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FACTORIZATIONS FOR nD POLYNOMIAL MATRICES*

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Abstract. In this paper, a constructive general matrix factorization scheme is developed for extracting a nontrivial factor from a given nD ($n > 2$) polynomial matrix whose maximal order minors satisfy certain conditions. It is shown that three classes of nD polynomial matrices admit this kind of general matrix factorization. It turns out that minor prime factorization and determinantal factorization are two interesting special cases of the proposal general factorization. As a consequence, the paper provides a partial solution to an open problem of minor prime factorization as well as to a recent conjecture on minor prime factorizability for nD polynomial matrices. Three illustrative examples are worked out in detail.
Key words: Multidimensional systems, nD polynomial matrices, matrix factorizations, reduced minors, minor primeness, Quillen–Suslin theorem.

1. Introduction

The problems of multivariate (nD) polynomial factorizations and nD polynomial matrix factorizations have attracted much attention over the past decades because of their wide applications in multidimensional (nD) circuits, systems, controls, signal processing, and other areas (see, e.g., [1]–[7], [10]–[19], [21]–[24], [28], [30], [32], [33]). For an arbitrary nD polynomial over the field of real numbers or the field of complex numbers, although the existence of its factorization into a product of irreducible polynomials has been known for a long time (see, e.g., [2]), constructive algorithms for carrying out such a factorization are available only for some classes of nD polynomials (see, e.g., [21], [22], [30]). A constructive

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method for testing and factorizing an nD polynomial into factors that are linear in one and the same variable has been developed in [21]. This method will be utilized later in our paper. Another algorithm for simple and group factorizations for nD polynomials has recently been presented in [22]. The development of a constructive method for the unique factorization of a general nD polynomial is still an active research area. However, there exist algorithms for extracting a greatest common divisor (g.c.d.) from a finite number of nD polynomials with coefficients from any field [2].

The factorization problems for nD polynomial matrices turn out to be even more difficult than their polynomial counterparts because unlike the ring of nD polynomials, which is a commutative ring, the ring of nD polynomial matrices is not commutative. For example, it is still unknown [32], [10], [19], [12, p. 63] how to extract a common factor from two nD ($n > 2$) polynomial matrices. In fact, this open problem is closely related to the factorization of a normal full rank nD polynomial matrix into a product of two nD polynomial matrices, with one of them being prime in some sense [23], [32], [7]. This prime factorization problem has long been solved for one-dimensional (1D) and two-dimensional (2D) polynomial matrices [23], [10], [11]. However, it is a challenging open problem for nD ($n > 2$) polynomial matrices [32], [7], [12, p. 63] because of some fundamental differences between nD polynomial matrices and their 1D and, 2D counterparts [32], [14], [7], [12, p. 63]. In this paper, we shall be concerned only with nD ($n > 2$) polynomial matrix factorizations, so in what follows, the term "nD" implies ($n > 2$) unless otherwise specified.

Recently, some progress has been made in solving the zero prime factorization problem for nD polynomial matrices [4], [6], [18]. By making use of Gröbner bases [5] for modules, Bose and Charoenlarpnopparut have proposed an algorithm for carrying out the zero prime factorization for nD polynomial matrices whose reduced minors are devoid of any common zeros [4], [6]. Lin conjectured that the absence of any common zeros in the reduced minors is a sufficient condition for the existence of zero prime factorization and also provided a partial solution to this conjecture [18]. Unfortunately, neither the algorithm proposed in [4], [6] nor the method presented in [18] is applicable for nD polynomial matrices whose reduced minors have some common zeros.

Lin [18] also posed another conjecture on minor prime factorizability of an nD polynomial matrix based on its reduced minors and the greatest common divisor of its maximal order minors (called maximal minors in this paper for simplicity). However, to our knowledge, this conjecture has not been answered up to now. Moreover, even when the existence of minor prime factorization is known for a given nD polynomial matrix, it is still nontrivial to carry out the actual factorization. In fact, Charoenlarpnopparut and Bose have recently raised the following open problem [6]:

Under the assumptions that the minor prime factorization of an nD polynomial *matrix exists and the reduced minors have common zeros, find an algorithm for computing this minor prime factorization.*

Another related matrix factorization problem is the determinantal factorization problem for a given nD ($n \geq 1$) polynomial matrix: to extract a factor with determinant equal to a divisor of the determinant of the given matrix. Again, it is well known [23], [10] that any 1D and 2D polynomial matrices admit determinantal factorizations. However, it has been pointed out [32], [13] that some nD ($n > 2$) polynomial matrices do not have determinantal factorizations. Thus, it is interesting to know whether or not a given nD polynomial matrix admits a determinantal factorization.

In this paper, we propose a general matrix factorization scheme for extracting a nontrivial factor from a given nD polynomial matrix whose maximal minors satisfy certain conditions. It is shown that three classes of nD polynomial matrices admit this kind of general matrix factorization. It turns out that minor prime factorization and determinantal factorization are two special cases of the proposal general factorization. As a consequence, the results obtained in this paper provide a partial solution to the open problem on minor prime factorization [6] and the conjecture on minor prime factorizability [18].

The organization of the paper is as follows. In Section 2, we recall some definitions and formulate the problems mathematically. A technique for general matrix factorizations for three classes of nD polynomial matrices is presented in Section 3. Its relationship with minor prime and determinantal factorizations is also pointed out in the same section. In Section 4, three illustrative examples are worked out in detail. The conclusions are given in Section 5.

2. Preliminaries and problem formulation

In the following, we shall denote $\mathbf{C}[\mathbf{z}] = \mathbf{C}[z_1, \ldots, z_n]$ the set of polynomials in complex variables $z_1, \ldots, z_n$ with coefficients in the field of complex numbers $\mathbf{C}$; $\mathbf{C}^{m \times l}[\mathbf{z}]$ the set of $m \times l$ matrices with entries in $\mathbf{C}[\mathbf{z}]$, etc.

Throughout this paper, the argument $(\mathbf{z})$ is omitted whenever its omission does not cause confusion. Without loss of generality, the dimension of a given matrix is assumed to be $m \times l$ with $m \geq l$. Only normal full rank matrices whose maximal minors are not all identically equal to zero are discussed. The new results can in fact be applied to matrices not of normal full rank after some minor modifications. We refer the reader to [18] for more details on this topic.

Definition 1 ([32]). Let $F \in \mathbf{C}^{m \times l}[\mathbf{z}]$. Then F is said to be:

(i) zero right prime (ZRP) if the $l \times l$ minors of F are zero coprime, i.e., devoid of any common zeros;
(ii) minor right prime (MRP) if the $l \times l$ minors of F are factor coprime, i.e., devoid of any common factors.

Remark 1. If F is a square matrix, i.e., $m = l$, the only $l \times l$ minor of F is its determinant. In such a case, MRP is equivalent to ZRP, and F is MRP if and only if F is a unimodular matrix, i.e., $\det F = k_0 \in \mathbf{C}^*$, where $\mathbf{C}^*$ denotes the set of all nonzero complex numbers.

Definition 2 ([14], [28]). Let $F \in \mathbf{C}^{m\times l}[\mathbf{z}]$ and let $a_1, \ldots, a_\beta$ denote the $l \times l$ minors of the matrix F, where $\beta = \binom{m}{l} = \frac{m!}{(m-l)!l!}$. Extracting the g.c.d., denoted by d, of $a_1, \ldots, a_\beta$ gives

$$a_i = d\, b_i, \qquad i = 1, \ldots, \beta. \tag{1}$$

Then, $b_1, \ldots, b_\beta$ are called the reduced minors of F.

