On the MIMO Channel Capacity for the General Channels

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Abstract

A general central limit theorem of the linear spectral statistics for sample covariance matrices has been established. Such a theorem is then used to obtain the asymptotic channel capacity of a multiple input multiple output system.

KEY WORDS: Capacity, Multiple input multiple output, random matrix theory, central limit theorem.

1 Introduction

Consider a MIMO (multiple input multiple output) system with n antennas at the transmitter and p antennas at the receiver. Such a channel can be modeled as

$$\mathbf{y} = \mathbf{Z}\mathbf{x} + \mathbf{n},$$

where $\mathbf{y} \in \mathbb{C}^p$ denotes the received signal vector, $\mathbf{x} \in \mathbb{C}^n$ denotes the transmitted signal vector, $\mathbf{Z}$ is a complex channel matrix of size $p \times n$ consisting of independent and identically distributed (i.i.d.) $\{Z_{ij}\}$ and $\mathbf{n} \in \mathbb{C}^p$ represents circularly-symmetric complex Gaussian noise with $\mathbb{E}(\mathbf{n}\mathbf{n}^*) = \mathbf{I}$. Here $\mathbf{n}^*$ denotes the transpose conjugate of $\mathbf{n}$. The signal power is constrained by $\mathbb{E}\mathbf{x}^*\mathbf{x} \le \rho$. We also assume that the realization of $\mathbf{Z}$ is known at the receiver but not at the transmitter and that $\mathbf{Z}, \mathbf{x}$ and $\mathbf{n}$ are independent.

It is of interest to evaluate the capacity of the above system. According to (Telatar (1999)) the MIMO instantaneous capacity is given by

$$C(\mathbf{Q}) = \log\det(\mathbf{I} + \mathbf{Z}\mathbf{Q}\mathbf{Z}^*), \tag{1.1}$$

where $\mathbf{Q}$ is the covariance matrix of $\mathbf{x}$ with $tr\mathbf{Q} \le \rho$ and the MIMO channel capacity is

$$C = \sup_{\mathbf{Q}\ge 0, tr\mathbf{Q}\le\rho} \mathbb{E}[C(\mathbf{Q})]. \tag{1.2}$$

Note that when the entries of $\mathbf{Z}$, $\{Z_{ij}\}$, are i.i.d complex Gaussian with mean zero and variance one, the distribution of $\mathbf{Z}$ is the same as that of $\mathbf{Z}\mathbf{U}$ where $\mathbf{U}$ is any constant unitary matrix U of size $n \times n$. As in Section 4.1 of Telatar (1999) one then has

$$C = \mathbb{E}\log\det(\mathbf{I} + \frac{\rho}{n}\mathbf{Z}\mathbf{Z}^*),$$

which implies that the optimizing Q is $\frac{\rho}{n}\mathbf{I}$. Therefore, as in Kamath and Hughes (2005) and Hachem et al. (2008), we consider the random variable

$$C_{\mathbf{Z}} = \log\det(\mathbf{I} + \frac{\rho}{n}\mathbf{Z}\mathbf{Z}^*), \tag{1.3}$$

referred to as Shannon's mutual information.

Another quantity associated with the capacity is the outage rate. Given an outage probability $0 < q < 1$, the outage rate is defined by

$$C_q = \sup\{R \geq 0 : P\Big(C_{\mathbf{Z}} < R\Big) \leq q\}, \tag{1.4}$$

the largest rate of reliable communication for a fixed outage probability.

As pointed out in Kamath and Hughes (2005) Shannon's mutual information and outage rate are often evaluated by simulations since there is no explicit expression for $C_{\mathbf{Z}}$ except a few special cases. To overcome this, Telatar (1999) obtained a closed-form asymptotic formula for the capacity ($\mathbb{E}C_{\mathbf{Z}}$) by using random matrix theory and assuming that $\frac{p}{n} \to c > 0$. The formula obtained is more informative than finite-array results. Furthermore, Kamath and Hughes (2005) investigates the fluctuation of $C_{\mathbf{Z}}$ around its limit and obtains its central limit theorem (CLT).

When doing so, Telatar (1999) considers the classical Rayleigh fading model that the entries of $\mathbf{Z}$ are assumed to be i.i.d. zero mean complex Gaussian with independent real and imaginary parts sharing the same variance. Having independent real and imaginary parts, which share the same variance, implies $\mathbb{E}Z_{11}^2 = 0$. Indeed, Kamath and Hughes (2005) and Hachem et al. (2008) also impose such a condition that $\mathbb{E}Z_{11}^2 = 0$ to develop CLT of $C_{\mathbf{Z}}$ when $p/n \to c > 0$. However, as pointed out by Fraidenraich et al. (2009) there is the case where the real and imaginary parts of the entries have different variances, their modulus being therefore distributed according to the Hoyt distribution (see Fraidenraich et al. (2009)). Indeed, the Hoyt distribution has found applications in the error performance evaluation of digital communication, outage analysis in cellular mobile radio system, or satellite channel modelling (see Fraidenraich et al. (2009)).

One of the aims of this work is to develop CLT of $C_{\mathbf{Z}}$ when $p/n \to c > 0$ without the constraint that $\mathbb{E}Z_{11}^2 = 0$. Specifically, we consider the matrix $\mathbf{Z} = \mathbf{Z}_n$ under the following basic assumptions.

(**a**) For each n, $Z_{ij} = Z_{ij}^{(n)}$, $1 \leq i \leq p, 1 \leq j \leq n$ are i.i.d complex variables. And the variables satisfy the following moment conditions

$$\mathbb{E}Z_{11} = 0,\ \mathbb{E}|Z_{11}|^2 = 1,\ |\mathbb{E}Z_{11}^2| = \Phi,\ \mathbb{E}|Z_{11}|^4 = \Psi. \tag{1.5}$$

(**b**) Assume $p =: p(n)$ and $p/n \to c \in (0, +\infty)$.

Then we have the following.

Theorem 1. *Suppose that* $\mathbf{Z}$ *satisfies conditions (***a***) and (***b***) above. Let*

$$\lambda_+ = -\frac{1}{2}(1 - \rho + \rho c) + \frac{1}{2}\sqrt{(1 - \rho + \rho c)^2 + 4\rho},$$
$$\lambda_- = -\frac{1}{2}(1 - \rho + \rho c) - \frac{1}{2}\sqrt{(1 - \rho + \rho c)^2 + 4\rho}.$$

We have

$$\log\det(\mathbf{I} + \frac{\rho}{n}\mathbf{Z}\mathbf{Z}^*) - \int \log(1 + \rho x)dF_{c_n}(x) \to N(\mu, \sigma^2), \tag{1.6}$$

where $c_n =: p/n$, $F_{c_n}(x)$ is the Marcenko and Pastur (M-P) law with parameter c_n,

$$\mu = \frac{(\Psi - \Phi^2 - 2)c}{2}\left(\frac{\lambda_+}{1+\lambda_+}\right)^2 + \frac{1}{2}\log[1 - \Phi^2 c\left(\frac{\lambda_+}{1+\lambda_+}\right)^2] \tag{1.7}$$

and

$$\sigma^2 = (\Psi - \Phi^2 - 2)c\left(\frac{\lambda_+}{1+\lambda_+}\right)^2 - \log\left(1 - \frac{\rho - \lambda_+}{\rho - \lambda_-}\right) + \log\frac{1}{1 - \Phi^2 c\left(\frac{\lambda_+}{1+\lambda_+}\right)^2}. \tag{1.8}$$

2 Random matrices

To establish Theorem 1, we generalize an important CLT of Bai and Silverstein (2004) in the field of random matrices. Random matrices have been used in wireless communication since Grant and Alexanders 1996 conference presentation [9] and it has proven to be a powerful tool.

Let $\mathbf{Z}_n = [Z_{ij}]_{p\times n}$ satisfying (**a**) and (**b**). We consider the sample covariance matrix $\mathbf{B}_n = \frac{1}{n}\mathbf{Z}_n\mathbf{Z}_n^*$ in this section. To present our main results, we first introduce some notation. Define the empirical spectral distribution (ESD) of an Hermitian matrix $\mathbf{A}$ by

$$F^{\mathbf{A}}(x) = \frac{1}{p}\sum_{k=1}^{p} I(\lambda_k \le x),$$

where $\lambda_k, k = 1, \cdots, p$ denote the eigenvalues of $\mathbf{A}$. Then $F^{\mathbf{B}_n}$ converges with probability one to the M-P law with the density function

$$f_c(x) = \begin{cases} (2\pi c x)^{-1}\sqrt{(b-x)(x-a)} & a \le x \le b. \\ 0 & \text{otherwise.} \end{cases}$$

It has point mass $1 - c^{-1}$ at the origin if $c > 1$, where $a = (1-\sqrt{c})^2$ and $b = (1+\sqrt{c})^2$. Moreover, we denote $c_n =: p/n$ and F_{c_n} to be the M-P law with parameter c_n. Set

$$G_n(x) =: p(F^{\mathbf{B}_n}(x) - F_{c_n}(x)).$$

For any test function $f(x)$ we write

$$L_n(f) = \int f(x)dG_n(x).$$

Theorem 2. *Assume that $\mathbf{Z}_n = [Z_{ij}]_{p\times n}$ satisfy the conditions (**a**) and (**b**). Let m be any fixed positive integer. Let $f_1, \ldots, f_m$ be functions analytic on an open region containing the interval $[I_{(0,1)}(c)(1-\sqrt{c})^2, (1+\sqrt{c})^2]$. Then the random vector*

$$\Big(L_n(f_1), L_n(f_2), \ldots, L_n(f_m)\Big)$$

forms a tight sequence in n and converges weakly to a Gaussian vector $(L_{f_1}, L_{f_2}, \ldots, L_{f_m})$ with mean vector $\boldsymbol{\mu}$ whose components are, for any $\ell = 1, 2, \ldots, m$,

$$\mathbb{E}L_{f_\ell} = -\frac{1}{2\pi i}\int_{\mathcal{C}} f_\ell(z)\frac{c\big(1+zs(z)\big)^3}{1 - c\big(1+zs(z)\big)^2}\Big[(\Psi - \Phi^2 - 2) + \frac{\Phi^2}{1 - \Phi^2 c\big(1+zs(z)\big)^2}\Big]dz \tag{2.1}$$

and covariance matrix $\widetilde{\boldsymbol{\Sigma}}$ whose units are, for any $\ell, \nu = 1, 2, \ldots, m$,

$$\begin{aligned}
& \operatorname{Cov}(L_{f_\ell}, L_{f_\nu}) \\
= \; & -\frac{1}{4\pi^2} c(\Psi - \Phi^2 - 2) \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} f_\ell(z_1) f_\nu(z_2) \big(z_1 s'(z_1) + s(z_1)\big) \big(z_2 s'(z_2) + s(z_2)\big) dz_1 dz_2 \\
& -\frac{1}{4\pi^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} f_\ell(z_1) f_\nu(z_2) \frac{\Phi^2 c\big(z_1 s'(z_1) + s(z_1)\big)\big(z_2 s'(z_2) + s(z_2)\big)}{\big(1 - \Phi^2 a(z_1, z_2)\big)^2} dz_1 dz_2 \\
& -\frac{1}{4\pi^2} \int_{\mathcal{C}_1} \int_{\mathcal{C}_2} f_\ell(z_1) f_\nu(z_2) \frac{c\big(z_1 s'(z_1) + s(z_1)\big)\big(z_2 s'(z_2) + s(z_2)\big)}{\big(1 - a(z_1, z_2)\big)^2} dz_1 dz_2, \qquad (2.2)
\end{aligned}$$

where

$$a(z_1, z_2) = c\big(1 + z_1 s(z_1)\big)\big(1 + z_2 s(z_2)\big).$$

The contours in (2.1) and (2.2) are contained in the analytic region of the functions $f_1, f_2, \ldots, f_m$ and enclose the support of $F_{c_n}(x)$ for all large n. Moreover, $\mathcal{C}_1$ and $\mathcal{C}_2$ are selected to be disjoint.

