Generalized Balanced Tournament Packings and Optimal Equitable Symbol Weight Codes for Power Line Communications

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Abstract: Generalized balanced tournament packings (GBTPs) extend the concept of generalized balanced tournament designs introduced by Lamken and Vanstone (1989). In this paper, we establish the connection between GBTPs and a class of codes called equitable symbol weight codes (ESWCs). The latter were recently demonstrated to optimize the performance against narrowband noise in a general coded modulation scheme for power line communications. By constructing classes of GBTPs, we establish infinite families of optimal ESWCs with code lengths greater than alphabet size and whose narrowband noise error-correcting capability to code length ratios do not diminish to zero as the length grows. © 2014 Wiley Periodicals, Inc. J. Combin. Designs 00: 1–32, 2014

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1. INTRODUCTION

Power line communications (PLCs) is a technology that enables the transmission of data over electric power lines. It was started in the 1910s for voice communication [28], and used in the 1950s in the form of ripple control for load and tariff management in power distribution. With the emergence of the Internet in the 1990s, research into broadband PLC gathered pace as a promising technology for Internet access and local area networking, since the electrical grid infrastructure provides "last mile" connectivity to premises and capillarity within premises. Recently, there has been a renewed interest in high-speed narrowband PLC due to applications in sustainable energy strategies, specifically in smart grids (see [15, 17, 25, 34]). However, power lines present a difficult communications environment and overcoming permanent narrowband disturbance has remained a challenging problem [2, 26, 30]. Vinck [30] addressed this problem through the use of a coded modulation scheme based on permutation codes. More recently, Chee et al. [8] extended Vinck's analysis to general block codes and motivated the study of *equitable symbol weight codes* (ESWCs).

Relatively little is known about optimal ESWCs, other than those that correspond to permutation codes, injection codes, and frequency permutation arrays. In particular, only six infinite families of optimal ESWCs with code length greater than alphabet size are known. These have all been constructed by Ding and Yin [13], and Huczynska and Mullen [18] as frequency permutation arrays and they meet the Plotkin bound. One drawback with the code parameters of these families is that the narrowband noise error-correcting capability to length ratio diminishes as length grows.

In this paper, we construct infinite families of optimal ESWCs whose code lengths are larger than alphabet size and whose narrowband noise error-correcting capability to length ratios tend to a positive constant as code length grows. These families of codes all attain the generalized Plotkin bound. Our results are based on the construction of equivalent combinatorial designs called generalized balanced tournament packings (GBTPs).

GBTPs extend the concept of generalized balanced tournament designs (GBTDs) introduced by Lamken and Vanstone [19]. GBTDs have been extensively studied [9, 11, 20–22, 33] and are useful in the constructions of resolvable, near-resolvable, doubly resolvable, and doubly near-resolvable balanced incomplete block designs [20, 23, 24]. Using the classical correspondence given by Semakov and Zinoviev [29] (see also [4, 12, 33]), we construct optimal families of ESWCs from certain families of GBTPs. We establish existence results for these families of GBTPs by borrowing standard recursion and direct construction methods from combinatorial design theory.

The paper is organized as follows. In Section 2, we introduce ESWCs and survey the known results on optimal codes. In Section 3, we introduce GBTPs and establish the equivalence between GBTPs and ESWCs. At the end of the section, we establish two classes of GBTPs that correspond to optimal ESWCs. In Sections 4–7, we settle the existence of these two classes of GBTPs. Section 4 outlines the general strategy, while Sections 5 and 6 provide recursive and direct constructions, respectively.

Some of the results of the paper have been initially reported at IEEE International Symposium on Information Theory 2012 [7], and the present paper contains detailed proofs and includes a new existence result on a family of GBTPs with block size 2 and 3.

2. PRELIMINARIES

2.1. Notation

For positive integer m and prime power q, denote the ring $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}_m$ and the finite field of q elements by $\mathbb{F}_q$. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Let $[m]$ denote the set $\{1, 2, \ldots, m\}$. We use angled brackets ($\langle$ and $\rangle$) for multisets. Disjoint set union is depicted using $\sqcup$. For sets A and B, an element $(a, b) \in A \times B$ is sometimes written as a_b for succinctness.

A *set system* is a pair $\mathfrak{S} = (X, \mathcal{A})$, where X is a finite set of *points* and $\mathcal{A} \subseteq 2^X$. Elements of $\mathcal{A}$ are called *blocks*. The *order* of $\mathfrak{S}$ is the number of points in X, and the *size* of $\mathfrak{S}$ is the number of blocks in $\mathcal{A}$. Let K be a set of nonnegative integers. The set system $(X, \mathcal{A})$ is said to be K*-uniform* if $|A| \in K$ for all $A \in \mathcal{A}$.

