Generalized Balanced Tournament Designs with Block Size Four*

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Submitted: Sep 16, 2012; Accepted: Jun 1, 2013; Published: Jun 7, 2013
Mathematics Subject Classifications: 05B05, 94B25

Abstract

In this paper, we remove the outstanding values m for which the existence of a GBTD$(4, m)$ has not been decided previously. This leads to a complete solution to the existence problem regarding GBTD$(4, m)$s.

Keywords: generalized balanced tournament design; holey generalized balanced tournament design; starter-adder

1 Introduction

A *set system* is a pair $\mathfrak{S} = (X, \mathcal{B})$, where X is a finite set of *points* and $\mathcal{B}$ is a collection of subsets of X. Elements of $\mathcal{B}$ are called *blocks*. The *order* of $\mathfrak{S}$ is $|X|$, the number of points. Let K be a set of positive integers. A set system $(X, \mathcal{B})$ is said to be K-*uniform* if $|B| \in K$ for all $B \in \mathcal{B}$. Let $(X, \mathcal{B})$ be a set system and $S \subseteq X$. A *partial* α-*parallel class* over $X \backslash S$ of $(X, \mathcal{B})$ is a set of blocks $\mathcal{A} \subseteq \mathcal{B}$ such that each point of $X \backslash S$ occurs in exactly α blocks of $\mathcal{A}$, and each point of S occurs in no block of $\mathcal{A}$. A *partial* α-*parallel class* over X is simply called an α-*parallel class*. We adopt the convention that if α is not specified, then it is taken to be one, so that a *parallel class* refers to a 1-parallel class. A set system $(X, \mathcal{B})$ is said to be *resolvable* if $\mathcal{B}$ can be partitioned into parallel classes.

*Research of Y. M. Chee, H. M. Kiah, and C. Wang is supported in part by the Singapore National Research Foundation under Research Grant NRF-CRP2-2007-03. Research of C. Wang is also supported in part by NSFC under Grants No. 11271280 and 10801064.

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A *balanced incomplete block design of order* v*, block size* k*, and index* λ, denoted by (v,k,λ)-BIBD, is a $\{k\}$-uniform set system $(X,\mathcal{B})$ of order v such that every 2-subset of X is contained in precisely λ blocks of $\mathcal{B}$. A resolvable $(km,k,k-1)$-BIBD $(X,\mathcal{B})$ is called a *generalized balanced tournament design* (GBTD), or simply a GBTD(k,m), if the $m(km-1)$ blocks of $\mathcal{B}$ are arranged in an $m\times(km-1)$ array such that

(i) the set of blocks in each column is a parallel class, and

(ii) each point of X is contained in at most k cells of each row.

GBTDs were introduced by Lamken [3] and are useful in the construction of many combinatorial designs, including resolvable, near-resolvable, doubly resolvable, and doubly near-resolvable balanced incomplete block designs (see [2, 3]). More recently, GBTDs have also found applications in near constant-composition codes [12], and codes for power line communications [1].

Schellenberg et al. [8] showed that a GBTD$(2,m)$ exists for all positive integers $m\neq 2$. Lamken [4] showed that a GBTD$(3,m)$ exists for all positive integers $m\neq 2$. For $k=4$, Yin et al. [12] obtained the following.

Theorem 1 (Yin et al. [12]). *A* GBTD$(4,m)$ *exists for all positive integers* $m\geqslant 5$*, except possibly when* $m\in\{28,32,33,34,37,38,39,44\}$.

The purpose of this paper is to remove all the remaining eight possible exceptions in Theorem 1 and to show that a GBTD$(4,m)$ exists for $m=4$ but not for $m\in\{2,3\}$.

Theorem 2. *For each* $m\in\{4,28,32,33,34,37,38,39,44\}$*, a GBTD*$(4,m)$ *exists. For* $m=2$ *and* 3*, a GBTD*$(4,m)$ *does not exist.*

A GBTD$(4,1)$ exists trivially. Combining Theorem 1 and Theorem 2, the existence of GBTD$(4,m)$ is now completely determined.

Theorem 3. *A* GBTD$(4,m)$ *exists if and only if* $m\geqslant 1$ *and* $m\neq 2,3$.

We first present a non-existence result.

Proposition 1.1. *A* GBTD$(k,2)$ *does not exist for all* $k\geqslant 2$.

Proof: Suppose $(X,\mathcal{B})$ is a $(2k,k,k-1)$-BIBD whose blocks are organized into a $2\times(2k-1)$ array to form a GBTD$(k,2)$. Since $(X,\mathcal{B})$ is a $(2k,k,k-1)$-BIBD, each point in X appears in exactly $2k-1$ blocks, and hence each point must appear either k times in the first row and $k-1$ times in the second row, or vice versa.

Consider a point $x\in X$ that appears k times in the first row and $k-1$ times in the second row. Let $y\in X$ be distinct from x. The cells in the first row can be classified as follows:

(i) Type-xy: a cell that contains both x and y.

(ii) Type-$x\bar{y}$: a cell that contains x but not y.

(iii) Type-$\bar{x}y$: a cell that contains y but not x.

(iv) Type-$\bar{x}\bar{y}$: a cell that contains neither x nor y.

Let α and β be the number of type-xy cells and type-$\bar{x}y$ cells in the first row, respecitvely. Then we have the table

T1=

	Type-xy	Type-$x\bar{y}$	Type-$\bar{x}y$	Type-$\bar{x}\bar{y}$
# cells in first row	α	$k-\alpha$	β	$k-1-\beta$
# cells in second row	$k-1-\beta$	β	$k-\alpha$	α

where the second row is obtained from the first by the following observation: if a cell is of type-xy (respectively, type-$x\bar{y}$, type-$\bar{x}y$, type-$\bar{x}\bar{y}$) in the first row, then the cell in the second row of the corresponding column is of type-$\bar{x}\bar{y}$ (respectively, type-$\bar{x}y$, type-$x\bar{y}$, type-xy). On the other hand, the total number of type-xy cells is $k-1$, since $(X, \mathcal{B})$ is a BIBD of index $k-1$. Hence, we have $\alpha + (k-1-\beta) = k-1$, implying $\alpha = \beta$.

Considering the number of occurrences of y in the first row and the number of occurrences of y in the second row of the GBTD give the inequalities

$$\begin{aligned} \alpha + \beta &\leqslant k, \\ 2k-1-\alpha-\beta &\leqslant k, \end{aligned}$$

from which, and $\alpha = \beta$ shown earlier, follow that

$$\alpha = \lfloor k/2 \rfloor.$$

Table T1 can therefore be revised to

T2=

	Type-xy	Type-$x\bar{y}$	Type-$\bar{x}y$	Type-$\bar{x}\bar{y}$
# cells in first row	$\lfloor k/2 \rfloor$	$\lceil k/2 \rceil$	$\lfloor k/2 \rfloor$	$\lceil k/2 \rceil - 1$
# cells in second row	$\lceil k/2 \rceil - 1$	$\lfloor k/2 \rfloor$	$\lceil k/2 \rceil$	$\lfloor k/2 \rfloor$

Counting in two ways the number of elements in the set

$$\{(y, C) : y \in X,\ y \neq x, \text{ and } C \text{ is a cell of type-}xy \text{ in the second row}\}.$$

gives

$$(2k-1)(\lceil k/2 \rceil - 1) = (k-1)^2,$$

which is a contradiction. $\square$

2 Existence of GBTD$(4, m)$s with $m = 3$ and 4

For a positive integer n, the set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. Let Σ be a set of q symbols. A *q-ary code of length n* over Σ is a subset $\mathcal{C} \subseteq \Sigma^n$. Elements of $\mathcal{C}$ are called *codewords*. The *size* of $\mathcal{C}$ is the number of codewords in $\mathcal{C}$. For $i \in [n]$, the ith coordinate of a codeword $\mathsf{u} \in \mathcal{C}$ is denoted u_i, so that $\mathsf{u} = (\mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_n)$.

