Based on this principle, two CMOS current gain cells have been proposed: one for differential inputs and the other for single-ended input. The main features of the proposed circuits are a large gain achievable and variable linearly via a small dc current as the gain is proportional to the aspect ratio of the MOS transistors and to the control current, constant bandwidth independent of the gain, very linear transfer characteristic, and very large input range. The linear controllability has been verified on the single-ended gain cell and the results show a THD of less than 0.5% for input up to 70% of the control current, a constant bandwidth of 1.4 MHz, a gain range from 0.75 to 132 via control current from 4 to 15μA, and a maximum gain-bandwidth product of 185 MHz.

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On Primitive Factorizations for 3-D Polynomial Matrices

Zhiping Lin

Abstract—**This paper presents a criterion for the existence of primitive factorizations for a class of 3-D polynomial matrices. The criterion can also be used to construct a primitive factorization, when it exists, for a 3-D polynomial matrix in this class. Two illustrative examples are also included.**

I. Introduction

During the last three decades, much attention has been directed to the development of multidimensional systems theory, which has applications in digital filtering, image processing, seismic data processing, some distributed-parameter systems, and other areas (see, e.g., [1]–[3]). While the 2-D systems theory is getting mature gradually, its n-D ($n \geq 3$) counterpart is not progressing at the same speed, because n-D ($n \geq 3$) systems are inherently more complex. A number of problems peculiar to the n-D ($n \geq 3$) systems theory remain completely or partially unsolved [1]–[5]. This paper is concerned with one of these problems: the primitive factorizations for 3-D polynomial matrices.

The existence of primitive factorizations for 2-D polynomial matrices was proved and algorithms were developed in [6], [7]. An important consequence of the feasibility of primitive factorization is the derivation of algorithms for the decomposition of a 2-D rational matrix into a factor (minor) coprime matrix fraction description (MFD) [6], [7], and the MFD approach has great potential in the analysis and synthesis of 2-D linear systems [2], [8]. Unfortunately, it was demonstrated via examples in [4] and [9] that there exist some n-D ($n \geq 3$) polynomial matrices that do not admit primitive factorizations. However, to the author's knowledge, a tractable criterion for the existence of primitive factorizations for n-D ($n \geq 3$) polynomial matrices is absent in the literature. Since the existence of primitive factorizations for n-D polynomial matrices is closely related to the existence of coprime MFD's of n-D rational matrices [4], [5], it is desirable to be able to test whether or not a given n-D polynomial matrix admits a primitive factorization. In this paper, we present a criterion for the existence of primitive factorizations for a class of 3-D polynomial matrices. The criterion can also be used to construct a primitive factorization, when it exists, for a 3-D polynomial matrix in this class. Two nontrivial examples from the literature are worked out using the criterion derived in this paper.

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The author is with the Department of Computer Science and Information Engineering, Shantou University, Shantou, Guangdong, China.

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II. Preliminaries

In this section, we recall some definitions and known results in [1], [4], [6], and [10], from which the new results in the next section are developed. In the following, we shall denote $C[z_1,\cdots,z_n] \equiv C[z]$ the set of polynomials in complex variables $z_1,\cdots,z_n$ with coefficients in the field of complex numbers C; $C[z_1][z_2,\cdots,z_n]$ the set of polynomials in $C[z]$, each written as a polynomial in $z_2,\cdots,z_n$ with coefficients in $C[z_1]$; $C^{m\times l}[z]$ the set of $m \times l$ matrices each of whose elements is in $C[z]$, etc.

Definition 1 [1]: Let $f(z) \in C[z]$ be written as

$$f(z) = \sum_{i_2=0}^{m_2} \cdots \sum_{i_n=0}^{m_n} f_{i_2\cdots i_n}(z_1) z_2^{i_2} \cdots z_n^{i_n} \in C[z_1][z_2,\cdots,z_n].$$

Then the greatest common divisor (g.c.d.) $d(z_1)$ of $f_{i_2\cdots i_n}(z_1)$ ($i_2 = 0,\cdots,m_2;\cdots;i_n = 0,\cdots,m_n$) is called the content of $f(z)$ with respect to $C[z_1]$. □

The next two definitions are straightforward generalizations from [1] and [6] for 2-D polynomial matrices to the n-D case.