Remark 1. *Theorem 2 is a generalization of the result of Bai and Silverstein (2004) by removing the assumptions $\mathbb{E}Z_{11}^2 = 0$ and $\mathbb{E}|Z_{11}|^4 = 2$ and also a generalization of that of Pan and Zhou (2008) by removing the assumptions $\mathbb{E}Z_{11}^2 = 0$ under the special setting of $\mathbf{T} = \mathbf{I}_p$. Moreover, our results can imply the classical real case ($\Psi = 3, \Phi = 1$) and the classical complex case ($\Psi = 2, \Phi = 0$) in Bai and Silverstein (2004) directly (by setting $\mathbf{T} = \mathbf{I}_p$ in Bai and Silverstein (2004) and using the relation (3.3) below). And from our result it can also be easily seen that the covariance function for the classical real case is 2 times of that for the classical complex case.*

3 Proof of Theorem 2

To prove Theorem 2, we will follow the main strategy in Bai and Silverstein (2004). A lot of existent results can be borrowed from their paper directly. Thus at first, we will introduce some necessary known results without proofs. To present them, we shall further introduce more concepts and notation below.

For any distribution function $G(x)$, its Stieltjes transform is defined to be

$$s_G(z) = \int \frac{1}{\lambda - z} dG(\lambda), \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C}, \Im z > 0\}.$$

For convenience, we denote the M-P law with parameter c and its Stieltjes transform by $F_c(x)$ and $s(z)$ respectively. It is well known that $s(z)$ satisfies the following equation

$$s(z) = \frac{1}{1 - z - c - czs(z)}. \qquad (3.1)$$

Now define $\underline{B}_n \equiv 1/n \mathbf{Z}_n^* \mathbf{Z}_n$ and denote its LSD and limiting Stieltjes transform by $\underline{F}_c$ and $\underline{s} = \underline{s}(z)$ respectively. Noting that $\underline{B}_n$ and B_n share the same non-zero eigenvalues, thus it is not difficult to see

$$\underline{F}_c = (1 - c) I_{[0,\infty)} + c F_c.$$

Consequently, we can derive the relation between $s(z)$ and $\underline{s}(z)$ as

$$\underline{s}(z) = -\frac{1-c}{z} + cs(z). \tag{3.2}$$

Moreover, by the equation (3.1) and the relation (3.2) we have

$$s(z) = \frac{1}{-z - z\underline{s}(z)}. \tag{3.3}$$

Hence

$$\underline{s}(z) = -\frac{1}{zs(z)} - 1. \tag{3.4}$$

Consequently, we also have

$$\frac{c\underline{s}^2(z)}{(1+\underline{s}(z))^2} = c(zs(z)+1)^2.$$

To lighten the notation we replace the symbols $s_{F^{\mathbf{B}_n}}(z), s_{F_{c_n}}(z), s_{F^{\underline{\mathbf{B}}_n}}(z), s_{\underline{F}_{c_n}}(z)$ by $s_n(z), s_n^0(z), \underline{s}_n(z), \underline{s}_n^0(z)$ in the sequel. Let v_0 be some small positive constant. Then basically, if $\Im z \geq v_0$ one has

$$|s_n(z)|, |s_n^0(z)|, |\underline{s}_n(z)|, |\underline{s}_n^0(z)|, |s(z)|, |\underline{s}(z)| \leq v_0^{-1},$$

and

$$s_n^0(z) = s(z) + o(1), \underline{s}_n^0(z) = \underline{s}^0(z) + o(1).$$

Now we set

$$M_n(z) = p[s_n(z) - s_n^0(z)].$$

By definition, we also have

$$M_n(z) = n[\underline{s}_n(z) - \underline{s}_n^0(z)].$$

To prove Theorem 2, the basic idea in Bai and Silverstein (2004) is to use the Cauchy integral formula

$$\int f(x)dG(x) = -\frac{1}{2\pi i}\int_{\mathcal{C}} f(z)s_G(z)dz$$

valid for any c.d.f G and any function f analytic on an open set containing the support of G. In our case

$$G(x) := G_n(x) = p(F^{\mathbf{B}_n}(x) - F_{c_n}(x)).$$

Note the support of $G_n(x)$ is random. Fortunately, it is well known that the extreme eigenvalues of $\mathbf{B}_n$ are highly concentrated around two edges of the support of the limiting M-P law $F_c(x)$. Then the contour $\mathcal{C}$ can be appropriately chosen. Moreover, it was shown in Bai and Silverstein (2004) that one can replace the process $\{M_n(z), \mathcal{C}\}$ by a slightly modified process $\{\widehat{M}_n(z), \mathcal{C}\}$. Below we present the definitions of the contour $\mathcal{C}$ and the modified process $\widehat{M}_n(z)$. Let x_r be any number greater than $(1+\sqrt{c})^2$. Let x_l be any negative number if $c \geq 1$. Otherwise we choose $x_l \in (0,(1-\sqrt{c})^2)$. Now let

$$\mathcal{C}_u = \{x + iv_0 : x \in [x_l, x_r]\}.$$

Then we define

$$\mathcal{C}^+ \equiv \{x_l + iv : v \in [0, v_0]\} \cup \mathcal{C}_u \cup \{x_r + iv : v \in [0, v_0]\},$$

and $\mathcal{C} = \mathcal{C}^+ \cup \overline{\mathcal{C}^+}$. Now we define the subsets $\mathcal{C}_n$ of $\mathcal{C}$ on which $M_n(\cdot)$ equals to $\widehat{M}_n(\cdot)$. Let $\{\varepsilon_n\}$ be a sequence decreasing to zero satisfying for some $\alpha \in (0,1)$,

$$\varepsilon_n \geq n^{-\alpha}.$$

Now we set

$$\mathcal{C}_l = \begin{cases} \{x_l + iv : v \in [n^{-1}\varepsilon_n, v_0]\} & \text{if } x_l > 0, \\ \{x_l + iv : v \in [0, v_0]\} & \text{if } x_l < 0, \end{cases}$$

and

$$\mathcal{C}_r = \{x_r + iv : v \in [n^{-1}\varepsilon, v_0]\}.$$

Then we define $\mathcal{C}_n = \mathcal{C}_l \cup \mathcal{C}_u \cup \mathcal{C}_r$. The process $\widehat{M}_n(z)$ is defined as

$$\widehat{M}_n(z) = \begin{cases} M_n(z) & \text{for } z \in \mathcal{C}_n, \\ M_n(x_r + in^{-1}\varepsilon_n) & \text{for } x = x_r, v \in [0, n^{-1}\varepsilon_n], \\ M_n(x_l + in^{-1}\varepsilon_n) & \text{for } x = x_l, v \in [0, n^{-1}\varepsilon_n]. \end{cases}$$

And we set $\widehat{M}_n(\bar{z}) = \overline{\widehat{M}_n(z)}$.

To prove Theorem 2, it has been shown in Bai and Silverstein (2004) that it suffices to prove the CLT for $\widehat{M}_n(z)$ with $z \in \mathcal{C}$. We state the result as the following lemma and then prove it.

Lemma 1. *Under the assumptions of Theorem 2, $\{\widehat{M}_n(\cdot)\}$ forms a tight sequence on $\mathcal{C}^+$. And $\{\widehat{M}_n(\cdot)$ converges weakly to a two-dimensional Gaussian process $\{M(\cdot)\}$ satisfying for $z \in \mathcal{C}^+$*

$$\mathbb{E}M(z) = \frac{c(1+zs(z))^3}{[1-c(1+zs(z))^2]}\left[(\Psi - \Phi^2 - 2) + \frac{\Phi^2}{1-\Phi^2 c(1+zs(z))^2}\right],$$

and for $z_i, z_j \in \mathcal{C}$

$$\begin{aligned} \operatorname{Cov}(M(z_i), M(z_j)) = \Gamma(z_i, z_j) \; &= \; (\Psi - \Phi^2 - 2)c(z_i s'(z_i) + s(z_i))(z_j s'(z_j) + s(z_j)) \\ &\quad + \frac{\Phi^2 c(z_i s'(z_i) + s(z_i))(z_j s'(z_j) + s(z_j))}{(1-\Phi^2 a(z_i, z_j))^2} \\ &\quad + \frac{c(z_i s'(z_i) + s(z_i))(z_j s'(z_j) + s(z_j))}{(1-a(z_i, z_j))^2}, \end{aligned}$$

where

$$a(z_i, z_j) = c(1 + z_i s(z_i))(1 + z_j s(z_j)).$$

By the discussions in Bai and Silverstein (2004), we see that Theorem 2 holds if Lemma 1 is proved. Thus the remaining work will be devoted to the proof of Lemma 1. To this end, we shall truncate the variables at first. By the discussion in Bai and Silverstein (2004), one can truncate $|Z_{ij}|$ at $\eta_n\sqrt{n}$ without altering the limiting behavior of $(L_n(f_1),\cdots,L_n(f_k))$. Here η_n is some positive number slowly converging to 0. For instance, one can choose $\eta_n \downarrow 0$ such that $\eta_n n^{1/5} \to \infty$. Moreover, one can further recentralize and renormalize the truncated variables to be with means zero and variances 1 without altering the limiting behavior of $(L_n(f_1),\cdots,L_n(f_k))$. Thus without loss of generality, we can always assume that $|Z_{ij}| \le \eta_n\sqrt{n}$ and

$$\mathbb{E}Z_{ij}=0,\quad \mathbb{E}|Z_{ij}|^2=1,\quad |\mathbb{E}Z_{ij}^2|=\Phi+o(1),\quad \mathbb{E}|Z_{ij}|^4=\Psi+o(1). \tag{3.5}$$

Write for $z\in\mathcal{C}_n$, $M_n(z)=M_n^{(1)}(z)+M_n^{(2)}(z)$, where

$$M_n^{(1)}(z)=p[s_n(z)-\mathbb{E}s_n(z)]$$

and

$$M_n^{(2)}(z)=p[\mathbb{E}s_n(z)-s_n^0(z)].$$

By the discussion in Bai and Silverstein (2004), it suffices to show the following four statements.

(**i**): *$M_n^{(1)}(z)$ is tight on $\mathcal{C}_n$.*
(**ii**): *Finite-dimensional convergence of $M_n^{(1)}(z)$ on $\mathcal{C}_n$.*
(**iii**): *$\{M_n^{(2)}(z)\}$ for $z\in\mathcal{C}_n$ is bounded and equicontinuous.*
(**iv**): *The following convergence holds.*

$$M_n^{(2)}(z)\to\mathbb{E}M(z),\quad z\in\mathcal{C}_n.$$

In Bai and Silverstein (2004), the proofs of (**i**) and (**iii**) do not rely on the additional assumptions of $\mathbb{E}Z_{11}^2=0$ and $\mathbb{E}|Z_{11}|^4=2$. So they still work under our assumptions. We will not present the proofs of these two statements here. Therefore, we will focus on the proofs of (**ii**) and (**iv**) in the sequel. At first, we deal with the convergence of $M_n^{(2)}(z)$, i.e. the statement (**iv**). Our result can be stated as the following proposition

Proposition 1. *Under the assumptions of Theorem 2, for $z\in\mathcal{C}_n$, one has*

$$M_n^{(2)}(z)=\frac{c(1+zs(z))^3}{[1-c(1+zs(z))^2]}\left[(\Psi-\Phi^2-2)+\frac{\Phi^2}{1-\Phi^2c(1+zs(z))^2}\right]+o(1).$$

Proof. For ease of presentation, we will omit the variable z from $s_n(z)$,$s_n^0(z)$,$\underline{s}_n(z)$,$\underline{s}_n^0(z)$, $s(z)$ and $\underline{s}(z)$ when there is no confusion. To derive the limit of

$$M_n^{(2)}(z)=p[\mathbb{E}s_n-s_n^0]=n[\mathbb{E}\underline{s}_n-\underline{s}_n^0],$$

we use the following equation provided in Bai and Silverstein (2006)(see (9.11.1) of Bai and Silverstein (2006)).