The *symbol weight* of $\mathsf{u} \in \Sigma^n$, denoted $\mathrm{swt}(\mathsf{u})$, is the maximum frequency of appearance of a symbol in u, that is,

$$\mathrm{swt}(\mathsf{u}) = \max_{\sigma \in \Sigma} |\{\mathsf{u}_i = \sigma : i \in [n]\}|.$$

A code has *constant symbol weight* w if all of its codewords have symbol weight exactly w. The *(Hamming) distance* between $\mathsf{u}, \mathsf{v} \in \Sigma^n$ is $d_{\mathrm{H}}(\mathsf{u}, \mathsf{v}) = |\{i \in [n] : \mathsf{u}_i = \mathsf{v}_i\}|$, the number of coordinates at which u and v differ. A code $\mathcal{C}$ is said to have *distance* d if $d_{\mathrm{H}}(\mathsf{u}, \mathsf{v}) \geqslant d$ for all distinct $\mathsf{u}, \mathsf{v} \in \mathcal{C}$. A q-ary code of length n, constant symbol weight w, and distance d is referred to as an *$(n, d, w)_q$-symbol weight code*. An $(n, d, w)_q$-symbol weight code with maximum size is said to be *optimal*.

Chee et al. [1] established the following relation between a GBTD and a symbol weight code.

Theorem 4 (Chee et al. [1])**.** *A* GBTD(k, m) *exists if and only if an optimal* $(km-1, k(m-1), k)_m$*-symbol weight code exists.*

In view of Theorem 4, to prove the nonexistence of a GBTD$(4, 3)$, it suffices to show that there does not exist a ternary code of length 11, constant symbol weight four, and size 12, that is of equidistance eight. Consider the *Gilbert graph* $G = (V, E)$, where $V = \{\mathsf{u} \in [3]^{11} : \mathrm{swt}(\mathsf{u}) = 4\}$ and two vertices $\mathsf{u}, \mathsf{v} \in V$ are adjacent in G if and only if $d_{\mathrm{H}}(\mathsf{u}, \mathsf{v}) = 8$. Then there exists a ternary code of length 11, constant symbol weight four, and size 12, that is of equidistance eight if and only if there exists a clique of size 12 in G. It is not hard to see that G is vertex-transitive so that we can just search for a clique of size 11 in G', the subgraph of G induced by the set of vertices $\{\mathsf{v} \in V : d_{\mathrm{H}}(\mathsf{u}, \mathsf{v}) = 8\}$ for some fixed $\mathsf{u} \in V$. This induced subgraph G' has 8001 vertices and 7233060 edges. We solve this clique-finding problem using `Cliquer`, an implementation of Östergård's clique-finding algorithm by Niskanen and Östergård [6]. The result is that the largest clique in G' has size 10. Consequently, we have the following.

Proposition 2.1. *There does not exist a* GBTD$(4, 3)$.

There exists, however, a GBTD$(4, 4)$. Unfortunately, a GBTD$(4, 4)$ is too large to be found by the method of clique-finding above. Instead, to curb the search space, we assume the existence of some automorphisms acting on the GBTD$(4, 4)$ to be found. Let $(X, \mathcal{B})$ be a GBTD$(4, 4)$, where $X = \mathbb{Z}_4 \times \mathbb{Z}_4$. If $B \subseteq X$ and $x \in X$, $B + x$ denotes the set $\{b + x : b \in B\}$. If A is an array over X and $x \in X$, $\mathsf{A} + x$ denotes the array obtained by adding x to every entry of A. For succinctness, a point $(x, y) \in \mathbb{Z}_4 \times \mathbb{Z}_4$ is sometimes written xy.

The GBTD(4, 4) we construct contains the 4×3 subarray

$$\mathsf{A}_0 = \begin{array}{|ccc|} \hline \{00,02,20,22\} & \{11,13,31,33\} & \{10,12,30,32\} \\ \{01,03,21,23\} & \{00,02,20,22\} & \{11,13,31,33\} \\ \{10,12,30,32\} & \{01,03,21,23\} & \{00,02,20,22\} \\ \{11,13,31,33\} & \{10,12,30,32\} & \{01,03,21,23\} \\ \hline \end{array}.$$

The blocks in A_0 contain exactly the 2-subsets $\{ab, cd\} \subseteq X$, where $a+c \equiv b+d \equiv 0 \bmod 2$, each thrice. The remaining 4×12 subarray of the GBTD(4, 4) is built from a set of 12 base blocks $\mathcal{S} = \{B_{i,j} : i \in [3] \text{ and } 0 \leqslant j \leqslant 3\}$ as follows. Let

$$\mathsf{A}_1 = \begin{array}{|ccc|} \hline B_{1,0} & B_{2,0} & B_{3,0} \\ B_{1,1} & B_{2,1} & B_{3,1} \\ B_{1,2} & B_{2,2} & B_{3,2} \\ B_{1,3} & B_{2,3} & B_{3,3} \\ \hline \end{array}.$$

Then the 4×12 subarray is given by

$$\begin{array}{|cccc|} \hline \mathsf{A}_1 & \mathsf{A}_1 + (0,1) & \mathsf{A}_1 + (0,2) & \mathsf{A}_1 + (0,3) \\ \hline \end{array}.$$

For

$$\begin{array}{|ccccc|} \hline \mathsf{A}_0 & \mathsf{A}_1 & \mathsf{A}_1 + (0,1) & \mathsf{A}_1 + (0,2) & \mathsf{A}_1 + (0,3) \\ \hline \end{array}$$

to be a GBTD(4, 4), the following conditions are imposed:

(i) $\bigcup_{j=0}^{3} B_{i,j} = \mathbb{Z}_4 \times \mathbb{Z}_4$, for $i \in [3]$, so that every column is a parallel class.

(ii) For each j, $0 \leqslant j \leqslant 3$, each element of $\mathbb{Z}_4$ appears exactly three times as a first coordinate among the elements of $\bigcup_{i=1}^{3} B_{i,j}$, so that every row contains each element of $\mathbb{Z}_4 \times \mathbb{Z}_4$ at most three times.

(iii) Let $k, l \in \mathbb{Z}_4$ and define $\Delta_{k,l}\mathcal{S}$ to be the multiset $\bigcup_{A \in \mathcal{S}} \{x - y : (k, x), (l, y) \in A\}$. Then

$$\Delta_{k,l}\mathcal{S} = \begin{cases} \{1,1,1,3,3,3\}, & \text{if } k = l \text{ or } k + l \equiv 0 \bmod 2; \\ \{0,0,0,1,1,1,2,2,2,3,3,3\}, & \text{otherwise.} \end{cases}$$

This ensures that every 2-subset of X not contained in any block in A_0 is contained in exactly three blocks in A_1, $\mathsf{A}_1 + (0,1)$, $\mathsf{A}_1 + (0,2)$, or $\mathsf{A}_1 + (0,3)$.

