Optimal Codes in the Enomoto-Katona Space

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Outline

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Functional Dependencies

Clearly, certain attributes can be determined from others. For example,

- {**Last name, Initials**} determines **Institute**.
- {**Paper**} determines **Date**.

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Definition

Let $X \subseteq A$ and $y \in A$. We say that y **functionally depends** on X, or $X \to y$, if no two rows of $R(A)$ agree in X but differs in y.

- $\{$**Last name, Initials**$\} \to$ **Institute**.
- $\{$**Paper**$\} \to$ **Date**.

(p, q)-Dependencies

So, $\{$**Paper**$\} \not\longrightarrow$ **Last Names**.

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Definition

Let $X \subseteq A$ and $y \in A$. For positive integers $p \leq q$, we say that y (p, q)-**depends** on X, or $X \xrightarrow{(p,q)} y$, if there do not exist $q + 1$ data items $\mathsf{d}_1, \mathsf{d}_2, \ldots, \mathsf{d}_{q+1}$ of $R(A)$ such that

(i) $|\{\mathsf{d}_i|\{x\} : 1 \leq i \leq q + 1\}| \leq p$ for each $x \in X$, and,

(ii) $|\{\mathsf{d}_i|\{y\} : 1 \leq i \leq q + 1\}| = q + 1$.

In particular, in our example, $\{$**Paper**$\} \xrightarrow{(1,5)}$ **Last Names**.

(p, q)-Dependencies

So, $\{$**Paper**$\} \not\longrightarrow$ **Last Names**.

Definition

Let $X \subseteq A$ and $y \in A$. For positive integers $p \leq q$, we say that y (p, q)-**depends** on X, or $X \overset{(p,q)}{\longrightarrow} y$, if there do not exist $q+1$ data items $\mathsf{d}_1, \mathsf{d}_2, \ldots, \mathsf{d}_{q+1}$ of $R(A)$ such that

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In particular, in our example, $\{$**Paper**$\} \overset{(1,5)}{\longrightarrow}$ **Last Names**.

- Functional dependency is equivalent to a $(1, 1)$-dependency
- When functional dependencies are not known, (p, q)-dependencies identified in a relational database can still be exploited for improving storage efficiency.

Implication Structure

Definition

Let $p \leq q$ be positive integers. For a table $R(A)$, define the operation $J_{R(A)}^{(p,q)} : 2^A \to 2^A$ so that for $X \subseteq A$, we have

$$J_{R(A)}^{(p,q)}(X) = \left\{ y \in A : X \xrightarrow{(p,q)} y \right\}.$$

We call $J_{R(A)}^{(p,q)}$ the (p,q)-**implication structure** of $R(A)$.

So,

- $J_{R(A)}^{(1,1)}(\{\textbf{Last Name, Initials}\}) = \{\textbf{Last Name, Initials, Institute}\}.$
- $J_{R(A)}^{(1,1)}(\{\textbf{Paper}\}) = \{\textbf{Date, Paper}\}.$
- $J_{R(A)}^{(1,5)}(\{\textbf{Paper}\}) = \{\textbf{Last Name, Initials, Institute, Date, Paper}\} = A.$

(p, q)-Representable Functions

Definition

A function $J : 2^A \to 2^A$ is said to be (p, q)-**representable** if there exists a table $R(A)$ such that $J^{(p,q)}_{R(A)} = J$.

Proposition (Armstrong, 1974)

The function $J : 2^A \to 2^A$ is $(1, 1)$-representable if and only if J is a closure operator on A.

Characterization for general p and q is given by Demetrovics *et al.*,1992.

(p, q)-Representable Functions

Objective

Given a function J, to determine the table $R(A)$ with the least number of rows such that $J_{R(A)}^{(p,q)} = J$.

In particular, for fixed k, consider the function

$$J_n^k(X) = \begin{cases} X, & \text{if } |X| < k \\ A, & \text{otherwise.} \end{cases}$$

Outline

Enomoto-Katona Space

- Let $2k \leq n$.
- Let X be a finite set of n elements.
- $\binom{X}{k}$ denote the set of all k-subsets of X.

