in a natural manner to the case when they are functions of t, where $F(t)$ and $\Delta(t)$ must satisfy (18) for all t.

V. Conclusions

Differential linear repetitive processes are a distinct class of 2-D continuous-discrete linear systems of both applications and systems theoretic interest. In applications, they arise in ILC schemes and in solution algorithms for nonlinear dynamic optimal control algorithms based on the maximum principle. Repetitive processes cannot be analyzed/controlled by direct application of existing systems theory and currently there is only a very limited literature on the specification and design of control schemes for them and essentially none on the class of processes considered in this paper.

The most significant new contribution in this paper is that an LMI formulation of stability along the pass (the stronger form of the two distinct stability concepts for these processes which will most often be required in applications) can be immediately used to design a powerful class of control laws for these processes which, crucially, have a well defined physical interpretation for applications areas such as ILC. These features are missing from alternative stability characterizations where the most that can be achieved is to test the resulting conditions using 1-D linear systems stability tests.

It is important to place the results of this paper in context; essentially, they represent the first systematic procedure for stability analysis and onward controller design, as opposed to just stability analysis only, for a very important and distinct class of 2-D linear systems using control laws which are well grounded in terms of the underlying process dynamics. One key area for which no results are currently available is the stability and control of differential linear repetitive processes in the presence of uncertainty in the model structure. Here, it has been shown that the LMI setting immediately allows significant progress to be made.

One counter argument here may be that the uncertainty structures used here are well known in the 1-D linear systems area. This is, in fact, true, but only in terms of some of the matrices in the defining repetitive process state–space model but, given the facts that: 1) no previous work has been done in this area and 2) these processes do have certain structural similarities with 1-D differential and discrete linear systems, this is not an unreasonable place to begin work. The most important conclusion to be drawn is, we argue, that it is indeed possible to control these processes in the presence of uncertainty in the defining model structure and that the results so obtained provide a useful benchmark for further work. Also, the numerics associated with the resulting conditions may not always be well behaved and this area also merits further attention.

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Comments on “Stability Tests of N-Dimensional Discrete Time Systems Using Polynomial Arrays”

Li Xu, Jiangqian Ying, Zhiping Lin, and Osami Saito

***Abstract*—In this brief, we wish to point out that the author of the above paper overlooked a mistake in the stability test procedure for N-dimensional (N-D, $N > 2$) systems proposed in the above paper, which made the polynomial array approach not general. It is shown that Hu’s test procedure applies only to a very restricted class of N-D stability test problems, and for a general case, instead of necessary and sufficient conditions it provides only sufficient conditions. A counterexample is also given.**

***Index Terms*—Multidimensional systems, polynomial array, stability test.**

I. Introduction and Problem Description

The purpose of this brief is to show that the author of [1] overlooked a mistake in the stability test procedure for N-dimensional (N-D, $N > 2$) systems proposed in [1], so that this procedure does not generally serve as a necessary and sufficient condition for N-D stability tests except for certain very restricted cases. As the usage of some notations in [1] is a little confusing, we first rephrase the related results of [1] here in a slightly different way.

Consider an N-D discrete system described by the transfer function

$$G(z_1,\ldots,z_N) = \frac{P(z_1,\ldots,z_N)}{F(z_1,\ldots,z_N)} \tag{1}$$

with $P(z_1,\ldots,z_N)$ and $F(z_1,\ldots,z_N)$ being N-D factor coprime polynomials, and assume that $G(z_1,\ldots,z_N)$ possesses no nonessential singularities of the second kind.