The general matrix factorization problem is now formulated as follows.

Let F be given in Definition 2, and let d_0 be a common divisor (not necessarily the g.c.d.) of $a_1, \ldots, a_\beta$, i.e., $a_i = d_0\, e_i$ with $e_i \in \mathbf{C}[\mathbf{z}]$ $(i = 1, \ldots, \beta)$. We say that F admits a general matrix factorization if F can be factorized as

$$F(\mathbf{z}) = F_0(\mathbf{z})\, G_0(\mathbf{z}), \tag{2}$$

such that $F_0 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, $G_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$, and $\det G_0 = d_0$.

It is obvious that the $l \times l$ minors of F_0 in (2) are equal to $e_1, \ldots, e_\beta$. If $m > l$ and $d_0 = d$ is the g.c.d. of $a_1, \ldots, a_\beta$, then $e_i = b_i$ $(i = 1, \ldots, \beta)$, and F_0 in (2) is MRP. In such a case, we say that F admits an MRP factorization [18]. It was conjectured in [18] that F admits an MRP factorization if $d, b_1, \ldots, b_\beta$ have no common zeros in $\mathbf{C}^n$.

If $m = l$, i.e., F is a square matrix, then $\beta = 1$, $a_1 = e_1\, d_0$, and F_0 in (2) is also a square matrix with $\det F_0 = e_1$. Notice that $\det G_0 = d_0$. In such a case, we say that F admits a determinant factorization.

In the next section, we show that the general matrix factorization (2) does exist for three classes of nD polynomial matrices.

3. Main results

Four lemmas are first required.

Lemma 1. *Let $F(z_1) \in \mathbf{C}^{m\times l}[z_1]$, and let $a(z_1)$ be the g.c.d. of the $l \times l$ minors of $F(z_1)$. If z_{11} is a simple zero of $a(z_1)$, i.e., $z_1 - z_{11}$ is a divisor of $a(z_1)$, but $(z_1 - z_{11})^2$ is not a divisor of $a(z_1)$, then* rank $F(z_{11}) = l - 1$.

Proof. By transforming $F(z_1)$ into its Smith form [8], the result follows immediately. □

Lemma 2. *Let $F(\mathbf{z}) \in \mathbf{C}^{m\times l}[\mathbf{z}]$. If* rank $F(\mathbf{z}) = l - 1$ *for every $(\mathbf{z}) \in \mathbf{C}^n$, then we can construct a ZRP vector $\mathbf{w} \in \mathbf{C}^{l\times 1}[\mathbf{z}]$ such that*

$$F(\mathbf{z})\mathbf{w}(\mathbf{z}) = [0, \ldots, 0]^T, \tag{3}$$

where $(\cdot)^T$ denotes transposition.

A proof can be given similarly as the proof for Theorem 1 of [16], thus it is omitted here. Notice that if $F \in \mathbf{C}^{m\times l}[z_2, \ldots, z_n]$, i.e., independent of z_1, then $\mathbf{w}$ in (3) can be chosen from $\mathbf{C}^{l\times 1}[z_2, \ldots, z_n]$ as well, i.e., independent of z_1.

Lemma 3. *Let* $\mathbf{w} \in \mathbf{C}^{l\times 1}[\mathbf{z}]$. *If* $\mathbf{w}(\mathbf{z})$ *is ZRP, then a square nD polynomial matrix* $U(\mathbf{z}) \in \mathbf{C}^{l\times l}[\mathbf{z}]$ *can be constructed such that* $\det U(\mathbf{z}) = 1$ *and* $\mathbf{w}(\mathbf{z})$ *is the first column of* $U(\mathbf{z})$.

The existence of $U(\mathbf{z})$ in Lemma 3 was proved independently by Quillen and Suslin in 1976 [27], [29], [33]. Their results on this problem became the famous Quillen–Suslin theorem, which is sometimes also referred to as the unimodular matrix completion problem (see, e.g., [33]). However, constructive solutions to the unimodular matrix completion problem for nD polynomial matrices were not available until the 1990s, when Logar-Sturmfels and Park–Woodburn published their algorithms in [20] and [25], respectively, using Gröbner bases [5]. Recently, a computationally more efficient method for the same problem has also been developed in [6].

Before stating the next lemma, it is necessary to explain the usage of some mathematical symbols. In this paper, when we state an nD polynomial matrix equation such as $F(\mathbf{z})\mathbf{w}(\mathbf{z}) = [0, \ldots, 0]^T$ in Lemma 2, we mean that the equality is for every $(\mathbf{z}) \in \mathbf{C}^n$, i.e., $F(\mathbf{z})\mathbf{w}(\mathbf{z}) = [0, \ldots, 0]^T$ has the same meaning as $F(\mathbf{z})\mathbf{w}(\mathbf{z}) \equiv [0, \ldots, 0]^T$. No confusion can arise in using one of these two expressions in other parts of the paper, therefore, we adopt the expression $F(\mathbf{z})\mathbf{w}(\mathbf{z}) = [0, \ldots, 0]^T$, which has been used commonly in the literature on nD polynomial matrix factorizations [1]–[7], [10]–[19], [23], [24], [28], [32], [33]. However, because an nD polynomial equation $g(\mathbf{z}) = 0$ could also mean solving $g(\mathbf{z}) = 0$ for some (but **not all**) $(\mathbf{z}) \in \mathbf{C}^n$, in the following lemma and its proof, we shall use the expression $g(\mathbf{z}) \equiv 0$ for the case when $g(\mathbf{z}) = 0$ for every $(\mathbf{z}) \in \mathbf{C}^n$. In the rest of the paper, we shall still use the conventional symbol "=" instead of "≡" when such a usage does not cause any confusion.

Lemma 4. *Let* $g \in \mathbf{C}[z_1, z_2, \ldots, z_n]$; $f \in \mathbf{C}[z_2, \ldots, z_n]$. *If* $g(f(z_2, \ldots, z_n), z_2, \ldots, z_n) = 0$ *for every* $(z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$, *i.e.,* $g(f(z_2, \ldots, z_n), z_2, \ldots, z_n) \equiv 0$, *then* $(z_1 - f(z_2, \ldots, z_n))$ *is a divisor of* $g(z_1, z_2, \ldots, z_n)$.

