1}, & \text{if } -a_2 < e_1 \leqslant 2a_2 - 1 \end{cases} \quad (18e)$$

with the constraint that $d_1 \neq 0$.

Proof: (Necessity): It is shown in the Appendix that a_1, b_1, c_1, d_1, and e_1 satisfy (6a)–(6d) and (10a)–(10d) if and only if they satisfy (18a)–(18e). It follows from Lemmas 1 and 2 that for both $G_2(z_1, z_2)$ and $G_2^{-1}(z_1, z_2)$ to be BIBO stable, it is necessary that a_1, b_1, c_1, d_1, and e_1 satisfy condition (18a)–(18e).

(Sufficiency): Suppose that a_1, b_1, c_1, d_1 and e_1 satisfy (18a)–(18e). Then, (6a)–(6d) and (10a)–(10d) hold (see the Appendix). By Lemma 1, $G_2(z_1, z_2)$ is BIBO stable. It remains to show that $G_2^{-1}(z_1, z_2)$ is also BIBO stable. By Lemma 2, $P_2(z_1, z_2) \neq 0$ in $\hat{U}^2$, which implies that $G_2^{-1}(z_1, z_2)$ has no poles in $\overline{U}^2$, nor second kind singularities in $\overline{U}^2$ except for one at (1,1). We also notice from (12) that $P_2(1, z_2) \neq 0\ \forall\, z_2 \neq 1$, i.e., $G_2^{-1}(z_1, z_2)$ has no second kind singularities of the form $(1, \gamma)$ outside $\overline{U}^2$. Therefore, to show that $G_2^{-1}(z_1, z_2)$ is BIBO stable, it suffices to show that (1,1) is a simple second kind singularity of $G^{-1}(z_1, z_2)$, and that condition (2) in Theorem 1 is satisfied. This will be done in two steps.

Step 1:

$$\begin{aligned} \left.\frac{\partial}{\partial z_1} P_2(z_1, z_2)\right|_{(1,1)} &= a_1 + c_1 + 2e_1 + 2d_1 \\ &= -\frac{1}{a_2}[a_2^2 + (a_2 - 1)^2 d_1 + e_1] + \frac{1}{a_2}[(1 - 3a_2) d_1 + (1 - a_2) e_1] + 2e_1 + 2d_1 \\ &= -[(a_2 - 1) d_1 + (a_2 - e_1)] \\ &\neq 0 \end{aligned}$$

since $d_1 < (a_2 - e_1)/(1 - a_2)$ (see (A13)):

$$\left.\frac{\partial}{\partial z_2} P_2(z_1, z_2)\right|_{(1,1)} = b_1 + c_1 + d_1 = -(1 + a_1 + e_1) \neq 0$$

(from (11)).

Thus according to [12], (1,1) is a simple second kind

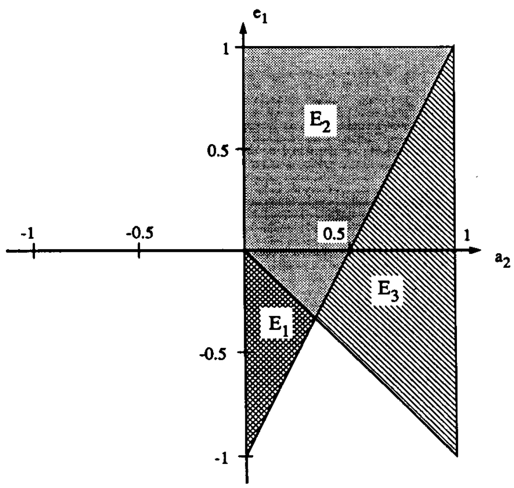

Fig. 1. The domain in the $a_2 - e_1$ plane for (18e).

singularity of $G_2^{-1}(z_1, z_2)$, and, therefore, Theorem 1 can be applied to check the BIBO stability of $G_2^{-1}(z_1, z_2)$.

Step 2: Since $R_{z_2}[Q_2, P_2] = -R_{z_2}[P_2, Q_2]$, by Lemma 1

$$m_1(R_{z_2}[Q_2, P_2]) = m_1(R_{z_2}[P_2, Q_2]) = 3.$$

Let us compute the z_2-resultant of $\boldsymbol{P}_2$ and P_2

$$R_{z_2}[\boldsymbol{P}_2, P_2] = \begin{vmatrix} z_1^2 + a_1 z_1 + e_1 & b_1 z_1^2 + c_1 z_1 + d_1 \\ b_1 + c_1 z_1 + d_1 z_1^2 & 1 + a_1 z_1 + e_1 z_1^2 \end{vmatrix}$$
$$= (e_1 - b_1 d_1) z_1^4 + (a_1 + a_1 e_1 - b_1 c_1 - c_1 d_1) z_1^3 + (1 + a_1^2 + e_1^2 - b_1^2 - c_1^2 - d_1^2) z_1^2 + (a_1 + a_1 e_1 - b_1 c_1 - c_1 d_1) z_1 + (e_1 - b_1 d_1). \quad (19)$$

To show that $G_2^{-1}(z_1, z_2)$ is BIBO stable, it suffices to show that $m_1(R_{z_2}[\boldsymbol{P}_2, P_2]) < 3$, which is ensured if

$$\frac{\partial^2}{\partial z_1^2}(R_{z_2}[\boldsymbol{P}_2, P_2])\bigg|_{z_1=1} = 2\big[6(e_1 - b_1 d_1) + 3(a_1 + a_1 e_1 - b_1 c_1 - c_1 d_1) + (1 + a_1^2 + e_1^2 - b_1^2 - c_1^2 - d_1^2)\big] \quad (20)$$

is not equal to zero.