2.2. Equitable Symbol Weight Codes

Let Σ be a set of q *symbols*. A q*-ary code* of *length* n over the *alphabet* Σ is a subset $\mathcal{C} \subseteq \Sigma^n$. Elements of $\mathcal{C}$ are called *codewords*. The *size* of $\mathcal{C}$ is the number of codewords in $\mathcal{C}$. For $i \in [n]$, the ith coordinate of a codeword $\mathsf{u} \in \mathcal{C}$ is denoted u_i, so that $\mathsf{u} = (\mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_n)$. Denote the *frequency* of symbol $\sigma \in \Sigma$ in codeword $\mathsf{u} \in \Sigma^n$ by $w_\sigma(\mathsf{u})$, that is, $w_\sigma(\mathsf{u}) = |\{\mathsf{u}_i = \sigma : i \in [n]\}|$.

An element $\mathsf{u} \in \Sigma^n$ is said to have *equitable symbol weight* if $w_\sigma(\mathsf{u}) \in \{\lfloor n/q \rfloor, \lceil n/q \rceil\}$ for any $\sigma \in \Sigma$. If all the codewords of $\mathcal{C}$ have equitable symbol weight, then the code $\mathcal{C}$ is called an ESWC. Consider the usual Hamming distance defined on codewords and codes and let d denote the minimum distance of a code $\mathcal{C}$. In addition, consider the following parameter.

Definition 2.1. Let $\mathcal{C}$ be a q-ary code with minimum distance d. The narrowband noise error-correcting capability of $\mathcal{C}$ is

$$c(\mathcal{C}) = \min\{e : E_{\mathcal{C}}(e) \geq d\},$$

where $E_{\mathcal{C}}$ is a function $E_{\mathcal{C}} : [q] \to [n]$, given by

$$E_{\mathcal{C}}(e) = \max_{\substack{\Gamma \subseteq \Sigma \\ |\Gamma| = e}} \max_{\mathsf{c} \in \mathcal{C}} \left\{ \sum_{\sigma \in \Gamma} w_\sigma(\mathsf{c}) \right\}.$$

Chee et al. [8] established that a code $\mathcal{C}$ can correct up to $c(\mathcal{C}) - 1$ narrowband noise errors and demonstrated that an ESWC maximizes the quantity $c(\mathcal{C})$, for fixed n, d, and q.

Henceforth, only ESWCs are considered. A q-ary ESWC of length n having minimum distance d is denoted ESWC$(n, d)_q$. Denote the maximum size of an ESWC$(n, d)_q$ by $A_q^{ESW}(n, d)$. Any ESWC$(n, d)_q$ of size $A_q^{ESW}(n, d)$ is said to be *optimal*. Taken as a q-ary code of length n, an optimal ESWC$(n, d)_q$ satisfies the generalized Plotkin bound [3].

Theorem 2.2 (Generalized Plotkin bound). *If there is an ESWC$(n,d)_q$ $\mathcal{C}$ of size M, then*

$$\binom{M}{2}d \le n\sum_{i=0}^{q-2}\sum_{j=i+1}^{q-1} M_iM_j, \tag{1}$$

where $M_i = \lfloor (M+i)/q \rfloor$. If q divides M and $\binom{M}{2}d = n\binom{q}{2}(M/q)^2$, then $\mathcal{C}$ is optimal.

In the rest of this paper, ESWCs whose sizes attain the generalized Plotkin bound are constructed. In particular, the following is established.

Theorem 2.3. *The following holds.*

(i)

$$A_q^{ESW}(2q-1, 2q-2) = \begin{cases} 3, & q = 2, \\ 2q, & q \ge 3. \end{cases}$$

(ii)

$$A_q^{ESW}(3q-1, 3q-3) = \begin{cases} 4, & q = 2, \\ 3q, & q \ge 3. \end{cases}$$

(iii)

$$A_q^{ESW}(4q-1, 4q-4) = \begin{cases} 4q-1, & q = 2, 3, \\ 4q, & q \ge 4. \end{cases}$$

(iv) If $q \ge 62$ or $q \in \{5-18, 30, 42, 46, 48-50, 54-57\}$,

$$A_q^{ESW}(5q-1, 5q-5) = 5q.$$

(v) If q is an odd prime power,

$$A_q^{ESW}(q^2-1, q^2-q) = q^2.$$

(vi)

$$A_q^{ESW}\left(\frac{3q-1}{2}, \frac{3q-3}{2}\right) = \begin{cases} 4q-6, & q = 3, 5, \\ 3q, & q \ge 7 \text{ is odd}. \end{cases}$$

(vii)

$$A_q^{ESW}(2q-3, 2q-4) = \begin{cases} 6q-12, & q = 3, 4, \\ 14, & q = 5, 6, \\ 2q+1, & q \ge 7, \text{ except possibly } q \in \{12, 13\}. \end{cases}$$

Observe that any ESWC $\mathcal{C}$ with the above parameters must have $c(\mathcal{C}) = q-1$. In Table I, we verify that $c(\mathcal{C})/n$ tends to a positive constant as q grows. In the same table, we compare with known families of optimal ESWC$(n,d)_q$.