Proof. If $g(z_1, z_2, \ldots, z_n)$ is independent of z_1, then $g(f(z_2, \ldots, z_n), z_2, \ldots, z_n) \equiv 0$ implies that $g(z_1, z_2, \ldots, z_n) \equiv 0$. Hence, $(z_1 - f(z_2, \ldots, z_n))$ is a divisor of $g(z_1, z_2, \ldots, z_n)$. Now assume that $g(z_1, z_2, \ldots, z_n)$ depends on z_1. We can write $g(z_1, z_2, \ldots, z_n)$ as

$$g(z_1, z_2, \ldots, z_n) = \sum_{i=0}^{r} h_i(z_2, \ldots, z_n) z_1^i, \tag{4}$$

where $r > 0$ is an integer and $h_i \in \mathbf{C}[z_2, \ldots, z_n]$ $(i = 0, 1, \ldots, r)$. Introduce a new variable t by

$$t = z_1 - f(z_2, \ldots, z_n). \tag{5}$$

We have

$$z_1 = t + f(z_2, \ldots, z_n). \tag{6}$$

Substituting (6) into (4) and arranging terms, we have

$$g(z_1, z_2, \ldots, z_n) = g_1(t, z_2, \ldots, z_n) = \sum_{i=0}^{r} s_i(z_2, \ldots, z_n)t^i, \tag{7}$$

where $s_i \in \mathbf{C}[z_2, \ldots, z_n]$ $(i = 0, 1, \ldots, r)$. From (5) and (7), it is obvious that $g(f(z_2, \ldots, z_n), z_2, \ldots, z_n) \equiv 0$ implies $g_1(0, z_2, \ldots, z_n) \equiv 0$, which in turns means that $s_0(z_2, \ldots, z_n) \equiv 0$. Therefore, it follows that

$$\begin{aligned} g(z_1, z_2, \ldots, z_n) = g_1(t, z_2, \ldots, z_n) &= \sum_{i=1}^{r} s_i(z_2, \ldots, z_n)t^i \\ &= t\sum_{i=0}^{r-1} s_i(z_2, \ldots, z_n)t^i. \end{aligned} \tag{8}$$

Recalling that $t = z_1 - f(z_2, \ldots, z_n)$, it follows immediately that $(z_1 - f(z_2, \ldots, z_n))$ is a divisor of $g(z_1, z_2, \ldots, z_n)$. □

Remark 2. For a 1D polynomial $g(z_1)$, it is well known that if $g(z_{10}) = 0$ for some complex number z_{10}, then $(z_1 - z_{10})$ is a divisor of $g(z_1)$. Lemma 4 can be considered as a generalization of this well-known fact.

We now present the main results of this paper in the following three theorems.

Theorem 1. *Let F be given in Definition 2. Assume that $d_0(\mathbf{z}) = z_1 - f(z_2, \ldots, z_n)$ is a common divisor of $a_1(\mathbf{z}), \ldots, a_\beta(\mathbf{z})$, i.e., $a_i(\mathbf{z}) = d_0(\mathbf{z})\, e_i(\mathbf{z})$ $(i = 1, \ldots, \beta)$. If $d_0(\mathbf{z})$, $e_1(\mathbf{z}), \ldots, e_\beta(\mathbf{z})$, have no common zeros, then* rank $F(f(z_2, \ldots, z_n), z_2, \ldots, z_n) = l - 1$ *for every $(z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$, and $F(\mathbf{z})$ admits a general matrix factorization in (2).*

Proof. Because $d_0(\mathbf{z}) = z_1 - f(z_2, \ldots, z_n)$ by assumption, it is obvious that $(f(z_2, \ldots, z_n), z_2, \ldots, z_n)$ is a zero of $d_0(\mathbf{z})$ for every $(z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$. Therefore, it is seen that rank $F(f(z_2, \ldots, z_n), z_2, \ldots, z_n) \leq l - 1$ for every $(z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$. We show by contradiction that rank $F(f(z_2, \ldots, z_n), z_2, \ldots, z_n)$ cannot be smaller than $l - 1$ for any $(z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$. Suppose that for some fixed $z_2 = z_{21}, \ldots, z_n = z_{n1}$,

$$\operatorname{rank} F(f(z_{21}, \ldots, z_{n1}), z_{21}, \ldots, z_{n1}) < l - 1. \tag{9}$$

Let $z_{11} = f(z_{21}, \ldots, z_{n1})$, which is a constant. Consider the 1D polynomial matrix $F(z_1, z_{21}, \ldots, z_{n1})$. Let $a'_1(z_1), \ldots, a'_\beta(z_1)$ denote the $l \times l$ minors of $F(z_1, z_{21}, \ldots, z_{n1})$. We have

$$\begin{aligned} a'_i(z_1) &= a_i(z_1, z_{21}, \ldots, z_{n1}) \\ &= (z_1 - f(z_{21}, \ldots, z_{n1}))e_i(z_1, z_{21}, \ldots, z_{n1}) \\ &= (z_1 - z_{11})e_i(z_1, z_{21}, \ldots, z_{n1}), \qquad i = 1, \ldots, \beta. \end{aligned} \tag{10}$$

Let $c(z_1)$ denote the g.c.d. of $e_1(z_1, z_{21}, \ldots, z_{n1}), \ldots, e_\beta(z_1, z_{21}, \ldots, z_{n1})$. It follows that $(z_1 - z_{11})\,c(z_1)$ is the g.c.d. of $a'_1(z_1), \ldots, a'_\beta(z_1)$. The assumption that $d_0(\mathbf{z})$, $e_1(\mathbf{z}), \ldots, e_\beta(\mathbf{z})$ have no common zeros implies that z_{11} cannot be a zero of $c(z_1)$. Otherwise, $(z_{11}, z_{21}, \ldots, z_{n1})$ would be a common zero of $d_0(\mathbf{z})$, $e_1(\mathbf{z}), \ldots, e_\beta(\mathbf{z})$, a contradiction. Hence, z_{11} is a simple zero of $(z_1 - z_{11})\,c(z_1)$. By Lemma 1,

$$\operatorname{rank} F(z_{11}, z_{21}, \ldots, z_{n1}) = l - 1, \tag{11}$$

or

$$\operatorname{rank} F(f(z_{21}, \ldots, z_{n1}), z_{21}, \ldots, z_{n1}) = l - 1. \tag{12}$$

Expressions (9) and (12) lead to a contradiction. Therefore, rank $F(f(z_2, \ldots, z_n), z_2, \ldots, z_n) = l - 1$ for every $(z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$.

By Lemma 2, we can construct a ZRP vector $\mathbf{w} \in \mathbf{C}^{l\times 1}[z_2, \ldots, z_n]$ such that

$$F(f(z_2, \ldots, z_n), z_2, \ldots, z_n)\mathbf{w}(z_2, \ldots, z_n) = [0, \ldots, 0]^T. \tag{13}$$

By Lemma 3, $\mathbf{w}(z_2, \ldots, z_n)$ can be completed into a unimodular matrix, i.e., we can construct a $U \in \mathbf{C}^{l\times l}[z_2, \ldots, z_n]$ such that $\det U = 1$ and $\mathbf{w}(z_2, \ldots, z_n)$ is the first column of $U(z_2, \ldots, z_n)$. From (13), we have

$$F(f(z_2, \ldots, z_n), z_2, \ldots, z_n)U(z_2, \ldots, z_n) = [0_{m,1}\ B], \tag{14}$$

where $0_{m,1}$ denotes the $m \times 1$ zero matrix, and $B \in \mathbf{C}^{m\times(l-1)}[z_2, \ldots, z_n]$. Let $Y(\mathbf{z}) = F(\mathbf{z})U(z_2, \ldots, z_n)$, and let $y_{j1}(\mathbf{z})$ be the jth element of the first column of $Y(\mathbf{z})$ $(j = 1, \ldots, m)$. From (14), we have that $y_{j1}(f(z_2, \ldots, z_n), z_2, \ldots, z_n) \equiv 0$ $(j = 1, \ldots, m)$. By Lemma 4, $(z_1 - f(z_2, \ldots, z_n))$ is a divisor of $y_{j1}(\mathbf{z})$ $(j = 1, \ldots, m)$, i.e., $(z_1 - f(z_2, \ldots, z_n))$ divides the first column of $Y(\mathbf{z})$. Hence, we have

$$Y(\mathbf{z}) = F(\mathbf{z})U(z_2, \ldots, z_n) = F_0(\mathbf{z})\,\underbrace{\operatorname{diag}\{(z_1 - f(z_2, \ldots, z_n)), 1, \ldots, 1\}}_{D(\mathbf{z})},$$

for some $F_0 \in \mathbf{C}^{m\times l}[\mathbf{z}]$. It follows that

$$F(\mathbf{z}) = F_0(\mathbf{z})\,G_0(\mathbf{z}),$$

where $G_0 = DU^{-1} \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G = (z_1 - f(z_2, \ldots, z_n)) = d_0$. □