A computer search found the following array A_1 that satisfies all the conditions above.

$$\mathsf{A}_1 = \begin{array}{|ccc|} \hline \{23,22,32,11\} & \{10,00,21,11\} & \{00,01,30,33\} \\ \{20,01,30,33\} & \{33,02,03,12\} & \{10,13,22,23\} \\ \{31,00,12,21\} & \{01,13,20,32\} & \{02,11,21,32\} \\ \{02,10,13,03\} & \{22,23,30,31\} & \{03,12,20,31\} \\ \hline \end{array}.$$

Consequently, we have the following.

Proposition 2.2. *There exists a* GBTD(4, 4).

3 Incomplete Holey GBTDs

Let $(X, \mathcal{B})$ be a set system, and let $\mathcal{G}$ be a partition of X into subsets, called *groups*. The triple $(X, \mathcal{G}, \mathcal{B})$ is a *group divisible design* (GDD) of index λ when every 2-subset of X not contained in a group appears in exactly λ blocks, and $|B \cap G| \leqslant 1$ for all $B \in \mathcal{B}$ and $G \in \mathcal{G}$. We denote a GDD $(X, \mathcal{G}, \mathcal{B})$ of index λ by (K, λ)-GDD if $(X, \mathcal{B})$ is K-uniform. The *type* of a GDD $(X, \mathcal{G}, \mathcal{B})$ is the multiset $[|G| : G \in \mathcal{G}]$. When more convenient, the exponential notation is used to describe the type of a GDD: a GDD of type $g_1^{t_1} g_2^{t_2} \cdots g_s^{t_s}$ is a GDD where there are exactly t_i groups of size g_i, $i \in [s]$.

Suppose further $\mathcal{G} = \{G_1, G_2, \ldots G_s\}$ and $\mathcal{H} = \{H_1, H_2, \ldots H_s\}$ is a collection of subsets of X with the property $H_i \subseteq G_i$, $0 \leqslant i \leqslant s$. Let $H = \bigcup_{i=1}^{s} H_i$. Then the quadruple $(X, \mathcal{G}, \mathcal{H}, \mathcal{B})$ is an *incomplete group divisible design* (IGDD) of index λ when every 2-subset of X not contained in a group or H appears in exactly λ blocks, and $|B \cap G| \leqslant 1$ and $|B \cap H| \leqslant 1$ for all $B \in \mathcal{B}$ and $G \in \mathcal{G}$. The *type* of an IGDD $(X, \{G_1, G_2, \ldots, G_s\}, \{H_1, H_2, \ldots, H_s\}, \mathcal{B})$ is the multiset $[(|G_i|, |H_i|) : 1 \leqslant i \leqslant s]$ and we use the exponential notation when more convenient.

Let k, g, u, and w be positive integers such that $k \mid g$ and $u \geqslant (k+1)w$. Let $R_i = \{r \in \mathbb{Z} : ig/k \leqslant r \leqslant (i+1)g/k - 1\}$. An *incomplete holey* GBTD with block size k and type $g^{(u,w)}$, denoted IHGBTD$\left(k, g^{(u,w)}\right)$, is a $(\{k\}, k-1)$-IGDD $(X, \{G_0, G_1, \ldots, G_{u-1}\}, \{\varnothing, \ldots, \varnothing, G_{u-w}, \ldots, G_{u-1}\}, \mathcal{B})$ of type $(g, 0)^{u-w}(g, g)^w$, whose blocks are arranged in a $(gu/k) \times g(u-1)$ array A, with rows and columns indexed by elements from the sets $\{0, 1, \ldots, gu/k - 1\}$ and $\{0, 1, \ldots, g(u-1) - 1\}$, respectively, such that the following properties are satisfied.

(i) The $(gw/k) \times g(w-1)$ subarray whose rows are indexed by $r \in R_i$, where $u - w \leqslant i \leqslant u - 1$, and columns indexed by c, where $g(u-w) \leqslant c \leqslant g(u-1) - 1$, is empty.

(ii) For each i, $0 \leqslant i \leqslant u - w - 1$, the blocks in row $r \in R_i$ form a partial k-parallel class over $X \setminus G_i$, and for each i, $u - w \leqslant i \leqslant u - 1$, the blocks in row $r \in R_i$ form a partial k-parallel class over $X \setminus \left(\bigcup_{j=u-w}^{w-1} G_j\right)$.

(iii) For each j, $0 \leqslant j \leqslant g(u-w) - 1$, the blocks in column j form a parallel class, and for each j, $g(u-w) \leqslant j \leqslant g(u-1) - 1$, the blocks in column j form a partial parallel class over $X \setminus \left(\bigcup_{i=u-w}^{w-1} G_j\right)$.

Each group acts as a *hole* of the design, since no block contains any 2-subset of a group. The design is also *incomplete* in the sense that the array A contains an empty $(gw/k) \times g(w-1)$ subarray.

We note that an IHGBTD$(k, g^{(u,1)})$ is a holey GBTD, HGBTD(k, g^u), as defined by Yin et al. [12]. The following was established by Yin et al. [12].

Proposition 3.1 (Yin et al. [12]). *If there exists an* HGBTD(k, k^u)*, then there exists a* GBTD(k, u).

Proposition 3.1 shows that a GBTD(k, u) can be obtained from an HGBTD(k, k^u). The next result shows how we can obtain an HGBTD(k, g^u) (and, in particular, an HGBTD(k, k^u)) from an IHGBTD$(k, g^{(u,w)})$ and an HGBTD(k, g^w).

Proposition 3.2. *If there exist an* IHGBTD$(k, g^{(u,w)})$ *and an* HGBTD(k, g^w)*, then there exists an* HGBTD(k, g^u).

Proof: When $w = 1$, an HGBTD(k, g^w) is empty and an IHGBTD$(k, g^{(u,w)})$ is just an HGBTD(k, g^u). So assume $w > 1$ and let $(X, \mathcal{G}, \mathcal{B})$ be an IHGBTD$(k, g^{(u,w)})$ with $\mathcal{G} = \{G_0, G_1, \ldots, G_{u-1}\}$. Fill in the empty subarray of this IHGBTD with an HGBTD(k, g^w), $(X', \mathcal{G}', \mathcal{B}')$, with $\mathcal{G}' = \{G_{u-w}, G_{u-w+1}, \ldots, G_{u-1}\}$ and $X' = \bigcup_{i=u-w}^{u-1} G_i$. The resulting array is a HGBTD(k, g^u), $(X, \mathcal{G}, \mathcal{B} \cup \mathcal{B}')$. □

4 Starter-Adder Construction for IHGBTD

The starter-adder technique first used by Mullin and Nemeth [5] to construct Room squares (also a combinatorial array) has been useful in constructing many types of designs with orthogonality properties, including GBTDs (see [3, 7, 10, 11, 12]). Here, we extend the technique to the construction of IHGBTDs. Since only IHGBTD$(k, g^{(u,w)})$ with $g = k$ are considered here, we describe our construction for this case.

Let Γ be an additive abelian group of order $k(u-w)$ with $u \geqslant (k+1)w$, and let $\Gamma_0 \subseteq \Gamma$ be a subgroup of order k. Fix a set, $\Delta = \{\delta_0 = 0, \delta_1, \ldots, \delta_{u-w-1}\} \subseteq \Gamma$, of representatives for the cosets of Γ_0 so that $\Gamma_i = \Gamma_0 + \delta_i$, $0 \leqslant i \leqslant u-w-1$, are the cosets of Γ_0. Let H be a set of kw points such that H and Γ are disjoint. Further, let H be partitioned into w subsets $H_0, H_1, \ldots, H_{w-1}$ of size k each.