TABLE I. Infinite families of optimal ESWC$(n, d)_q$.

ESWC$(n,d)_q$ $\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$c(\mathcal{C})$	$\lim_{q\to\infty} c(\mathcal{C})/n$	Remarks
$(n,n)_q$ for $q \geq 2$	q	$\min\{n,q\}$	–	easy
$(3,2)_q$ for $q \geq 3$	$q(q-1)$	2	0	Injection code [14]
$(4,2)_q$ for $q \geq 4, q \neq 7$	$q(q-1)(q-2)$	2	0	Injection code [14]
$(n,1)_q$ for $n < q$	$q(q-1)\cdots(q-n+1)$	1	$1/n$	Injection code, easy
$(qn,2)_q$ for $q \geq 2$	$(qn)!/(n!)^q$	2	0	Frequency permutation array, easy
$(q,3)_q$ for $q \geq 3$	$q!/2$	3	0	Permutation code, easy
$(q,q-1)_q$ for prime powers q	$q(q-1)$	$q-1$	1	Injection code [10]
$(n,n-1)_q$ for q sufficiently large and $n \leq q$	$q(q-1)$	$n-1$	$1-1/n$	Injection code [14]
$(q,q-2)_q$ for prime powers $q-1$	$q(q-1)(q-2)$	$q-2$	1	Permutation code [16]
$(q(q+1),q^2)_q$ for prime powers q	q^2	q	0	Frequency permutation array [12]
$(\frac{q(kq^2-1)}{k-1}, \frac{kq^2(q-1)}{k-1})_q$ for prime powers q, $2 \leq k \leq 5$, $(k,q) \neq (5,9)$	kq^2	q	0	Frequency permutation array [13]
$(\frac{\mu q^{s-t}(q^{2s-t}-1)}{q^t-1}, \frac{\mu q^{2s-t}(q^{s-t}-1)}{q^t-1})_{q^{s-t}}$ for prime powers q, $1 \leq t < s$, $\mu = \prod_{i=1}^{t-1} \frac{q^{s-i}-1}{q^i-1}$	q^{2s-t}	q^{s-t}	0	Frequency permutation array [13]
$(q^s(q^{2s+c}-1), q^{2s+c}(q^s-1))_{q^s}$, for prime powers q, and $s, c \geq 1$	q^{2s+c}	q^s	0	Frequency permutation array [13]
$(\binom{kq}{k}, \frac{kq-k}{kq-1}\binom{kq}{k})_q$ for $q, k \geq 1$	kq	$q-1$	0	Frequency permutation array [18]
$(2q^2-q, 2q^2-2q)_q$ for even q, $q \notin \{2,6\}$	$2q$	q	0	Frequency permutation array [18]
$(2q-1,2q-2)_q$ for $q \geq 3$	$2q$	$q-1$	$1/2$	Theorem 2.3
$(3q-1,3q-3)_q$ for $q \geq 3$	$3q$	$q-1$	$1/3$	Theorem 2.3
$(4q-1,4q-4)_q$ for $q \geq 4$	$4q$	$q-1$	$1/4$	Theorem 2.3
$(5q-1,5q-5)_q$ for $q \geq 62$	$5q$	$q-1$	$1/5$	Theorem 2.3
$(q^2-1,q^2-q)_q$ for $q \geq 4$	q^2	$q-1$	0	Theorem 2.3
$(\frac{3q-1}{2}, \frac{3q-3}{2})_q$ for $q \geq 7$ and q odd	$3q$	$q-1$	$2/3$	Theorem 2.3
$(2q-3,2q-4)_q$ for $q \geq 14$	$2q+1$	$q-2$	$1/2$	Theorem 2.3

In particular, only six infinite nontrivial families of optimal codes with $n > q$ are known. However, code parameters for these six families are such that their relative narrowband noise error-correcting capability to length ratios diminish to zero as q grows. This is undesirable for narrowband noise correction for PLC. Hence, Theorem 2.3 provides infinite families of optimal ESWCs with code lengths larger than alphabet size and whose relative narrowband noise capability to length ratios tend to a positive constant as length grows.