Theorem 1 shows that the general matrix factorization (2) always exists for the class of nD polynomial matrices with $d_0(\mathbf{z}) = (z_1 - f(z_2, \ldots, z_n))$ and $d_0(\mathbf{z})$, $e_1(\mathbf{z}), \ldots, e_\beta(\mathbf{z})$ devoid of any common zero. Moreover, from the preceding proof and references [16], [20], [25], it is seen that the proposed factorization scheme is constructive and can be performed in a finite number of steps, provided that $d_0(\mathbf{z})$ is already known to be of the form $d_0(\mathbf{z}) = (z_1 - f(z_2, \ldots, z_n))$. However, as pointed out in the Introduction, although algorithms exist [2] for extracting a g.c.d. from a finite number of nD polynomials with coefficients from any field, it is still unknown how to factor an arbitrary nD polynomial with coefficients from the field of complex (or real) numbers into a product of irreducible polynomials constructively [21], [22], [30]. It is interesting that the problem of nD polynomial matrix factorizations is closely related to that of nD polynomial factorizations. We point out here that algorithms have been available for factorizing an nD polynomial with coefficients from the ring of integers into a product of irreducible polynomials [31], [2].

Because constructive methods are available for factorizing an nD polynomial $d_0(\mathbf{z})$ as $d_0(\mathbf{z}) = \prod_{k=1}^{p}(z_1 - f_k(z_2, \ldots, z_n))$, when such a factorization exists [21], it is natural to ask whether or not Theorem 1 can be generalized to this case with $p > 1$. The answer is yes when $f_1, \ldots, f_p$ are distinguished constants or when they depend only on one variable, say, z_2. For the latter case, an additional condition is also required. We first consider the former case.

Theorem 2. *Let $F(\mathbf{z})$ be given in Definition 2. Assume that $d_0(\mathbf{z}) = d_0(z_1) = \prod_{k=1}^{p}(z_1 - z_{1k})$, where $z_{11}, \ldots, z_{1p}$ are p distinguished complex numbers, is a common divisor of $a_1(\mathbf{z}), \ldots, a_\beta(\mathbf{z})$, i.e., $a_i(\mathbf{z}) = d_0(z_1)\, e_i(\mathbf{z})$ $(i = 1, \ldots, \beta)$. If $d_0(z_1)$, $e_1(\mathbf{z}), \ldots, e_\beta(\mathbf{z})$ have no common zeros, then $F(\mathbf{z})$ admits a general matrix factorization in* (2).

Proof. From the assumption,

$$\begin{aligned} a_i(\mathbf{z}) &= d_0(z_1)\, e_i(\mathbf{z}) \\ &= \prod_{k=1}^{p}(z_1 - z_{1k})\, e_i(\mathbf{z}) \\ &= (z_1 - z_{11})\, e_i'(\mathbf{z}), \end{aligned} \tag{15}$$

where $e_i' = \prod_{k=2}^{p}(z_1 - z_{1k})\, e_i$. The assumption that $d_0(z_1)$, $e_1(\mathbf{z}), \ldots, e_\beta(\mathbf{z})$ have no common zeros implies that the set of $(n-1)$D polynomials $e_1(z_{11}, z_2, \ldots, z_n), \ldots, e_\beta(z_{11}, z_2, \ldots, z_n)$ have no common zeros. Because $z_{11}, \ldots, z_{1p}$ are distinguished, it is clear that $\prod_{k=2}^{p}(z_{11} - z_{1k})\, e_1(z_{11}, z_2, \ldots, z_n)$, $\ldots, \prod_{k=2}^{p}(z_{11} - z_{1k})\, e_\beta(z_{11}, z_2, \ldots, z_n)$ have no common zeros. Consequently, $(z_1 - z_{11})$, $e_1'(\mathbf{z}), \ldots, e_\beta'(\mathbf{z})$ have no common zeros either. By Theorem 1, $F(\mathbf{z})$ can be factorized as

$$F(\mathbf{z}) = F_1(\mathbf{z})\, G_1(\mathbf{z}), \tag{16}$$

such that $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, $G_1 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_1 = (z_1 - z_{11})$.

Repeating this procedure for $F_1(\mathbf{z})$ with respect to $(z_1 - z_{12})$, we have

$$F_1(\mathbf{z}) = F_2(\mathbf{z})\, G_2(\mathbf{z}), \tag{17}$$

or

$$F(\mathbf{z}) = F_2(\mathbf{z})\, G_2(\mathbf{z})\, G_1(\mathbf{z}), \tag{18}$$

such that $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det G_2 = (z_1 - z_{12})$.

Repeating the same procedure continuously with respect to $(z_1 - z_{1k})$, $k = 3, \ldots, p$, we finally have

$$F(\mathbf{z}) = F_p(\mathbf{z})\, G_p(\mathbf{z}) \cdots G_2(\mathbf{z})\, G_1(\mathbf{z}), \tag{19}$$

or

$$F(\mathbf{z}) = F_0(\mathbf{z})\, G_0(\mathbf{z}), \tag{20}$$

where $F_0 = F_p \in \mathbf{C}^{m \times l}[\mathbf{z}]$, $G_0 = \prod_{k=1}^{p} G_k \in \mathbf{C}^{l \times l}[\mathbf{z}]$ with $\det G_0 = \prod_{k=1}^{p}(z_1 - z_{1k}) = d_0(z_1)$. $\square$

We next generalize Theorem 1 for another class of nD polynomial matrices, where $d(\mathbf{z})$ is of the form $d(\mathbf{z}) = d(z_1, z_2) = \prod_{k=1}^{p}(z_1 - f_k(z_2))$, with $f_k \in \mathbf{C}[z_2]$ ($k = 1, \ldots, p$) and $f_t(z_2) \not\equiv f_j(z_2)$, for any pair of $t, j \in \{1, \ldots, p\}, t \neq j$. Note that unlike the case where $f_1, \ldots, f_p$ are distinguished constants, the method of Theorem 2 cannot be directly applied here without an additional assumption as some of $(z_1 - f_1(z_2)), \ldots, (z_1 - f_p(z_2))$ may share common zeros even when $f_t(z_2) \not\equiv f_j(z_2)$, for any pair of $t, j \in \{1, \ldots, p\}, t \neq j$. Consider a simple example where $e_1 = 1 + z_3(z_1 - z_2)$, $e_2 = 1 + z_3(z_1 + z_2 - 2)$, $e_3 = e_4 = 0$, and $d_0 = (z_1 - z_2)(z_1 + z_2 - 2)$ with evidently $f_1(z_2) = z_2$, $f_2(z_2) = -z_2 + 2$. Clearly, $d_0, e_1, \ldots, e_4$ have no common zeros, and $f_1(z_2) \not\equiv f_2(z_2)$. However, $(z_1 - z_2), (z_1 + z_2 - 2)\, e_1(z_1, z_2, z_3), \ldots, (z_1 + z_2 - 2)\, e_4(z_1, z_2, z_3)$ have an infinite number of common zeros at $(1, 1, z_3)$. The difficulty arises at the common zeros of $(z_1 - f_1(z_2))$ and $(z_1 - f_2(z_2))$, or equivalently at the case when $f_1(z_2) = f_2(z_2)$, i.e., $z_2 = -z_2 + 2$, which results in $z_2 = 1$.