Definition 2: Let $A(z) \in C^{m\times l}[z]$ be of full normal rank, with $m \geq l$. Then $A(z)$ is said to be primitive in $C[z_1][z_2,\cdots,z_n]$ if for all fixed $z_{11} \in C$, $A(z_{11}, z_2,\cdots,z_n) \in C^{m\times l}[z_2,\cdots,z_n]$ is of full normal rank. □

Definition 3: Let $A(z) \in C^{m\times l}[z]$ be of full normal rank, with $m \geq l$; $e(z) \in C[z]$ the g.c.d. of all the $l \times l$ minors of $A(z)$ and $g(z_1) \in C[z_1]$ the content of $e(z)$. We say that $A(z)$ has a primitive factorization in $C[z_1][z_2,\cdots,z_n]$ if $A(z) = L(z)R(z)$ for some $L(z) \in C^{m\times l}[z]$, $R(z) \in C^{l\times l}[z]$ with det $R(z) = g(z_1)$, and $L(z)$ being primitive in $C[z_1][z_2,\cdots,z_n]$. □

Definition 4 [4]: Let $M(z) \in C^{m\times l}[z]$, with $m \geq l$. Then $M(z)$ is said to be zero right-prime (ZRP) if all the $l \times l$ minors of $M(z)$ do not have a common zero. □

Proposition 1 [4]: Let $A(z_1, z_2) \in C^{m\times l}[z_1, z_2]$ be of normal rank $r < \min\{m, l\}$. Then $A(z_1, z_2)$ can be factored as

$A(z_1, z_2) = A_1(z_1, z_2)A_2(z_1, z_2)$, with $A_1 \in C^{m\times r}[z_1, z_2]$, $A_2 \in C^{r\times l}[z_1, z_2]$. □

Proposition 2 [10]: Let $A(z) \in C^{m\times l}[z]$ with $m > l$. If $A(z)$ in ZRP, there exists $B \in C^{m\times(m-l)}[z]$, such that the $m \times m$ matrix $U(z) = [A(z)\;\; B(z)]$ is a unimodular matrix. □

III. The Main Results

The main results are presented after a lemma. Theorems 1 and 2 are about factoring a 3-D polynomial matrix $A(z_1, z_2, z_3)$ into a product of two 3-D polynomial matrices $A_1(z_1, z_2, z_3)R_1(z_1, z_2, z_3)$, with $\det R_1 = (z_1 - z_{11})$. Theorem 3 is a criterion for the existence of primitive factorizations for a class of 3-D polynomial matrices $A(z_1, z_2, z_3)$.

Lemma 1: Let $A(z_1) \in C^{l\times l}[z_1]$ and $a(z_1) = \det A(z_1) \not\equiv 0$. If $z_{11} \in C$ is a simple zero of $a(z_1)$, then rank $A(z_{11}) = l - 1$.

Proof: A proof can be carried out easily by transforming $A(z_1)$ to the Smith canonic form and making use of the fact that z_{11} is a simple zero of $a(z_1)$. The details are omitted here. □

Theorem 1: Let $A \in C^{l\times l}[z_1, z_2, z_3]$, and $a(z_1) = \det A(z_1, z_2, z_3) \in C[z_1]$, with $a(z_1) \not\equiv 0$. If z_{11} is a simple zero of $a(z_1)$, there exist unimodular matrices $U_1, U_2 \in C^{l\times l}[z_2, z_3]$, with $\det U_1 = \det U_2 = 1$, such that

$$U_1(z_2, z_3)A(z_1, z_2, z_3) = \mathrm{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}A_1(z_1, z_2, z_3) \quad (1)$$

and

$$A(z_1, z_2, z_3)U_2(z_2, z_3) = A_2(z_1, z_2, z_3)\,\mathrm{diag}\{(z_1 - z_{11}), 1, \ldots, 1\} \quad (2)$$

for some $A_1, A_2 \in C^{l\times l}[z_1, z_2, z_3]$.