These optimal ESWCs are constructed from GBTPs using the classical correspondence given by Semakov and Zinoviev [29].[1] We remark that GBTPs extend the concept of GBTDs and consequently Theorem 2.3 (i)–(v) follows directly from known classes of GBTDs. We explain the connection in detail in the next section.

3. CONSTRUCTIONS OF ESWCs

We first determine $A_q^{ESW}(n, d)$ for small values of n, q, and d. With the exception of $A_6^{ESW}(9, 8)$, an exhaustive computer search established the following values of $A_q^{ESW}(n, d)$. For $A_6^{ESW}(9, 8)$, an ESWC$(9, 8)_6$ of size 14 was found via computer search. Since an ESWC$(9, 8)_6$ of size 15 cannot exist by the generalized Plotkin bound, it follows that $A_6^{ESW}(9, 8) = 14$. We record the results of the computations in the following proposition and the corresponding optimal codes can be found at [5].

Proposition 3.1. *The following holds:*

$$\begin{array}{lll} A_2^{ESW}(3,2)=3 & A_2^{ESW}(5,3)=4 & A_2^{ESW}(7,4)=7 \\ A_3^{ESW}(3,2)=6 & A_3^{ESW}(4,3)=6 & A_3^{ESW}(11,8)=11 \\ A_4^{ESW}(5,4)=12 & A_5^{ESW}(7,6)=14 & A_6^{ESW}(9,8)=14. \end{array}$$

The rest of the paper establishes the remaining values in Theorem 2.3. To do so, we define a class of combinatorial designs that is equivalent to ESWCs.

3.1. ESWCs and GBTPs

Let λ, v be positive integers and K be a set of nonnegative integers. A (v, K, λ)*-packing* is a K-uniform set system of order v such that every pair of distinct points is contained in at most λ blocks. The value λ is called the *index* of the packing. A *parallel class* (or *resolution class*) of a packing is a subset of the blocks that partitions the set of points X. If the set of blocks can be partitioned into parallel classes, then the packing is *resolvable*, and denoted by RP(v, K, λ). An RP(v, K, λ) is called a *maximum resolvable packing*, denoted by MRP(v, K, λ), if it contains maximum possible number of parallel classes.

Furthermore, an MRP$(v, \{k\}, \lambda)$ is called a *resolvable* $(v, \{k\}, \lambda)$*-balanced incomplete block design*, or RBIBD(v, k, λ) in short, if every pair of distinct points is contained in exactly λ blocks. A simple computation gives the size of an RBIBD(v, k, λ) to be $\frac{\lambda v(v-1)}{k(k-1)}$.

We define the combinatorial object of study in this paper. We note that this definition is a generalization of GBTDs to packings and various indices.

[1] Bogdanova et al. [4] gave a survey of connection between equidistant codes and designs. Using this correspondence, Ding and Yin [12] constructed optimal constant-composition codes, while Yin et al. [33] constructed near-constant-composition codes.

Definition 3.2. Let $(X, \mathcal{A})$ be an RP(v, K, λ) with n parallel classes. Then $(X, \mathcal{A})$ is called a GBTP if the blocks of $\mathcal{A}$ are arranged into an $m \times n$ array satisfying the following conditions:

(i) every point in X is contained in exactly one cell of each column,
(ii) every point in X is contained in either $\lceil n/m \rceil$ or $\lfloor n/m \rfloor$ cells of each row.

We denote such a GBTP by GBTP$_\lambda(K; v, m \times n)$.

Unless otherwise stated, the rows of a GBTP$_\lambda(K; v, m \times n)$ are indexed by $[m]$ and the columns by $[n]$.

In a GBTP$_\lambda(K; v, m \times n)$, given point x and column j, there is a unique row that contains the point x in column j. Hence, for each point $x \in X$ of a GBTP$_\lambda(K; v, m \times n)$ $(X, \mathcal{A})$, we may correspond the codeword $\mathsf{c}(x) = (r_1, r_2, \ldots, r_n) \in [m]^n$, where r_j is the row in which point x appears in column j. It is obvious that $\mathcal{C} = \{\mathsf{c}(x) : x \in X\}$ is an m-ary code of length n over the alphabet $[m]$. We note that this correspondence is precisely the one used by Semakov and Zinoviev [29] to show the equivalence between *equidistant codes* and resolvable balanced incomplete block designs.

For distinct points $x, y \in X$, the distance between $\mathsf{c}(x)$ and $\mathsf{c}(y)$ is the number of columns for which x and y are not both contained in the same row. Since there are at most λ blocks containing both x and y, and that no two such blocks can occur in the same column of the GBTP$_\lambda(K; v, m \times n)$, the distance between $\mathsf{c}(x)$ and $\mathsf{c}(y)$ is at least $n - \lambda$.