$$(\mathbb{E}\underline{s}_n-\underline{s}_n^0)\left(1-\frac{\frac{c_n\underline{s}_n^0}{(1+\mathbb{E}\underline{s}_n)(1+\underline{s}_n^0)}}{-z+\frac{c_n}{1+\mathbb{E}\underline{s}_n}-R_n}\right)=\mathbb{E}\underline{s}_n\underline{s}_n^0R_n, \tag{3.6}$$

where

$$R_n = c_n n^{-1} \sum_{j=1}^{n} \mathbb{E}\beta_j d_j (\mathbb{E}\underline{s}_n)^{-1},$$
$$d_j = d_j(z) = \frac{1}{\mathbb{E}\underline{s}_n + 1}(-\mathbf{r}_j^*(\mathbf{B}_{(j)} - z\mathbf{I})^{-1}\mathbf{r}_j + n^{-1}tr(\mathbf{B}_n - z\mathbf{I})^{-1}),$$
$$\beta_j^{-1} = 1 + \mathbf{r}_j^*(\mathbf{B}_{(j)} - z\mathbf{I})^{-1}\mathbf{r}_j.$$

Here $\mathbf{r}_j$ is the j-th column of $1/\sqrt{n}\mathbf{Z}_n$ and $\mathbf{B}_{(j)} = \mathbf{B}_n - \mathbf{r}_j\mathbf{r}_j^*$. Some arguments in Chapter 9 of Bai and Silverstein (2006) (see pages 272-273) show that

$$\sup_{z\in\mathcal{C}_n} |\mathbb{E}\underline{s}_n(z) - \underline{s}(z)| \to 0, \quad \sup_{z\in\mathcal{C}_n} |\underline{s}_n^0(z) - \underline{s}(z)| \to 0,$$
$$\sup_{z\in\mathcal{C}_n} |R_n| \to 0, \quad n \to \infty. \tag{3.7}$$

Combining the fact

$$-z + \frac{c}{1+\underline{s}(z)} = \frac{1}{\underline{s}(z)} \tag{3.8}$$

with $\underline{s}_n^0(z) \to \underline{s}(z)$, we obtain

$$\frac{\frac{c_n \underline{s}_n^0}{(1+\mathbb{E}\underline{s}_n)(1+\underline{s}_n^0)}}{-z + \frac{c_n}{1+\mathbb{E}\underline{s}_n} - R_n} \to \frac{c\underline{s}^2}{(1+\underline{s})^2}. \tag{3.9}$$

Thus by (3.6), (3.7) and (3.9), it suffices to evaluate the quantity nR_n. Therefore, our task is to estimate

$$c_n \sum_{j=1}^{n} \mathbb{E}\beta_j d_j \tag{3.10}$$

in the sequel. To this end, we shall further define some notations. Let

$$\mathbf{D} := \mathbf{D}(z) = \mathbf{B}_n - z\mathbf{I}, \quad \mathbf{D}_j := \mathbf{D}_j(z) = \mathbf{D} - \mathbf{r}_j\mathbf{r}_j^* = \mathbf{B}_{(j)} - z\mathbf{I},$$
$$\mathbf{D}_{ij} := \mathbf{D}_{ij}(z) = \mathbf{D} - \mathbf{r}_i\mathbf{r}_i^* - \mathbf{r}_j\mathbf{r}_j^*.$$

and

$$\beta_j = \frac{1}{1+\mathbf{r}_j^*\mathbf{D}_j^{-1}\mathbf{r}_j}, \quad \bar{\beta}_j = \frac{1}{1+n^{-1}tr\mathbf{D}_j^{-1}},$$
$$b_n(z) = \frac{1}{1+n^{-1}\mathbb{E}tr\mathbf{D}_1^{-1}}, \quad \hat{\gamma}_j = \mathbf{r}_j^*\mathbf{D}_j^{-1}\mathbf{r}_j - n^{-1}tr\mathbf{D}_j^{-1}.$$

Using the identity $\beta_j = \bar{\beta}_j - \bar{\beta}_j^2\hat{\gamma}_j + \bar{\beta}_j^2\beta_j\hat{\gamma}_j^2$, then some routine arguments as those in Bai and Silverstein (2006) (see the last line of page 273 therein) lead us to the estimation

$$c_n \sum_{j=1}^{n} \mathbb{E}\beta_j d_j = \frac{1}{\underline{s}_n + 1}\sum_{j=1}^{n} \mathbb{E}\bar{\beta}_j^2\hat{\gamma}_j^2 - \frac{1}{\underline{s}_n + 1}\sum_{j=1}^{n} \mathbb{E}\beta_j^2\mathbf{r}_j^*\mathbf{D}_j^{-2}\mathbf{r}_j + o(1). \tag{3.11}$$

It was shown in Bai and Silverstein (2004) that both β_j and $\bar{\beta}_j$ can be estimated by $-z\underline{s}(z)$ in the evaluation of the quantity (3.10). In fact one has

$$\mathbb{E}\beta_j = b_n + O(n^{-1}), \quad \mathbb{E}|\beta_j - b_n|^2 \le Kn^{-1}, \quad b_n = -z\underline{s}(z) + o(1), \tag{3.12}$$

where K is some positive constant. Moreover, $b_n(z)$ and $|(zI - b_n(z))^{-1}|$ are bounded. For details of these results, we refer to (4.14) of Bai and Silverstein (2004) and the discussion above it. The boundness of $b_n(z)$ can be found in page 581 of Bai and Silverstein (2004). Consequently, one has

$$\frac{1}{\underline{s}_n + 1}\mathbb{E}\beta_j^2 \mathbf{r}_j^* \mathbf{D}_j^{-2} \mathbf{r}_j = \frac{1}{n}\frac{z^2\underline{s}^2}{\underline{s}+1}\mathbb{E}tr\mathbf{D}_j^{-2} + o(1). \tag{3.13}$$

Moreover,

$$\frac{1}{\underline{s}_n + 1}\sum_{j=1}^{n}\mathbb{E}\bar{\beta}_j^2\hat{\gamma}_j^2 = \frac{z^2\underline{s}^2}{\underline{s}+1}\sum_{j=1}^{n}\mathbb{E}\hat{\gamma}_j^2(1+o(1)) \tag{3.14}$$

Thus a key step is to estimate the values of $\mathbb{E}\hat{\gamma}_j^2$ for $j = 1, \cdots, n$. We see from definition that $\mathbb{E}\hat{\gamma}_j^2$ is in the form of

$$\begin{aligned} &\mathbb{E}(\mathbf{r}_j^*\mathbf{P}\mathbf{r}_j - tr\mathbf{P})(\mathbf{r}_j^*\mathbf{Q}\mathbf{r}_j - tr\mathbf{Q}) \\ &= \sum_{i=1}^{p}(\mathbb{E}|Z_{11}|^4 - |\mathbb{E}Z_{11}^2|^2 - 2)\mathbb{E}p_{ii}q_{ii} + |\mathbb{E}Z_{11}^2|^2\mathbb{E}tr\mathbf{P}\mathbf{Q}' + \mathbb{E}tr\mathbf{P}\mathbf{Q}. \end{aligned} \tag{3.15}$$

Here $\mathbf{P} = (p_{ij}), \mathbf{Q} = (q_{ij})$ are $p \times p$ matrices independent of $\mathbf{r}_j$.

Letting $\mathbf{P} = \mathbf{Q} = \mathbf{D}_j^{-1}$, we have

$$\mathbb{E}\hat{\gamma}_j^2 = \frac{1}{n^2}\left(\sum_{i=1}^{p}(\Psi - \Phi^2 - 2)\mathbb{E}[\mathbf{D}_j^{-1}]_{ii}^2 + \Phi^2\mathbb{E}tr\mathbf{D}_j^{-1}(\mathbf{D}_j^{-1})' + \mathbb{E}tr\mathbf{D}_j^{-2}\right).$$

We claim that

$$\mathbb{E}[\mathbf{D}_j^{-1}]_{ii}^2 = s^2(z) + o(1), \quad \frac{1}{n}\mathbb{E}tr\mathbf{D}_j^{-2} = cs'(z) + o(1). \tag{3.16}$$

Indeed, the first equation can be seen from the estimation (4.1) in Pan and Zhou (2008) (by choosing $\mathbf{T} = \mathbf{I}_p$ therein). And the second one follows from

$$\sup_{z\in\mathcal{C}_n}\mathbb{E}|\frac{1}{p}tr\mathbf{D}_j^{-2} - s'(z)| = \sup_{z\in\mathcal{C}_n}\mathbb{E}|\int\frac{dF^{\mathbf{B}_{(j)}}(x)}{(x-z)^2} - \int\frac{dF_c(x)}{(x-z)^2}| = o(1).$$

Thus the main task is to calculate the limit of the quantity

$$\frac{1}{n}\mathbb{E}tr\mathbf{D}_j^{-1}(\mathbf{D}_j^{-1})', 1 \le j \le n.$$

Now we set

$$\beta_{ij} =: \beta_{ij}(z) = \frac{1}{1 + \mathbf{r}_i^*\mathbf{D}_{ij}^{-1}\mathbf{r}_i}.$$

And we use the following decomposition, which has been presented in Bai and Silverstein (2004)(see (2.9) therein):

$$\mathbf{D}_j^{-1}(z) = -(z - \frac{n-1}{n} b_n(z))^{-1}\mathbf{I}$$
$$+\sum_{i\neq j} \beta_{ij}(z)(z - \frac{n-1}{n} b_n(z))^{-1}\mathbf{r}_i\mathbf{r}_i^*\mathbf{D}_{ij}^{-1}(z)$$
$$-\frac{n-1}{n} b_n(z)(z - \frac{n-1}{n} b_n(z))^{-1}\mathbf{D}_j^{-1}(z)$$
$$=: -(z - \frac{n-1}{n} b_j(z))^{-1}\mathbf{I} + b_n(z)\mathbf{A}(z) + \mathbf{B}(z) + \mathbf{C}(z), \tag{3.17}$$

where the matrices $\mathbf{A}(z), \mathbf{B}(z), \mathbf{C}(z)$ are

$$\mathbf{A}(z) = (z - \frac{n-1}{n} b_n(z))^{-1} \sum_{i\neq j} (\mathbf{r}_i\mathbf{r}_i^* - n^{-1}\mathbf{I})\mathbf{D}_{ij}^{-1}(z),$$
$$\mathbf{B}(z) = (z - \frac{n-1}{n} b_n(z))^{-1} \sum_{i\neq j} (\beta_{ij}(z) - b_n(z))\mathbf{r}_i\mathbf{r}_i^*\mathbf{D}_{ij}^{-1}(z),$$
$$\mathbf{C}(z) = n^{-1} b_n(z)(z - \frac{n-1}{n} b_n(z))^{-1} \sum_{i\neq j} (\mathbf{D}_{ij}^{-1} - \mathbf{D}_j^{-1}(z)).$$

Note that by the fact that $b_n(z)$ and $|(z - b_n(z))^{-1}|$ are bounded, one see for sufficiently large n, $|(z - \frac{n-1}{n} b_n(z))^{-1}|$ is also bounded. Moreover, by (4.15) and (4.16) of Bai and Silverstein (2004), one has for any $p \times p$ matrix $\mathbf{M}$, there exist

$$\mathbb{E}|tr\mathbf{B}(z)\mathbf{M}| \leq K(\mathbb{E}||\mathbf{M}||^4)^{1/4} n^{1/2}, \quad |tr\mathbf{C}(z)\mathbf{M}| \leq K(\mathbb{E}||\mathbf{M}||^2)^{1/2}. \tag{3.18}$$

When $\mathbf{M}$ is non-random, we also have for any j,

$$\mathbb{E}|tr\mathbf{A}(z)\mathbf{M}| \leq K||\mathbf{M}||n^{1/2}. \tag{3.19}$$

Using this inequality, by the bounds provided in (3.1) of Bai and Silverstein (2004) and the decomposition (3.17), it is not difficult to see

$$\frac{1}{n}\mathbb{E}tr\mathbf{D}_j^{-1}(z)(\mathbf{D}_j^{-1}(z))' = c_n(z - \frac{n-1}{n} b_n(z))^{-2} + \frac{1}{n}\mathbb{E}tr b_n(z)\mathbf{A}(z)(\mathbf{D}_j^{-1})'(z) + o(1)$$
$$= c_n(z + z\underline{s}(z))^{-2} + \frac{1}{n} b_n(z)\mathbb{E}tr\mathbf{A}(z)(\mathbf{D}_j^{-1})'(z) + o(1)$$
$$= c_n(z + z\underline{s}(z))^{-2} + b_n(z)(z - \frac{n-1}{n} b_n(z))^{-1}$$
$$\times \frac{1}{n}\mathbb{E}tr(\sum_{i\neq j}(\mathbf{r}_i\mathbf{r}_i^* - n^{-1}\mathbf{I})\mathbf{D}_{ij}^{-1}(z))(\mathbf{D}_j^{-1})'(z) + o(1). \tag{3.20}$$