Proof: Write $A(z_1, z_2, z_3)$ as

$$A(z_1, z_2, z_3) = A^{(k)}(z_2, z_3)(z_1 - z_{11})^k + \cdots + A^{(1)}(z_2, z_3)(z_1 - z_{11}) + A^{(0)}(z_2, z_3).$$

We show by contradiction that rank $A(z_{11}, z_2, z_3) = l - 1$ for every $(z_2, z_3) \in C^2$. Suppose for some fixed $z_2 = z_{21}$, $z_3 = z_{31}$,

$$\mathrm{rank}\ A(z_{11}, z_{21}, z_{31}) < l - 1. \quad (3)$$

Since $A(z_1, z_{21}, z_{31}) \in C^{l\times l}[z_1]$ and $\det A(z_1, z_{21}, z_{31}) = a(z_1)$, applying Lemma 1 to $A(z_1, z_{21}, z_{31})$ gives

$$\mathrm{rank}\ A(z_{11}, z_{21}, z_{31}) = l - 1. \quad (4)$$

Equations (3) and (4) lead to a contradiction. Therefore, rank $A(z_{11}, z_2, z_3) =$ rank $A^{(0)}(z_2, z_3) = l - 1$ for every $(z_2, z_3) \in C^2$. By Proposition 1, $A^{(0)}(z_2, z_3)$ can be factored as

$$A^{(0)}(z_2, z_3) = A_1^{(0)}(z_2, z_3)A_2^{(0)}(z_2, z_3)$$

for some $A_1^{(0)}(z_2, z_3) \in C^{l\times(l-1)}[z_2, z_3]$, $A_2^{(0)}(z_2, z_3) \in C^{(l-1)\times l}[z_2, z_3]$. Since rank $A^{(0)}(z_2, z_3) = l - 1$ for every $(z_2, z_3) \in C^2$, $A_1^{(0)}(z_2, z_3)$ must be ZRP. By Proposition 2, we can find $\boldsymbol{a} \in C^{l\times 1}[z_2, z_3]$ such that $V_1(z_2, z_3) = [\boldsymbol{a}(z_2, z_3)\; A_1^{(0)}(z_2, z_3)]$ is a unimodular matrix. Let $U_1(z_2, z_3) = V_1^{-1}(z_2, z_3)$. Clearly, $U_1 \in C^{l\times l}[z_2, z_3]$ is a unimodular matrix and

$$U_1(z_2, z_3)[\boldsymbol{a}(z_2, z_3) A_1^{(0)}(z_2, z_3)] = I_l$$

or

$$U_1(z_2, z_3)A_1^{(0)}(z_2, z_3) = \begin{bmatrix} 0 \cdots 0 \\ I_{l-1} \end{bmatrix}.$$

Hence,

$$U_1(z_2, z_3)A(z_{11}, z_2, z_3) = U_1(z_2, z_3)A^{(0)}(z_2, z_3) = \begin{bmatrix} 0 \cdots 0 \\ X \end{bmatrix}.$$

Therefore,

$$U_1(z_2, z_3)A(z_1, z_2, z_3) = \mathrm{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}A_1(z_1, z_2, z_3)$$

for some $A_1 \in C^{l\times l}[z_1, z_2, z_3]$.

Analogously, there exists a unimodular matrix $U_2 \in C^{l\times l}[z_2, z_3]$, such that

$$A(z_1, z_2, z_3)U_2(z_2, z_3) = A_2(z_1, z_2, z_3)\,\mathrm{diag}\{(z_1 - z_{11}, 1, \cdots, 1)\}$$

for some $A_2 \in C^{l\times l}[z_1, z_2, z_3]$.

Theorem 2: Let $A \in C^{m\times l}[z_1, z_2, z_3]$ be of full normal rank, with $m \geq l$; $e(z_1, z_2, z_3)$ the g.c.d. of all the $l \times l$ minors of $A(z_1, z_2, z_3)$, $g(z_1)$ the content of $e(z_1, z_2, z_3)$, and z_{11} a zero of $g(z_1)$. Then, $A(z_1, z_2, z_3)$ admits a factorization:

$$A(z_1, z_2, z_3) = A_1(z_1, z_2, z_3)R_1(z_1, z_2, z_3) \quad (5)$$

for some $A_1 \in C^{m\times l}[z_1, z_2, z_3]$, $R_1 \in C^{l\times l}[z_1, z_2, z_3]$ with $\det R_1 = (z_1 - z_{11})$, iff there exists a ZRP vector $\boldsymbol{w}_1 \in C^{l\times 1}[z_2, z_3]$ such that

$$A(z_{11}, z_2, z_3)\boldsymbol{w}_1(z_2, z_3) = [0, \cdots, 0]^T. \quad (6)$$

Proof: (*Sufficiency*): Suppose that there exists a ZRP vector $\boldsymbol{w}_1 \in C^{l\times 1}[z_2, z_3]$ such that