Next, we determine $w_i(\mathsf{c}(x))$, for $x \in X$ and $i \in [m]$. From the construction of $\mathsf{c}(x)$, the number of times a symbol i appears in $\mathsf{c}(x)$ is the number of cells in row i that contains x. By the definition of a GBTP$_\lambda(K; v, m \times n)$, this number belongs to $\{\lfloor n/m \rfloor, \lceil n/m \rceil\}$. Hence, $\mathcal{C}$ is an ESWC of size v. Finally, this construction of an ESWC from a GBTP can easily be reversed. We record these observations as:

Theorem 3.3. *Let K be set of non-negative integers. Then a* GBTP$_\lambda(K; v, m \times n)$ *exists if and only if an ESWC$(n, n - \lambda)_m$ of size v exists.*

We note that the correspondence between GBTPs and ESWCs was observed by Yin et al. [33, Theorem 2.2]. However, in the latter paper, the class of codes constructed is called near-constant-composition codes (NCCCs). Indeed, an NCCC is a special class of ESWC and one observes that an ESWC$(n, d)_q$ is an NCCC when $n + 1 \equiv 0 \bmod q$.

Example 3.1. Consider the GBTP$_1(\{2, 3\}, 6, 3 \times 4)$ below.

	{1,4}	{2,6}	{3,5}
{1,2,3}	{2,5}	{3,4}	{1,6}
{4,5,6}	{3,6}	{1,5}	{2,4}

Each point $x \in [6]$ gives a codeword $\mathsf{c}(x) = (r_1, r_2, \ldots, r_5)$, where r_j is the row in which point x appears in column j. Hence, we have

$$\mathsf{c}(1) = (2, 1, 3, 2), \qquad \mathsf{c}(2) = (2, 2, 1, 3), \qquad \mathsf{c}(3) = (2, 3, 2, 1),$$
$$\mathsf{c}(4) = (3, 1, 2, 3), \qquad \mathsf{c}(5) = (3, 2, 3, 1), \qquad \mathsf{c}(6) = (3, 3, 1, 2).$$

The code $\mathcal{C} = \{\mathsf{c}(1), \mathsf{c}(2), \mathsf{c}(3), \mathsf{c}(4), \mathsf{c}(5), \mathsf{c}(6)\}$ is an ESWC$(4, 3)_3$ of size 6.

Theorem 3.3 sets up the equivalence between GBTPs and ESWCs. In general, a GBTP may not correspond to an optimal ESWC. However, in the following, we look at specific K's to derive families of optimal ESWCs.

3.2. Optimal ESWCs from GBTDs

A GBTP$_\lambda$ $\left(\{k\}; km, m \times \frac{\lambda(km-1)}{k-1}\right)$ is called a GBTD, denoted by GBTD$_\lambda(k, m)$. In this case, we check that each pair of distinct points is contained in exactly λ blocks and every point is contained in either $\lceil \frac{\lambda(km-1)}{m(k-1)} \rceil$ or $\lfloor \frac{\lambda(km-1)}{m(k-1)} \rfloor$ cells of each row.

Applying Theorem 3.3, an ESWC$(\frac{\lambda(km-1)}{k-1}, \frac{\lambda k(m-1)}{k-1})_m$ of size km exists and the corresponding code is optimal by generalized Plotkin bound. So, we have the following.

Theorem 3.4. *A GBTD$_\lambda(k, m)$ exists if and only if an optimal ESWC$(\frac{\lambda(km-1)}{k-1}, \frac{\lambda k(m-1)}{k-1})_m$ of size km exists and attains the generalized Plotkin bound.*

We remark that our definition of a GBTD extends that of Lamken and Vanstone [19], which corresponds in our definition to the case when $\lambda = k - 1$. The following summarizes the state-of-the-art results on the existence of GBTD$_{k-1}(k, m)$.

Theorem 3.5 (Lamken [19–22], Yin et al. [33], Chee et al. [9], Dai et al. [11])**.** *The following holds.*

- *(i) A GBTD$_1(2, m)$ exists if and only if $m = 1$ or $m \geq 3$.*
- *(ii) A GBTD$_2(3, m)$ exists if and only if $m = 1$ or $m \geq 3$.*
- *(iii) A GBTD$_3(4, m)$ exists if and only if $m = 1$ or $m \geq 4$.*
- *(iv) A GBTD$_4(5, m)$ exists if $m \geq 62$ or $m \in \{5 - 18, 30, 42, 46, 48 - 50, 54 - 57\}$.*
- *(v) A GBTD$_{k-1}(k, k)$ exists if k is an odd prime power.*

Theorem 2.3 (i)–(v) is now an immediate consequence of Theorems 3.4, 3.5, and Proposition 3.1. The existence of GBTD$_\lambda(k, m)$ when $\lambda \neq k - 1$ has not been previously investigated. The smallest open case is when $k = 3$ and $\lambda = 1$, which is the case dealt with in this paper.