Substituting (A15) in the Appendix into (20) leads to

$$\frac{\partial^2}{\partial z_1^2}(R_{z_2}[\boldsymbol{P}_2, P_2])\bigg|_{z_1=1}$$
$$= 2\big[6(e_1 - b_1 d_1) + 3(a_1 + a_1 e_1 - b_1 c_1 - c_1 d_1) + 2(b_1 c_1 + b_1 d_1 + c_1 d_1 - a_1 - e_1 - a_1 e_1)\big]$$
$$= -2\big[(b_1 + d_1) c_1 - (1 + e_1) a_1 + 4(b_1 d_1 - e_1)\big]$$
$$= -\frac{2}{a_2}\big[(a_2 - 1) d_1 + (a_2 - e_1)\big]^2 \quad \text{(from (A29))}$$
$$\neq 0$$

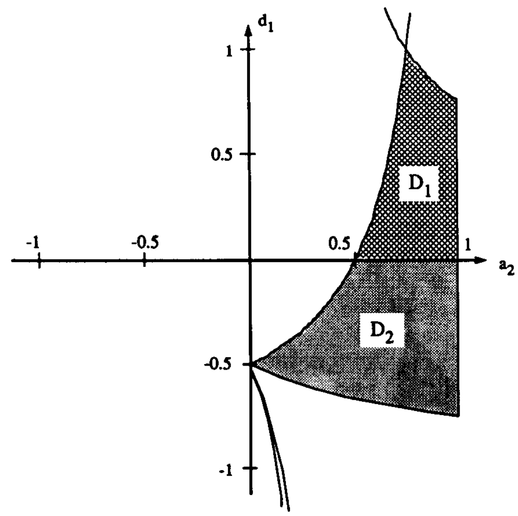

Fig. 2. The admissible regions in the $a_2 - d_1$ plane given by (18e) for $e_1 = 0.5$.

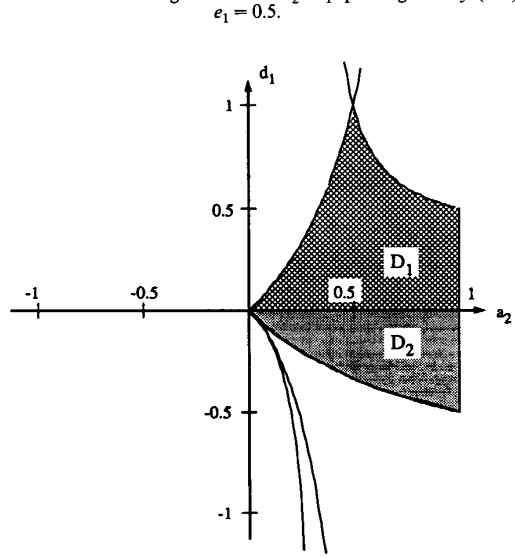

Fig. 3. The admissible regions in the $a_2 - d_1$ plane given by (18e) for $e_1 = 0$.

since $d_1 < (a_2 - e_1)/(1 - a_2)$ (see (A13)). Therefore,

$$m_1(R_{z_2}[\boldsymbol{P}_2, P_2]) < 3 = m_1(R_{z_2}[Q_2, P_2]).$$

By Theorem 1, $G_2^{-1}(z_1, z_2)$ is BIBO stable, and the proof is completed.

Remark 3: It can be seen from Lemma 1 that for $G_2(z_1, z_2)$ to be BIBO stable, the absolute values of a_1, b_1, c_1, d_1, and e_1 are not necessarily bounded. However, it can be shown from Theorem 2 that for both $G_2(z_1, z_2)$ and $G_2^{-1}(z_1, z_2)$ to be BIBO stable, the absolute values of b_1, d_1, and e_1 are bounded by 1, while those of a_1 and c_1 by 2. The derivation is straightforward but rather tedious, and is omitted here.

The domain in the $a_2 - e_1$ plane for (18e) is sketched in Fig. 1, in which (18d) is given by the union of the regions E_1, E_2 and E_3. Equation (18e) is depicted in Figs. 2–4 for $e_1 = 0.5$, 0, and -0.5, respectively; and in Figs. 5 and 6 for

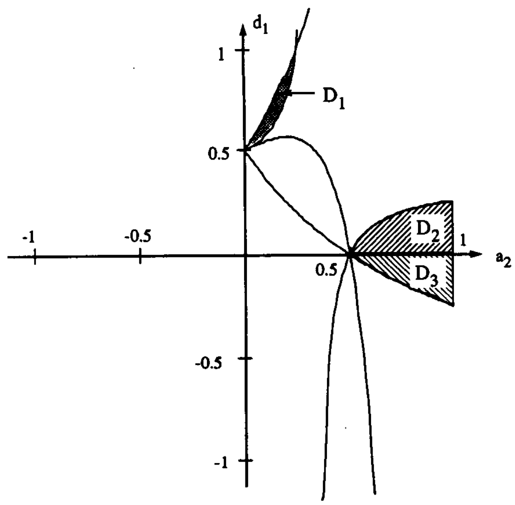

Fig. 4. The admissible regions in the a_2-d_1 plane given by (18e) for $e_1=-0.5$.