$$A(z_{11}, z_2, z_3)\boldsymbol{w}_1(z_2, z_3) = [0, \cdots, 0]^T.$$

By Proposition 2, we can find $B \in C^{l\times(l-1)}[z_2, z_3]$, such that $U_1(z_2, z_3) = [\boldsymbol{w}_1(z_2, z_3)\; B(z_2, z_3)]$ is a unimodular matrix with $\det U_1 = 1$. Thus

$$A(z_{11}, z_2, z_3)U_1(z_2, z_3) = \begin{bmatrix} 0 & \\ \vdots & X \\ 0 & \end{bmatrix}$$

or

$$A(z_1, z_2, z_3)U_1(z_2, z_3) = A_1(z_1, z_2, z_3)\,\mathrm{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}$$

for some $A_1 \in C^{m\times l}[z_1, z_2, z_3]$. Hence,

$$A(z_1, z_2, z_3) = A_1(z_1, z_2, z_3)\,\mathrm{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}U_1^{-1}(z_2, z_3).$$

Therefore,

$$A(z_1, z_2, z_3) = A_1(z_1, z_2, z_3)R_1(z_1, z_2, z_3)$$

where $R_1 = \mathrm{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}U_1^{-1}(z_2, z_3) \in C^{l\times l}[z_1, z_2, z_3]$, with $\det R_1 = (z_1 - z_{11})$.

(*Necessity*): Suppose that $A(z_1, z_2, z_3)$ can be factored as

$$A(z_1, z_2, z_3) = A_1(z_1, z_2, z_3)R_1(z_1, z_2, z_3) \quad (7)$$

for some $A_1 \in C^{m\times l}[z_1, z_2, z_3]$, $R_1 \in C^{l\times l}[z_1, z_2, z_3]$ with $\det R_1 = (z_1 - z_{11})$. By Theorem 1, there exists a unimodular matrix $V_2(z_2, z_3) \in C^{l\times l}[z_2, z_3]$ such that

$$R_1(z_1, z_2, z_3)V_2(z_2, z_3) = V_1(z_1, z_2, z_3)\,\mathrm{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}$$

or

$$R_1(z_1, z_2, z_3) = V_1(z_1, z_2, z_3)\,\mathrm{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}V_2^{-1}(z_2, z_3) \quad (8)$$

for some $V_1 \in C^{l\times l}[z_1, z_2, z_3]$. From (7) and (8) we obtain

$$A(z_1, z_2, z_3) = A_1(z_1, z_2, z_3)V_1(z_1, z_2, z_3) \cdot \mathrm{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}V_2^{-1}(z_2, z_3)$$

or

$$A(z_1, z_2, z_3)V_2(z_2, z_3) = A_1(z_1, z_2, z_3)V_1(z_1, z_2, z_3) \cdot \operatorname{diag}\{(z_1 - z_{11}), 1, \cdots, 1\}.$$

Hence,

$$A(z_{11}, z_2, z_3)V_2(z_2, z_3) = \begin{bmatrix} 0 & \\ \vdots & X \\ 0 & \end{bmatrix}.$$

Let $w_1(z_2, z_3)$ denote the first column of $V_2(z_2, z_3)$. Clearly, $w_1(z_2, z_3)$ is ZRP and

$$A(z_{11}, z_2, z_3)w_1(z_2, z_3) = [0, \cdots, 0]^T. \qquad \square$$

Theorem 3: Let $A \in C^{m \times l}[z_1, z_2, z_3]$ be of full normal rank, with $m \geq l$; $e(z_1, z_2, z_3)$ the g.c.d. of all the $l \times l$ minors of $A(z_1, z_2, z_3)$; $g(z_1)$ the content of $e(z_1, z_2, z_3)$, and $g(z_1) = \prod_{i=1}^{p}(z_1 - z_{1i})$ where $z_{1k} \neq z_{1j}$, for $k \neq j$. Then $A(z_1, z_2, z_3)$ admits a primitive factorization

$$A(z_1, z_2, z_3) = L(z_1, z_2, z_3)R(z_1, z_2, z_3) \tag{9}$$

for some $L \in C^{m \times l}[z_1, z_2, z_3]$, and $R \in C^{l \times l}[z_1, z_2, z_3]$ with det $R = g(z_1)$, iff for every $i = 1, \cdots, p$, there exists a ZRP vector $w_i \in C^{l \times 1}[z_2, z_3]$ such that

$$A(z_{1i}, z_2, z_3)w_i(z_2, z_3) = [0, \cdots, 0]^T. \tag{10}$$

Moreover, if a primitive factorization (9) exists, $L(z_1, z_2, z_3)$ is unique (modulo a right unimodular matrix), and $R(z_1, z_2, z_3)$ is unique (modulo a left unimodular matrix).