It follows, readily from the fact that a GBTD$_1(3, m)$ is also an RBIBD$(3m, 3, 1)$, that a necessary condition for a GBTD$_1(3, m)$ to exist is that m must be odd. We note from Proposition 3.1 that $A_3^{ESW}(4, 3) = 6$ and $A_5^{ESW}(7, 6) = 14$, which do not meet the Plotkin bound. Hence, the corresponding designs GBTD$_1(3, 3)$ and GBTD$_1(3, 5)$ do not exist by Theorem 3.4.

Hence, a GBTD$_1(3, m)$ can exist only if m is odd and $m \notin \{3, 5\}$. In Sections 4–7, we prove that this necessary condition is also sufficient for the existence of GBTD$_1(3, m)$. A direct consequence of this is Theorem 2.3 (vi).

3.3. Optimal ESWCs from GBTP$_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$

Theorem 3.4 constructs optimal ESWCs from GBTDs. In this subsection, we make slight variations to obtain another infinite family of optimal ESWCs.

Consider a GBTP$_1(\{2, 3\}; v, m \times n)$. If there is exactly one block of size 3 in each resolution class, then we denote the GBTP by GBTP$_1(\{2, 3^*\}; v, m \times n)$. A simple computation then shows $v = 2m + 1$. Now we establish the following construction for optimal ESWCs.

Theorem 3.6. *Let $m \geq 7$. If there exists a $GBTP_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$, then there exists an optimal $ESWC(2m - 3, 2m - 4)_m$ of size $2m + 1$, which attains the generalized Plotkin bound.*

Proof. By Theorem 3.3, we have a ESWC$(2m - 3, 2m - 4)_m$ of size $2m + 1$. It remains to verify its optimality.

Suppose otherwise that there exists an ESWC$(2m - 3, 2m - 4)_m$ of size $2m + 2$. Consider (1) in Theorem 2.2. On the left-hand side, we have

$$\binom{2m+2}{2} \cdot (2m - 4) = 4m^3 - 2m^2 - 10m - 4.$$

Since $\lfloor \frac{2m+2+i}{m} \rfloor = 2$ for $0 \leq i \leq m - 3$ and $\lfloor \frac{2m+2+(m-2)}{m} \rfloor = \lfloor \frac{2m+2+(m-1)}{m} \rfloor = 3$, the term on the right hand is

$$\begin{aligned} &(2m-3)\left(\left(\sum_{i=0}^{m-3} 4(m-3-i) + 12\right) + 9\right) \\ &\quad = (2m-3)(4m(m-2) - 2(m-3)(m-2) + 9) \\ &\quad = 4m^3 - 2m^2 - 12m + 9. \end{aligned}$$

But for $m \geq 7$,

$$4m^3 - 2m^2 - 10m - 4 > 4m^3 - 2m^2 - 12m + 9,$$

contradicting (1). Hence, an ESWC$(2m - 3, 2m - 4)_m$ of size $2m + 2$ does not exist and the result follows. □

In the rest of this paper, we construct a $\mathrm{GBTP}_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$ for $m \geq 4$, except possibly $m \in \{12, 13\}$. This with Theorem 3.6 and Proposition 3.1 gives Theorem 2.3 (vii).

4. PROOF STRATEGY OF THEOREM 2.3 (vi) AND THEOREM 2.3 (vii)

For the rest of the paper, we determine with finite possible exceptions the existence of $\mathrm{GBTD}_1(3, m)$ and $\mathrm{GBTP}_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$. Our proof is technical and rather complex. However, it follows the general strategy of the previous work [11, 21, 33]. This section outlines the general strategy used, and introduces some required combinatorial designs.

As with most combinatorial designs, using direct constructions to settle their existence is often difficult. Instead, we use recursive constructions, building big designs from smaller ones. Direct methods are used to construct a large enough set of small designs on which the recursions can work to generate all larger designs. For our recursive techniques to work, the GBTPs must possess more structure than stipulated in its definition. First, we consider $\mathrm{GBTD}_1(3, m)$s that are $*$*colorable*, which are defined below.