Using the fact

$$\mathbf{D}_j^{-1}(z) - \mathbf{D}_{ij}^{-1}(z) = -\mathbf{D}_{ij}^{-1}(z)\mathbf{r}_i\mathbf{r}_i^*\mathbf{D}_{ij}^{-1}(z)\beta_{ij}(z), \tag{3.21}$$

we can write

$$\mathbb{E}tr(\sum_{i\neq j}(\mathbf{r}_i\mathbf{r}_i^* - n^{-1}\mathbf{I})\mathbf{D}_{ij}^{-1}(z))(\mathbf{D}_j^{-1})' = A_1(z) + A_2(z),$$

where

$$A_1(z) = -\mathbb{E}tr(\sum_{i\neq j}\mathbf{r}_i\mathbf{r}_i^*\mathbf{D}_{ij}^{-1}(z))(\beta_{ij}(z)\mathbf{D}_{ij}^{-1}(z)\mathbf{r}_i\mathbf{r}_i^*\mathbf{D}_{ij}^{-1}(z))',$$
$$A_2(z) = -\mathbb{E}tr\sum_{i\neq j}n^{-1}\mathbf{D}_{ij}^{-1}(z)(\mathbf{D}_j^{-1}(z) - \mathbf{D}_{ij}^{-1}(z))'.$$

Similar to (4.18) of Bai and Silverstein (2004), we have

$$|A_2(z)| \leq K.$$

Thus it suffices to estimate $A_1(z)$. Note that by definition we have

$$A_1(z) = -\mathbb{E}\sum_{i\neq j}\beta_{ij}(z)\mathbf{r}_i^*\mathbf{D}_{ij}^{-1}(z)(\mathbf{D}_{ij}^{-1}(z))'\bar{\mathbf{r}}_i\mathbf{r}_i'(\mathbf{D}_{ij}^{-1}(z))'\mathbf{r}_i. \tag{3.22}$$

To evaluate (3.22), we need the following lemma.

Lemma 2. *For non-random $p \times p$ matrices $\mathbf{A}_k, k = 1, \cdots, m$ and $\mathbf{B}_l, l = 1, \cdots, q$, there exists*

$$\left|\mathbb{E}\left(\prod_{k=1}^m \mathbf{r}_t'\mathbf{A}_k\mathbf{r}_t\prod_{l=1}^q[\mathbf{r}_t^*\mathbf{B}_l\bar{\mathbf{r}}_t - n^{-1}\mathbb{E}(\bar{Z}_{11})^2tr\mathbf{B}_l]\right)\right|$$
$$\leq Kn^{-(1\wedge q)}\eta_n^{(2q-4)\vee 0}\prod_{k=1}^m||\mathbf{A}_k||\prod_{l=1}^q||\mathbf{B}_l||.$$

And the inequality also holds if we replace $(\mathbf{r}_t, Z_{11})$ by $(\bar{\mathbf{r}}_t, \bar{Z}_{11})$.

Remark 2. *The proof of Lemma 2 is nearly the same as that of (9.9.6) of Bai and Silverstein (2006), and thus here we omit it.*

With the aid of Lemma 2 and (3.12), it is not difficult to see that

$$\begin{aligned}&\mathbb{E}\beta_{ij}(z)\mathbf{r}_i^*\mathbf{D}_{ij}^{-1}(z)(\mathbf{D}_{ij}^{-1}(z))'\bar{\mathbf{r}}_i\mathbf{r}_i'(\mathbf{D}_{ij}^{-1}(z))'\mathbf{r}_i\\ =\ & b_n(z)\frac{\Phi^2}{n^2}\mathbb{E}tr\mathbf{D}_{ij}^{-1}(z)(\mathbf{D}_{ij}^{-1}(z))'\mathbb{E}tr\mathbf{D}_{ij}^{-1}(z) + o(1)\end{aligned} \tag{3.23}$$

by using the bounds provided in (3.1) of Bai and Silverstein (2004). Above we used the fact that

$$\mathbb{E}Z_{11}^2\mathbb{E}\bar{Z}_{11}^2 = \Phi^2.$$

Further, by (3.1)-(3.3), (3.6) and (4.3) of Bai and Silverstein (2004), one has both

$$\mathbb{E}tr\mathbf{D}_{ij}^{-1}(z) - \mathbb{E}tr\mathbf{D}_j^{-1}(z)$$

and

$$\mathbb{E}tr\mathbf{D}_{ij}^{-1}(z)(\mathbf{D}_{ij}^{-1}(z))' - \mathbb{E}tr\mathbf{D}_j^{-1}(z)(\mathbf{D}_j^{-1}(z))'$$

are bounded. Thus we also have

$$\begin{aligned}&\mathbb{E}\beta_{ij}(z)\mathbf{r}_i^*\mathbf{D}_{ij}^{-1}(z)(\mathbf{D}_{ij}^{-1}(z))'\bar{\mathbf{r}}_i\mathbf{r}_i'(\mathbf{D}_{ij}^{-1}(z))'\mathbf{r}_i\\ =\ & b_n(z)\frac{\Phi^2}{n^2}\mathbb{E}tr\mathbf{D}_j^{-1}(z)(\mathbf{D}_j^{-1}(z))'\mathbb{E}tr\mathbf{D}_j^{-1}(z) + o(1).\end{aligned}$$

Consequently, we have

$$A_1(z) + \frac{n-1}{n^2}\Phi^2 b_n(z)\mathbb{E}tr\mathbf{D}_j^{-1}(z)(\mathbf{D}_j^{-1}(z))'\mathbb{E}tr\mathbf{D}_j^{-1}(z) = o(1).$$

Therefore, by using (3.12) one can get

$$\frac{1}{n}\mathbb{E}Tr\mathbf{D}_j^{-1}(z)(\mathbf{D}_j^{-1}(z))'(1 - c\Phi^2\frac{z^2\underline{s}^2(z)}{(z+z\underline{s}(z))^2}) = c(z+z\underline{s}(z))^{-2} + o(1),$$

which implies

$$\begin{aligned}\frac{1}{n}\mathbb{E}Tr\mathbf{D}_j^{-1}(z)(\mathbf{D}_j^{-1}(z))' &= \frac{c}{(z+z\underline{s}(z))^2 - \Phi^2 cz^2\underline{s}^2(z)} + o(1)\\ &= \frac{cs^2(z)}{1-\Phi^2 c(1+zs(z))^2} + o(1).\end{aligned}$$

Here we used the fact that $0 \le \Phi \le 1$ and

$$|c\frac{\underline{s}^2(z)}{(1+\underline{s}(z))^2}| < 1,$$

which is implied by (4.5) of Bai and Silverstein (2004). Consequently one has

$$n\hat{\gamma}_j^2 = (\Psi - \Phi^2 - 2)cs^2(z) + \frac{c\Phi^2 s^2(z)}{1-\Phi^2 c(1+zs(z))^2} + cs'(z) + o(1).$$

Then by (3.11)-(3.14), we have

$$c_n\sum_{j=1}^n \mathbb{E}\beta_j d_j = \frac{z^2\underline{s}^2}{\underline{s}+1}((\Psi - \Phi^2 - 2)cs^2(z) + \frac{c\Phi^2 s^2(z)}{1-\Phi^2 c(1+zs(z))^2}) + o(1).$$

Therefore, by the discussions in (3.6)-(3.10) we have

$$\begin{aligned}&n(\mathbb{E}s_n(z) - s_n^0(z))\\ &= c\frac{\frac{z^2\underline{s}^3}{\underline{s}+1}}{(1-\frac{c\underline{s}^2}{(1+\underline{s})^2})}\left[(\Psi - \Phi^2 - 2)s^2(z) + \frac{\Phi^2 s^2(z)}{1-\Phi^2 c(1+zs(z))^2}\right] + o(1)\\ &= \frac{c(1+zs(z))^3}{[1-c(1+zs(z))^2]}\left[(\Psi - \Phi^2 - 2) + \frac{\Phi^2}{1-\Phi^2 c(1+zs(z))^2}\right] + o(1),\end{aligned}$$

where at the last step above we have used the relation (3.4). □

Next, we come to prove the finite dimensional convergence of $\{M_n^{(1)}(z); z \in \mathcal{C}_n\}$, i.e. the statement (**ii**).

Proposition 2. *Under the conditions of Theorem 2, for any set of k points $\{z_s, s = 1, \cdots, k\}$ of $\mathcal{C}_n$, the random vector $(\{M_n^{(1)}(z_1)\cdots, M_n^{(1)}(z_k)\})$ converges weakly to a k-dimensional zero-mean Gaussian distribution with covariance matrix given by*

$$\begin{aligned}\Gamma(z_i, z_j) &= (\Psi - \Phi^2 - 2)c(z_i s'(z_i) + s(z_i))(z_j s'(z_j) + s(z_j))\\ &\quad + \frac{\Phi^2 c(z_i s'(z_i) + s(z_i))(z_j s'(z_j) + s(z_j))}{(1-\Phi^2 a(z_i, z_j))^2}\\ &\quad + \frac{c(z_i s'(z_i) + s(z_i))(z_j s'(z_j) + s(z_j))}{(1-a(z_i, z_j))^2},\end{aligned} \tag{3.24}$$

where

$$a(z_i, z_j) = c(1 + z_i s(z_i))(1 + z_j s(z_j)).$$

Proof. By resorting to the CLT of martingale, Bai and Silverstein (2004) showed that it suffices to develop the limit (in probability) of the following quantity. That is

$$\sum_{j=1}^{n} \mathbb{E}_{j-1}[Y_j(z_1)Y_j(z_2)],$$

where

$$Y_j(z) = -\mathbb{E}_j \frac{d}{dz}\bar{\beta}_j(z)\hat{\gamma}_j(z).$$

It was shown in Bai and Silverstein (2004) that it suffices to evaluate the limit of

$$\frac{\partial^2}{\partial z_2 \partial z_1} \sum_{j=1}^{n} \mathbb{E}_{j-1}[\mathbb{E}_j(\bar{\beta}_j(z_1)\hat{\gamma}_j(z_1))\mathbb{E}_j(\bar{\beta}_j(z_2)\hat{\gamma}_j(z_2))].$$

only for $z_1, z_2 \in \mathcal{C}_u$. Thus in the sequel, we always assume that $\Im z_1, \Im z_2 = v_0 > 0$. On $\mathcal{C}_u$, we have some more precise bounds such as

$$\begin{aligned} &|(z - \frac{n-1}{n} b_n(z))^{-1}| \le \frac{1 + \frac{p}{nv_0}}{v_0}, \\ &\mathbb{E}|\beta_{ij}(z) - b_n(z)|^2 \le K/n \end{aligned} \tag{3.25}$$

with some positive constant K. A further step taken in Bai and Silverstein (2004)(see (2.7) of Bai and Silverstein (2004)) leads us to estimate the following quantity instead. That is

$$\frac{\partial^2}{\partial z_2 \partial z_1} \sum_{j=1}^{n} b_n(z_1)b_n(z_2)\mathbb{E}_{j-1}[\mathbb{E}_j(\hat{\gamma}_j(z_1))\mathbb{E}_j(\hat{\gamma}_j(z_2))]. \tag{3.26}$$