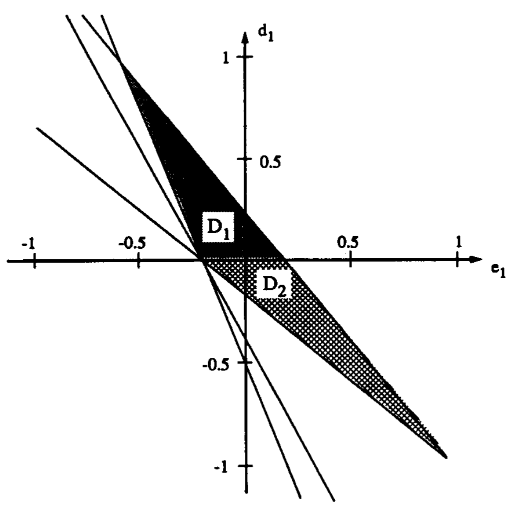

Fig. 5. The admissible regions in the e_1-d_1 plane given by (18e) for $a_2=0.2$.

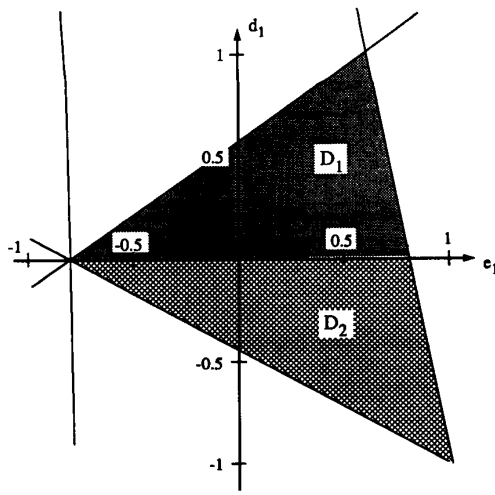

Fig. 6. The admissible regions in the e_1-d_1 plane given by (18e) for $a_2=0.8$.

$a_2=0.2$ and 0.8, respectively. It is interesting to see from Figs. 1 and 4 that for a fixed e_1 such that $-1<e_1<-1/3$, the regions in the a_2-d_1 plane such that both $G_2(z_1,z_2)$ and $G_2^{-1}(z_1,z_2)$ are BIBO stable are disjointed.

Before closing this section, let us illustrate Theorem 2 by reconsidering Example 1. Rewrite $G_1(z_1,z_2)$ as

$$G_1(z_1,z_2)=\frac{4}{3}\frac{1-\frac{3}{4}z_1-\frac{3}{4}z_1z_2+\frac{1}{2}z_1^2z_2}{1-\frac{2}{3}z_1-\frac{1}{3}z_2}$$

which is of the same form as in (5). Since the point $(a_2,e_1)=(2/3,0)$ is in E_2 of Fig. 1, (18d) holds. Observing Fig. 3, we see that the point $(2/3,1/2)$ lies inside D_1, i.e., (18e) is satisfied. Straightforward calculation shows that $a_1=-3/4$, $b_1=0$ and $c_1=-3/4$ satisfy (18a)–(18c) for $a_2=2/3$, $d_1=1/2$ and $e_1=0$. Therefore, by Theorem 2, $G_1(z_1,z_2)$ and $G_1^{-1}(z_1,z_2)$ are both BIBO stable, as has been established previously.

IV. Characterization of a Class of BIBO Stable 2-D Functions having No Stable Inverses

It has been shown in the previous section that some BIBO stable 2-D transfer functions having simple second kind singularities on T^2 can admit BIBO stable inverses. In this section, we concentrate on a special class of 2-D real transfer functions, and show that a BIBO stable function in this class cannot have a BIBO stable inverse. The following theorem contains the main result of this section.

Theorem 3: Let

$$G_3(z_1,z_2)=k\frac{[P_3(z_1,z_2)]^r}{[Q_3(z_1,z_2)]^t}=k\frac{(1+a_1z_1+b_1z_2+c_1z_1z_2)^r}{(1+a_2z_1+b_2z_2+c_2z_1z_2)^t} \quad (21)$$

where P_3 and Q_3 are relatively prime, $k\neq 0$ and r,t are positive integers. Assume that $G_3(z_1,z_2)$ has some second kind singularities on T^2 and is BIBO stable. Then, $G_3^{-1}(z_1,z_2)$ cannot be BIBO stable.

The proof of the above theorem is essentially based on Lemmas 3–5 given as follows.

Lemma 3: $G_3(z_1,z_2)$ as defined in Theorem 3 has a unique second kind singularity in $\overline{U}^2$ at $(\pm 1,\pm 1)$.