$0_00_1\infty$ $\clubsuit$	$2_04_03_1$ $\clubsuit$	$6_14_01_0$ $\diamondsuit$	$2_11_16_1$ $\clubsuit$	$5_11_02_0$ $\heartsuit$	$4_13_11_1$ $\heartsuit$	$5_14_12_1$ $\diamondsuit$
$6_15_13_1$ $\clubsuit$	$1_01_1\infty$ $\clubsuit$	$3_05_04_1$ $\clubsuit$	$3_03_1\infty$ $\diamondsuit$	$5_00_06_1$ $\diamondsuit$	$6_01_00_1$ $\diamondsuit$	$0_02_01_1$ $\heartsuit$
$1_03_02_1$ $\clubsuit$	$0_16_14_1$ $\clubsuit$	$2_02_1\infty$ $\diamondsuit$	$4_06_05_1$ $\clubsuit$	$1_16_03_0$ $\diamondsuit$	$5_05_1\infty$ $\heartsuit$	$3_11_05_0$ $\diamondsuit$
$4_12_06_0$ $\clubsuit$	$5_13_00_0$ $\clubsuit$	$1_10_15_1$ $\diamondsuit$	$0_15_02_0$ $\heartsuit$	$4_04_1\infty$ $\diamondsuit$	$2_10_04_0$ $\heartsuit$	$6_06_1\infty$ $\heartsuit$
$1_14_05_0$ $\clubsuit$	$2_15_06_0$ $\diamondsuit$	$3_16_00_0$ $\clubsuit$	$4_10_01_0$ $\diamondsuit$	$3_12_10_1$ $\heartsuit$	$6_12_03_0$ $\clubsuit$	$0_13_04_0$ $\diamondsuit$

FIGURE 1. A 3-$*$colorable RBIBD(15, 3, 1) $(X, \mathcal{A})$, where $X = (\mathbb{Z}_7 \times \mathbb{Z}_2) \cup \{\infty\}$. The set of colors used is $\{\clubsuit, \diamondsuit, \heartsuit\}$. $(X, \mathcal{A})$ has property Π as 1_0 is a witness for $\clubsuit$ and ∞ is a witnesses for both $\diamondsuit$ and $\heartsuit$ in row 1. For succintness, a block $\{x, y, z\}$ is written xyz.

4.1. c-$*$colorable GBTDs

We generalize the notion of factored GBTDs (FGBTDs) introduced by Lamken [22]. FGBTDs are crucial in the k-tupling construction for GBTDs of index $k-1$. However, when the index is 1, we extend this notion to $*$-colorability.

Definition 4.1. Let c be positive. A c-$*$colorable RBIBD(v, k, λ) is an RBIBD(v, k, λ) with the property that its $\frac{\lambda v(v-1)}{k(k-1)}$ blocks can be arranged in a $\frac{v}{k} \times \frac{\lambda(v-1)}{k-1}$ array, and each block can be colored with one of c colors so that

(i) each point appears exactly once in each column, and
(ii) in each row, blocks of the same color are pairwise disjoint.

Definition 4.2. A GBTD$_\lambda(k, m)$ is c-$*$colorable if each of its blocks can be colored with one of c colors so that in each row, blocks of the same color are pairwise disjoint.

Definition 4.3. A k-$*$colorable RBIBD$(v, k, 1)$ is k-$*$colorable with property Π if there exists a row r such that for each color i, there exists a point (called a witness for i) that is not contained in any block in row r that is colored i.

A GBTD$_1(k, m)$ that is c-$*$*colorable with property* Π is similarly defined.

Example 4.1. The RBIBD(15, 3, 1) in Fig. 1 is 3-$*$colorable with property Π.

Proposition 4.4. *If an* RBIBD$(v, k, 1)$ *is* $(k-1)$-$*$*colorable, then it is* k-$*$*colorable with property* Π.

Proof. Consider a $(k-1)$-$*$colorable RBIBD$(v, k, 1)$ with colors $c_1, c_2, \ldots, c_{k-1}$. There must exists a point, say x, that appears only once in the first row. Recolor the block that contains this point with color c_k. This new coloring shows that the RBIBD$(v, k, 1)$ is k-$*$colorable with property Π, since for the first row, the point x is a witness for colors $c_1, c_2, \ldots, c_{k-1}$, and any point not in the block colored by c_k is a witness for c_k. $\square$

Example 4.2. The GBTD$_1(3, 9)$ in Fig. 2 is 2-$*$colorable and is therefore 3-$*$colorable with property Π by Proposition 4.4.