The necessary and sufficient condition for N-D system (1) to be BIBO stable is that $F(\boldsymbol{z}_N) \triangleq F(z_1,\ldots,z_N)$ is devoid of zeros in the closed-unit polydisk, i.e.,

$$F(\boldsymbol{z}_N) \neq 0, \qquad \text{for } \bigcap_{p=1}^{N} |z_p| \leq 1. \tag{2}$$

Further, it is well known that this condition is equivalent to a set of tests given by

$$F(\boldsymbol{z}_m) \triangleq F(z_1,\ldots,z_m) \neq 0, \quad \text{for } \bigcap_{p=1}^{m-1} |z_p| = 1, |z_m| \leq 1, \quad m = 1,2,\ldots,N \tag{3}$$

where $F(\boldsymbol{z}_m)$ is obtained by setting $z_i = 0$ in $F(\boldsymbol{z}_N)$ for $i > m$.

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TABLE I
POLYNOMIAL ARRAY FOR Fz_N ON $\bigcap_{i=1}^{N-1} |z_i| = 1$

Row(i)	z_N^0	z_N^1	$\cdots$	$z_N^{n_N-1}$	$z_N^{n_N}$
0	$F^{0,0}(z_{N-1})$	$F^{0,1}(z_{N-1})$	$\cdots$	$F^{0,n_N-1}(z_{N-1})$	$F^{0,n_N}(z_{N-1})$
1	$F^{1,0}(z_{N-1})$	$F^{1,1}(z_{N-1})$	$\cdots$	$F^{1,n_N-1}(z_{N-1})$	
$\vdots$			$\vdots$		
n_N-1	$F^{n_N-1,0}(z_{N-1})$	$F^{n_N-1,1}(z_{N-1})$			
n_N	$F^{n_N,0}(z_{N-1})$				

For the tests of (3), it obviously suffices to consider only the case for $m = N$, i.e.,

$$F(z_N) \neq 0, \qquad \text{for } \bigcap_{p=1}^{N-1} |z_p| = 1, \quad |z_N| \leq 1 \tag{4}$$

as the others can be done similarly.

Regard $F(z_N)$ as a one-dimensional (1-D) polynomial in z_N having coefficients of polynomials in $z_1, \ldots, z_{N-1}$, i.e.,

$$\begin{aligned} F(z_N) &= \sum_{k_1=0}^{n_1} \cdots \sum_{k_N=0}^{n_N} f_{k_1,\ldots,k_N} z_1^{k_1} \cdots z_N^{k_N} \\ &= \sum_{k_N=0}^{n_N} F_{k_N}(z_{N-1}) z_N^{k_N}. \end{aligned} \tag{5}$$

Then, applying the well-known 1-D Marden–Jury table to (5) and noting that $\bar{z}_i = 1/z_i$, $i = 1, \ldots, N-1$ on $\bigcap_{p=1}^{N-1} |z_p| = 1$, the polynomial array in Table I can be obtained [1] where

$$F^{0,j}(z_{N-1}) = F_j(z_{N-1}), \quad j = 0, 1, \ldots, n_N \tag{6}$$

$$\begin{aligned} &F^{i,j}(z_{N-1}) \\ &\quad = \begin{vmatrix} \dfrac{F^{i-1,0}(z_{N-1})}{F^{i-1,n_N-i+1}(z_{N-1})} & \dfrac{F^{i-1,n_N-i+1}}{\overline{F^{i-1,j}(z_{N-1})}} \end{vmatrix} \\ &\quad = \sum_{k_1=-m_{i,1}}^{m_{i,1}} \cdots \sum_{k_{N-1}=-m_{i,N-1}}^{m_{i,N-1}} f_{k_1,\ldots,k_{N-1}}^{i,j} z_1^{k_1} \cdots z_{N-1}^{k_{N-1}}, \\ &\qquad i = 1, 2, \ldots, n_N, j = 0, 1, \ldots, n_N - i \end{aligned} \tag{7}$$

$$m_{i,p} = n_p 2^{i-1}, i = 1, \ldots, n_N, \quad p = 1, 2, \ldots, N-1 \tag{8}$$

and $\overline{F^{i-1,j}(z_{N-1})}$ denotes the conjugate polynomial of $F^{i-1,j}(z_{N-1})$, etc.