To determine the limit (in probability) of (3.26), we use (3.15) again. It is not difficult to see

$$\begin{aligned} &n\mathbb{E}_{j-1}[\mathbb{E}_j(\hat{\gamma}_j(z_1))\mathbb{E}_j(\hat{\gamma}_j(z_2))] \\ &= \frac{1}{n}\Bigg(\sum_{i=1}^{p} (\Psi - \Phi^2 - 2)[\mathbb{E}_j \mathbf{D}_j^{-1}(z_1)]_{ii}[\mathbb{E}_j \mathbf{D}_j^{-1}(z_2)]_{ii} \\ &+\Phi^2 tr\mathbb{E}_j(\mathbf{D}_j^{-1}(z_1))\mathbb{E}_j(\mathbf{D}_j^{-1}(z_2))' + tr\mathbb{E}_j(\mathbf{D}_j^{-1}(z_1))\mathbb{E}_j(\mathbf{D}_j^{-1}(z_2)) \Bigg). \end{aligned} \tag{3.27}$$

By Pan and Zhou (2008), we have

$$[\mathbb{E}_j \mathbf{D}_j^{-1}(z)]_{ii} \xrightarrow{P} -\frac{1}{z\underline{s}(z) + z} = s(z). \tag{3.28}$$

Moreover we also have the following fact proved by Bai and Silverstein (2004),

$$\frac{1}{n} tr\mathbb{E}_j(\mathbf{D}_j^{-1}(z_1))\mathbb{E}_j(\mathbf{D}_j^{-1}(z_2)) = \frac{cs(z_1)s(z_2)}{1 - \frac{j-1}{n}c(1 + z_1 s(z_1))(1 + z_2 s(z_2))} + o_{L_1}(1). \tag{3.29}$$

Here and after $o_{L_1}(1)$ represents some random variable $\xi := \xi_n$ satisfying $\mathbb{E}|\xi| \to 0$ as $n \to \infty$. Then it suffices to prove the following lemma

Lemma 3. *Under the assumptions of Theorem 2, for $z_1, z_2 \in \mathcal{C}_u$, one has*

$$\frac{1}{n}tr\mathbb{E}_j(\mathbf{D}_j^{-1}(z_1))\mathbb{E}_j(\mathbf{D}_j^{-1}(z_2))' = \frac{cs(z_1)s(z_2)}{1-\frac{j-1}{n}\Phi^2 c(1+z_1 s(z_1))(1+z_2 s(z_2))} + o_{L_1}(1).$$

Proof. To prove Lemma 3, we begin with the decomposition (3.17). Then by using the bounds (3.18) and (3.19) and the fact that $||\mathbf{D}_j(z_2)||$ is bounded in $\mathcal{C}_u$ we have

$$\begin{aligned}
&\frac{1}{n}tr\mathbb{E}_j(\mathbf{D}_j^{-1}(z_1))(\mathbf{D}_j^{-1}(z_2))' \\
= \quad & -\frac{1}{n}(z_1 - \frac{n-1}{n}b_n(z_1))^{-1}tr\mathbf{D}_j^{-1}(z_2) + \frac{1}{n}b_n(z_1)tr\mathbb{E}_j(\mathbf{A}(z_1))(\mathbf{D}_j^{-1}(z_2))' + o_{L_1}(1).
\end{aligned} \tag{3.30}$$

Note that the main task is to estimate the second term of the right hand side of (3.30). Again we can use the definition of $\mathbf{A}(z)$ and (3.21) to write

$$tr\mathbb{E}_j(\mathbf{A}(z_1))(\mathbf{D}_j^{-1}(z_2))' = B_1(z_1,z_2) + B_2(z_1,z_2) + B_3(z_1,z_2),$$

where

$$\begin{aligned}
B_1(z_1,z_2) &= -(z_1 - \frac{n-1}{n}b_n(z_1))^{-1} \\
&\quad \times tr(\sum_{i<j}\mathbf{r}_i\mathbf{r}_i^*[\mathbb{E}_j\mathbf{D}_{ij}^{-1}(z))](\beta_{ij}(z_2)\mathbf{D}_{ij}^{-1}(z_2)\mathbf{r}_i\mathbf{r}_i^*\mathbf{D}_{ij}^{-1}(z_2))', \\
B_2(z_1,z_2) &= -(z_1 - \frac{n-1}{n}b_n(z_1))^{-1} \\
&\quad \times tr\sum_{i<j}n^{-1}[\mathbb{E}_j\mathbf{D}_{ij}^{-1}(z_1)](\mathbf{D}_j^{-1}(z_2) - \mathbf{D}_{ij}^{-1}(z_2))', \\
B_3(z_1,z_2) &= (z_1 - \frac{n-1}{n}b_n(z_1))^{-1}tr\sum_{i<j}(\mathbf{r}_i\mathbf{r}_i^* - n\mathbf{I})[\mathbb{E}_j\mathbf{D}_{ij}^{-1}(z_1)](\mathbf{D}_{ij}^{-1}(z_2))'.
\end{aligned}$$

Similar to the discussions on the terms of (2.14) of Bai and Silverstein (2004), we can see that

$$|B_2(z_1,z_2)| \le K, \quad \mathbb{E}|B_3(z_1,z_2)| \le Kn^{1/2}.$$

Consequently, it suffices to estimate $B_1(z_1,z_2)$ in the sequel. In other words, we have

$$\begin{aligned}
&\frac{1}{n}b_n(z_1)tr\mathbb{E}_j(A(z_1))(\mathbf{D}_j^{-1}(z_2))' \\
&= -\frac{1}{n}b_n(z_1)(z_1 - \frac{n-1}{n}b_n(z_1))^{-1} \\
&\times \sum_{i<j}\beta_{ij}(z_2)\mathbf{r}_i^*[\mathbb{E}_j\mathbf{D}_{ij}^{-1}(z_1)][\mathbf{D}_{ij}^{-1}(z_2)]'\bar{\mathbf{r}}_i\mathbf{r}_i'[\mathbf{D}_{ij}^{-1}(z_2)]'\mathbf{r}_i + o_{L_1}(1).
\end{aligned}$$

Similar to (3.23), by using Lemma 2 and (3.25) , it is not difficult to get

$$\begin{aligned}
&\mathbb{E}|\beta_{ij}(z_2)\mathbf{r}_i^*[\mathbb{E}_j\mathbf{D}_{ij}^{-1}(z_1)][\mathbf{D}_{ij}^{-1}(z_2)]'\bar{\mathbf{r}}_i\mathbf{r}_i'[\mathbf{D}_{ij}^{-1}(z_2)]'\mathbf{r}_i \\
&-b_n(z_2)n^{-2}\Phi^2 tr[\mathbb{E}_j\mathbf{D}_{ij}^{-1}(z_1)][\mathbf{D}_{ij}^{-1}(z_2)]'tr\mathbf{D}_{ij}^{-1}(z_2)| \le Kn^{-1/2}.
\end{aligned} \tag{3.31}$$

Consequently one has

$$\begin{aligned}
&\frac{1}{n} b_n(z_1) tr \mathbb{E}_j(\mathbf{A}(z_1))(\mathbf{D}_j^{-1}(z_2))' \\
&= -\frac{\Phi^2}{n^3} b_n(z_1) b_n(z_2)(z_1 - \frac{n-1}{n} b_n(z_1))^{-1} \\
&\times \sum_{i<j} tr[\mathbb{E}_j \mathbf{D}_{ij}^{-1}(z_1)][\mathbf{D}_{ij}^{-1}(z_2)]' tr[\mathbf{D}_{ij}^{-1}(z_2)] + o_{L_1}(1).
\end{aligned} \tag{3.32}$$

By (2.2) of Bai and Silverstein (2004) we can replace $\mathbf{D}_{ij}$ by $\mathbf{D}_j$ in (3.32), and thus obtain

$$\begin{aligned}
&\frac{1}{n} b_n(z_1) tr \mathbb{E}_j(\mathbf{A}(z_1))(\mathbf{D}_j^{-1}(z_2))' \\
&= -\frac{\Phi^2}{n^3} b_n(z_1) b_n(z_2)(z_1 - \frac{n-1}{n} b_j(z_1))^{-1} \\
&\times (j-1) tr[\mathbb{E}_j \mathbf{D}_j^{-1}(z_1)][\mathbf{D}_j^{-1}(z_2)]' tr[\mathbf{D}_j^{-1}(z_2)] + o_{L_1}(1).
\end{aligned} \tag{3.33}$$

Combining (3.30) and (3.33) we can get

$$\begin{aligned}
&\frac{1}{n} tr \mathbb{E}_j(\mathbf{D}_j^{-1}(z_1)) \mathbb{E}_j(\mathbf{D}_j^{-1}(z_2))' \\
&\times \left(1 + \frac{(j-1)\Phi^2}{n^2} b_n(z_1) b_n(z_2)(z_1 - \frac{n-1}{n} b_n(z_1))^{-1} tr \mathbf{D}_j^{-1}(z_2)\right) \\
&= -\frac{1}{n}(z_1 - \frac{n-1}{n} b_n(z_1))^{-1} tr \mathbf{D}_j^{-1}(z_2) + o_{L_1}(1),
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{n} tr \mathbb{E}_j(D_j^{-1}(z_1)) \mathbb{E}_j(\mathbf{D}_j^{-1}(z_2))' \\
&\times \left(1 + \frac{(j-1)\Phi^2}{n} b_n(z_1) b_n(z_2)(z_1 - \frac{n-1}{n} b_n(z_1))^{-1} c_n s(z_2) + o_{L_1}(1)\right) \\
&= -\frac{1}{n}(z_1 - \frac{n-1}{n} b_n(z_1))^{-1} c_n s(z_2) + o_{L_1}(1).
\end{aligned}$$

By taking the relations (3.3) and (3.12) into account, we can further write

$$\begin{aligned}
&\frac{1}{n} tr \mathbb{E}_j(\mathbf{D}_j^{-1}(z_1)) \mathbb{E}_j(\mathbf{D}_j^{-1}(z_2))' \\
&\times \left(1 - c\frac{j-1}{n} \Phi^2 z_1 \underline{s}(z_1) z_2 \underline{s}(z_2)(z_1 + z_1 \underline{s}(z_1))^{-1}(z_2 + z_2 \underline{s}(z_2))^{-1} + o_{L_1}(1)\right) \\
&= c(z_1 + z_1 \underline{s}(z_1))^{-1}(z_2 + z_2 \underline{s}(z_2))^{-1} + o_{L_1}(1).
\end{aligned}$$

Note that by (2.19) of Bai and Silverstein (2004) and the fact that $\Phi \leq 1$, we have

$$|c\Phi^2 \underline{s}(z_1) \underline{s}(z_2)(1 + \underline{s}(z_1))^{-1}(1 + \underline{s}(z_2))^{-1}| < 1.$$

Thus finally we have

$$
\begin{aligned}
&\frac{1}{n} tr \mathbb{E}_j(\mathbf{D}_j^{-1}(z_1))\mathbb{E}_j(\mathbf{D}_j^{-1}(z_2))' \\
&= c(z_1 + z_1\underline{s}(z_1))^{-1}(z_2 + z_2\underline{s}(z_2))^{-1} \\
&\times \left(1 - c\frac{j-1}{n}\Phi^2 z_1\underline{s}(z_1)z_2\underline{s}(z_2)(z_1 + z_1\underline{s}(z_1))^{-1}(z_2 + z_2\underline{s}(z_2))^{-1}\right)^{-1} + o_{L_1}(1) \\
&= \frac{cs(z_1)s(z_2)}{1 - \frac{j-1}{n}\Phi^2 c(1 + z_1 s(z_1))(1 + z_2 s(z_2))} + o_{L_1}(1).
\end{aligned}
$$

Thus we complete the proof of Lemma 3. □

By using (3.27),(3.28), (3.29) and Lemma 3 we have

$$
\begin{aligned}
& n\mathbb{E}_{j-1}[\mathbb{E}_j(\hat{\gamma}_j(z_1))\mathbb{E}_j(\hat{\gamma}_j(z_2))] \\
= \;& (\Psi - \Phi^2 - 2)cs(z_1)s(z_2) + \frac{\Phi^2 cs(z_1)s(z_2)}{1 - \frac{j-1}{n}\Phi^2 c(1 + z_1 s(z_1))(1 + z_2 s(z_2))} \\
& + \frac{cs(z_1)s(z_2)}{1 - \frac{j-1}{n}c(1 + z_1 s(z_1))(1 + z_2 s(z_2))} + o_{L_1}(1).
\end{aligned}
$$