We take $X = \Gamma \bigcup H$ to be the point set of an IHGBTD$(k, k^{(u,w)})$. An *intransitive starter* for an IHGBTD$(k, k^{(u,w)})$, with groups $\{G_0, G_1, \ldots, G_{u-1}\}$, where

$$G_i = \begin{cases} \Gamma_i, & \text{if } 0 \leqslant i \leqslant u-w-1; \\ H_{i-u+w}, & \text{if } u-w \leqslant i \leqslant u-1, \end{cases}$$

is defined as a quadruple $(X, \mathcal{S}, \mathcal{R}, \mathcal{C})$ satisfying the properties:

(i) $(X, \mathcal{S})$, $(X, \mathcal{R})$, and $(X, \mathcal{C})$ are $\{k\}$-uniform set systems of size $u-w$, w, and $w-1$, respectively;

(ii) among the blocks in $\mathcal{S}$, kw of them intersects H in one point, that is, $|\{B \in \mathcal{S} : |B \cap H| = 1\}| = kw$;

(iii) blocks in $\mathcal{R}$ are each disjoint from H;

(iv) blocks in $\mathcal{C}$ are each disjoint from H, and $\bigcup_{i=0}^{u-w-1}(\delta_i + C) = \Gamma$, for each $C \in \mathcal{C}$.

(v) $\mathcal{S} \bigcup \mathcal{R}$ is a partition of X;

(vi) the difference list from the base blocks of $\mathcal{S} \bigcup \mathcal{R} \bigcup \mathcal{C}$ contains every element of $\Gamma \setminus \Gamma_0$ precisely $k-1$ times, and no element in Γ_0.

Suppose $\mathcal{S} = \{B_0, B_1, \ldots, B_{u-w-1}\}$. Then a corresponding *adder* $\Omega(\mathcal{S})$ for $\mathcal{S}$ is a permutation $\Omega(\mathcal{S}) = (\omega_0, \omega_1, \ldots, \omega_{u-w-1})$ of the $u-w$ elements of the representative system Δ satisfying the following property:

(vii) the multiset $\left(\bigcup_{i=0}^{u-w-1}(B_i + \omega_i)\right) \bigcup \left(\bigcup_{C\in\mathcal{C}} C\right)$ contains exactly k elements (not necessarily distinct) from Γ_j for $1 \leqslant j \leqslant u-w-1$, and is disjoint from Γ_0. We remark that when $B \in \mathcal{S}$ and $B \cap H = \{\infty\}$, or $B = \{\infty, b_1, b_2, \ldots, b_{k-1}\}$, the block $B + \gamma$ is defined to be $\{\infty, b_1 + \gamma, b_2 + \gamma, \ldots, b_{k-1} + \gamma\}$ for $\gamma \in \Gamma$.

The result below shows how to construct an IHGBTD from an intransitive starter and its corresponding adder.

Proposition 4.1. *Let Γ be an additive abelian group of order $k(u-w)$ with $u \geqslant (k+1)w$ and Γ_0 be a subgroup of order k. Define X and the groups G_i $(0 \leqslant i \leqslant u-1)$ as above. If there exists an intransitive starter $(X, \mathcal{S}, \mathcal{R}, \mathcal{C})$ with groups $\{G_i : 0 \leqslant i \leqslant u-1\}$, a corresponding adder $\Omega(\mathcal{S})$, then there exists an IHGBTD$(k, k^{(u,w)})$.*

Proof: Retain the notations in the definition of intransitive starter and adder. Suppose

$$\mathcal{A} = \{A + \gamma : \gamma \in \Gamma, A \in \mathcal{S} \cup \mathcal{R} \cup \mathcal{C}\},$$

then $(X, \{G_0, G_1, \ldots, G_{u-1}\}, \{\varnothing, \ldots, \varnothing, H_0, \ldots, H_{w-1}\}, \mathcal{A})$ forms a $(\{k\}, k-1)$-IGDD of type $(k,0)^{u-w}(k,k)^w$ by Condition (vi) in the definition of intransitive starter. Therefore, it remains to arrange the blocks in a $u \times k(u-1)$ array.

First, consider the blocks $\mathcal{S}$. Consider a $(u-w) \times (u-w)$ array S, whose rows and columns are indexed with the elements in Δ. Now identify the elements in Δ with elements in the quotient group Γ/Γ_0, so that the binary operation $\mathring{+}$ on Δ is defined by the additive operation on Γ/Γ_0. In addition, for $\delta \in \Delta$, denote the additive inverse of δ by $\mathring{-}\delta$. That is, $\delta \mathring{+} (\mathring{-}\delta) = \delta_0$.

So, for $0 \leqslant i, j \leqslant u-w-1$, we place the block $B_i + \delta_j$ at the cell $(\delta_j \mathring{-} \delta_l, \delta_j)$ if $\delta_l = \omega_i$. Note that this placement is well defined because $\Omega(\mathcal{S})$ is a permutation of Δ. Let $\Gamma_0 = \{\gamma_0 = 0, \gamma_1, \cdots, \gamma_{k-1}\}$. Form a $(u-w) \times k(u-w)$ array $\widehat{\mathsf{S}}$ from the square S by concatenating k squares $\mathsf{D} + \gamma_i$ where $0 \leqslant i \leqslant k-1$ as follows.

$$\widehat{\mathsf{S}} = \boxed{\mathsf{S} \quad \mathsf{S} + \gamma_1 \quad \cdots \quad \mathsf{S} + \gamma_{k-1}}$$

Next, let $\mathcal{R} = \{R_1, R_2, \ldots, R_w\}$ and construct a $w \times k(u-w)$ array $\widehat{\mathsf{R}}$ in the following way:

$$\widehat{\mathsf{R}} = \boxed{\mathsf{R} \quad \mathsf{R} + \gamma_1 \quad \cdots \quad \mathsf{R} + \gamma_{k-1}},$$

where the $w \times w$ subarray R is given by

$$\mathsf{R} = \begin{array}{|cccc|} \hline R_1 & R_1 + \delta_1 & \cdots & R_1 + \delta_{u-w-1} \\ R_2 & R_2 + \delta_1 & \cdots & R_2 + \delta_{u-w-1} \\ \vdots & \vdots & \ddots & \vdots \\ R_w & R_w + \delta_1 & \cdots & R_w + \delta_{u-w-1} \\ \hline \end{array}.$$

Similarly, let $\mathcal{C} = \{C_0, C_1, \ldots, C_{w-2}\}$, and we construct a $(u-w) \times k(w-1)$ array $\widehat{\mathsf{C}}$.

$$\widehat{\mathsf{C}} = \boxed{\mathsf{C}_0 \quad \mathsf{C}_1 \quad \cdots \quad \mathsf{C}_{w-2}},$$

where each $(u-w) \times k$ subarray C_i, $0 \leqslant i \leqslant w-2$, is given by

$$\mathsf{C}_i = \begin{array}{|cccc|}\hline C_i & C_i+\gamma_1 & \cdots & C_i+\gamma_{k-1} \\ C_i+\delta_1 & C_i+\delta_1+\gamma_1 & \cdots & C_i+\delta_1+\gamma_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ C_i+\delta_{u-w-1} & C_i+\delta_{u-w-1}+\gamma_1 & \cdots & C_i+\delta_{u-w-1}+\gamma_{k-1} \\ \hline\end{array}.$$

Finally, let

$$\mathsf{A} = \begin{array}{|c|c|}\hline \widehat{\mathsf{S}} & \widehat{\mathsf{C}} \\ \hline \widehat{\mathsf{R}} & \\ \hline\end{array},$$

and it is readily verified that the placement results in an IHGBTD$(k, k^{(u,w)})$. □

5 Proof of Theorem 1.2

We first remove all the eight remaining values in Theorem 1.