It is shown in [1] that condition (4) holds if and only if

$$F^{i,0}(z_{N-1}) > 0 \quad \text{on } \bigcap_{p=1}^{N-1} |z_p| = 1, \quad i = 1, 2, \ldots, n_N \tag{9}$$

and further, condition (9) is equivalent to

$$F^{i,0}(\mathbf{1}) = F^{i,0}(1, \ldots, 1) > 0 \tag{10}$$

and

$$F^{i,0}(z_{N-1}) \neq 0 \quad \text{on } \bigcap_{p=1}^{N-1} |z_p| = 1, \quad i = 1, 2, \ldots, n_N. \tag{11}$$

Since $F^{i,0}(z_{N-1})$ is self-inversive in z_{N-1}, $F^{i,0}(z_{N-1}) \neq 0$ on $\bigcap_{p=1}^{N-1} |z_p| = 1$ if and only if $F^{i,0}(z_{N-1})$ has half of its zero clusters (or zeros) located inside $|z_{N-1}| < 1$ for all (points on) $\bigcap_{p=1}^{N-2} |z_p| = 1$, which can be tested again by using the Marden–Jury table. To avoid the singular case in the table due to the self-inversive property of $F^{i,0}(z_{N-1})$, an auxiliary polynomial $H^{i,0}(z_{N-1})$ has to be constructed first, as follows:

$$\begin{aligned} &H^{i,0}(z_{N-1}) \\ &\quad = z_{N-1}^{l_{N-1}-1} \left[\frac{\partial \left\{ z_1^{l_1/2} \cdots z_{N-1}^{l_{N-1}/2} F^{i,0}(z_{N-1}) \right\}}{\partial z_{N-1}} \right]_{z_{N-1}^{-1}} \end{aligned} \tag{12}$$

where $l_p = 2m_{i,p} = 2^i n_p$, $p = 1, \ldots, N-1$. The problem is now reduced to verifying if $H^{i,0}(z_{N-1})$ has $l_{N-1}/2$ zero clusters in $|z_{N-1}| < 1$ for all $\bigcap_{p=1}^{N-2} |z_p| = 1$.

Let $F_H^{i,0}(z_{N-2})$ be the entries of the first column in the Marden–Jury table for $H^{i,0}(z_{N-1})$ and define

$$\begin{aligned} P_r(z_{N-2}) = \prod_{i=1}^{r} F_H^{i,0}(z_{N-2}) \quad &\text{on } \bigcap_{p=1}^{N-2} |z_p| = 1, \\ &r = 1, 2, \ldots, l_{N-1} - 1. \end{aligned} \tag{13}$$

Then, the number of zero clusters of $F_H^{i,0}(z_{N-2})$ located in $|z_{N-1}| < 1$ for all $\bigcap_{p=1}^{N-2} |z_p| = 1$ is equal to the number of products having negative sign in (13) [1].

Note that in [1], instead of $F^{i,0}(z_{N-1})$, $H^{i,0}(z_{N-1})$, and $F_H^{i,0}(z_{N-2})$, the notations of $F^{i,0}(z_N)$, $H^{i,0}(z_N)$, and $F^{i,0}(z_{N-1})$ are used, which may cause a confusion with the $F^{i,0}(z_{N-1})$ used in (9).

It is claimed in [1] that, in order to determine the sign of (13), each of the factors $F_H^{i,0}(z_{N-2})$ should have the same sign for all $\bigcap_{p=1}^{N-2} |z_p| = 1$, i.e., $F_H^{i,0}(z_{N-2}) \neq 0$ for $\bigcap_{p=1}^{N-2} |z_p| = 1$. Failure in this condition would indicate that $H^{i,0}(z_{N-1}) = 0$ and, thus, $F^{i,0}(z_{N-1}) = 0$ at some point on $\bigcap_{p=1}^{N-1} |z_p| = 1$ which violates the main stability condition (4), implying that the system under test is unstable.