Proof: Rewrite $Q_3(z_1,z_2)$ in (21) as

$$Q_3(z_1,z_2)=(1+a_2z_1)[1+f_3(z_1)z_2] \quad (22)$$

where $f_3(z_1)\triangleq(b_2+c_2z_1)/(1+a_2z_1)$. The assumption that $G_3(z_1,z_2)$ is BIBO stable implies that $|a_2|<1$ and $Q_3(z_1,z_2)\neq 0$ in the region $\{\overline{U}^2-T^2\}$, which in turn implies

$$|f_3(z_1)| \leqslant 1 \qquad \forall\ |z_1| = 1. \tag{23}$$

By means of the bilinear transformation $z_1 = (s_1 - 1)/(s_1 + 1)$, (23) is equivalent to

$$\left| f_3\left(\frac{j\omega_1 - 1}{j\omega_1 + 1} \right) \right|^2 \leqslant 1, \qquad -\infty \leqslant \omega_1 \leqslant +\infty. \tag{24}$$

Direct calculation gives

$$\left| f_3\left(\frac{j\omega_1 - 1}{j\omega_1 + 1} \right) \right|^2 = \frac{(b_2 + c_2)^2 \omega_1^2 + (b_2 - c_2)^2}{(1 + a_2)^2 \omega_1^2 + (1 - a_2)^2}. \tag{25}$$

There are four possible cases where (24) can be satisfied.

Case 1: $|b_2 - c_2| < |1 - a_2|$ and $|b_2 + c_2| < |1 + a_2|$.

From (25), we have

$$\left| f_3\left(\frac{j\omega_1 - 1}{j\omega_1 + 1} \right) \right|^2 < 1, \qquad -\infty \leqslant \omega_1 \leqslant +\infty.$$

Hence,

$$|f_3(z_1)| < 1 \qquad \forall\ |z_1| = 1. \tag{26}$$

By the maximum modulus principle [17], (26) implies that

$$|f_3(z_1)| < 1 \qquad \forall\ |z_1| \leqslant 1. \tag{27}$$

It follows from (22) that $Q_3(z_1, z_2) \neq 0$ in $\bar{U}^2$, contradicting the assumption that $G_3(z_1, z_2)$ has some second kind singularities on T^2.

Case 2: $|b_2 - c_2| = |1 - a_2|$ and $|b_2 + c_2| = |1 + a_2|$.

It follows from (25) that

$$\left| f_3\left(\frac{j\omega_1 - 1}{j\omega_1 + 1} \right) \right|^2 = 1, \qquad -\infty \leqslant \omega_1 \leqslant +\infty.$$

Thus

$$|f_3(z_1)| = 1 \qquad \forall\ |z_1| = 1. \tag{28}$$

However, this implies from (22) that $Q_3(z_1, z_2)$ has an infinite number of zeros on T^2, contradicting the assumption that $G_3(z_1, z_2)$ is BIBO stable.

Case 3: $|b_2 - c_2| < |1 - a_2|$ and $|b_2 + c_2| = |1 + a_2|$.

Then,

$$\lim_{\omega_1 \to \infty} \left| f_3\left(\frac{j\omega_1 - 1}{j\omega_1 + 1} \right) \right|^2 = 1 \tag{29a}$$

and

$$\left| f_3\left(\frac{j\omega_1 - 1}{j\omega_1 + 1} \right) \right|^2 < 1, \qquad -\infty < \omega_1 < +\infty. \tag{29b}$$

That is,

$$|f_3(z_1)| = 1, \qquad z_1 = 1 \tag{30a}$$

and

$$|f_3(z_1)| < 1 \qquad \forall\ |z_1| = 1,\ z_1 \neq 1. \tag{30b}$$

Applying the maximum modulus principle to (30) yields

$$|f_3(z_1)| < 1 \qquad \forall\ |z_1| \leqslant 1,\ z_1 \neq 1. \tag{31}$$

It follows from (22) that

$$Q_3(z_1, z_2) \neq 0 \qquad \forall\ (z_1, z_2) \in \bar{U}^2, \qquad z_1 \neq 1. \tag{32}$$

On the other hand, taking into account that $G_3(z_1, z_2)$ has real coefficients, we have from (30a) that $f_3(1) = \pm 1$. Hence,

$$Q_3(1, z_2) = (1 + a_2)[1 + f_3(1) z_2] \neq 0 \qquad \forall\ z_2 \neq -f_3(1). \tag{33}$$

Combining (32) and (33) yields

$$\begin{cases} Q_3(z_1, z_2) = 0, & \text{if } z_1 = 1,\ z_2 = -f_3(1) = \pm 1; \\ Q_3(z_1, z_2) \neq 0, & \text{elsewhere in } \bar{U}^2. \end{cases}$$

Case 4: $|b_2 - c_2| = |1 - a_2|$ and $|b_2 + c_2| < |1 + a_2|$.

Similarly as in Case 3, it can be shown that

$$\begin{cases} Q_3(z_1, z_2) = 0, & \text{if } z_1 = -1,\ z_2 = -f_3(-1) = \pm 1; \\ Q_3(z_1, z_2) \neq 0, & \text{elsewhere in } \bar{U}^2. \end{cases}$$

It follows from the above discussion that $Q_3(z_1, z_2)$ has a unique zero in $\bar{U}^2$ at $(\pm 1, \pm 1)$, which is the only second kind singularity of $G_3(z_1, z_2)$ in $\bar{U}^2$.