Lemma 5. *For $(u,w) \in \{(28,5), (32,5), (33,6)\}$, an IHGBTD$\big(4, 4^{(u,w)}\big)$ exists.*

Proof: We apply Proposition 4.1 to construct the desired IHGBTDs. Take

$$\begin{aligned}
\Gamma &= \mathbb{Z}_{u-w} \times \mathbb{Z}_4, \\
\Gamma_0 &= \{0\} \times \mathbb{Z}_4, \\
\Delta &= \{(0,0), (1,0), \ldots, (u-w-1,0)\}, \text{ and} \\
H &= \bigcup_{i=0}^{w-1} H_i, \text{ where } H_i = \{\infty_i, \infty_{i+w}, \infty_{i+2w}, \infty_{i+3w}\} \text{ for } 0 \leqslant i \leqslant w-1.
\end{aligned}$$

For each pair $(u,w) \in \{(28,5), (32,5), (33,6)\}$, the desired intransitive starter and corresponding adder are displayed below. Here we write the element (a,b) of Γ as a_b for succinctness.

When $(u,w) = (28,5)$:

$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$
$\{4_1, 3_0, 7_0, 0_0\}$	17_0	$\{5_0, 19_0, 12_1, 1_2\}$	12_0	$\{18_0, 13_3, 16_3, 8_1\}$	19_0
$\{\infty_0, 3_1, 12_2, 11_3\}$	1_0	$\{\infty_1, 14_3, 6_0, 10_3\}$	21_0	$\{\infty_2, 14_1, 9_1, 20_1\}$	20_0
$\{\infty_3, 19_1, 10_1, 22_2\}$	7_0	$\{\infty_4, 3_3, 1_3, 2_2\}$	18_0	$\{\infty_5, 0_2, 15_1, 1_0\}$	15_0
$\{\infty_6, 1_1, 6_3, 9_3\}$	2_0	$\{\infty_7, 14_0, 11_1, 0_1\}$	10_0	$\{\infty_8, 0_3, 17_2, 21_2\}$	22_0
$\{\infty_9, 4_3, 8_0, 21_0\}$	6_0	$\{\infty_{10}, 13_1, 19_3, 16_2\}$	9_0	$\{\infty_{11}, 4_2, 21_3, 17_1\}$	5_0
$\{\infty_{12}, 17_0, 5_2, 21_1\}$	16_0	$\{\infty_{13}, 5_1, 20_2, 11_2\}$	4_0	$\{\infty_{14}, 22_0, 2_3, 16_0\}$	14_0
$\{\infty_{15}, 18_3, 20_3, 2_0\}$	0_0	$\{\infty_{16}, 12_3, 2_1, 22_3\}$	3_0	$\{\infty_{17}, 5_3, 7_1, 17_3\}$	8_0
$\{\infty_{18}, 6_2, 9_0, 19_2\}$	13_0	$\{\infty_{19}, 7_2, 8_3, 22_1\}$	11_0		

$$\begin{array}{llll} \mathcal{C} = & \{18_0, 11_1, 5_3, 6_2\}, & \{18_2, 8_3, 19_0, 6_1\}, & \{14_3, 12_0, 3_2, 7_1\}, \\ & \{5_2, 7_1, 16_3, 11_0\}. & & \\ \mathcal{R} = & \{3_2, 18_2, 16_1, 10_2\}, & \{8_2, 15_0, 20_0, 13_2\}, & \{13_0, 9_2, 18_1, 15_3\}, \\ & \{6_1, 7_3, 14_2, 15_2\}, & \{12_0, 10_0, 4_0, 11_0\}. & \end{array}$$

When $(u, w) = (32, 5)$:

$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$
$\{4_2, 17_2, 16_1, 22_2\}$	16_0	$\{3_1, 4_1, 1_0, 9_1\}$	11_0	$\{4_3, 26_3, 22_0, 10_3\}$	0_0
$\{14_1, 6_0, 26_0, 3_0\}$	12_0	$\{\infty_0, 3_3, 24_2, 25_1\}$	7_0	$\{\infty_1, 2_2, 12_0, 1_3\}$	6_0
$\{\infty_2, 0_1, 26_1, 20_2\}$	4_0	$\{\infty_3, 25_0, 15_0, 23_0\}$	15_0	$\{\infty_4, 13_0, 21_2, 16_0\}$	3_0
$\{\infty_5, 5_0, 19_3, 12_1\}$	24_0	$\{\infty_6, 6_3, 14_3, 13_2\}$	1_0	$\{\infty_7, 1_2, 2_0, 0_0\}$	21_0
$\{\infty_8, 0_2, 10_0, 19_0\}$	14_0	$\{\infty_9, 15_2, 18_2, 0_3\}$	2_0	$\{\infty_{10}, 6_1, 5_2, 2_3\}$	17_0
$\{\infty_{11}, 12_3, 25_2, 11_3\}$	22_0	$\{\infty_{12}, 10_1, 21_3, 17_3\}$	18_0	$\{\infty_{13}, 17_0, 9_0, 20_3\}$	20_0
$\{\infty_{14}, 20_0, 3_2, 16_3\}$	5_0	$\{\infty_{15}, 12_2, 21_1, 8_2\}$	9_0	$\{\infty_{16}, 18_1, 11_0, 15_3\}$	10_0
$\{\infty_{17}, 1_1, 15_1, 17_1\}$	8_0	$\{\infty_{18}, 9_2, 16_2, 23_2\}$	13_0	$\{\infty_{19}, 14_2, 18_3, 21_0\}$	25_0

$$\begin{array}{llll} \mathcal{C} = & \{1_3, 26_0, 16_1, 17_2\}, & \{5_3, 14_1, 24_2, 12_0\}, & \{19_2, 25_0, 17_1, 13_3\}, \\ & \{6_2, 8_0, 11_3, 13_1\}. & & \\ \mathcal{R} = & \{5_1, 11_1, 24_3, 20_1\}, & \{24_1, 18_0, 7_0, 6_2\}, & \{22_1, 25_3, 8_0, 13_3\}, \\ & \{19_2, 7_2, 2_1, 23_3\}, & \{7_1, 9_3, 26_2, 4_0\}. & \end{array}$$

When $(u, w) = (33, 6)$:

$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$
$\{22_0, 0_1, 23_0, 21_3\}$	13_0	$\{25_3, 4_3, 15_1, 20_1\}$	4_0	$\{7_3, 2_2, 23_3, 1_0\}$	7_0
$\{\infty_0, 21_1, 3_0, 22_2\}$	18_0	$\{\infty_1, 0_0, 14_3, 10_1\}$	6_0	$\{\infty_2, 12_3, 8_0, 16_1\}$	8_0
$\{\infty_3, 6_1, 23_2, 9_1\}$	23_0	$\{\infty_4, 4_0, 8_2, 14_2\}$	2_0	$\{\infty_5, 14_1, 2_3, 6_0\}$	17_0
$\{\infty_6, 21_2, 24_2, 11_2\}$	9_0	$\{\infty_7, 5_0, 2_1, 25_1\}$	20_0	$\{\infty_8, 11_1, 22_1, 12_1\}$	22_0
$\{\infty_9, 0_2, 7_2, 19_2\}$	15_0	$\{\infty_{10}, 13_0, 16_0, 14_0\}$	24_0	$\{\infty_{11}, 11_0, 15_0, 18_1\}$	3_0
$\{\infty_{12}, 7_0, 9_0, 26_1\}$	19_0	$\{\infty_{13}, 25_0, 7_1, 10_0\}$	21_0	$\{\infty_{14}, 18_0, 25_2, 26_3\}$	26_0
$\{\infty_{15}, 4_2, 15_2, 13_3\}$	16_0	$\{\infty_{16}, 17_1, 20_0, 11_3\}$	5_0	$\{\infty_{17}, 20_2, 9_3, 12_0\}$	14_0
$\{\infty_{18}, 26_2, 5_2, 17_2\}$	12_0	$\{\infty_{19}, 24_0, 13_1, 10_3\}$	1_0	$\{\infty_{20}, 1_3, 10_2, 12_2\}$	11_0
$\{\infty_{21}, 3_2, 15_3, 24_1\}$	25_0	$\{\infty_{22}, 5_1, 18_3, 21_0\}$	10_0	$\{\infty_{23}, 17_0, 24_3, 26_0\}$	0_0