Definition

Enomoto-Katona Space (2001)

- The set of all unordered pairs of disjoint k-subsets of X is given by

$$\mathcal{E}(X,k) = \left\{ \{A,B\} \subseteq \binom{X}{k} : A \cap B = \varnothing \right\}$$

- "Codewords" of $\mathcal{E}(X,k)$ are called *set-pairs*.

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- "Codewords" of $\mathcal{E}(X,k)$ are called *set-pairs*.
- The function $\mathrm{d}_{\mathcal{E}} : \mathcal{E}(X,k) \times \mathcal{E}(X,k) \to \{0,1,\ldots,2k\}$ is given by

$$\mathrm{d}_{\mathcal{E}}(\{A,B\},\{S,T\}) = \min\{|A \setminus S| + |B \setminus T|, |A \setminus T| + |B \setminus S|\}$$

- Then $(\mathcal{E}(X,k), \mathrm{d}_{\mathcal{E}})$ is a metric space called the *Enomoto-Katona space*.

Enomoto-Katona Space - An example

Let $X = \mathbb{Z}/4\mathbb{Z}$ and $k = 2$.
$\mathcal{E}(X, 2)$ consists of the following set-pairs:

$$\{\{0,1\},\{2,3\}\}, \quad \{\{0,2\},\{1,3\}\}, \quad \{\{0,3\},\{1,2\}\}.$$

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$$\mathrm{d}_{\mathcal{E}}(\{\{0,1\},\{2,3\}\},\{\{0,2\},\{1,3\}\})$$

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$$= \min\{|\{0,1\} \setminus \{0,2\}| + |\{2,3\} \setminus \{1,3\}|, |\{0,1\} \setminus \{1,3\}| + |\{2,3\} \setminus \{0,2\}|\}$$

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For example,

$$\begin{aligned} &\mathrm{d}_{\mathcal{E}}(\{\{0,1\},\{2,3\}\},\{\{0,2\},\{1,3\}\}) \\ &= \min\{|\{0,1\} \setminus \{0,2\}| + |\{2,3\} \setminus \{1,3\}|, |\{0,1\} \setminus \{1,3\}| + |\{2,3\} \setminus \{0,2\}|\} \\ &= \min\{1+1, 1+1\} = 2. \end{aligned}$$

Enomoto-Katona Code

Enomoto-Katona Code

An *Enomoto-Katona code* (or *EK code*, in short), is a set $\mathcal{C} \subseteq \mathcal{E}(X, k)$. More specifically, $\mathcal{C}$ is an EK code of *length* n, *weight* k, and *distance* d, or (n, k, d)-EK code, if

$$\mathrm{d}_{\mathcal{E}}(\mathbf{u}, \mathbf{v}) \geq d \text{ for all distinct } \mathbf{u}, \mathbf{v} \in \mathcal{C}.$$

Let $X = \mathbb{Z}/4\mathbb{Z}$ and $k = 2$.
Let $\mathcal{C}$ consists of the following set-pairs:

$$\{\{0,1\},\{2,3\}\}, \quad \{\{0,2\},\{1,3\}\}, \quad \{\{0,3\},\{1,2\}\}.$$

Then $\mathcal{C}$ is a $(4, 2, 2)$-EK code.

Constructing a table with an EK code

Consider the following $(9, 2, 3)$-EK code $\mathcal{C}$, where $X = \mathbb{Z}/9\mathbb{Z}$.

$$
\begin{aligned}
&c_1 = \{\{0,1\},\{2,4\}\}, \quad &c_2 = \{\{1,2\},\{3,5\}\}, \quad &c_3 = \{\{2,3\},\{4,6\}\},\\
&c_4 = \{\{3,4\},\{5,7\}\}, \quad &c_5 = \{\{4,5\},\{6,8\}\}, \quad &c_6 = \{\{5,6\},\{7,0\}\},\\
&c_7 = \{\{6,7\},\{8,1\}\}, \quad &c_8 = \{\{7,8\},\{0,2\}\}, \quad &c_9 = \{\{8,0\},\{1,3\}\}.
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Each set-pair $\{A, B\}$ constructs a column in the following manner:

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
0									
1									
2									
3									
4									
5									
6									
7									
8									

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Each set-pair $\{A, B\}$ constructs a column in the following manner:

- place 1 at rows indexed by elements of A,

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
0	1								1
1	1	1							
2		1	1						
3			1	1					
4				1	1				
5					1	1			
6						1	1		
7							1	1	
8								1	1

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Consider the following $(9, 2, 3)$-EK code $\mathcal{C}$, where $X = \mathbb{Z}/9\mathbb{Z}$.

$$c_1 = \{\{0,1\},\{2,4\}\}, \quad c_2 = \{\{1,2\},\{3,5\}\}, \quad c_3 = \{\{2,3\},\{4,6\}\},$$
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Each set-pair $\{A, B\}$ constructs a column in the following manner:

- place 1 at rows indexed by elements of A,
- place 2 at rows by elements of B,

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
0	1					2		2	1
1	1	1					2		2
2	2	1	1					2	
3		2	1	1					2
4	2		2	1	1				
5		2		2	1	1			
6			2		2	1	1		
7				2		2	1	1	
8					2		2	1	1

Constructing a table with an EK code

Consider the following $(9, 2, 3)$-EK code $\mathcal{C}$, where $X = \mathbb{Z}/9\mathbb{Z}$.

$c_1 = \{\{0, 1\}, \{2, 4\}\}$, $c_2 = \{\{1, 2\}, \{3, 5\}\}$, $c_3 = \{\{2, 3\}, \{4, 6\}\}$,
$c_4 = \{\{3, 4\}, \{5, 7\}\}$, $c_5 = \{\{4, 5\}, \{6, 8\}\}$, $c_6 = \{\{5, 6\}, \{7, 0\}\}$,
$c_7 = \{\{6, 7\}, \{8, 1\}\}$, $c_8 = \{\{7, 8\}, \{0, 2\}\}$, $c_9 = \{\{8, 0\}, \{1, 3\}\}$.

Each set-pair $\{A, B\}$ constructs a column in the following manner:

- place 1 at rows indexed by elements of A,
- place 2 at rows by elements of B,
- place distinct elements from $\mathbb{Z}_{\geq 3}$ for the remaining rows.

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
0	1	3	3	3	3	2	3	2	1
1	1	1	4	4	4	3	2	3	2
2	2	1	1	5	5	4	4	2	3
3	3	2	1	1	6	5	5	4	2
4	2	4	2	1	1	6	6	5	4
5	4	2	5	2	1	1	7	6	5
6	5	5	2	6	2	1	1	7	6
7	6	6	6	2	7	2	1	1	7
8	7	7	7	7	2	7	2	1	1

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4	2	4	2	1	1	6	6	5	4
5	4	2	5	2	1	1	7	6	5
6	5	5	2	6	2	1	1	7	6
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Check that the implication structure $J^{(1,1)}$ is J_9^2.

Central Problem

The maximum size of an (n, k, d)-EK code is denoted by $C(n, k, d)$.

An (n, k, d)-EK code of size $C(n, k, d)$ is said to be *optimal*.

The central problem is to determine $C(n, k, d)$.

Problem Status

Theorem (Brightwell and Katona, 2001)

For $1 \le d \le 2k \le n$,

$$C(n,k,d) \le \frac{n(n-1)\cdots(n-2k+d)}{2k(k-1)\cdots\lceil (d+1)/2\rceil \cdot k(k-1)\lfloor (d+1)/2\rfloor}.$$

In fact,

$$C(n,k,d) = \Theta(n^{2k-d+1}) \text{ for fixed } k \text{ and } d.$$

Theorem (Bollobás, Katona, Leader)

$$\lim_{n\to\infty} \frac{C(n,k,d)}{n^{2k-d+1}} = \frac{1}{2k(k-1)\cdots\lceil (d+1)/2\rceil \cdot k(k-1)\lfloor (d+1)/2\rfloor}.$$

Problem Status

Best upper bound $C(n,k,d)$ currently known:

Theorem (Quistorff, 2009)

Suppose $k-d+1 \leq e \leq \min\{k, 2k-d\}$. *Then*

$$C(n,k,d) \leq \left\lfloor \frac{\binom{n}{e}}{2\binom{k}{e}} \left\lfloor \frac{\binom{n-e}{2k-d-e+1}}{\binom{k}{2k-d-e+1}} \right\rfloor \right\rfloor .$$

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Only the following exact values of $C(n,k,d)$ are known.