It is based on this claim that the following conclusion was given in [1]: the zero distribution problem of (13) could be reduced to a subproblem of verifying the zero distribution of $H^{i,0}(z_{N-1})$ on z_{N-1}-plane for just a fixed point on $\bigcap_{p=1}^{N-2} |z_p| = 1$, say, $(z_1, z_2, \ldots, z_{N-2}) = (1, 1, \ldots, 1)$, and another subproblem of testing that $F_H^{i,0}(z_{N-2}) \neq 0$ on $\bigcap_{p=1}^{N-2} |z_p| = 1$ for $i = 1, 2, \ldots, l_{N-1} - 1$. To be more precise, this conclusion can be summarized as the following proposition.

Proposition 1: Let $F_H^{i,0}(z_{N-2})$ and $P_r(z_{N-2})$ be defined as in (12) and (13), respectively. Then, condition (11) holds true, which is equivalent to that $F^{i,0}(z_{N-1})$, or $H^{i,0}(z_{N-1})$, has $l_{N-1}/2$ zero clusters in $|z_{N-1}| < 1$ for all $\bigcap_{p=1}^{N-2} |z_p| = 1$, if and only if

1) $v_p(1) = l_{N-1}/2$, where $v_p(1)$ is the number of $P_r(\mathbf{1}) = \prod_{i=1}^{r} F_H^{i,0}(\mathbf{1}) < 0$, $r = 1, 2, \ldots, l_{N-1} - 1$;
2) $F_H^{i,0}(z_{N-2}) \neq 0$, for $\bigcap_{p=1}^{N-2} |z_p| = 1$, $i = 1, 2, \ldots, l_{N-1} - 1$.

Condition 1) is easy to test, while condition 2) could be considered in the same argument used for condition (11). Repeating this process until the involved entries are reduced to 1-D polynomials, the N-D stability test problem could finally be reduced to just some 1-D positivity tests. Based on these arguments, an N-D stability test procedure is given in [1].

However, the above claim and further, the conclusion, i.e., the results stated in Proposition 1, are in fact not correct, and consequently, instead of necessary and sufficient conditions the approach from [1] provides in general only sufficient conditions. Detailed discussions on this problem and a counterexample are given in Section II.

II. DISCUSSIONS AND COUNTEREXAMPLE

For a given 1-D polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, let $a^{i,j}$, $i = 0, 1, \ldots, n$, $j = 0, 1, \ldots, n-i$, be the entries of its Marden–Jury table. In view of Theorems (44,1), (45,1), and (45,2) of [2], it can be seen that $a^{i,0} = 0$, for some $i \leq n$, either in the case where only

$a^{i,0} = 0$ or in the case where $a^{i,j} = 0$ for $j = 0, 1, \ldots, m$, $1 \le m \le n - i$, does not necessarily imply that $f(z)$ has zeros on $|z| = 1$. That is, $f(z)$ may have no zeros on $|z| = 1$ even if a singular case occurs in its Marden–Jury table. In other words, $a^{i,0} \ne 0$ is not necessary for $f(z) \ne 0$ on $|z| = 1$. One can simply verify this property by using, for example, an arbitrary self-inversive polynomial having no zeros on $|z| = 1$.

Applying the above property to the Marden–Jury table for an $(N-1)$-D polynomial $H^{i,0}(z_{N-1})$ by viewing it as a 1-D polynomial having coefficients of $(N-2)$-D polynomials, as stated in Section I, one can readily conclude that the necessity of condition 2) in Proposition 1, i.e., $F_H^{i,0}(z_{N-2}) \ne 0$ on $\bigcap_{p=1}^{N-2} |z_p| = 1$ is necessary for $H^{i,0}(z_{N-1}) \ne 0$, or equivalently, $F^{i,0}(z_{N-1}) \ne 0$, on $\bigcap_{p=1}^{N-1} |z_p| = 1$, does not hold. Therefore, the claim of [1] is not correct and the procedure given there can only serve as a sufficient N-D stability test in general. The failure of a test by this procedure does not necessarily mean that the tested system is unstable.