$$\begin{array}{llll} \mathcal{C} = & \{3_3, 10_1, 5_2, 15_0\}, & \{8_3, 14_1, 9_2, 18_0\}, & \{12_0, 10_3, 26_2, 5_1\}, \\ & \{21_2, 11_1, 23_0, 9_3\}, & \{15_1, 5_2, 12_3, 3_0\}. & \\ \mathcal{R} = & \{6_3, 2_0, 18_2, 19_0\}, & \{8_3, 9_2, 3_1, 1_2\}, & \{17_3, 3_3, 4_1, 22_3\}, \\ & \{19_3, 13_2, 6_2, 5_3\}, & \{16_3, 23_1, 1_1, 19_1\}, & \{20_3, 16_2, 8_1, 0_3\}. \end{array}$$

□

Lemma 6. *For $(u, w) \in \{(34, 6), (44, 8)\}$, an IHGBTD$\big(4, 4^{(u,w)}\big)$ exists.*

Proof: As with Lemma 5, we apply Proposition 4.1 to construct the desired IHGBTDs. Take

$$\Gamma = \mathbb{Z}_{2(u-w)} \times \mathbb{Z}_2,$$
$$\Gamma_0 = \{0, u-w\} \times \mathbb{Z}_2,$$
$$\Delta = \{(0,0), (1,0), \cdots, (u-w-1, 0)\}, \text{ and}$$
$$H = \bigcup_{i=0}^{w-1} H_i, \text{ where } H_i = \{\infty_i, \infty_{i+w}, \infty_{i+2w}, \infty_{i+3w}\} \text{ for } 0 \leqslant i \leqslant w-1.$$

The desired intransitive starter and corresponding adder for $(u, w) \in \{(34, 6), (44, 8)\}$ are displayed below. Here we write the element (a, b) of Γ as a_b for succinctness.

When $(u, w) = (34, 6)$:

$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$
$\{41_1, 16_0, 6_0, 15_0\}$	20_0	$\{36_0, 9_0, 33_1, 13_1\}$	16_0	$\{37_0, 18_0, 26_1, 4_1\}$	0_0
$\{16_1, 2_1, 4_0, 3_1\}$	3_0	$\{\infty_0, 20_1, 24_0, 42_0\}$	23_0	$\{\infty_1, 22_1, 30_0, 39_1\}$	11_0
$\{\infty_2, 14_0, 31_1, 1_1\}$	10_0	$\{\infty_3, 48_0, 45_0, 8_0\}$	25_0	$\{\infty_4, 25_1, 48_1, 14_1\}$	4_0
$\{\infty_5, 8_1, 30_1, 20_0\}$	12_0	$\{\infty_6, 6_1, 21_0, 44_1\}$	2_0	$\{\infty_7, 40_1, 33_0, 52_1\}$	1_0
$\{\infty_8, 45_1, 21_1, 28_1\}$	18_0	$\{\infty_9, 27_0, 28_0, 34_1\}$	17_0	$\{\infty_{10}, 42_1, 35_1, 37_1\}$	22_0
$\{\infty_{11}, 3_0, 22_0, 12_0\}$	19_0	$\{\infty_{12}, 44_0, 35_0, 39_0\}$	14_0	$\{\infty_{13}, 36_1, 7_0, 9_1\}$	7_0
$\{\infty_{14}, 15_1, 53_1, 51_1\}$	6_0	$\{\infty_{15}, 53_0, 11_0, 51_0\}$	15_0	$\{\infty_{16}, 50_0, 55_1, 10_1\}$	9_0
$\{\infty_{17}, 52_0, 32_1, 17_1\}$	13_0	$\{\infty_{18}, 55_0, 29_1, 25_0\}$	5_0	$\{\infty_{19}, 0_1, 7_1, 41_0\}$	27_0
$\{\infty_{20}, 12_1, 31_0, 47_0\}$	8_0	$\{\infty_{21}, 17_0, 27_1, 47_1\}$	21_0	$\{\infty_{22}, 19_0, 23_0, 29_0\}$	24_0
$\{\infty_{23}, 34_0, 40_0, 50_1\}$	26_0				

$$\begin{aligned}
\mathcal{C} = \;& \{27_1, 10_0, 44_1, 51_0\}, \quad \{35_1, 15_0, 50_0, 14_1\}, \quad \{16_1, 51_1, 54_0, 27_0\}, \\
& \{24_1, 12_0, 37_0, 21_1\}, \quad \{39_0, 2_1, 45_1, 50_0\}. \\
\mathcal{R} = \;& \{13_0, 26_0, 38_0, 24_1\}, \quad \{54_1, 23_1, 46_1, 49_1\}, \quad \{1_0, 49_0, 18_1, 43_0\}, \\
& \{10_0, 2_0, 11_1, 54_0\}, \quad \{46_0, 19_1, 43_1, 5_0\}, \quad \{38_1, 32_0, 5_1, 0_0\}.
\end{aligned}$$

When $(u, w) = (44, 8)$:

$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$
$\{32_0, 69_1, 36_1, 53_1\}$	20_0	$\{42_1, 65_1, 0_0, 43_1\}$	1_0	$\{39_1, 27_1, 45_1, 51_1\}$	3_0
$\{22_1, 39_0, 55_1, 33_1\}$	11_0	$\{\infty_0, 67_0, 40_1, 54_0\}$	22_0	$\{\infty_1, 23_0, 10_1, 34_1\}$	25_0
$\{\infty_2, 18_0, 67_1, 36_0\}$	28_0	$\{\infty_3, 25_1, 10_0, 28_1\}$	16_0	$\{\infty_4, 63_1, 6_0, 37_0\}$	29_0
$\{\infty_5, 16_0, 44_0, 2_0\}$	35_0	$\{\infty_6, 28_0, 50_1, 35_1\}$	10_0	$\{\infty_7, 43_0, 46_1, 32_1\}$	9_0
$\{\infty_8, 69_0, 52_1, 2_1\}$	13_0	$\{\infty_9, 37_1, 66_0, 71_1\}$	26_0	$\{\infty_{10}, 70_1, 21_1, 24_1\}$	8_0
$\{\infty_{11}, 71_0, 15_1, 47_0\}$	32_0	$\{\infty_{12}, 59_0, 19_1, 6_1\}$	23_0	$\{\infty_{13}, 9_0, 47_1, 20_0\}$	7_0
$\{\infty_{14}, 52_0, 46_0, 60_1\}$	24_0	$\{\infty_{15}, 17_0, 60_0, 22_0\}$	0_0	$\{\infty_{16}, 64_0, 54_1, 12_0\}$	17_0
$\{\infty_{17}, 49_0, 9_1, 53_0\}$	4_0	$\{\infty_{18}, 68_0, 0_1, 56_1\}$	15_0	$\{\infty_{19}, 27_0, 12_1, 4_1\}$	27_0
$\{\infty_{20}, 65_0, 68_1, 23_1\}$	2_0	$\{\infty_{21}, 20_1, 18_1, 8_0\}$	31_0	$\{\infty_{22}, 59_1, 17_1, 44_1\}$	14_0
$\{\infty_{23}, 1_0, 70_0, 26_1\}$	12_0	$\{\infty_{24}, 57_1, 11_1, 13_0\}$	21_0	$\{\infty_{25}, 16_1, 5_0, 7_0\}$	18_0
$\{\infty_{26}, 58_1, 4_0, 57_0\}$	5_0	$\{\infty_{27}, 41_1, 13_1, 31_1\}$	19_0	$\{\infty_{28}, 64_1, 56_0, 30_1\}$	30_0
$\{\infty_{29}, 19_0, 48_0, 21_0\}$	6_0	$\{\infty_{30}, 48_1, 58_0, 50_0\}$	33_0	$\{\infty_{31}, 40_0, 49_1, 5_1\}$	34_0