Because of Lemma 3, it may be assumed in the rest of this section that $G_3(z_1, z_2)$ in (21) has $(1,1)$ as its only second kind singularity on T^2.

Lemma 4: Let

$$G_3(z_1, z_2) = k \frac{[P_3(z_1, z_2)]^r}{[Q_3(z_1, z_2)]^t}$$

where P_3 and Q_3 are defined as in Theorem 3. Assume that $(1,1)$ is the only second kind singularity of $G_3(z_1, z_2)$ on T^2, and G_3 is BIBO stable. Then, $G_3(z_1, z_2)$ has no simple or multiple second kind singularities of the form $(1, \gamma)$ outside $\bar{U}^2$, and $(1,1)$ is a simple second kind singularity of the function $[P_3(z_1, z_2)]^r/[Q_3(z_1, z_2)]$.

Proof: The assumption that $(1,1)$ is the second kind singularity of $G_3(z_1, z_2)$ implies $Q_3(1,1) = 1 + a_2 + b_2 + c_2 = 0$, from which it follows

$$a_2 + c_2 = -(1 + b_2) \neq 0 \tag{34}$$

and

$$b_2 + c_2 = -(1 + a_2) \neq 0 \tag{35}$$

since $|a_2| < 1$, $|b_2| < 1$ for stability of $G_3(z_1, z_2)$. Thus

$$Q_3(1, z_2) = (1 + a_2)(1 - z_2) \neq 0 \qquad \forall\ z_2 \neq 1.$$

Therefore, $G_3(z_1, z_2)$ has no second kind singularities of the form $(1, \gamma)$ outside $\bar{U}^2$.

Direct computation leads to

$$\left. \frac{\partial}{\partial z_1} Q_3(z_1, z_2) \right|_{(1,1)} = a_2 + c_2 \neq 0 \text{ (from (34))}$$

and

$$\left. \frac{\partial}{\partial z_2} Q_3(z_1, z_2) \right|_{(1,1)} = b_2 + c_2 \neq 0 \text{ (from (35))}.$$

Thus according to [12], $(1,1)$ is a simple second kind singularity of the function $[P_3(z_1, z_2)]^r/[Q_3(z_1, z_2)]$, thereby proving Lemma 4.

Lemma 5: Let $G_3(z_1, z_2) = k[P_3(z_1, z_2)]^r/[Q_3(z_1, z_2)]^t$, where P_3 and Q_3 are defined as in Theorem 3. Assume that (1,1) is the only second kind singularity of $G_3(z_1, z_2)$ on T^2, and G_3 is BIBO stable. Then, $r > t$.

Proof:

$$Q_3(z_1, z_2) = z_1 z_2 + a_2 z_2 + b_2 z_1 + c_2,$$

$$R_{z_2}[P_3, Q_3] = \begin{vmatrix} b_1 + c_1 z_1 & 1 + a_1 z_1 \\ b_2 + c_2 z_1 & 1 + a_2 z_1 \end{vmatrix} = (c_1 a_2 - c_2 a_1) z_1^2 + (a_2 b_1 + c_1 - a_1 b_2 - c_2) z_1 + b_1 - b_2 \quad (36)$$

and

$$R_{z_2}[Q_3, Q_3] = \begin{vmatrix} z_1 + a_2 & b_2 z_1 + c_2 \\ b_2 + c_2 z_1 & 1 + a_2 z_1 \end{vmatrix} = (a_2 - b_2 c_2) z_1^2 + (1 + a_2^2 - b_2^2 - c_2^2) z_1 + (a_2 - b_2 c_2). \quad (37)$$

The assumption that P_3 and Q_3 are relatively prime implies that $R_{z_2}[P_3, Q_3] \not\equiv 0$. Hence,

$$m_1(R_{z_2}[P_3, Q_3]) \leqslant 2 \quad (38)$$

since the degree of $R_{z_2}[P_3, Q_3]$ in z_1 is less or equal to two.

Next, the fact that (1,1) is a common zero of Q_3 and Q_3 implies

$$R_{z_2}[Q_3, Q_3] = (z_1 - 1)(x_1 z_1 - x_2) \quad (39)$$

for some real numbers x_1 and x_2. Comparing the coefficients of (37) and (39) gives $x_1 = x_2 = (a_2 - b_2 c_2)$, or $R_{z_2}[Q_3, Q_3] = (a_2 - b_2 c_2)(z_1 - 1)^2$. The assumption that $G_3(z_1, z_2)$ is BIBO stable requires that Q_3 and Q_3 are relatively prime [12], which implies $(a_2 - b_2 c_2) \neq 0$. Therefore,

$$m_1(R_{z_2}[Q_3, Q_3]) = 2. \quad (40)$$

Since we have shown in Lemma 4 that $G_3(z_1, z_2)$ has no second kind singularities of the form $(1, \gamma)$ outside $\bar{U}^2$, and that (1,1) is a simple second kind singularity of the function $[P_3(z_1, z_2)]^r/[Q_3(z_1, z_2)]$, according to Theorem 1, $G_3(z_1, z_2)$ is BIBO stable if and only if

$$t m_1(R_{z_2}[Q_3, Q_3]) < m_1(R_{z_2}[P_3^r, Q_3])$$

or

$$t m_1(R_{z_2}[Q_3, Q_3]) < r m_1(R_{z_2}[P_3, Q_3]).$$

From (38) and (40), this is possible only if $r > t$, and the proof is completed.