In the sequel, we assume that $f_t(z_2) \not\equiv f_j(z_2)$, for any pair of $t, j \in \{1, \ldots, p\}$, $t \neq j$, and define

$$\mathcal{V} = \{z_2 \in \mathbf{C} \mid f_t(z_2) = f_j(z_2), \text{ for any pair of } t, j \in \{1, \ldots, p\}, t \neq j\}. \tag{21}$$

Because $f_1(z_2), \ldots, f_p(z_2)$ are 1D polynomials and $f_t(z_2) \not\equiv f_j(z_2)$, for any pair of $t, j \in \{1, \ldots, p\}, t \neq j$, the number of points contained in $\mathcal{V}$ is finite. These points are denoted by $z_{21}, \ldots, z_{2P}$.

Theorem 3. *Let $F(\mathbf{z})$ be given in Definition* 2. *Assume that $d_0(\mathbf{z}) = d_0(z_1, z_2) = \prod_{k=1}^{p}(z_1 - f_k(z_2))$ is a common divisor of $a_1(\mathbf{z}), \ldots, a_\beta(\mathbf{z})$, i.e., $a_i(\mathbf{z}) = d_0(z_1, z_2)\, e_i(\mathbf{z})$ ($i = 1, \ldots, \beta$), and $f_t(z_2) \not\equiv f_j(z_2)$, for any pair of $t, j \in \{1, \ldots, p\}, t \neq j$. Let $\mathcal{V}$ be defined by* (21). *If $d_0(z_1, z_2)$, $e_1(\mathbf{z}), \ldots, e_\beta(\mathbf{z})$ have no common zeros,* **and** *for $k = 1, \ldots, p$ and $r = 1, \ldots, P$,* rank $F(f_k(z_{2r}), z_{2r}, z_3, \ldots, z_n) = l - 1$ *for every $(z_3, \ldots, z_n) \in \mathbf{C}^{n-2}$, then $F(\mathbf{z})$ admits a general matrix factorization in* (2).

Proof. From the assumption,

$$\begin{aligned} a_i(\mathbf{z}) &= d_0(z_1, z_2)\, e_i(\mathbf{z}) \\ &= \prod_{k=1}^{p}(z_1 - f_k(z_2))\, e_i(\mathbf{z}) \\ &= (z_1 - f_1(z_2))\, e_i'(\mathbf{z}), \qquad i = 1, \ldots, \beta, \end{aligned} \tag{22}$$

where $e_i' = \prod_{k=2}^{p}(z_1 - f_k(z_2))\, e_i$.

Consider first an arbitrary but fixed $z_{20} \in \mathbf{C} \backslash \mathcal{V}$. It is clear that $(z_1 - f_1(z_{20}))$ and $\prod_{k=2}^{p}(z_1 - f_k(z_{20}))$ have no common zeros as $f_1(z_{20}) \neq f_k(z_{20})$ for

$k = 2, \ldots, p$. It follows that $(z_1 - f_1(z_{20}))$, $e'_1(z_1, z_{20}, z_3, \ldots, z_n), \ldots, e'_\beta(z_1, z_{20}, z_3, \ldots, z_n)$ have no common zeros. Arguing similarly as in the proof for Theorem 1, it is easy to show that rank $F(f_1(z_{20}), z_{20}, z_3, \ldots, z_n) = l - 1$ for every $(z_3, \ldots, z_n) \in \mathbf{C}^{n-2}$. Now consider an arbitrary but fixed $z_{2r} \in \mathcal{V}$ $(r = 1, \ldots, P)$. By assumption, rank $F(f_1(z_{2r}), z_{2r}, z_3, \ldots, z_n) = l - 1$ for every $(z_3, \ldots, z_n) \in \mathbf{C}^{n-2}$.

Therefore, we have rank $F(f_1(z_2), z_2, \ldots, z_n) = l-1$ for every $(z_2, \ldots, z_n) \in \mathbf{C}^{n-1}$. By Lemma 2, we can construct a ZRP vector $\mathbf{w}_1 \in \mathbf{C}^{l\times 1}[z_2, \ldots, z_n]$ such that

$$F(f_1(z_2), z_2, \ldots, z_n)\mathbf{w}_1(z_2, \ldots, z_n) = [0, \ldots, 0]^T. \tag{23}$$

Arguing similarly as in the proof of Theorem 1, it is easy to show that $F(\mathbf{z})$ can be factorized as

$$F(\mathbf{z}) = F_1(\mathbf{z})\, G_1(\mathbf{z}) \tag{24}$$

with $F_1 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, $G_1 \in \mathbf{C}^{l\times l}[\mathbf{z}]$, and $\det G_1 = (z_1 - f_1(z_2))$. Now $\prod_{k=2}^{p}(z_1 - f_k(z_2))$ is a common divisor of the $l \times l$ minors of $F_1(\mathbf{z})$. Repeating the above procedure for $F_1(\mathbf{z})$, we have

$$F_1(\mathbf{z}) = F_2(\mathbf{z})\, G_2(\mathbf{z}), \tag{25}$$

with $F_2 \in \mathbf{C}^{m\times l}[\mathbf{z}]$, $G_2 \in \mathbf{C}^{l\times l}[\mathbf{z}]$, and $\det G_2 = (z_1 - f_2(z_2))$. Combining (24) and (25) gives

$$F(\mathbf{z}) = F_2(\mathbf{z})\, G_2(\mathbf{z})\, G_1(\mathbf{z}). \tag{26}$$

Repeating this process for p times, we finally arrive at

$$\begin{aligned} F(\mathbf{z}) &= F_p(\mathbf{z})G_p(\mathbf{z}) \cdots G_2(\mathbf{z})\, G_1(\mathbf{z}) &\quad (27)\\ &= F_0(\mathbf{z})\, G_0(\mathbf{z}), &\quad (28)\end{aligned}$$

with $F_0 = F_p \in \mathbf{C}^{m\times l}[\mathbf{z}]$, $G_0 = \prod_{k=1}^{p} G_k \in \mathbf{C}^{l\times l}[\mathbf{z}]$, and $\det G_0 = \prod_{k=1}^{p}(z_1 - f_k(z_2)) = d_0(z_1, z_2)$. □

Remark 3. It is necessary to assume that $f_t(z_2) \not\equiv f_j(z_2)$, for any pair of $t, j \in \{1, \ldots, p\}$, $t \neq j$. Otherwise, assume that $f_t(z_2) \equiv f_j(z_2)$ for some $t, j \in \{1, \ldots, p\}$. Then the expression of equation (22) becomes

$$a_i(\mathbf{z}) = (z_1 - f_t(z_2))\, e'_i(\mathbf{z}), \qquad i = 1, \ldots, \beta,$$

where $e'_i = (z_1 - f_t(z_2)) \prod_{k=1,k\neq t,j}^{p}(z_1 - f_k(z_2))\, e_i(z)$. Because $(z_1 - f_t(z_2))$ is a common divisor of $(z_1 - f_t(z_2))$, $e'_1(\mathbf{z}), \ldots, e'_\beta(\mathbf{z})$, it is then obvious that $(z_1 - f_t(z_2))$, $e'_1(\mathbf{z}), \ldots, e'_\beta(\mathbf{z})$ have many common zeros, and hence the results of Theorem 1 cannot be applied here. At present, we do not know how to solve this problem when $f_t(z_2) \equiv f_j(z_2)$ for some $t, j \in \{1, \ldots, p\}$, $t \neq j$.