$$\begin{aligned}
\mathcal{C} = \;& \{2_1, 3_1, 22_0, 69_0\}, \quad \{28_1, 69_0, 19_1, 62_0\}, \quad \{41_1, 4_0, 20_1, 59_0\}, \\
& \{57_0, 12_1, 4_0, 55_1\}, \quad \{41_0, 21_1, 32_1, 8_0\}, \quad \{7_1, 13_0, 14_1, 28_0\}, \\
& \{33_1, 21_0, 28_1, 52_0\}. \\
\mathcal{R} = \;& \{66_1, 3_1, 25_0, 29_1\}, \quad \{38_0, 34_0, 3_0, 24_0\}, \quad \{55_0, 15_0, 62_0, 45_0\}, \\
& \{62_1, 61_0, 42_0, 29_0\}, \quad \{51_0, 35_0, 30_0, 26_0\}, \quad \{61_1, 1_1, 14_0, 38_1\}, \\
& \{14_1, 11_0, 31_0, 63_0\}, \quad \{7_1, 33_0, 8_1, 41_0\}.
\end{aligned}$$

□

Lemma 7. *For each* $(u, w) \in \{(37, 6), (38, 7), (39, 6)\}$*, an IHGBTD*$\big(4, 4^{(u,w)}\big)$ *exists.*

Proof: As with Lemma 5, we apply Proposition 4.1. Take

$$\begin{aligned}\Gamma &= \mathbb{Z}_{u-w} \times \mathbb{Z}_2 \times \mathbb{Z}_2,\\ \Gamma_0 &= \{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2\\ \Delta &= \{((0,0,0),(1,0,0),\cdots,(u-w-1,0,0))\},\ \text{and}\\ H &= \bigcup_{i=0}^{w-1} H_i,\ \text{where } H_i = \{\infty_i, \infty_{i+w}, \infty_{i+2w}, \infty_{i+3w}\} \text{ for } 0 \leqslant i \leqslant w-1.\end{aligned}$$

The desired intransitive starter and corresponding adder for $(u,w) \in \{(37,6),(38,7),(39,6)\}$ are displayed below. Here we write the element (a,b,c) of Γ as a_{bc} for succinctness.

When $(u,w)=(37,6)$:

$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$
$\{6_{00},25_{00},3_{00},7_{11}\}$	30_{00}	$\{20_{10},13_{00},23_{11},27_{01}\}$	28_{00}	$\{12_{00},13_{01},19_{11},17_{00}\}$	2_{00}
$\{20_{11},19_{00},9_{00},1_{11}\}$	17_{00}	$\{29_{11},26_{11},2_{11},0_{01}\}$	3_{00}	$\{21_{10},11_{10},1_{10},27_{10}\}$	21_{00}
$\{9_{01},27_{11},4_{10},16_{11}\}$	11_{00}	$\{\infty_0,26_{01},28_{01},5_{00}\}$	4_{00}	$\{\infty_1,14_{10},3_{11},25_{11}\}$	29_{00}
$\{\infty_2,21_{00},11_{11},23_{01}\}$	24_{00}	$\{\infty_3,21_{11},5_{10},18_{00}\}$	7_{00}	$\{\infty_4,28_{11},10_{11},20_{01}\}$	0_{00}
$\{\infty_5,28_{10},25_{01},15_{11}\}$	25_{00}	$\{\infty_6,0_{10},2_{01},7_{10}\}$	14_{00}	$\{\infty_7,29_{01},10_{10},22_{00}\}$	12_{00}
$\{\infty_8,3_{01},12_{11},19_{10}\}$	8_{00}	$\{\infty_9,30_{01},27_{00},8_{11}\}$	27_{00}	$\{\infty_{10},19_{01},21_{01},2_{00}\}$	23_{00}
$\{\infty_{11},4_{11},22_{11},7_{00}\}$	20_{00}	$\{\infty_{12},26_{00},6_{01},4_{00}\}$	19_{00}	$\{\infty_{13},28_{00},22_{01},14_{01}\}$	22_{00}
$\{\infty_{14},2_{10},16_{01},22_{10}\}$	13_{00}	$\{\infty_{15},4_{01},29_{00},7_{01}\}$	18_{00}	$\{\infty_{16},24_{00},8_{01},5_{11}\}$	16_{00}
$\{\infty_{17},18_{11},1_{01},15_{10}\}$	1_{00}	$\{\infty_{18},17_{01},23_{10},8_{00}\}$	26_{00}	$\{\infty_{19},24_{10},16_{00},8_{10}\}$	10_{00}
$\{\infty_{20},3_{10},18_{01},24_{01}\}$	5_{00}	$\{\infty_{21},30_{11},24_{11},18_{10}\}$	9_{00}	$\{\infty_{22},0_{11},14_{11},23_{00}\}$	15_{00}
$\{\infty_{23},6_{10},15_{01},29_{10}\}$	6_{00}				

$$\begin{aligned}\mathcal{C} = \ & \{30_{10},13_{00},7_{11},8_{01}\}, && \{7_{01},2_{10},28_{11},17_{00}\}, && \{6_{11},9_{01},10_{00},13_{10}\},\\ & \{30_{10},28_{01},18_{00},17_{11}\}, && \{30_{01},26_{00},8_{11},6_{10}\}.\\ \mathcal{R} = \ & \{14_{00},30_{00},13_{10},0_{00}\}, && \{9_{10},16_{10},15_{00},11_{00}\}, && \{10_{00},25_{10},17_{10},30_{10}\},\\ & \{20_{00},5_{01},9_{11},1_{00}\}, && \{26_{10},12_{10},13_{11},17_{11}\}, && \{12_{01},11_{01},10_{01},6_{11}\}.\end{aligned}$$

When $(u,w)=(38,7)$:

$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$
$\{28_{00},29_{00},22_{11},27_{00}\}$	8_{00}	$\{20_{11},23_{11},11_{11},5_{11}\}$	6_{00}	$\{18_{00},27_{10},8_{01},30_{00}\}$	21_{00}
$\{\infty_0,30_{01},13_{00},5_{01}\}$	3_{00}	$\{\infty_1,28_{01},3_{01},23_{01}\}$	20_{00}	$\{\infty_2,27_{11},8_{10},24_{11}\}$	25_{00}
$\{\infty_3,0_{11},4_{11},6_{00}\}$	11_{00}	$\{\infty_4,4_{00},9_{00},8_{00}\}$	26_{00}	$\{\infty_5,16_{11},29_{10},10_{01}\}$	12_{00}
$\{\infty_6,26_{00},29_{01},21_{01}\}$	0_{00}	$\{\infty_7,27_{01},16_{00},18_{10}\}$	19_{00}	$\{\infty_8,7_{01},23_{00},13_{11}\}$	1_{00}
$\{\infty_9,30_{11},6_{10},16_{10}\}$	28_{00}	$\{\infty_{10},13_{01},24_{10},22_{00}\}$	14_{00}	$\{\infty_{11},2_{00},20_{00},12_{11}\}$	13_{00}
$\{\infty_{12},11_{00},23_{10},12_{10}\}$	16_{00}	$\{\infty_{13},1_{10},15_{00},14_{11}\}$	18_{00}	$\{\infty_{14},18_{11},10_{10},12_{01}\}$	22_{00}
$\{\infty_{15},3_{00},25_{00},17_{00}\}$	27_{00}	$\{\infty_{16},12_{00},26_{11},22_{10}\}$	29_{00}	$\{\infty_{17},1_{01},17_{01},10_{00}\}$	9_{00}
$\{\infty_{18},0_{00},19_{11},20_{10}\}$	23_{00}	$\{\infty_{19},24_{00},2_{11},4_{10}\}$	10_{00}	$\{\infty_{20},5_{00},2_{10},1_{11}\}$	17_{00}
$\{\infty_{21},25_{10},7_{10},0_{01}\}$	15_{00}	$\{\infty_{22},17_{10},20_{01},19_{10}\}$	30_{00}	$\{\infty_{23},14_{00},21_{11},7_{00}\}$	7_{00}
$\{\infty_{24},0_{10},4_{01},11_{01}\}$	5_{00}	$\{\infty_{25},9_{11},19_{01},21_{10}\}$	4_{00}	$\{\infty_{26},9_{01},24_{01},25_{11}\}$	2_{00}
$\{\infty_{27},14_{01},25_{01},30_{10}\}$	24_{00}				

$$\begin{aligned}\mathcal{C} = \ & \{14_{00},29_{11},25_{01},30_{10}\}, && \{20_{10},9_{11},7_{01},5_{00}\}, && \{4_{01},25_{00},28_{11},12_{10}\},\\ & \{13_{00},24_{10},1_{01},22_{11}\}, && \{7_{10},6_{01},20_{11},10_{00}\}, && \{24_{01},6_{10},1_{00},16_{11}\}.\\ \mathcal{R} = \ & \{8_{11},5_{10},19_{00},15_{10}\}, && \{26_{01},7_{11},13_{10},17_{11}\}, && \{9_{10},15_{11},6_{01},1_{00}\},\\ & \{26_{10},14_{10},21_{00},28_{10}\}, && \{22_{01},18_{01},10_{11},15_{01}\}, && \{3_{11},2_{01},16_{01},29_{11}\},\\ & \{3_{10},28_{11},11_{10},6_{11}\}.\end{aligned}$$

When $(u,w)=(39,6)$:

$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$	$\mathcal{S}$	$\Omega(\mathcal{S})$
$\{28_{10},29_{10},26_{10},2_{00}\}$	23_{00}	$\{24_{01},10_{11},9_{01},17_{00}\}$	13_{00}	$\{3_{00},29_{00},6_{00},21_{01}\}$	0_{00}
$\{11_{01},30_{01},10_{00},7_{11}\}$	10_{00}	$\{9_{11},26_{00},21_{00},20_{01}\}$	11_{00}	$\{30_{00},32_{00},0_{01},8_{00}\}$	8_{00}
$\{22_{01},8_{10},18_{00},27_{01}\}$	9_{00}	$\{21_{10},30_{11},24_{00},4_{11}\}$	5_{00}	$\{32_{01},27_{10},18_{01},25_{00}\}$	25_{00}
$\{\infty_0,28_{01},16_{00},12_{11}\}$	32_{00}	$\{\infty_1,1_{01},18_{10},16_{01}\}$	20_{00}	$\{\infty_2,9_{10},6_{11},4_{01}\}$	3_{00}
$\{\infty_3,15_{00},32_{10},6_{10}\}$	19_{00}	$\{\infty_4,32_{11},30_{10},1_{10}\}$	27_{00}	$\{\infty_5,29_{01},8_{11},31_{00}\}$	16_{00}
$\{\infty_6,26_{01},14_{11},23_{00}\}$	18_{00}	$\{\infty_7,28_{00},13_{01},24_{10}\}$	15_{00}	$\{\infty_8,24_{11},31_{01},13_{10}\}$	31_{00}
$\{\infty_9,27_{00},18_{11},12_{10}\}$	28_{00}	$\{\infty_{10},25_{11},13_{11},19_{11}\}$	22_{00}	$\{\infty_{11},5_{10},4_{00},0_{00}\}$	30_{00}
$\{\infty_{12},7_{00},13_{00},19_{01}\}$	6_{00}	$\{\infty_{13},2_{10},16_{11},25_{01}\}$	26_{00}	$\{\infty_{14},17_{01},7_{01},11_{10}\}$	7_{00}
$\{\infty_{15},15_{01},19_{10},2_{11}\}$	17_{00}	$\{\infty_{16},22_{00},12_{00},1_{00}\}$	4_{00}	$\{\infty_{17},0_{10},14_{01},5_{00}\}$	1_{00}
$\{\infty_{18},15_{11},2_{01},14_{00}\}$	12_{00}	$\{\infty_{19},4_{10},3_{01},23_{11}\}$	2_{00}	$\{\infty_{20},3_{10},16_{10},17_{10}\}$	14_{00}
$\{\infty_{21},3_{11},19_{00},25_{10}\}$	29_{00}	$\{\infty_{22},5_{11},11_{00},22_{11}\}$	24_{00}	$\{\infty_{23},10_{10},22_{10},23_{01}\}$	21_{00}

$$\begin{aligned}\mathcal{C} = \ & \{10_{11},15_{10},23_{00},13_{01}\}, \quad \{22_{11},4_{01},20_{00},27_{10}\}, \quad \{12_{10},16_{11},8_{00},4_{01}\}, \\ & \{23_{11},12_{01},1_{00},9_{10}\}, \quad \{20_{00},30_{01},23_{10},28_{11}\}. \\ \mathcal{R} = \ & \{20_{11},6_{01},28_{11},5_{01}\}, \quad \{29_{11},12_{01},11_{11},31_{11}\}, \quad \{31_{10},10_{01},15_{10},7_{10}\}, \\ & \{9_{00},27_{11},14_{10},20_{00}\}, \quad \{23_{10},0_{11},20_{10},8_{01}\}, \quad \{26_{11},1_{11},21_{11},17_{11}\}.\end{aligned}$$

□

Proof of Theorem 2: We first construct a GBTD$(4,m)$ for any $m \in N$, where $N = \{28,32,33,34,37,38,39,44\}$.

For each $w \in \{5,6,7,8\}$, an HGBTD$(4,4^w)$ is given by Yin *et al.* [12]. For each $m \in N$, apply Theorem 3.2, with IHGBTDs from Lemma 5, Lemma 6 and Lemma 7 and corresponding HGBTD$(4,4^w)$'s where $w \in \{5,6,7,8\}$ as ingredients, to produce the desired HGBTD$(4,4^m)$. Hence, the desired GBTD$(4,m)$ follows from Proposition 3.1.

Combining Proposition 1.1, Proposition 2.1 and Proposition 2.2, we complete the proof.

□

Acknowledgement We are grateful to the anonymous reviewers for their helpful comments.

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