Further analysis reveals that the necessity of the test of [1] only holds for certain very restricted cases. In particular, when $N = 3$, $n_3 = 1$, $n_2 = 1$, i.e., $F(z_3)$ is linear in z_3 and z_2, one will have in the corresponding Marden–Jury table only one entry $F^{1,0}(z_1, z_2)$ with $m_{1,2} = n_2 2^{1-1} = 1$. Then, the degree of $H^{1,0}(z_1, z_2)$ in z_2 is $l_2 - 1 = 2m_{1,2} - 1 = 1$, where $H^{1,0}(z_1, z_2)$ is constructed from $F^{1,0}(z_1, z_2)$ in the way of (12). Again, only one entry $F_H^{1,0}(z_1, z_2)$ will occur in the Marden–Jury table for $H^{1,0}(z_1, z_2)$, and to ensure that $H^{1,0}(z_1, z_2)$ has one zero cluster in $|z_2| < 1$ for $|z_1| = 1$, $F_H^{1,0}(z_1)$ must be negative, i.e., $F_H^{1,0}(z_1) < 0$, for all $|z_1| = 1$. This in turn is equivalent to the conditions that $P_1(z_1) = F_H^{1,0}(z_1) < 0$ at $z_1 = 1$ and $F_H^{1,0}(z_1) \ne 0$ for all $|z_1| = 1$, which can be tested by verifying the number of the zeros of $F_H^{1,0}(z_1)$ in $|z_1| < 1$. It means that, in this case, the stability test procedure of [1] is sufficient and necessary. In fact, this case has been investigated through numerical examples in [3].

When $n_2 > 1$ in the above case, the degree of $H^{1,0}(z_1, z_2)$ in z_2 will be larger than 1 and the Marden–Jury table for it will have more than one polynomial entry in the first column, i.e., $F_H^{i,0}(z_1)$ with $i > 1$. More importantly, the condition $F_H^{i,0}(z_1) \ne 0$ is no longer necessary, as discussed previously, and we have to consider in general a sign variation problem for the set of 1-D polynomials $F_H^{i,0}(z_1)$, $i = 1, 2, \ldots, l_2 - 1$, which is usually a more complicated problem and needs a careful treatment of the possible singular cases. Nevertheless, as $F_H^{i,0}(z_1)$ are 1-D polynomials, many available computer software programs such as Maple, MATLAB, etc., can be utilized to solve this problem, as shown in the counterexample below. Also, due to the self-inversive property, one can convert $F_H^{i,0}(z_1)$ to polynomials $\tilde{F}_H^{i,0}(x_1)$ in real variable $x_1 = \left(z_1 + z_1^{-1}\right)/2$, and investigate the zero distribution of $\tilde{F}_H^{i,0}(x_1)$ on $-1 \le x \le 1$ by any of the available 1-D methods [4].

To determine the stability of N-D systems with $N > 3$, tests for positivity on a set of polynomials of three or more variables are necessary. This in turn requires the investigation of sign variation of a set of polynomials of two or more variables for all values on $\bigcap_{i=1}^{m} |z_i| = 1$, $m \ge 2$ [3], [4]. Obviously, the test procedure of [1] does not provide a solution to such a general case as pointed out in the above. Therefore, the main difficulties for the general N-D stability test problem still remain to be challenged.

In the following, we present a counterexample to the test procedure of [1].

Counterexample: Test the stability of the 3-D system given by

$$G(z_3) = \frac{1}{F(z_3)} = \frac{1}{(z_1^2 + z_2^2 + 4)(z_1 + z_2 + z_3 + 5)}.$$

Since $|z_1^2 + z_2^2| \le |z_1^2| + |z_2^2| = |z_1|^2 + |z_2|^2 \le 2$ and $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3| \le 3$ on $\cap_{i=1}^{3} |z_i| \le 1$, it is obvious that $F(z_3) \ne 0$ on $\cap_{i=1}^{3} |z_i| \le 1$, i.e., the system $G(z_3)$ is stable.