As a consequence of Lemma 5, a proof of Theorem 3 is now sketched. The inverse of $G_3(z_1, z_2)$ in (21) is given by

$$G_3^{-1}(z_1, z_2) = \frac{1}{k} \frac{[Q_3(z_1, z_2)]^t}{[P_3(z_1, z_2)]^r} = \frac{1}{k} \frac{(1 + a_2 z_1 + b_2 z_2 + c_2 z_1 z_2)^t}{(1 + a_1 z_1 + b_1 z_2 + c_1 z_1 z_2)^r}$$

which is of the same form as $G_3(z_1, z_2)$. By Lemma 5, the assumption that $G_3(z_1, z_2)$ is BIBO stable implies $r > t$. Now applying Lemma 5 to $G_3^{-1}(z_1, z_2)$, we conclude that $G_3^{-1}(z_1, z_2)$ cannot be BIBO stable.

V. Conclusions

In this paper, we have studied the open problem regarding the BIBO stability of inverse 2-D digital filters in the presence of nonessential singularities of the second kind on T^2. Using a recently proposed method [12] for testing the BIBO stability of 2-D functions having simple second kind singularities on T^2, we have constructed BIBO stable 2-D functions having such singularities that are also inverse BIBO stable, thereby disproving a conjecture posed by Bose [9, p. 245]. A necessary and sufficient condition has been derived for the class of 2-D functions of the form

$$G(z_1, z_2) = k \frac{1 + a_1 z_1 + e_1 z_1^2 + b_1 z_2 + c_1 z_1 z_2 + d_1 z_1^2 z_2}{1 - a_2 z_1 - b_2 z_2}$$

to be BIBO stable as well as inverse stable, where $a_2 > 0$, $b_2 > 0$ and $a_2 + b_2 = 1$. It is believed that this result can be useful in the investigation of minimum phase stable 2-D transfer functions [9], as well as in the synthesis of stable inverse 2-D digital filters.

We have also shown that the class of BIBO stable 2-D real transfer functions of the form

$$G(z_1, z_2) = k \frac{(1 + a_1 z_1 + b_1 z_2 + c_1 z_1 z_2)^r}{(1 + a_2 z_1 + b_2 z_2 + c_2 z_1 z_2)^t}$$

having nonessential singularities of the second kind on T^2 cannot admit BIBO stable inverses.

Appendix
Derivation of (18a)–(18e)

In this appendix, we derive all the solutions of a_1, b_1, c_1, d_1, and e_1 that satisfy the following conditions:

$$1 + a_1 + b_1 + c_1 + d_1 + e_1 = 0 \quad \text{(A1)}$$

$$\frac{d_1 - a_2 c_1 + b_2 e_1}{a_2 d_1} = 3 \quad \text{(A2)}$$

$$\frac{a_2 b_1 - b_2 a_1 - c_1}{a_2 d_1} = 3 \quad \text{(A3)}$$

$$\frac{b_2 + b_1}{a_2 d_1} = 1 \quad \text{(A4)}$$

$$|e_1| < 1 \quad \text{(A5)}$$

$$|a_1| - e_1 < 1 \quad \text{(A6)}$$

$$(1 - a_1 + e_1)^2 - (b_1 + d_1 - c_1)^2 > 0 \quad \text{(A7)}$$

$$1 + a_1^2 + e_1^2 - b_1^2 - c_1^2 - d_1^2 - 6e_1 + 6b_1 d_1 \geqslant 0 \quad \text{(A8)}$$

where $a_2 > 0$, $b_2 > 0$, $a_2 + b_2 = 1$, and $d_1 \neq 0$.

Substituting $b_2=1-a_2$ into (A1–4) and simplifying gives

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -a_2 & 1-3a_2 & 1-a_2 \\ a_2-1 & a_2 & -1 & -3a_2 & 0 \\ 0 & 1 & 0 & -a_2 & 0 \end{bmatrix}\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ e_1 \end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ 0 \\ a_2-1 \end{bmatrix}. \tag{A9}$$

Using standard algebraical techniques, the solutions to (A9) can be obtained as follows:

$$a_1=-\frac{1}{a_2}\left[a_2^2+(1-a_2)^2d_1+e_1\right] \tag{A10}$$

$$b_1=a_2d_1+(a_2-1) \tag{A11}$$

$$c_1=\frac{1}{a_2}\left[(1-3a_2)d_1+(1-a_2)e_1\right]. \tag{A12}$$

The ranges of d_1 and e_1 will be determined such that (A5)–(A8) are satisfied.