Proof: (*Sufficiency*): Suppose that the condition (10) holds. A primitive factorization (9) can be carried out via the following steps.

Step 1: By assumption, there exists a ZRP vector $w_1 \in C^{l \times 1}[z_2, z_3]$ such that

$$A(z_{11}, z_2, z_3)w_1(z_2, z_3) = [0, \cdots, 0]^T.$$

Applying Theorem 2 to $A(z_1, z_2, z_3)$ gives

$$A(z_1, z_2, z_3) = A_1(z_1, z_2, z_3)R_1(z_1, z_2, z_3) \tag{11}$$

for some $A_1 \in C^{m \times l}[z_1, z_2, z_3]$, $R_1 \in C^{l \times l}[z_1, z_2, z_3]$ with det $R(z_1, z_2, z_3) = (z_1 - z_{11})$.

Step 2: By assumption, there exists a ZRP vector $w_2 \in C^{l \times 1}[z_2, z_3]$ such that

$$A(z_{12}, z_2, z_3)w_2(z_2, z_3) = [0, \cdots, 0]^T.$$

Recalling (11) gives

$$A_1(z_{12}, z_2, z_3)R_1(z_{12}, z_2, z_3)w_2(z_2, z_3) = [0, \cdots, 0]^T$$

or

$$A_1(z_{12}, z_2, z_3)w_2'(z_2, z_3) = [0, \cdots, 0]^T$$

where $w_2'(z_2, z_3) = R_1(z_{12}, z_2, z_3)w_2(z_2, z_3)$.

Since $z_{12} \neq z_{11}$ by assumption, det $R_1(z_{12}, z_2, z_3) = z_{12} - z_{11} \neq 0$, and hence $R_1(z_{12}, z_2, z_3)$ is a unimodular matrix. Thus, $w_2'(z_2, z_3)$ is a ZRP vector. By Theorem 2, $A_1(z_1, z_2, z_3)$ can be factored as

$$A_1(z_1, z_2, z_3) = A_2(z_1, z_2, z_3)R_2(z_1, z_2, z_3) \tag{12}$$

for some $A_2 \in C^{m \times l}[z_1, z_2, z_3]$, $R_2 \in C^{l \times l}[z_1, z_2, z_3]$ with det $R_2(z_1, z_2, z_3) = (z_1 - z_{12})$. Combining (11) and (12) gives

$$A(z_1, z_2, z_3) = A_2(z_1, z_2, z_3)R_2(z_1, z_2, z_3)R_1(z_1, z_2, z_3) \tag{13}$$

Step p: We have

$$A(z_1, z_2, z_3) = A_p(z_1, z_2, z_3)R_p(z_1, z_2, z_3) \cdots R_2(z_1, z_2, z_3)R_1(z_1, z_2, z_3)$$

for some $A_p \in C^{m \times l}[z_1, z_2, z_3]$, $R_i \in C^{l \times l}[z_1, z_2, z_3]$ with det $R_i(z_1, z_2, z_3) = (z_1 - z_{1i})$, for $i = 1, \ldots, p$. Let $R(z_1, z_2, z_3) = R_p(z_1, z_2, z_3) \cdots R_2(z_1, z_2, z_3)R_1(z_1, z_2, z_3)$, and $L(z_1, z_2, z_3) = A_p(z_1, z_2, z_3)$. We obtain

$$A(z_1, z_2, z_3) = L(z_1, z_2, z_3)R(z_1, z_2, z_3)$$

with det $R(z_1, z_2, z_3) = \prod_{i=1}^{p}(z_1 - z_{1i}) = g(z_1)$.