It is easy to see that $F(z_1) = (z_1^2 + 4)(z_1 + 5)$ and $F(z_2) = (z_1^2 + z_2^2 + 4)(z_1 + z_2 + 5)$ satisfy condition (3) and the Marden–Jury table for $F(z_3)$ on $|z_1| = 1, |z_2| = 1$ has, due to (7), only the entry

$$F^{1,0}(z_2) = \left(z_1^2 + z_2^2 + 4\right)\left(\frac{1}{z_1^2} + \frac{1}{z_2^2} + 4\right) \times \left[(z_1 + z_2 + 5)\left(\frac{1}{z_1} + \frac{1}{z_2} + 5\right) - 1\right].$$

As $F^{1,0}(1,1) = 6^2(7^2 - 1) > 0$, all we need to do to test the stability of $G(z_3)$ is to verify whether condition (11) is true, i.e., $F^{1,0}(z_2) \ne 0$ on $|z_1| = 1, |z_2| = 1$. The auxiliary polynomial $H^{1,0}(z_2)$ corresponding to $F^{1,0}(z_2)$ is calculated based on (12) as follows:

$$\begin{aligned} H^{1,0}(z_2) &= z_2^5 \left[\frac{\partial\{z_1^3 z_2^3 F^{1,0}(z_2)\}}{\partial z_2}\right]_{z_2^{-1}} \\ &= a_5(z_1)z_2^5 + a_4(z_1)z_2^4 + a_3(z_1)z_2^3 \\ &\quad + a_2(z_1)z_2^2 + a_1(z_1)z_2 + a_0(z_1) \end{aligned}$$

where

$$\begin{aligned} a_5(z_1) &= 5z_1^6 + 26z_1^5 + 25z_1^4 + 104z_1^3 + 20z_1^2 \\ a_4(z_1) &= 8z_1^6 + 50z_1^5 + 38z_1^4 + 220z_1^3 + 16z_1^2 + 40z_1 \\ a_3(z_1) &= 60z_1^6 + 312z_1^5 + 330z_1^4 + 1404z_1^3 + 330z_1^2 \\ &\quad + 312z_1 + 60 \\ a_2(z_1) &= 80z_1^5 + 32z_1^4 + 440z_1^3 + 76z_1^2 + 100z_1 + 16 \\ a_1(z_1) &= 100z_1^4 + 520z_1^3 + 125z_1^2 + 130z_1 + 25 \\ a_0(z_1) &= 120z_1^3 + 24z_1^2 + 30z_1 + 6. \end{aligned}$$

It can be confirmed by, e.g., Maple that the polynomial entries in the first column of the Marden–Jury table for $H^{1,0}(z_2)$ have zeros on $|z_1| = 1$, i.e.,

$$\begin{aligned} F_H^{1,0}(z_1) &= 0, && \text{for } z_1 = z_{1a} \\ F_H^{2,0}(z_1) &= 0, && \text{for } z_1 = z_{1b}, \bar{z}_{1b} \\ F_H^{3,0}(z_1) &= 0, && \text{for } z_1 = z_{1a} \\ F_H^{4,0}(z_1) &= 0, && \text{for } z_1 = z_{1a}, z_{1b}, \bar{z}_{1b} \\ F_H^{5,0}(z_1) &= 0, && \text{for } z_1 = z_{1a}, z_{1b}, \bar{z}_{1b} \end{aligned}$$

where

$$\begin{aligned} z_{1a} &= 1 \\ z_{1b}, \bar{z}_{1b} &= -0.955\,254 \pm 0.295\,786I \quad (\text{with } |z_{1b}| = |\bar{z}_{1b}| = 1) \end{aligned}$$