By (3.12), one has

$$
\begin{aligned}
& \sum_{j=1}^{n} b_n(z_1)b_n(z_2)\mathbb{E}_{j-1}[\mathbb{E}_j(\hat{\gamma}_j(z_1))\mathbb{E}_j(\hat{\gamma}_j(z_2))] \\
&= c(1 + z_1 s(z_1))(1 + z_2 s(z_2))\Bigg[(\Psi - \Phi^2 - 2) \\
&+ \frac{1}{n}\sum_{j=1}^{n} \frac{\Phi^2}{1 - \frac{j-1}{n}\Phi^2 c(1 + z_1 s(z_1))(1 + z_2 s(z_2))} \\
&+ \frac{1}{n}\sum_{j=1}^{n} \frac{1}{1 - \frac{j-1}{n}c(1 + z_1 s(z_1))(1 + z_2 s(z_2))}\Bigg] + o_{L_1}(1).
\end{aligned}
$$

Setting

$$
a(z_1, z_2) = c(1 + z_1 s(z_1))(1 + z_2 s(z_2)),
$$

we obtain

$$
\begin{aligned}
& \sum_{j=1}^{n} b_n(z_1)b_n(z_2)\mathbb{E}_{j-1}[\mathbb{E}_j(\hat{\gamma}_j(z_1))\mathbb{E}_j(\hat{\gamma}_j(z_2))] \\
&= a(z_1, z_2)\left[(\Psi - \Phi^2 - 2) + \int_0^1 \frac{\Phi^2}{1 - t\Phi^2 a(z_1, z_2)}dt + \int_0^1 \frac{1}{1 - ta(z_1, z_2)}dt\right] + o_{L_1}(1).
\end{aligned}
$$

Consequently one has

$$
\begin{aligned}
& \frac{\partial^2}{\partial z_1 \partial z_2}\sum_{j=1}^{n} b_n(z_1)b_n(z_2)\mathbb{E}_{j-1}[\mathbb{E}_j(\hat{\gamma}_j(z_1))\mathbb{E}_j(\hat{\gamma}_j(z_2))] \\
&= (\Psi - \Phi^2 - 2)\frac{\partial^2}{\partial z_1 \partial z_2}a(z_1, z_2) + \frac{\partial}{\partial z_2}\frac{\Phi^2 \frac{\partial}{\partial z_1}a(z_1, z_2)}{1 - \Phi^2 a(z_1, z_2)} + \frac{\partial}{\partial z_2}\frac{\frac{\partial}{\partial z_1}a(z_1, z_2)}{1 - a(z_1, z_2)} + o_{L_1}(1).
\end{aligned}
$$

By an elementary calculation one has

$$\frac{\partial}{\partial z_2}\frac{\Phi^2\frac{\partial}{\partial z_1}a(z_1,z_2)}{1-\Phi^2 a(z_1,z_2)}=\frac{\Phi^2 c(z_1 s'(z_1)+s(z_1))(z_2 s'(z_2)+s(z_2))}{(1-\Phi^2 a(z_1,z_2))^2}.$$

Finally we obtain

$$\begin{aligned}&\frac{\partial^2}{\partial z_1\partial z_2}\sum_{j=1}^n b_n(z_1)b_n(z_2)\mathbb{E}_{j-1}[\mathbb{E}_j(\hat\gamma_j(z_1))\mathbb{E}_j(\hat\gamma_j(z_2))]\\&=(\Psi-\Phi^2-2)c(z_1 s'(z_1)+s(z_1))(z_2 s'(z_2)+s(z_2))\\&+\frac{\Phi^2 c(z_1 s'(z_1)+s(z_1))(z_2 s'(z_2)+s(z_2))}{(1-\Phi^2 a(z_1,z_2))^2}\\&+\frac{c(z_1 s'(z_1)+s(z_1))(z_2 s'(z_2)+s(z_2))}{(1-a(z_1,z_2))^2}+o_{L_1}(1).\end{aligned}$$

Thus we complete the proof of Proposition 2. □

4 Proof of Theorem 1

For convenience, we will focus on the case of $c\le 1$ at first. At the end, we will extend the result to the case of $c>1$ simply by a reciprocal relation. In this section, we will choose the contours $\mathcal{C}$, $\mathcal{C}_1$, $\mathcal{C}_2$ in Theorem 2 to cross the real axis in the interval $(-\rho^{-1},0)$ and (b,∞), where $b=(1+\sqrt{c})^2$. And we will set $f_\ell(z)=f_\nu(z)=\log(1+\rho z)$ in the sequel.

We start with (1.7). We use the following elementary relations

$$1+zs(z)=\frac{\underline{s}(z)}{1+\underline{s}(z)},\quad \underline{s}'(z)=\frac{\underline{s}^2(z)}{1-c(\frac{\underline{s}(z)}{1+\underline{s}(z)})^2}$$

which can be derived from (3.4) and (3.8) easily. Therefore, we have

$$\begin{aligned}\mathbb{E}L_{f_\ell}&=-(\Psi-\Phi^2-2)\frac{1}{2\pi i}\int_{\mathcal{C}}f_\ell(z)\frac{c(\frac{\underline{s}(z)}{1+\underline{s}(z)})^3}{1-c(\frac{\underline{s}(z)}{1+\underline{s}(z)})^2}dz\\&\quad-\frac{1}{2\pi i}\int_{\mathcal{C}}f_\ell(z)\frac{\Phi^2 c(\frac{\underline{s}(z)}{1+\underline{s}(z)})^3}{(1-c(\frac{\underline{s}(z)}{1+\underline{s}(z)})^2)(1-\Phi^2 c(\frac{\underline{s}(z)}{1+\underline{s}(z)})^2)}dz\\&=(\Psi-\Phi^2-2)\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\frac{c\underline{s}}{(1+\underline{s})^3}d\underline{s}\\&\quad+\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\frac{\Phi^2 c\underline{s}}{(1+\underline{s})^3\left(1-\Phi^2 c\frac{\underline{s}^2}{(1+\underline{s})^2}\right)}d\underline{s},\end{aligned}\tag{4.1}$$

where $\tilde{\mathcal{C}}$ is a contour crossing the real axis in the interval $(-(1+\sqrt{c})^{-1},0)$ and $(\underline{s}(-\rho^{-1}),\infty)$. The choice of $\tilde{\mathcal{C}}$ depends on the fact that

$$(\underline{s}(-\rho^{-1}),\underline{s}(0-))=(\underline{s}(-\rho^{-1}),\infty),\quad(\underline{s}(b),\underline{s}(\infty))=(-(1+\sqrt{c})^{-1},0).$$

At first, we come to calculate the first term on the r.h.s of (4.1). Note that

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\frac{c\underline{s}}{(1+\underline{s})^3}d\underline{s}\\
=&\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\frac{c}{(1+\underline{s})^2}d\underline{s}-\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\frac{c}{(1+\underline{s})^3}d\underline{s}\\
=&\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}(\log(1+\rho z(\underline{s})))'_{\underline{s}}\frac{c}{1+\underline{s}}d\underline{s}-\frac{1}{4\pi i}\int_{\tilde{\mathcal{C}}}(\log(1+\rho z(\underline{s})))'_{\underline{s}}\frac{c}{(1+\underline{s})^2}d\underline{s}\\
=&\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\frac{c}{1+\underline{s}}\cdot\frac{\rho[(1+\underline{s})^2-c\underline{s}^2]}{\underline{s}(1+\underline{s})[\underline{s}(1+\underline{s})-\rho(1+\underline{s})+\rho c\underline{s}]}d\underline{s}\\
&-\frac{1}{4\pi i}\int_{\tilde{\mathcal{C}}}\frac{c}{(1+\underline{s})^2}\cdot\frac{\rho[(1+\underline{s})^2-c\underline{s}^2]}{\underline{s}(1+\underline{s})[\underline{s}(1+\underline{s})-\rho(1+\underline{s})+\rho c\underline{s}]}d\underline{s}\\
=&\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\frac{c\rho[(1+\underline{s})^2-c\underline{s}^2]}{\underline{s}(1+\underline{s})^2(\underline{s}-\lambda_+)(\underline{s}-\lambda_-)}d\underline{s}-\frac{1}{4\pi i}\int_{\tilde{\mathcal{C}}}\frac{c\rho[(1+\underline{s})^2-c\underline{s}^2]}{\underline{s}(1+\underline{s})^3(\underline{s}-\lambda_+)(\underline{s}-\lambda_-)}d\underline{s} \qquad (4.2)
\end{aligned}$$

where

$$\begin{aligned}
\lambda_+&=-\frac{1}{2}(1-\rho+\rho c)+\frac{1}{2}\sqrt{(1-\rho+\rho c)^2+4\rho},\\
\lambda_-&=-\frac{1}{2}(1-\rho+\rho c)-\frac{1}{2}\sqrt{(1-\rho+\rho c)^2+4\rho}
\end{aligned}$$

are two solutions of the equation

$$\underline{s}(1+\underline{s})-\rho(1+\underline{s})+\rho c\underline{s}=0.$$

It is not difficult to check that $\lambda_-\le -1$ and

$$\lambda_+(\lambda_++1)=\rho(1-c)\lambda_++\rho,\quad \lambda_+\lambda_-=-\rho,\quad 1+2\lambda_++\rho(1-c)=\lambda_+-\lambda_-, \qquad (4.3)$$

thus

$$(\lambda_++1)^2-c\lambda_+^2=\rho^{-1}\lambda_+(\lambda_++1)(\lambda_+-\lambda_-). \qquad (4.4)$$

Now by the choice of $\tilde{\mathcal{C}}$, it suffices to consider the poles 0 and $\lambda_+=\underline{s}(-\rho^{-1})$ in two integrals in (4.2). Therefore,

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\frac{c\underline{s}}{(1+\underline{s})^3}d\underline{s}
&=\left[\frac{c\rho}{\lambda_+\lambda_-}+\frac{c\rho\left((1+\lambda_+)^2-c\lambda_+^2\right)}{\lambda_+(1+\lambda_+)^2(\lambda_+-\lambda_-)}\right]\\
&\quad-\frac{1}{2}\left[\frac{c\rho}{\lambda_+\lambda_-}+\frac{c\rho\left((1+\lambda_+)^2-c\lambda_+^2\right)}{\lambda_+(1+\lambda_+)^3(\lambda_+-\lambda_-)}\right]\\
&=-\frac{c}{2}+\frac{c}{1+\lambda_+}-\frac{1}{2}\frac{c}{(1+\lambda_+)^2}. \qquad (4.5)
\end{aligned}$$

Now we come to deal with the second term on the r.h.s of (4.1).