Theorem (Bollobás, Katona, Leader)

$$C(n,2,3) = \frac{n(n-1)}{8}, \qquad \text{if } n \equiv 1 \text{ or } 9 \bmod 72,$$

$$C(n,3,5) = \frac{n(n-1)}{18}, \qquad \text{if } n \equiv 1 \text{ or } 19 \bmod 342.$$

Our Contributions

Main Theorem

For any fixed $k \geq 2$, we have

$$C(n, k, 2k-1) = \left\lfloor \frac{n}{2k} \left\lfloor \frac{n-1}{k} \right\rfloor \right\rfloor$$

for all sufficiently large n satisfying

(i) $n \equiv 1 \bmod k$ and $n(n-1) \equiv 0 \bmod 2k^2$, or

(ii) $n \equiv 0 \bmod k$.

Exact values

We determine

(i) the value of $C(n, 2, d)$ for all n and $1 \leq d \leq 4$.

(ii) the value of $C(n, 3, 5)$ for $n \equiv 0 \bmod 3$ and $n \equiv 1 \bmod 9$ with finite exceptions.

Our Contributions

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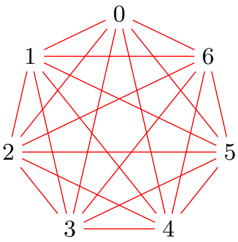
Our main tool is **combinatorial design theory**.

In particular, decomposition of edge-colored complete graphs.

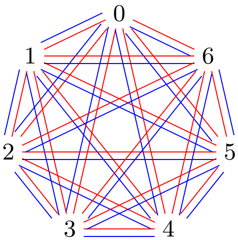
Outline

Complete Graph K_n



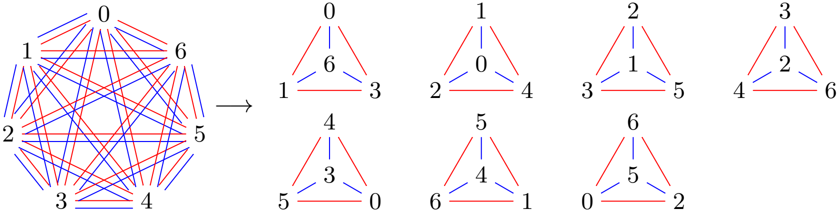
A complete graph K_n on n vertices has an edge between any two vertices.

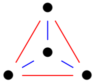
Edge-Colored Complete Graph $K_n^{(r)}$



A complete graph $K_n^{(r)}$ on n vertices has an edge of each of r colors between any two vertices.

Decomposition of Edge-Colored Graphs



A  –decomposition of $K_7^{(2)}$.

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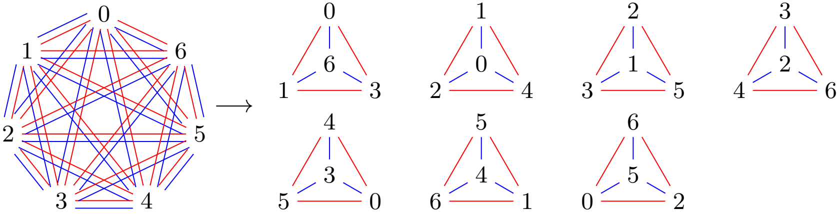
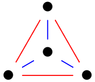
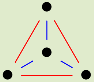


A  –decomposition of $K_7^{(2)}$.

Observations

- Each subgraph in the decomposition is isomorphic to  .