It can be seen from (A10) that condition (A6) is equivalent to

$$\left|a_2^2+(1-a_2)^2d_1+e_1\right|<(1+e_1)a_2$$

or

$$-\frac{(a_2+e_1)(1+a_2)}{(1-a_2)^2}<d_1<\frac{a_2-e_1}{1-a_2}. \tag{A13}$$

Letting $F_1\triangleq(1-a_1+e_1)^2-(b_1+d_1-c_1)^2$, (A7) is equivalent to

$$F_1>0. \tag{A14}$$

From (A1), we have

$$(b_1+c_1+d_1)^2=(1+a_1+e_1)^2$$

or

$$\begin{aligned}1+a_1^2+e_1^2-b_1^2-c_1^2-d_1^2\\ =2(b_1c_1+b_1d_1+c_1d_1-a_1-e_1-a_1e_1).\end{aligned} \tag{A15}$$

Thus

$$\begin{aligned}F_1&=\left(1+a_1^2+e_1^2+2e_1-2a_1-2a_1e_1\right)\\ &\quad-\left(b_1^2+d_1^2+c_1^2+2b_1d_1-2b_1c_1-2c_1d_1\right)\\ &=1+a_1^2+e_1^2-b_1^2-c_1^2-d_1^2\\ &\quad+2(b_1c_1-b_1d_1+c_1d_1-a_1+e_1-a_1e_1)\\ &=2(b_1c_1+b_1d_1+c_1d_1-a_1-e_1-a_1e_1)\\ &\quad+2(b_1c_1-b_1d_1+c_1d_1-a_1+e_1-a_1e_1)\\ &=4\left[(b_1+d_1)c_1-(1+e_1)a_1\right].\end{aligned} \tag{A16}$$

Substituting (A10)–(A12) into (A16) and simplifying yields

$$\begin{aligned}(b_1+d_1)c_1-(1+e_1)a_1\\ =\frac{1}{a_2}\{(a_2+1)(1-3a_2)d_1^2\\ +2(a_2+e_1)(1-a_2)d_1+(a_2+e_1)^2\}.\end{aligned} \tag{A17}$$

Hence,

$$\begin{aligned}F_1=\frac{4}{a_2}\{(a_2+1)(1-3a_2)d_1^2\\ +2(a_2+e_1)(1-a_2)d_1+(a_2+e_1)^2\}.\end{aligned} \tag{A18}$$

We now investigate in detail the conditions for (A13) and (A14) to hold. The condition (A5), or $|e_1|<1$, is implicitly assumed in the following discussion. There are three cases to consider.

Case 1: $0<a_2<1/3$

We have from (A18)

$$F_1=\frac{4}{a_2}(1-3a_2)(1+a_2)\left(d_1+\frac{a_2+e_1}{1+a_2}\right)\left(d_1+\frac{a_2+e_1}{1-3a_2}\right). \tag{A19}$$

There are three subcases to discuss.

Case 1a: $e_1+a_2<0$

It can be easily seen from (A19) that $F_1>0$ if and only if

$$-\infty<d_1<-\frac{a_2+e_1}{1+a_2}\quad\text{or}\quad\frac{a_2+e_1}{3a_2-1}<d_1<+\infty. \tag{A20}$$

Straightforward computation gives

$$-\frac{(a_2+e_1)(1+a_2)}{(1-a_2)^2}>-\frac{a_2+e_1}{1+a_2} \tag{A21a}$$

and

$$\begin{cases}\dfrac{a_2-e_1}{1-a_2}\leqslant\dfrac{a_2+e_1}{3a_2-1}, & \text{if } e_1\leqslant 2a_2-1\\[2ex] \dfrac{a_2-e_1}{1-a_2}>\dfrac{a_2+e_1}{3a_2-1}, & \text{if } e_1>2a_2-1\end{cases} \tag{A21b}$$

in this subcase. It follows from (A21a) and (A21b) that for both (A13) and (A14) (or (A20)) to hold, e_1 and d_1 must satisfy

$$e_1>2a_2-1\quad\text{and}\quad\frac{a_2+e_1}{3a_2-1}<d_1<\frac{a_2-e_1}{1-a_2}.$$

Case 1b: $e_1+a_2=0$

We have

$$F_1=\frac{4}{a_2}(a_2+1)(1-3a_2)d_1^2.$$

Hence, (A13) and (A14) are satisfied if and only if

$$0<d_1<\frac{a_2-e_1}{1-a_2}.$$

Case 1c: $e_1+a_2>0$

It can be seen from (A19) that $F_1>0$ if and only if

$$-\infty<d_1<\frac{a_2+e_1}{3a_2-1}\quad\text{or}\quad-\frac{a_2+e_1}{1+a_2}<d_1<+\infty. \tag{A22}$$

Since

$$\frac{a_2+e_1}{3a_2-1}<-\frac{(a_2+e_1)(1+a_2)}{(1-a_2)^2}<-\frac{a_2+e_1}{1+a_2}<\frac{a_2-e_1}{1-a_2}$$

in this subcase, (A13) and (A14) together require

$$-\frac{a_2-e_1}{1+a_2}<d_1<\frac{a_2-e_1}{1-a_2}.$$

Case 2: $a_2=1/3$

In this case, (A13) and (A14) become

$$-(1+3e_1)<d_1<\frac{1+3e_1}{2} \tag{A23}$$

and

$$F_1=\frac{4}{3}(1+3e_1)(4d_1+1+3e_1)>0. \tag{A24}$$

It is straightforward to show that (A23) and (A24) hold if and only if e_1 and d_1 satisfy

$$-\frac{1}{3}<e_1<1 \quad \text{and} \quad -\frac{1+3e_1}{4}<d_1<\frac{1+3e_1}{2}.$$

Case 3: $1/3<a_2<1$

Again, there are three subcases to discuss.