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\frac{\Phi^2 c\underline{s}}{(1+\underline{s})^3\left(1-\Phi^2 c\frac{\underline{s}^2}{(1+\underline{s})^2}\right)}d\underline{s}\\
=&\ \frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\left[\frac{1}{1+\underline{s}}-\frac{1+\underline{s}-\Phi^2 c\underline{s}}{(1+\underline{s})^2-\Phi^2 c\underline{s}^2}\right]d\underline{s}\\
=&\ \frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\left[\frac{1}{1+\underline{s}}-\frac{1}{2}\left(\log[(1+\underline{s})^2-\Phi^2 c\underline{s}^2]\right)'_{\underline{s}}\right]d\underline{s}\\
=&\ \frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\frac{1}{2}\log[(1+\underline{s})^2-\Phi^2 c\underline{s}^2]\left(\log(1+\rho z(\underline{s}))\right)'_{\underline{s}}\,ds-\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\underline{s})\left(\log(1+\rho z(\underline{s}))\right)'_{\underline{s}}\,d\underline{s}\\
=&\ \frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\frac{1}{2}\frac{\rho[(1+\underline{s})^2-c\underline{s}^2]\log[(1+\underline{s})^2-\Phi^2 c\underline{s}^2]}{\underline{s}(1+\underline{s})(\underline{s}-\lambda_+)(\underline{s}-\lambda_-)}ds-\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\frac{\rho[(1+\underline{s})^2-c\underline{s}^2]\log(1+\underline{s})}{\underline{s}(1+\underline{s})(\underline{s}-\lambda_+)(\underline{s}-\lambda_-)}ds.
\end{aligned}
$$

Above we have chosen $\tilde{\mathcal{C}}$ such that $\log[(1+\underline{s})^2-\Phi^2 c\underline{s}^2]$ is analytic in the region enclosed by $\tilde{\mathcal{C}}$. Such $\tilde{\mathcal{C}}$ does exit since we have $\Phi^2\le 1$ and the assumption that $c\le 1$. Therefore,

$$
\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}}\log(1+\rho z(\underline{s}))\frac{\Phi^2 c\underline{s}}{(1+\underline{s})^3\left(1-\Phi^2 c\frac{\underline{s}^2}{(1+\underline{s})^2}\right)}d\underline{s}=\frac{1}{2}\log[(1+\lambda_+)^2-\Phi^2 c\lambda_+^2]-\log(1+\lambda_+).\quad(4.6)
$$

Then (4.5) and (4.6) together imply that

$$
\mathbb{E}L_{f_\ell}=(\Psi-\Phi^2-2)\left[-\frac{c}{2}+\frac{c}{1+\lambda_+}-\frac{1}{2}\frac{c}{(1+\lambda_+)^2}\right]+\frac{1}{2}\log[(1+\lambda_+)^2-\Phi^2 c\lambda_+^2]-\log(1+\lambda_+).
$$

Now we start to verify (1.8). Note that we have

$$
zs'(z)+s(z)=\frac{\underline{s}'(z)}{(1+\underline{s}(z))^2},\quad a(z_1,z_2)=c\frac{\underline{s}(z_1)\underline{s}(z_2)}{(1+\underline{s}(z_1))(1+\underline{s}(z_2))}.
$$

Therefore, one has

$$
\frac{\Phi^2 c(z_1s'(z_1)+s(z_1))(z_2s'(z_2)+s(z_2))}{(1-\Phi^2 a(z_1,z_2))^2}=\frac{\Phi^2 c\frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(1+\underline{s}(z_1))^2(1+\underline{s}(z_2))^2}}{(1-\Phi^2 c\frac{\underline{s}(z_1)\underline{s}(z_2)}{(1+\underline{s}(z_1))(1+\underline{s}(z_2))})^2}\quad(4.7)
$$

and analogously,

$$
\frac{c(z_1s'(z_1)+s(z_1))(z_2s'(z_2)+s(z_2))}{(1-a(z_1,z_2))^2}=\frac{c\frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(1+\underline{s}(z_1))^2(1+\underline{s}(z_2))^2}}{(1-c\frac{\underline{s}(z_1)\underline{s}(z_2)}{(1+\underline{s}(z_1))(1+\underline{s}(z_2))})^2}.\quad(4.8)
$$

Actually, for (4.8), we can simplify it further as follows. Note that

$$
a(z_1,z_2)=\frac{\underline{s}(z_1)\underline{s}(z_2)}{\underline{s}(z_2)-\underline{s}(z_1)}\left(\frac{c}{1+\underline{s}(z_1)}-\frac{c}{1+\underline{s}(z_2)}\right).
$$

Using the equation

$$
\underline{s}(z)=\frac{1}{-z+\frac{c}{1+\underline{s}(z_2)}},
$$

we can get

$$a(z_1, z_2) = 1 + \frac{\underline{s}(z_1)\underline{s}(z_2)}{\underline{s}(z_2) - \underline{s}(z_1)}(z_1 - z_2). \tag{4.9}$$

Then by (4.8), it is not difficult to see that

$$\frac{c(z_1 s'(z_1) + s(z_1))(z_2 s'(z_2) + s(z_2))}{(1 - a(z_1, z_2))^2} = \frac{\partial}{\partial z_2}\left(\frac{\frac{\partial}{\partial z_1} a(z_1, z_2)}{1 - a(z_1, z_2)}\right).$$

Now by (4.9) we have

$$\frac{\frac{\partial}{\partial z_1} a(z_1, z_2)}{1 - a(z_1, z_2)} = -\frac{\underline{s}'(z_1)}{\underline{s}(z_1)} - \frac{1}{z_1 - z_2} - \frac{\underline{s}'(z_1)}{\underline{s}(z_2) - \underline{s}(z_1)}.$$

Consequently, we have

$$\frac{c(z_1 s'(z_1) + s(z_1))(z_2 s'(z_2) + s(z_2))}{(1 - a(z_1, z_2))^2} = \frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(\underline{s}(z_2) - \underline{s}(z_1))^2} - \frac{1}{(z_1 - z_2)^2}.$$

Rewrite (4.7) as

$$\frac{\Phi^2 c(z_1 s'(z_1) + s(z_1))(z_2 s'(z_2) + s(z_2))}{(1 - \Phi^2 a(z_1, z_2))^2} = \frac{\Phi^2 c\underline{s}'(z_1)\underline{s}'(z_2)}{((1 + \underline{s}(z_1))(1 + \underline{s}(z_2)) - \Phi^2 c\underline{s}(z_1)\underline{s}(z_2))^2}.$$

Therefore, one has

$$\begin{aligned}
& \operatorname{Cov}(L_{f_\ell}, L_{f_\nu}) \\
= \; & -\frac{1}{4\pi^2}(\Psi - \Phi^2 - 2)\int_{\mathcal{C}_1}\int_{\mathcal{C}_2} f_\ell(z_1) f_\nu(z_2) c \frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(1 + \underline{s}(z_1))^2(1 + \underline{s}(z_2))^2} dz_1 dz_2 \\
& -\frac{1}{4\pi^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2} f_\ell(z_1) f_\nu(z_2) \frac{\Phi^2 c\underline{s}'(z_1)\underline{s}'(z_2)}{\big((1 + \underline{s}(z_1))(1 + \underline{s}(z_2)) - \Phi^2 c\underline{s}(z_1)\underline{s}(z_2)\big)^2} dz_1 dz_2 \\
& -\frac{1}{4\pi^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2} f_\ell(z_1) f_\nu(z_2) \frac{\underline{s}'(z_1)\underline{s}'(z_2)}{(\underline{s}(z_2) - \underline{s}(z_1))^2} dz_1 dz_2 \\
= \; & -\frac{1}{4\pi^2}(\Psi - \Phi^2 - 2)\int_{\tilde{\mathcal{C}}_1}\int_{\tilde{\mathcal{C}}_2} f_\ell(z(\underline{s}_1)) f_\nu(z(\underline{s}_2)) \frac{c}{(1 + \underline{s}_1)^2(1 + \underline{s}_2)^2} d\underline{s}_1 d\underline{s}_2 \\
& -\frac{1}{4\pi^2}\int_{\tilde{\mathcal{C}}_1}\int_{\tilde{\mathcal{C}}_2} f_\ell(z(\underline{s}_1)) f_\nu(z(\underline{s}_2)) \frac{\Phi^2 c}{\big((1 + \underline{s}_1)(1 + \underline{s}_2) - \Phi^2 c\underline{s}_1\underline{s}_2\big)^2} d\underline{s}_1 d\underline{s}_2 \\
& -\frac{1}{4\pi^2}\int_{\tilde{\mathcal{C}}_1}\int_{\tilde{\mathcal{C}}_2} f_\ell(z(\underline{s}_1)) f_\nu(z(\underline{s}_2)) \frac{1}{(\underline{s}_2 - \underline{s}_1)^2} d\underline{s}_1 d\underline{s}_2,
\end{aligned} \tag{4.10}$$

where the contours $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ cross the real axis in the interval $(-(1+\sqrt{c})^{-1}, 0)$ and $(\underline{s}(-\rho^{-1}), \infty)$ and $\tilde{\mathcal{C}}_1$ is inside $\tilde{\mathcal{C}}_2$. Note that when $f_\ell(x) = f_\nu(x) = \log(1 + \rho x)$, the last term on the r.h.s. of (4.10) has been calculated in Kamath and Hughes (2005), that is

$$-\frac{1}{4\pi^2}\int_{\tilde{\mathcal{C}}_1}\int_{\tilde{\mathcal{C}}_2} \log(1 + \rho z(\underline{s}_1)) \log(1 + \rho z(\underline{s}_2)) \frac{1}{(\underline{s}_2 - \underline{s}_1)^2} d\underline{s}_1 d\underline{s}_2 = -\log\left(1 - \frac{\rho - \lambda_+}{\rho - \lambda_-}\right).$$

Now we come to calculate the first term of the r.h.s. of (4.10). It suffices to deal with

$$\begin{aligned}&\int_{\tilde{\mathcal{C}}_1} \frac{f_\ell(z(\underline{s}_1))}{(1+\underline{s}_1)^2} d s_1 = \int_{\tilde{\mathcal{C}}_1} \frac{\log(1+\rho z(\underline{s}_1))}{(1+\underline{s}_1)^2} d\underline{s}_1\\ &= \int_{\tilde{\mathcal{C}}_1} \frac{\rho[\underline{s}_1^{-2} - c(1+\underline{s}_1)^{-2}]}{(1+\underline{s}_1)[1-\rho\underline{s}_1^{-1} + \rho c(1+\underline{s}_1)^{-1}]} d\underline{s}_1\\ &= \int_{\tilde{\mathcal{C}}_1} \frac{\rho[(1+\underline{s}_1)^2 - c\underline{s}_1^2]}{\underline{s}_1(1+\underline{s}_1)^2(\underline{s}_1-\lambda_+)(\underline{s}_1-\lambda_-)} d\underline{s}_1\end{aligned}$$

Again, it suffices to consider the poles 0 and λ_+, thus

$$\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}_1} \frac{f_\ell(z(\underline{s}_1)}{(1+\underline{s}_1))^2} d s_1 = \frac{\rho}{\lambda_+\lambda_-} + \frac{\rho[(1+\lambda_+)^2 - c\lambda_+^2]}{\lambda_+(1+\lambda_+)^2(\lambda_+-\lambda_-)} = \frac{1}{1+\lambda_+} - 1,$$

where the last step follows from (4.3) and (4.4). Therefore, we have

$$\begin{aligned}&-\frac{1}{4\pi^2}(\Psi - \Phi^2 - 2)\int_{\tilde{\mathcal{C}}_1}\int_{\tilde{\mathcal{C}}_2} f_\ell(z(\underline{s}_1)) f_\nu(z(\underline{s}_2)) \frac{c}{(1+\underline{s}_1))^2(1+\underline{s}_2)^2} d\underline{s}_1 d\underline{s}_2\\ &= (\Psi - \Phi^2 - 2)c\left(\frac{\lambda_+}{1+\lambda_+}\right)^2.\end{aligned}$$

It remains to calculate the second term of the r.h.s. of (4.10)

$$-\frac{1}{4\pi^2}\int_{\tilde{\mathcal{C}}_1}\int_{\tilde{\mathcal{C}}_2} \log(1+\rho z(\underline{s}_1))\log(1+\rho z(\underline{s}_2)) \frac{\Phi^2 c}{\big((1+\underline{s}_1)(1+\underline{s}_2) - \Phi^2 c\underline{s}_1\underline{s}_2\big)^2} d\underline{s}_1 d\underline{s}_2.$$

At first, using integral by parts we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}_1} \frac{\Phi^2 c\log(1+\rho z(\underline{s}_1))}{\big((1+\underline{s}_1)(1+\underline{s}_2) - \Phi^2 c\underline{s}_1\underline{s}_2\big)^2} d\underline{s}_1\\ =\;& \frac{1}{2\pi i}\frac{\Phi^2 c\rho}{1+\underline{s}_2 - \Phi^2 c\underline{s}_2}\int_{\tilde{\mathcal{C}}_1} \frac{\underline{s}_1^{-2} - c(1+\underline{s}_1)^{-2}}{\big((1+\underline{s}_1)(1+\underline{s}_2) - \Phi^2 c\underline{s}_1\underline{s}_2\big)(1-\rho\underline{s}_1^{-1} + \rho c(1+\underline{s}_1)^{-1})} d\underline{s}_1\\ =\;& \frac{1}{2\pi i}\frac{\Phi^2 c\rho}{1+\underline{s}_2 - \Phi^2 c\underline{s}_2}\int_{\tilde{\mathcal{C}}_1} \frac{(1+\underline{s}_1)^2 - c\underline{s}_1^2}{\underline{s}_1(1+\underline{s}_1)\big((1+\underline{s}_1)(1+\underline{s}_2) - \Phi^2 c\underline{s}_1\underline{s}_2\big)\big(\underline{s}_1(1+\underline{s}_1) - \rho(1+\underline{s}_1) + \rho c\underline{s}_1\big)} d\underline{s}_1\\ =\;& \frac{1}{2\pi i}\frac{\Phi^2 c\rho}{(1+\underline{s}_2 - \Phi^2 c\underline{s}_2)^2}\int_{\tilde{\mathcal{C}}_1} \frac{(1+\underline{s}_1)^2 - c\underline{s}_1^2}{\underline{s}_1(1+\underline{s}_1)\big(\underline{s}_1 + \frac{1+\underline{s}_2}{1+\underline{s}_2 - \Phi^2 c\underline{s}_2}\big)(\underline{s}_1-\lambda_+)(\underline{s}_1-\lambda_-)} d\underline{s}_1. \qquad (4.11)\end{aligned}$$