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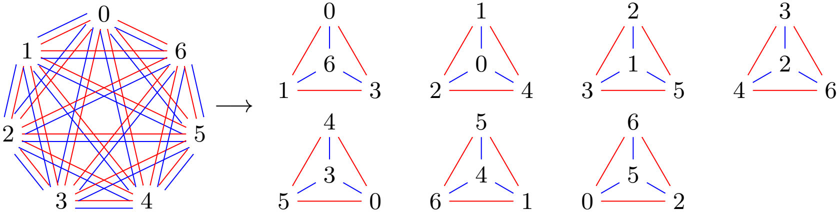
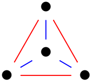


A  –decomposition of $K_7^{(2)}$.

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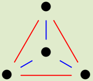
- Each subgraph in the decomposition is isomorphic to  .
- Every edge of $K_7^{(2)}$ belongs to exactly one subgraph in the decomposition.

Existence of G-Decompositions of $K_n^{(r)}$

Theorem (Lamken, Wilson, 2000)

Let G be an edge-colored graph with r colors and m edges of each of r different colors. There exists a constant n_0 such that there is G-decomposition of $K_n^{(r)}$ for all $n \geq n_0$ satisfying both

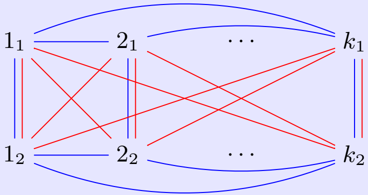
$$n - 1 \equiv 0 \bmod \alpha(G),$$
$$n(n-1) \equiv 0 \bmod 2m,$$

where $\alpha(G)$ is a parameter dependent on G.

$C(n, k, 2k-1)$ for sufficiently large n

Definition

Let G_k be the following graph.



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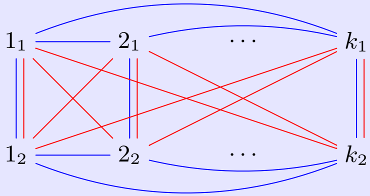

Proposition

If there is a G_k-decomposition of K_n, then there is an $(n, k, 2k-1)$-EK code of size $n(n-1)/2k^2$, which is optimal by Quistorff bound.

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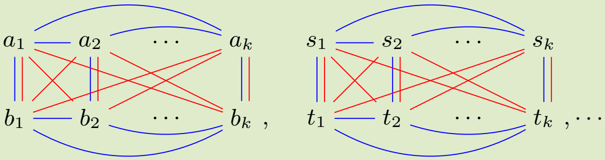
Proof Sketch

Suppose we have a G_k-decomposition of K_n:



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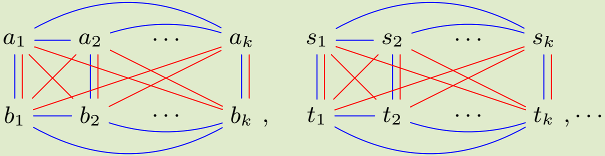

Obtain our code by taking the set-pairs:

$$\{\{a_1, a_2, \ldots, a_k\}, \{b_1, b_2, \ldots, b_k\}\}, \quad \{\{s_1, s_2, \ldots, s_k\}, \{t_1, t_2, \ldots, t_k\}\}, \ldots$$

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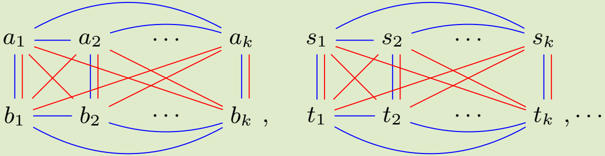

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Check that this is indeed a $(n, k, 2k-1)$-EK code. □

$C(n, k, 2k-1)$ for sufficiently large n

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Main Theorem (i)

For any fixed $k \geq 2$,

$$C(n, k, 2k-1) = \frac{n(n-1)}{2k^2}$$

for all sufficiently large n satisfying

$$n \equiv 1 \bmod k \text{ and } n(n-1) \equiv 0 \bmod 2k^2.$$

Outline

Conclusion

- Motivated by relational databases, we looked at codes in the Enomoto-Katona space.
- Showed that $C(n, k, 2k-1)$ attains the Quistorff bound for infinitely many n.
- Direct application of decomposition of edge-colored graphs.
- Other combinatorial design tools like t-*wise balanced designs* and *Wilson's fundamental construction* enable us to determine other values of $C(n, k, d)$.

Thank you for your attention!