and

$$\begin{aligned} F_H^{1,0}(z_1) &= -100\left(z_1^4 + z_1^{-4}\right) - 320\left(z_1^3 + z_1^{-3}\right) \\ &\quad + 415\left(z_1^2 + z_1^{-2}\right) - 1680\left(z_1 + z_1^{-1}\right) + 3370 \\ F_H^{2,0}(z_1) &= 225\,600\left(z_1^8 + z_1^{-8}\right) + 812\,160\left(z_1^7 + z_1^{-7}\right) \\ &\quad - 8\,406\,192\left(z_1^6 + z_1^{-6}\right) - 53\,891\,040\left(z_1^5 + z_1^{-5}\right) \\ &\quad - 178\,449\,024\left(z_1^4 + z_1^{-4}\right) - 550\,027\,200\left(z_1^3 + z_1^{-3}\right) \\ &\quad - 1\,021\,440\,912\left(z_1^2 + z_1^{-2}\right) \\ &\quad - 1\,669\,789\,920\left(z_1 + z_1^{-1}\right) - 2\,070\,668\,544. \end{aligned}$$

$F_H^{3,0}(z_1) \sim F_H^{5,0}(z_1)$ are omitted as they are too long to be included and it suffices, for our purpose here, to just have, e.g., $F_H^{1,0}(z_1) = 0$ for some $|z_1| = 1$.

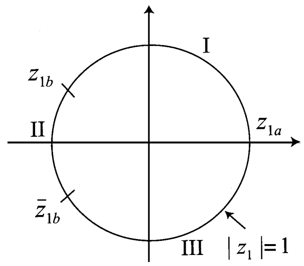

Fig. 1. Regions on $|z_1| = 1$ partitioned by z_{1a}, z_{1b} and $\bar{z}_{1b}$.

Due to the test given in [1], it would be concluded that the system $G(z_3)$ was not stable at this point as $F_H^{i,0}(z_1)$ has zeros on $|z_1| = 1$. However, further investigation shows that $H^{1,0}(z_2)$ has three zeros in $|z_2| < 1$ when $z_1 = z_{1a}$, $z_1 = z_{1b}$, and $z_1 = \bar{z}_{1b}$, i.e.,

$$\begin{aligned} H^{1,0}(z_{1a}, z_2) &= 180z_2^5 + 372z_2^4 + 2808z_2^3 + 744z_2^2 + 900z_2 + 180 \\ &= 0 \end{aligned}$$

at the points: $\{z_{2a} = -0.916\,662 + 3.742\,114I,\ z_{2b} = \bar{z}_{2a},\ z_{2c} = -0.208\,366,\ z_{2d} = -0.012\,489 + 0.568\,474I,\ z_{2e} = \bar{z}_{2d}\}$ with $|z_{2a}| = |z_{2b}| > 1$, $|z_{2c}| < 1$, $|z_{2d}| = |z_{2e}| < 1$, and

$$\begin{aligned} H^{1,0}(z_{1b}, z_2) = &- (41.993\,102 + 67.976\,986I)z_2^5 \\ &- (153.187\,792 + 181.8951I)z_2^4 \\ &- (753.734\,031 + 951.456\,737I)z_2^3 \\ &- (284.029\,541 + 381.492\,500I)z_2^2 \\ &- (282.820\,099 + 282.170\,289I)z_2 \\ &- 77.371\,705 - 89.372\,696I \\ = &\ 0 \end{aligned}$$