(*Necessity*): Suppose that $A(z_1, z_2, z_3)$ can be factored as

$$A(z_1, z_2, z_3) = L(z_1, z_2, z_3)R(z_1, z_2, z_3)$$

for some $L \in C^{m \times l}[z_1, z_2, z_3]$, and $R \in C^{l \times l}[z_1, z_2, z_3]$ with det $R = \prod_{i=1}^{p}(z_1 - z_{1i}) = g(z_1)$. Since z_{1i} ($i = 1, \cdots, p$) is a simple zero of $g(z_1)$, by Theorem 1, we can find a unimodular matrix $U_i \in C^{l \times l}[z_2, z_3]$, with det $U_i = 1$, such that

$$R(z_1, z_2, z_3)U_i(z_2, z_3) = R_i'(z_1, z_2, z_3) \operatorname{diag}\{(z_1 - z_{1i}), 1, \cdots, 1\}$$

or

$$R(z_1, z_2, z_3) = R_i'(z_1, z_2, z_3) \cdot \operatorname{diag}\{(z_1 - z_{1i}), 1, \cdots, 1\}U_i^{-1}(z_2, z_3)$$

for some $R_i' \in C^{l \times l}[z_1, z_2, z_3]$. Thus,

$$A(z_1, z_2, z_3) = L(z_1, z_2, z_3)R_i'(z_1, z_2, z_3) \cdot \operatorname{diag}\{(z_1 - z_{1i}), 1, \cdots, 1\}U_i^{-1}(z_2, z_3)$$

or

$$A(z_1, z_2, z_3)U_i(z_2, z_3) = L(z_1, z_2, z_3)R_i'(z_1, z_2, z_3) \cdot \operatorname{diag}\{(z_1 - z_{1i}), 1, \cdots, 1\}.$$

Let $w_i(z_2, z_3)$ denote the first column of $U_i(z_2, z_3)$. Clearly, $w_i(z_2, z_3)$ is a ZRP vector and satisfies

$$A(z_{1i}, z_2, z_3)w_i(z_2, z_3) = [0, \cdots, 0]^T.$$

Finally, if the primitive factorization (9) exists, the uniqueness of $L(z_1, z_2, z_3)$ (modulo a right unimodular matrix) and $R(z_1, z_2, z_3)$ (modulo a left unimodular matrix) can be proved similarly as it is for the 2-D case [6], [7]. The proof is omitted here to save space. $\square$

Remark 1: With minor modification, Theorems 2 and 3 can be applied to the case where $A(z_1, z_2, z_3) \in C^{m \times l}[z_1, z_2, z_3]$ is of full normal rank, with $m \leq l$. It should also be pointed out that Theorem 2 is not a special case of Theorem 3, since z_{11} in Theorem 2 is not necessarily a simple zero of $g(z_1)$. $\square$

IV. Examples

In this section, we present two nontrivial examples from the literature to illustrate the usefulness of the results derived in the last section. As it will be seen, the criterion for the existence of primitive factorizations for the class of 3-D polynomial matrices as given in Theorem 3 is quite simple to apply.

Example 1 [4]: Let

$$A(z_1, z_2, z_3) = \begin{bmatrix} z_2 z_3 - z_1^3 & z_2^2 - z_1 z_3 \\ z_3^2 - z_1^2 z_2 & z_2 z_3 - z_1^3 \end{bmatrix}.$$

We have $a(z_1, z_2, z_3) = \det A(z_1, z_2, z_3) = z_1(z_1^5 - 3z_1^2 z_2 z_3 + z_1 z_2^3 + z_3^3)$, and z_1 is the content of $a(z_1, z_2, z_3)$. Theorem 3 is now applied to testing whether or not $A(z_1, z_2, z_3)$ admits a primitive factorization:

$$A(z_1, z_2, z_3) = L(z_1, z_2, z_3)R(z_1, z_2, z_3)$$

with det $R(z_1, z_2, z_3) = z_1$. For $z_1 = 0$, we have

$$A(0, z_2, z_3) = \begin{bmatrix} z_2z_3 & z_2^2 \\ z_3^2 & z_2z_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \end{bmatrix}[z_3 \quad z_2].$$

Consider all the 2×1 vectors $w(z_2, z_3)$, such that

$$A(0, z_2, z_3)w(z_2, z_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Clearly, $w(z_2, z_3)$ is parametrized by

$$w(z_2, z_3) = \begin{bmatrix} z_2b(z_2, z_3) \\ -z_3b(z_2, z_3) \end{bmatrix}$$

where $b \in C[z_2, z_3]$ is arbitrary. Obviously, $w(z_2, z_3)$ is not ZRP for any choice of $b \in C[z_2, z_3]$. Therefore, $A(z_1, z_2, z_3)$ does not admit a primitive factorization (9). □

It can be seen that the above proof is more elementary and simpler than the one given in [4].