Case 3a: $e_1+a_2<0$

It is easy to see from (A19) that (A14) is equivalent to

$$\frac{a_2+e_1}{3a_2-1}<d_1<-\frac{a_2+e_1}{1+a_2}. \tag{A25}$$

However, since

$$-\frac{a_2+e_1}{1+a_2}<-\frac{(a_2+e_1)(1+a_2)}{(1-a_2)^2}$$

there exists no d_1 that can satisfy (A13) and (A25) simultaneously.

Case 3b: $e_1+a_2=0$

We have

$$F_1=\frac{4}{a_2}(1-3a_2)(1+a_2)d_1^2\leqslant 0 \qquad \forall\, d_1$$

and thus (A14) cannot be satisfied.

Case 3c: $e_1+a_2>0$

In this subcase, (A14) is equivalent to

$$-\frac{a_2+e_1}{1+a_2}<d_1<\frac{a_2+e_1}{3a_2-1}. \tag{A26}$$

Since

$$-\frac{(a_2+e_1)(1+a_2)}{(1-a_2)^2}<-\frac{a_2+e_1}{1+a_2}$$

and

$$\begin{cases} \dfrac{a_2+e_1}{3a_2-1}\leqslant\dfrac{a_2-e_1}{1-a_2}, & \text{if } e_1\leqslant 2a_2-1\\[2mm] \dfrac{a_2-e_1}{1-a_2}<\dfrac{a_2+e_1}{3a_2-1}, & \text{if } e_1>2a_2-1\end{cases}$$

in this subcase, it follows that (A13) and (A26) hold if and only if d_1 satisfies

$$\begin{cases} -\dfrac{a_2+e_1}{1+a_2}<d_1<\dfrac{a_2+e_1}{3a_2-1}, & \text{if } e_1\leqslant 2a_2-1\\[2mm] -\dfrac{a_2+e_1}{1+a_2}<d_1<\dfrac{a_2-e_1}{1-a_2}, & \text{if } e_1>2a_2-1.\end{cases}$$

Summarizing the results obtained in the above case studies, we now derive close-form solutions of a_1, b_1, c_1, d_1, and e_1 that satisfy (A1)–(A7), i.e., if and only if a_1, b_1 and c_1 satisfy (A10)–(A12) where e_1 and d_1 satisfy (A27) and (A28) given as follows:

$$\begin{cases} 2a_2-1<e_1<1, & \text{if } 0<a_2<\dfrac{1}{3}\\[2mm] -a_2<e_1<1, & \text{if } \dfrac{1}{3}\leqslant a_2<1\end{cases} \tag{A27}$$

and

$$\begin{cases} \dfrac{a_2+e_1}{3a_2-1}<d_1<\dfrac{a_2-e_1}{1-a_2}, & \text{if } 2a_2-1<e_1<-a_2\\[2mm] -\dfrac{a_2+e_1}{1+a_2}<d_1<\dfrac{a_2-e_1}{1-a_2}, & \text{if } e_1\geqslant -a_2 \text{ and } e_1>2a_2-1\\[2mm] -\dfrac{a_2+e_1}{1+a_2}<d_1<\dfrac{a_2+e_1}{3a_2-1}, & \text{if } -a_2<e_1\leqslant 2a_2-1\end{cases} \tag{A28}$$

with the constraint that $d_1\neq 0$.

Finally, it is shown that (A8) holds when a_1, b_1, and c_1 satisfy (A10)–(A12). Let $F_2\triangleq 1+a_1^2+e_1^2-b_1^2-c_1^2-d_1^2-6e_1+6b_1d_1$. Recalling (A15) gives

$$\begin{aligned}F_2&=2(b_1c_1+b_1d_1+c_1d_1-a_1-e_1-a_1e_1)-6e_1+6b_1d_1\\&=2\left[(b_1+d_1)c_1-(1+e_1)a_1+4(b_1d_1-e_1)\right].\end{aligned}$$

From (A11) and (A17), we have

$$\begin{aligned}&(b_1+d_1)c_1-(1+e_1)a_1+4(b_1d_1-e_1)\\&=\frac{1}{a_2}\left[(a_2+1)(1-3a_2)d_1^2+2(a_2+e_1)\right.\\&\left.\cdot(1-a_2)d_1+(a_2+e_1)^2\right]+4\left[(a_2d_1+a_2-1)d_1-e_1\right]\\&=\frac{1}{a_2}\left[(a_2-1)^2d_1^2+2(a_2-e_1)(a_2-1)d_1+(a_2-e_1)^2\right]\\&=\frac{1}{a_2}\left[(a_2-1)d_1+(a_2-e_1)\right]^2.\end{aligned} \tag{A29}$$

Hence,

$$F_2=\frac{2}{a_2}\left[(a_2-1)d_1+(a_2-e_1)\right]^2\geqslant 0.$$

Therefore, we have shown that a_1, b_1, c_1, d_1, and e_1 satisfy (A1)–(A8), with the constraint $d_1\neq 0$, if and only if they satisfy (A10)–(A12) and (A27), (A28) or (18a)–(18e) in Theorem 2.

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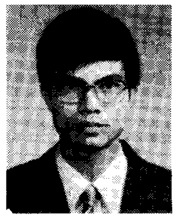

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