Remark 4. To test whether $d_0(z_1, z_2)$ can be factorized as $d_0(z_1, z_2) = \prod_{k=1}^{p}(z_1 - f_k(z_2))$ and to carry out the above factorization of $d_0(z_1, z_2)$ when it exists, we can use either the method proposed in [21], or in the case when the coefficients are from the ring of integers or the field of rational numbers, a computationally more efficient software package such as SINGULAR [9]. To find the solutions of $f_t(z_2) = f_j(z_2)$, we can first transform this problem to finding the roots of the equation $f_{tj}(z_2) = f_t(z_2) - f_j(z_2) = 0$, and then employ any one of the many existing root-finding methods (see, e.g., [26]).

We point out that for the three classes of nD polynomial matrices discussed in Theorems 1–3, we have not only proved the existence of general matrix factorization (2), but also given constructive methods for carrying out the actual factorizations, as evident from the proofs. Notice that the additional condition that for $k = 1, \ldots, p$ and $r = 1, \ldots, P$, rank $F(f_k(z_{2r}), z_{2r}, z_3, \ldots, z_n) = l - 1$ for every $(z_3, \ldots, z_n) \in \mathbf{C}^{n-2}$ can be tested in a finite number of steps because there are only a finite number of points in $\mathcal{V}$, and we only need to show that all the $(l-1) \times (l-1)$ minors of $F(f_k(z_{2r}), z_{2r}, z_3, \ldots, z_n)$ are zero coprime, which can be accomplished by using Gröbner bases [5].

Can Theorem 1 be further generalized to the case where $d(\mathbf{z}) = \prod_{k=1}^{p}(z_1 - f_k(z_2, \ldots, z_n))$ with f_k depending on more than one variable? Our answer is yes theoretically, but no practically. Consider the simple case where $p = 2$, $d(z_2, z_3) = (z_1 - f_1(z_2, z_3))(z_1 - f_2(z_2, z_3))$. Similarly as in (21), let

$$\mathcal{V}' = \{(z_2, z_3) \in \mathbf{C}^2 \mid f_1(z_2, z_3) = f_2(z_2, z_3)\}. \tag{29}$$

Because $f_1(z_2, z_3)$ and $f_2(z_2, z_3)$ are 2D polynomials, the number of points in $\mathcal{V}'$ is now *infinite* even if $f_1(z_2, z_3) \not\equiv f_2(z_2, z_3)$. Hence, it is computationally intractable to test whether or not for an arbitrary but fixed $(z_{21}, z_{31}) \in \mathcal{V}'$, rank $F(f_k(z_{21}, z_{31}), z_{21}, z_{31}, z_4, \ldots, z_n) = l - 1$ for every $(z_4, \ldots, z_n)$ and for $k = 1, 2$. Therefore, it remains an open problem for obtaining the required general matrix factorization (2) for an arbitrary nD polynomial matrix even when the existence is proved.

The matrix factorization technique developed in Theorems 1–3 is very general in that it deals with both rectangular and square matrices, and it assumes that d_0 is a common divisor (but not necessarily the g.c.d.) of the maximal minors of F. As mentioned earlier, this general matrix factorization scheme can be specialized to two important cases: MRP factorization and determinantal factorization, as presented in the following.

Corollary 1. *Let $F(\mathbf{z})$ be given in Definition 2. Assume that $d(\mathbf{z})$ is a g.c.d. of $a_1(\mathbf{z}), \ldots, a_\beta(\mathbf{z})$, i.e., $a_i(\mathbf{z}) = d(\mathbf{z})\, b_i(\mathbf{z})$ $(i = 1, \ldots, \beta)$ and is given in one of the following three forms:*

(i) $d(\mathbf{z}) = z_1 - f(z_2, \ldots, z_n)$;
(ii) $d(\mathbf{z}) = d(z_1) = \prod_{k=1}^{p}(z_1 - z_{1k})$, *with* $z_{1t} \neq z_{1j}$, *for any pair of* $t, j \in \{1, \ldots, p\}, t \neq j$;
(iii) $d(\mathbf{z}) = d(z_1, z_2) = \prod_{k=1}^{p}(z_1 - f_k(z_2))$ *(with an additional assumption that* $f_t(z_2) \not\equiv f_j(z_2)$, *for any pair of* $t, j \in \{1, \ldots, p\}, t \neq j$, *and for* $k = 1, \ldots, p$ *and* $r = 1, \ldots, P$, rank $F(f_k(z_{2r}), z_{2r}, z_3, \ldots, z_n) = l - 1$ *for every* $(z_3, \ldots, z_n) \in \mathbf{C}^{n-2}$*).*

If $d(\mathbf{z})$, $b_1(\mathbf{z}), \ldots, b_\beta(\mathbf{z})$ have no common zeros, then $F(\mathbf{z})$ admits the following MRP factorization:

$$F(\mathbf{z}) = F_0(\mathbf{z})\, G_0(\mathbf{z}), \tag{30}$$

such that $F_0 \in \mathbf{C}^{m\times l}[\mathbf{z}]$ is MRP, $G_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$, and $\det G_0 = d(\mathbf{z})$.

Corollary 2. *Let $F \in \mathbf{C}^{l\times l}[\mathbf{z}]$ and $\det F(\mathbf{z}) = e_1(\mathbf{z})\, d_0(\mathbf{z})$. Assume that $d_0(\mathbf{z})$ is one of the following three forms:*

(i) $d_0(\mathbf{z}) = z_1 - f(z_2, \ldots, z_n)$;

(ii) $d_0(\mathbf{z}) = d(z_1) = \prod_{k=1}^{p}(z_1 - z_{1k})$, *with $z_{1t} \neq z_{1j}$, for any pair of $t, j \in \{1, \ldots, p\}, t \neq j$;*

(iii) $d_0(\mathbf{z}) = d(z_1, z_2) = \prod_{k=1}^{p}(z_1 - f_k(z_2))$ *(with an additional assumption that $f_t(z_2) \not\equiv f_j(z_2)$, for any pair of $t, j \in \{1, \ldots, p\}, t \neq j$, and for $k = 1, \ldots, p$ and $r = 1, \ldots, P$,* $\operatorname{rank} F(f_k(z_{2r}), z_{2r}, z_3, \ldots, z_n) = l - 1$ *for every $(z_3, \ldots, z_n) \in \mathbf{C}^{n-2}$).*

If $d_0(\mathbf{z})$ and $e_1(\mathbf{z})$ have no common zeros, then $F(\mathbf{z})$ admits the following determinantal factorization:

$$F(\mathbf{z}) = F_0(\mathbf{z})\, G_0(\mathbf{z}), \tag{31}$$

such that $F_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with $\det F_0 = e_1(\mathbf{z})$, *and $G_0 \in \mathbf{C}^{l\times l}[\mathbf{z}]$ with* $\det G_0 = d_0(\mathbf{z})$.