Example 2 [1][1]: Let

$$A(z_1, z_2, z_3) = \begin{bmatrix} z_1 & 0 \\ z_2 & z_2z_3 \end{bmatrix}.$$

We have $a(z_1, z_2, z_3) = \det A(z_1, z_2, z_3) = z_1z_2z_3$, and z_1 is the content of $a(z_1, z_2, z_3)$. It was stated in [1] that $A(z_1, z_2, z_3)$ cannot have a primitive factorization:

$$A(z_1, z_2, z_3) = L(z_1, z_2, z_3)R(z_1, z_2, z_3) \qquad (9)$$

with det $R(z_1, z_2, z_3) = z_1$. However, by using Theorem 3, it will be shown that this statement is not true. For $z_1 = 0$,

$$A(0, z_2, z_3) = \begin{bmatrix} 0 & 0 \\ z_2 & z_3 \end{bmatrix}.$$

Let $w(z_2, z_3) = \begin{bmatrix} z_3 \\ -1 \end{bmatrix}$, we have

$$A(0, z_2, z_3)w(z_2, z_3) = \begin{bmatrix} 0 & 0 \\ z_2 & z_2z_3 \end{bmatrix}\begin{bmatrix} z_3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let $U(z_2, z_3) = \begin{bmatrix} z_3 & 1 \\ -1 & 0 \end{bmatrix}$. Then,

$$A(0, z_2, z_3)U(z_2, z_3) = \begin{bmatrix} 0 & 0 \\ 0 & z_2 \end{bmatrix}.$$

Thus

$$A(z_1, z_2, z_3)U(z_2, z_3) = \begin{bmatrix} z_1 & 0 \\ z_2 & z_2z_3 \end{bmatrix}\begin{bmatrix} z_3 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} z_3 & z_1 \\ 0 & z_2 \end{bmatrix}\begin{bmatrix} z_1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So

$$A(z_1, z_2, z_3) = \underbrace{\begin{bmatrix} z_3 & z_1 \\ 0 & z_2 \end{bmatrix}}_{L(z_1, z_2, z_3)} \underbrace{\begin{bmatrix} z_1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} z_3 & 1 \\ -1 & 0 \end{bmatrix}^{-1}}_{R(z_1, z_2, z_3)}$$

where $L, R \in C^{2\times 2}[z_1, z_2, z_3]$, with det $R = z_1$. Therefore, $A(z_1, z_2, z_3)$ does admit a primitive factorization (9). □

Remark 2: The criterion for the existence of primitive factorizations presented in Theorem 3 works for the class of 3-D polynomial matrices $A(z_1, z_2, z_3)$ whose associated content $g(z_1)$ having only simple zeroes. It is natural to ask whether or not one can generalize this criterion to the n-D ($n > 3$) case. The generalization is not straightforward since an extension of Proposition 1 to the n-D ($n \geq 3$) case cannot be made in general [4], and further investigation is required. It may also be interesting to study the applicability of the developed criterion when $g(z_1)$ has multiple zeroes.

[1]The variables z_1 and z_3 have been exchanged for convenience of exposition here.

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An Exact Expression for the Noise Voltage Across a Resistor Shunted by a Capacitor

Vittal P. Pyati

***Abstract*—The mean square voltage developed across a noisy resistor (R) with a white spectral density shunted by a capacitor (C) is known to be (kT/C), where k is Boltzmann's constant and T the absolute temperature of R. A puzzling aspect of this simple result is that the value of the resistor (which is the real source of noise) appears to play no part and gives no clue on the outcome in the limiting cases of $R \to 0$ or $C \to 0$. The difficulty is resolved by first noting that due to quantum effects, the simple white noise model breaks down at frequencies of 10^9 Hz or so. Using a quantum-corrected expression for the spectral density, the problem of the RC network has been reworked exactly. It is verified that the new expression yields correct results for all values of the parameters. A Taylor series expansion of the output in terms of the noise equivalent bandwidth of the RC circuit shows that the old result**

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The author is with the Department of Electrical and Computer Engineering, Air Force Institute of Technology, Wright-Patterson Air Force Base, OH 45433.

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