at the points: $\{z_{2a} = -1.441\,362 - 3.266\,240I,\ z_{2b} = -1.146\,045 + 3.724\,524I,\ z_{2c} = -0.322\,843 - 0.025\,447I,\ z_{2d} = -0.051\,896 - 0.570\,818I,\ z_{2e} = 0.017\,809 - 0.572\,627I\}$ with $|z_{2a}| > 1$, $|z_{2b}| > 1$, $|z_{2c}| < 1$, $|z_{2d}| < 1$, and $|z_{2e}| < 1$. The case for $H^{1,0}(\bar{z}_{1b}, z_2)$ is similar to $H^{1,0}(z_{1b}, z_2)$. Therefore, it is concluded that $F^{1,0}(z_1, z_2) \neq 0$ on $|z_2| = 1$ at least for $z_1 = z_{1a}, z_{1b}$ and $\bar{z}_{1b}$. This can be verified in a more direct way by substituting $z_{1a}, z_{1b}, \bar{z}_{1b}$ into $F^{1,0}(z_1, z_2)$ and finding their zeros, respectively. Also, one can convert $F_H^{i,0}(z_1)$ to $\tilde{F}_H^{i,0}(x_1)$ by using $x_1 = (z_1 + z_1^{-1})/2$ and verify that they have the zeros of $x_1 = 1$, $x_1 = -0.955\,254$ in $-1 \leq x_1 \leq 1$ which correspond to z_{1a} and $z_{1b}, \bar{z}_{1b}$, respectively. However, all these detailed elaborations are omitted here for brevity.

To determine whether $F^{1,0}(z_1, z_2) \neq 0$ on $|z_2| = 1$ for the other points on $|z_1| = 1$, we have to investigate the sign variation of $F_H^{i,0}(z_1)$ and $P_r^{i,0}(z_1) = \prod_{i=1}^{r} F_H^{i,0}(z_1)$, $r = 1, \ldots, 5$, on each region of $|z_1| = 1$ partitioned by z_{1a}, z_{1b} and $\bar{z}_{1b}$ as shown in Fig. 1. As $F_H^{i,0}(z_1)$, $i = 1, \ldots, 5$, are 1-D polynomials and have, respectively, the same sign on each region, it is not difficult to obtain the results shown in Table II. From the results we see that, though different sign variations occur for different regions on $|z_1| = 1$, the numbers of negative signs for these regions are all the same, i.e., are all 3, which means that $H^{1,0}(z_1, z_2)$ has three zeros in $|z_2| < 1$ for $|z_1| = 1$, i.e., $H^{1,0}(z_1, z_2) \neq 0$ and $F^{1,0}(z_1, z_2) \neq 0$ on $|z_1| = 1$, $|z_2| = 1$. Further, combining with the fact that $F^{1,0}(1, 1) > 0$, we see that $F(z_3)$ satisfies condition (4), thus $G(z_3)$ is stable. It should be noted that if we use $\tilde{F}_H^{i,0}(x_1)$ instead of $F_H^{i,0}(z_1)$, the sign variation of $\tilde{F}_H^{i,0}(x_1)$ on $-1 \leq x_1 \leq 1$ can be verified more easily.

TABLE II
SIGN VARIATIONS OF $P_r(z_1)$ ON $|z_1| = 1$

	P_1	P_2	P_3	P_4	P_5	Number of "–"
I	+	–	+	–	–	3
II	+	+	–	–	–	3
III	+	–	+	–	–	3

III. CONCLUDING REMARKS

It has been shown that the author of [1] overlooked a mistake in the N-D $(N > 2)$ stability test procedure given in [1], which made the polynomial array approach not general. It has been pointed out in this brief that the test procedure of [1] in fact applies only to a very restricted class of N-D stability test problems. As a result, for the stability test of a general N-D system, the polynomial array approach proposed in [1] provides only sufficient conditions instead of necessary and sufficient conditions.

A numerical counterexample has also been given to support our arguments, which has reinforced that singular cases must be treated carefully when applying a table (or polynomial array) approach to the N-D stability testing problem. Therefore, we believe that there are still many difficulties at the present stage for generally reducing a complicated zero distribution problem of an N-D polynomial to a comparatively simple positivity test problem for 1-D polynomials. Finding an effective and efficient method for N-D $(N > 2)$ stability test remains challenging.

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