Remark 5. We point out that in Theorems 1–3 as well as in Corollaries 1 and 2, the condition that $d_0(\mathbf{z}), e_1(\mathbf{z}), \ldots, e_\beta(\mathbf{z})$ (or $d(\mathbf{z}), b_1(\mathbf{z}), \ldots, b_\beta(\mathbf{z})$) have no common zeros is only a sufficient (but in general not a necessary) condition for $F(\mathbf{z})$ to admit a general matrix factorization (2). It is therefore of interest to find a necessary and sufficient condition. This is a topic for future research.

4. Examples

In this section, we present three examples to illustrate the proposed general matrix factorization scheme for three classes of nD polynomial matrices. The first two examples are concerned with MRP factorization, and the third example involves determinantal factorization. The constructive feature of our new methods will be emphasized.

Example 1. Let

$$F(z_1, z_2, z_3) = \begin{bmatrix} 2z_1z_2 + z_1^2 - 2z_2 - 1 & -z_1z_3 + z_3 \\ 2z_2^3 + z_1z_2^2 + 2z_1z_2 + 4z_2^2 + 2z_1 + 4z_2 + 1 & -z_2^2z_3 - 2z_2z_3 + z_2 - 2z_3 + 1 \\ 2z_2^2 + z_1z_2 - 2z_2 + 2z_2z_3 + z_1z_3 + z_3 - z_1 - 2 & -z_2z_3 - z_3^2 + z_3 + 1 \end{bmatrix}.$$

It can be checked that the g.c.d. of all the 2×2 minors of F is $d(z_1, z_2, z_3) = z_1 - z_2z_3 + 2z_2 - z_3 + 1$, and the reduced minors are $b_1 = (z_1 - 1)(z_2 + 1)$, $b_2 = -z_2z_3 - z_3 + 2z_2 + 3$, and $b_3 = z_1 - 1$. Clearly, b_1, b_2, b_3 have many common zeros, e.g.,

$(1, 0, 3)$. However, we observe that d, b_1, b_2, b_3 are devoid of any common zeros, and $d(z_1, z_2, z_3)$ is of the form $z_1 - f(z_2, z_3)$, with $f(z_2, z_3) = z_2z_3 - 2z_2 + z_3 - 1$. By Theorem 1 or Corollary 1, $F(z_1, z_2, z_3)$ admits an MRP factorization. In the following, we illustrate the steps for obtaining such a factorization.

Substituting $z_1 = f(z_2, z_3)$ into $F(z_1, z_2, z_3)$. After simplification, we have

$$F(f(z_2, z_3), z_2, z_3) = \begin{bmatrix} z_3(z_2+1)(z_2z_3 - 2z_2 + z_3 - 2) & -z_3(z_2z_3 - 2z_2 + z_3 - 2) \\ -(z_2+1)(-z_2^2z_3 - 2z_2z_3 + z_2 - 2z_3 + 1) & -z_2^2z_3 - 2z_2z_3 + z_2 - 2z_3 + 1 \\ -(z_2+1)(-z_2z_3 - z_3^2 + z_3 + 1) & -z_2z_3 - z_3^2 + z_3 + 1 \end{bmatrix}.$$

Let

$$U = \begin{bmatrix} 1 & 0 \\ (z_2+1) & 1 \end{bmatrix}.$$

We have

$$F(f(z_2, z_3), z_2, z_3)U = \begin{bmatrix} 0 & -z_3(z_2z_3 - 2z_2 + z_3 - 2) \\ 0 & -z_2^2z_3 - 2z_2z_3 + z_2 - 2z_3 + 1 \\ 0 & -z_2z_3 - z_3^2 + z_3 + 1 \end{bmatrix},$$

or

$$F(z_1, z_2, z_3)U = \underbrace{\begin{bmatrix} z_1 - 1 & -z_1z_3 + z_3 \\ z_2^2 + 2z_2 + 2 & -z_2^2z_3 - 2z_2z_3 + z_2 - 2z_3 + 1 \\ z_2 + z_3 - 1 & -z_2z_3 - z_3^2 + z_3 + 1 \end{bmatrix}}_{F_0} \underbrace{\begin{bmatrix} z_1 - z_2z_3 + 2z_2 - z_3 + 1 & 0 \\ 0 & 1 \end{bmatrix}}_{D},$$

or

$$F(z_1, z_2, z_3) = F_0(z_1, z_2, z_3)\, G_0(z_1, z_2, z_3),$$

where

$$G_0 = DU^{-1} = \begin{bmatrix} z_1 - z_2z_3 + 2z_2 - z_3 + 1 & 0 \\ -(z_2+1) & 1 \end{bmatrix}.$$

Clearly, $\det G_0 = z_1 - z_2z_3 + 2z_2 - z_3 + 1 = d$, and we have obtained the desired MRP factorization for $F(z_1, z_2, z_3)$.

We next consider the case where $d(z_1) = \prod_{k=1}^{p}(z_1 - z_{1k})$, with $z_{1i} \neq z_{1j}$, $i \neq j$.

Example 2. Let

$$F(z_1, z_2, z_3) = \begin{bmatrix} z_1^2z_3 + z_2z_3 + z_3 + 1 & z_1^3z_3 + 2z_1^2z_3 + z_1z_2z_3 + 2z_2z_3 + z_1z_3 + 2z_3 + z_1 + 2 \\ z_1 + z_2 + 1 & z_1^2 + z_1z_2 + 3z_1 + 2z_2 + 2 \\ 0 & z_1^2 - z_1 \end{bmatrix}.$$

It is easy to check that the g.c.d. of all the 2×2 minors of F is $d(z_1) = z_1^2 - z_1 = z_1(z_1 - 1)$, and the reduced minors are $b_1 = 0$, $b_2 = z_1 + z_2 + 1$, and $b_3 = z_1^2z_3 + z_2z_3 + z_3 + 1$. Clearly, b_1, b_2, b_3 have many common zeros, e.g., $(-1, 0, -0.5)$. However, d, b_1, b_2, b_3 are devoid of any common zeros, and $d(\mathbf{z})$ is of the form $d(\mathbf{z}) = d(z_1) = \prod_{k=1}^{p}(z_1 - z_{1k})$, with $z_{1i} \neq z_{1j}, i \neq j$. By

Theorem 2 or Corollary 1, F admits an MRP factorization. In the following, we illustrate the steps for obtaining such a factorization.