Let $\tilde{D}_1$ be the region enclosed by $\tilde{\mathcal{C}}_1$. Actually we can always choose appropriate $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ such that

$$\underline{s}_1 + \frac{1+\underline{s}_2}{1+\underline{s}_2 - \Phi^2 c\underline{s}_2} \neq 0, \quad \text{for all} \quad \underline{s}_1 \in \tilde{D}_1, \quad \underline{s}_2 \in \tilde{\mathcal{C}}_2.$$

To see this, we denote $t = 1 - \Phi^2 c$ for simplicity. If there are some $\underline{s}_1 \in \tilde{D}_1, \underline{s}_2 \in \tilde{\mathcal{C}}_2$ such that

$$\underline{s}_1 = -\frac{1+\underline{s}_2}{1+t\underline{s}_2}$$

we also have

$$\underline{s}_2 = -\frac{1+\underline{s}_1}{1+t\underline{s}_1}. \tag{4.12}$$

Note that it is easy to choose $\tilde{\mathcal{C}}_1$ such that in $\tilde{D}_1$ we always have

$$\Re(-\frac{1+\underline{s}_1}{1+t\underline{s}_1}) \le -(1+\sqrt{c})^{-1}.$$

Then it is easy to construct $\tilde{\mathcal{C}}_2$ enclosing $\tilde{\mathcal{C}}_1$ such that (4.12) does not hold for any $\underline{s}_2 \in \tilde{\mathcal{C}}_2$.

Analogously, we only need to consider the poles 0 and λ_+ in the integral in (4.11). Therefore, we have

$$\begin{aligned} & \frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}_1} \frac{\Phi^2 c\log(1+\rho z(\underline{s}_1))}{\big((1+\underline{s}_1)(1+\underline{s}_2)-\Phi^2 c\underline{s}_1\underline{s}_2\big)^2} d\underline{s}_1 \\ = & \frac{\Phi^2 c\rho}{(1+\underline{s}_2-\Phi^2 c\underline{s}_2)^2}\left[\frac{(1+\lambda_+)^2 - c\lambda_+^2}{\lambda_+(1+\lambda_+)\big(\lambda_+ + \frac{1+\underline{s}_2}{1+\underline{s}_2-\Phi^2 c\underline{s}_2}\big)(\lambda_+-\lambda_-)} + \frac{1}{\frac{1+\underline{s}_2}{1+\underline{s}_2-\Phi^2 c\underline{s}_2}\lambda_+\lambda_-}\right] \\ = & \frac{\Phi^2 c}{\lambda_+(1+\underline{s}_2-\Phi^2 c\underline{s}_2)^2 + (1+\underline{s}_2)(1+\underline{s}_2-\Phi^2 c\underline{s}_2)} - \frac{\Phi^2 c}{(1+\underline{s}_2-\Phi^2 c\underline{s}_2)(1+\underline{s}_2)}. \end{aligned}$$

Now recall the notation $t = 1-\Phi^2 c$, we can write

$$\begin{aligned} & \frac{1}{2\pi i}\int_{\tilde{\mathcal{C}}_1} \frac{\Phi^2 c\log(1+\rho z(\underline{s}_1))}{\big((1+\underline{s}_1)(1+\underline{s}_2)-\Phi^2 c\underline{s}_1\underline{s}_2\big)^2} d\underline{s}_1 \\ = & A_1\frac{1}{(\underline{s}_2-\tilde{\lambda}_+)(\underline{s}_2-\tilde{\lambda}_-)} + A_2\frac{1}{(\underline{s}_2+1)(\underline{s}_2+t^{-1})}, \end{aligned}$$

where

$$A_1 = \frac{1-t}{\lambda_+ t^2 + t}, \quad A_2 = -\frac{1-t}{t} \tag{4.13}$$

and

$$\begin{aligned} \tilde{\lambda}_+ &= \frac{-(1+2\lambda_+ t + t) + \sqrt{(1+2\lambda_+ t+t)^2 - 4(\lambda_+ +1)(\lambda_+ t^2 + t)}}{2(\lambda_+ t^2 + t)} \\ \tilde{\lambda}_- &= \frac{-(1+2\lambda_+ t + t) - \sqrt{(1+2\lambda_+ t+t)^2 - 4(\lambda_+ +1)(\lambda_+ t^2 + t)}}{2(\lambda_+ t^2 + t)}. \end{aligned}$$

Note that

$$\sqrt{(1+2\lambda_+ t+t)^2 - 4(\lambda_+ +1)(\lambda_+ t^2 + t)} = 1-t,$$

thus we have

$$\tilde{\lambda}_+ = -\frac{1+\lambda_+}{1+t\lambda_+}, \quad \tilde{\lambda}_- = -\frac{1}{t}.$$

Hence, when $c \le 1$ we have $0 < t \le 1$, thus $\tilde{\lambda}_+, \tilde{\lambda}_- \le -1 < -(1+\sqrt{c})^{-1}$. Therefore, we have

$$\begin{aligned}
& -\frac{1}{4\pi^2}\int_{\tilde{\mathcal{C}}_1}\int_{\tilde{\mathcal{C}}_2} \frac{\Phi^2 c\log(1+\rho z(\underline{s}_1))\log(1+\rho z(\underline{s}_2))}{\left((1+\underline{s}_1)(1+\underline{s}_2)-\Phi^2 c\underline{s}_1\underline{s}_2\right)^2} d\underline{s}_1 d\underline{s}_2 \\
= \;& \frac{A_1}{2\pi i}\int_{\tilde{\mathcal{C}}_2} \frac{\log(1+\rho z(\underline{s}_2))}{(\underline{s}_2-\tilde{\lambda}_+)(\underline{s}_2-\tilde{\lambda}_-)} d\underline{s}_2 + \frac{A_2}{2\pi i}\int_{\tilde{\mathcal{C}}_2} \frac{\log(1+\rho z(\underline{s}_2))}{(\underline{s}_2+1)(\underline{s}_2+t^{-1})} d\underline{s}_2 \\
= \;& (\tilde{\lambda}_- - \tilde{\lambda}_+)^{-1}\frac{A_1}{2\pi i}\int_{\tilde{\mathcal{C}}_2} \log[(\underline{s}_2-\tilde{\lambda}_+)/(\underline{s}_2-\tilde{\lambda}_-)](\log(1+\rho z(\underline{s}_2)))' d\underline{s}_2 \\
& -(t^{-1}-1)^{-1}\frac{A_2}{2\pi i}\int_{\tilde{\mathcal{C}}_2} \log[(\underline{s}_2+1)/(\underline{s}_2+t^{-1})](\log(1+\rho z(\underline{s}_2)))' d\underline{s}_2 \\
= \;& (\tilde{\lambda}_- - \tilde{\lambda}_+)^{-1}\frac{\rho A_1}{2\pi i}\int_{\tilde{\mathcal{C}}_2} \frac{\log[(\underline{s}_2-\tilde{\lambda}_+)/(\underline{s}_2-\tilde{\lambda}_-)]\left((\underline{s}_2+1)^2 - c\underline{s}_2^2\right)}{\underline{s}_2(\underline{s}_2+1)(\underline{s}_2-\lambda_+)(\underline{s}_2-\lambda_-)} d\underline{s}_2 \\
& -(t^{-1}-1)^{-1}\frac{\rho A_2}{2\pi i}\int_{\tilde{\mathcal{C}}_2} \frac{\log[(\underline{s}_2+1)/(\underline{s}_2+t^{-1})]\left((\underline{s}_2+1)^2 - c\underline{s}_2^2\right)}{\underline{s}_2(\underline{s}_2+1)(\underline{s}_2-\lambda_+)(\underline{s}_2-\lambda_-)} d\underline{s}_2 \\
= \;& (\tilde{\lambda}_- - \tilde{\lambda}_+)^{-1}\rho A_1\left(\frac{\log[(\lambda_+-\tilde{\lambda}_+)/(\lambda_+-\tilde{\lambda}_-)]\left((\lambda_++1)^2-c\lambda_+^2\right)}{\lambda_+(\lambda_++1)(\lambda_+-\lambda_-)} + \frac{\log[\tilde{\lambda}_+/\tilde{\lambda}_-]}{\lambda_+\lambda_-}\right) \\
& -(t^{-1}-1)^{-1}\rho A_2\left(\frac{\log[(\lambda_++1)/(\lambda_++t^{-1})]\left((\lambda_++1)^2-c\lambda_+^2\right)}{\lambda_+(\lambda_++1)(\lambda_+-\lambda_-)} + \frac{\log t}{\lambda_+\lambda_-}\right). \qquad (4.14)
\end{aligned}$$

Substituting (4.4) and (4.13) into (4.14) we have

$$\begin{aligned}
& -\frac{1}{4\pi^2}\int_{\tilde{\mathcal{C}}_1}\int_{\tilde{\mathcal{C}}_2} \frac{\Phi^2 c\log(1+\rho z(\underline{s}_1))\log(1+\rho z(\underline{s}_2))}{\left((1+\underline{s}_1)(1+\underline{s}_2)-\Phi^2 c\underline{s}_1\underline{s}_2\right)^2} d\underline{s}_1 d\underline{s}_2 \\
= \;& -(\tilde{\lambda}_- - \tilde{\lambda}_+)^{-1}A_1\log\frac{\tilde{\lambda}_+(\lambda_+-\tilde{\lambda}_-)}{\tilde{\lambda}_-(\lambda_+-\tilde{\lambda}_+)} - (t^{-1}-1)^{-1}A_2\log\frac{\lambda_++1}{1+t\lambda_+}
\end{aligned}$$

Note that

$$(\tilde{\lambda}_- - \tilde{\lambda}_+)^{-1}A_1 = (t^{-1}-1)^{-1}A_2 = \frac{\Phi^2 c}{t-1} = -1.$$

Finally we can get

$$\begin{aligned}
& -\frac{1}{4\pi^2}\int_{\tilde{\mathcal{C}}_1}\int_{\tilde{\mathcal{C}}_2} \frac{\Phi^2 c\log(1+\rho z(\underline{s}_1))\log(1+\rho z(\underline{s}_2))}{\left((1+\underline{s}_1)(1+\underline{s}_2)-\Phi^2 c\underline{s}_1\underline{s}_2\right)^2} d\underline{s}_1 d\underline{s}_2 \\
= \;& \log[t\tilde{\lambda}_+^2(\lambda_+-\tilde{\lambda}_-)/(\lambda_+-\tilde{\lambda}_+)] = \log[(1+\lambda_+)^2/(1+2\lambda_++(1-\Phi^2 c)\lambda_+^2)]. \qquad (4.15)
\end{aligned}$$

Note that we have proved Theorem 1 for $c \le 1$ through the above discussions. For $c > 1$, by the simple fact that

$$\log\det(\mathbf{I}+\frac{\rho}{n}\mathbf{Z}\mathbf{Z}^*) = \log\det(\mathbf{I}+\frac{\rho c}{p}\mathbf{Z}^*\mathbf{Z}),$$

we can use $(c^{-1}, \rho c)$ to replace (c, ρ) in the above discussion. Then it is easy to check that Theorem 1 still holds when $c > 1$ by the definition of λ_+, λ_-. Therefore, we conclude the proof of Theorem 1.

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