For $z_1 = 0$, we have

$$F(0, z_2, z_3) = \begin{bmatrix} z_2z_3 + z_3 + 1 & 2z_2z_3 + 2z_3 + 2 \\ z_2 + 1 & 2z_2 + 2 \\ 0 & 0 \end{bmatrix}.$$

Let

$$U_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}.$$

We have

$$F(0, z_2, z_3)U_1 = \begin{bmatrix} 0 & z_2z_3 + z_3 + 1 \\ 0 & z_2 + 1 \\ 0 & 0 \end{bmatrix},$$

or

$$F(z_1, z_2, z_3)U_1 = \underbrace{\begin{bmatrix} -(z_1^2z_3 + z_2z_3 + z_3 + 1) & z_1^2z_3 + z_2z_3 + z_3 + 1 \\ -(z_1 + z_2 + 1) & z_1 + z_2 + 1 \\ -z_1 + 1 & 0 \end{bmatrix}}_{F_1} \underbrace{\begin{bmatrix} z_1 & 0 \\ 0 & 1 \end{bmatrix}}_{D_1},$$

or

$$F(z_1, z_2, z_3) = F_1(z_1, z_2, z_3)\, G_1(z_1, z_2, z_3), \tag{32}$$

where

$$G_1 = D_1U_1^{-1} = \begin{bmatrix} 0 & -z_1 \\ 1 & 2 \end{bmatrix}.$$

Clearly, $\det G_1 = z_1$. Continuing in the same manner, we can factorize $F_1(z_1, z_2, z_3)$ as

$$F_1(z_1, z_2, z_3) = F_2(z_1, z_2, z_3)\, G_2(z_1, z_2, z_3), \tag{33}$$

where

$$F_2 = \begin{bmatrix} 0 & z_1^2z_3 + z_2z_3 + z_3 + 1 \\ 0 & z_1 + z_2 + 1 \\ -1 & 0 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} z_1 - 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

Combining (32) and (33) gives

$$F(z_1, z_2, z_3) = F_0(z_1, z_2, z_3)G_0(z_1, z_2, z_3),$$

where $F_0 = F_2$ is MRP, $G_0 = G_2\, G_1 = \begin{bmatrix} 0 & -z_1(z_1 - 1) \\ 1 & z_1 + 2 \end{bmatrix}$, with $\det G_0 = d = z_1(z_1 - 1)$.

Finally, we present another example to illustrate Theorem 3 or Corollary 2 for determinant factorization for a square matrix.

Example 3. Let

$$F(z_1, z_2, z_3) = \begin{bmatrix} z_1^2 z_3 - z_1 z_2 z_3 + z_1 z_2 - z_2^2 - z_1 + 2z_2 & -z_2^2 z_3 + z_1 z_2 z_3 - z_1 z_2 + z_1^2 - 2z_1 + 3z_2 \\ -z_1^2 z_3 - z_1 z_2 z_3 + 3z_1 z_3 + z_2 - 2 & -z_2^2 z_3 - z_1 z_2 z_3 + 3z_2 z_3 + z_1 - 2 \end{bmatrix}.$$

It is easy to check that $\det F = e_1 d_0$, where $d_0 = (z_1 - z_2)(z_1 + z_2 - 2) = (z_1 - f_1(z_2))(z_1 - f_2(z_2))$ and $e_1 = 1 + z_3(z_1 - z_2)(z_1 + z_2 - 2) = 1 + z_3 d_0$. Hence, d_0 and e_1 are zero coprime. We need to compute the set $\mathcal{V}$ first. For this simple example, the only solution to the equation $z_2 = 2 - z_2$ is when $z_2 = 1$. Hence, $z_2 = 1$ is the only point in $\mathcal{V}$. We have

$$F(f_1(z_2), z_2, z_3) = \begin{bmatrix} z_2 & z_2 \\ -2z_2^2 z_3 + 3z_2 z_3 + z_2 - 2 & -2z_2^2 z_3 + 3z_2 z_3 + z_2 - 2 \end{bmatrix},$$

or

$$F(f_1(1), 1, z_3) = \begin{bmatrix} 1 & 1 \\ z_3 - 1 & z_3 - 1 \end{bmatrix}.$$

It is obvious that rank $F(f_1(1), 1, z_3) = 2 - 1 = 1$ for any $z_3 \in \mathbf{C}$. Hence, rank $F(f_1(z_2), z_2, z_3) = 1$ for every $(z_2, z_3) \in \mathbf{C}^2$. It can be similarly tested that rank $F(f_2(z_2), z_2, z_3) = 1$ for every $(z_2, z_3) \in \mathbf{C}^2$. Therefore, by Theorem 3 or Corollary 2, F admits a determinantal factorization.

Let

$$U_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

We have

$$F(f_1(z_2), z_2, z_3)U_1 = \begin{bmatrix} 0 & z_2 \\ 0 & -2z_2^2 z_3 + 3z_2 z_3 + z_2 - 2 \end{bmatrix},$$

or

$$F(z_1, z_2, z_3)U_1 = \underbrace{\begin{bmatrix} z_1 z_3 - z_1 - z_2 z_3 + z_2 + 1 & -z_2^2 z_3 + z_1 z_2 z_3 - z_1 z_2 + z_1^2 - 2z_1 + 3z_2 \\ -z_1 z_3 - z_2 z_3 + 3z_3 - 1 & -z_2^2 z_3 - z_1 z_2 z_3 + 3z_2 z_3 + z_1 - 2 \end{bmatrix}}_{F_1} \underbrace{\begin{bmatrix} (z_1 - z_2) & 0 \\ 0 & 1 \end{bmatrix}}_{D_1}$$

or

$$F(z_1, z_2, z_3) = F_1(z_1, z_2, z_3)\, G_1(z_1, z_2, z_3), \tag{34}$$

where

$$G_1 = D_1 U_1^{-1} = \begin{bmatrix} (z_1 - z_2) & 0 \\ 1 & 1 \end{bmatrix}.$$

Clearly, $\det G_1 = z_1 - z_2$. Continuing in the same manner, we can factorize $F_1(z_1, z_2, z_3)$ as

$$F_1(z_1, z_2, z_3) = F_2(z_1, z_2, z_3)\, G_2(z_1, z_2, z_3), \tag{35}$$

where

$$F_2 = \begin{bmatrix} z_2 - z_1 & z_1z_3 - z_1 - z_2z_3 + z_2 + 1 \\ -1 & -z_1z_3 - z_2z_3 + 3z_3 - 1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} 0 & -z_1 - z_2 + 2 \\ 1 & z_2 \end{bmatrix}.$$

Combining (34) and (35) gives

$$F(z_1, z_2, z_3) = F_0(z_1, z_2, z_3)G_0(z_1, z_2, z_3),$$

where $F_0 = F_2$, with $\det F_0 = e_1 = 1 + z_3(z_1 - z_2)(z_1 + z_2 - 2)$,

$$G_0 = G_2\,G_1 = \begin{bmatrix} -z_1 - z_2 + 2 & -z_1 - z_2 + 2 \\ z_1 & z_2 \end{bmatrix},$$

with $\det G_0 = d_0 = (z_1 - z_2)(z_1 + z_2 - 2)$.

5. Conclusions

In this paper, we have considered the open problem of minor prime factorization [6] and the conjecture on minor prime factorizability [18], as well as the open problem of determinantal factorization [32], [13] for nD ($n > 2$) polynomial matrices. We have presented a constructive general matrix factorization scheme for extracting a nontrivial factor from a given nD polynomial matrix whose maximal minors satisfy certain conditions. It has been shown that three classes of nD polynomial matrices admit this kind of general matrix factorization. It turns out that minor prime factorization and determinantal factorization are two interesting special cases of the proposal general factorization. As a consequence, we have provided a partial solution to the open problem of minor prime factorization [6] and the conjecture on minor prime factorizability [18] for nD polynomial matrices. Three illustrative examples have also been worked out in detail.

Finally, we admit that the new results can only deal with the classes of nD polynomial matrices discussed in this paper. The general matrix factorization for an arbitrary nD polynomial matrix remains a challenging and important open problem (see also [12, p. 63]).

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