We note that a 3-$*$colorable RBIBD and a 3-$*$colorable RBIBD with property Π are crucial in the tripling construction of a GBTD$_1(3, m)$ and a special GBTD$(_1(3, m)$, respectively (see Proposition 5.1). This is an adaptation of the k-tupling construction for GBTDs with index $k-1$ [22, Theorem 3.1]. However, we note certain differences. An FGBTD by definition is necessary a GBTD, while $*$-colorability and property Π are

A	B

where A is the array

$1_0 7_1 \infty_2$ $\clubsuit$	$6_0 3_2 \infty_1$ $\clubsuit$	$0_0 4_1 6_2$ $\clubsuit$	$6_0 7_1 0_2$ $\diamondsuit$	$7_0 0_1 1_2$ $\clubsuit$	$5_0 5_1 5_2$ $\diamondsuit$	$1_1 4_2 \infty_0$ $\clubsuit$
$5_0 2_2 \infty_1$ $\diamondsuit$	$2_0 0_1 \infty_2$ $\clubsuit$	$7_0 4_2 \infty_1$ $\clubsuit$	$1_0 5_1 7_2$ $\clubsuit$	$4_0 4_1 4_2$ $\diamondsuit$	$0_0 1_1 2_2$ $\clubsuit$	$7_0 5_1 \infty_2$ $\diamondsuit$
$3_1 6_2 \infty_0$ $\clubsuit$	$4_1 7_2 \infty_0$ $\diamondsuit$	$3_0 1_1 \infty_2$ $\clubsuit$	$0_0 5_2 \infty_1$ $\clubsuit$	$2_0 6_1 0_2$ $\clubsuit$	$3_0 7_1 1_2$ $\diamondsuit$	$1_0 2_1 3_2$ $\clubsuit$
$3_0 4_1 5_2$ $\clubsuit$	$1_0 1_1 1_2$ $\diamondsuit$	$5_1 0_2 \infty_0$ $\clubsuit$	$4_0 2_1 \infty_2$ $\diamondsuit$	$5_0 3_1 \infty_2$ $\clubsuit$	$2_0 7_2 \infty_1$ $\clubsuit$	$4_0 0_1 2_2$ $\clubsuit$
$6_0 2_1 4_2$ $\diamondsuit$	$4_0 5_1 6_2$ $\clubsuit$	$5_0 6_1 7_2$ $\diamondsuit$	$6_1 1_2 \infty_0$ $\clubsuit$	$1_0 6_2 \infty_1$ $\diamondsuit$	$6_0 4_1 \infty_2$ $\clubsuit$	$3_0 0_2 \infty_1$ $\clubsuit$
$0_0 0_1 0_2$ $\diamondsuit$	$5_0 2_1 0_2$ $\clubsuit$	$4_0 7_1 3_2$ $\clubsuit$	$2_0 1_1 6_2$ $\clubsuit$	$7_1 2_2 \infty_0$ $\diamondsuit$	$1_0 6_1 4_2$ $\clubsuit$	$0_0 3_1 7_2$ $\clubsuit$
$7_0 6_1 3_2$ $\clubsuit$	$7_0 3_1 5_2$ $\diamondsuit$	$6_0 3_1 1_2$ $\clubsuit$	$5_0 0_1 4_2$ $\clubsuit$	$3_0 2_1 7_2$ $\clubsuit$	$0_1 3_2 \infty_0$ $\diamondsuit$	$2_0 7_1 5_2$ $\clubsuit$
$2_0 5_1 1_2$ $\clubsuit$	$0_0 7_1 4_2$ $\clubsuit$	$2_0 2_1 2_2$ $\diamondsuit$	$7_0 4_1 2_2$ $\clubsuit$	$6_0 1_1 5_2$ $\clubsuit$	$4_0 3_1 0_2$ $\clubsuit$	$6_0 6_1 6_2$ $\diamondsuit$
$4_0 1_1 7_2$ $\clubsuit$	$3_0 6_1 2_2$ $\clubsuit$	$1_0 0_1 5_2$ $\clubsuit$	$3_0 3_1 3_2$ $\diamondsuit$	$0_0 5_1 3_2$ $\clubsuit$	$7_0 2_1 6_2$ $\clubsuit$	$5_0 4_1 1_2$ $\clubsuit$

where B is the array

$0_0 6_1 \infty_2$ $\diamondsuit$	$4_0 6_1 5_2$ $\clubsuit$	$1_0 2_0 4_0$ $\diamondsuit$	$2_0 3_0 5_0$ $\clubsuit$	$2_2 7_2 0_2$ $\clubsuit$	$2_1 3_1 5_1$ $\clubsuit$
$2_1 5_2 \infty_0$ $\clubsuit$	$5_0 7_1 6_2$ $\clubsuit$	$3_1 4_1 6_1$ $\clubsuit$	$0_1 1_1 3_1$ $\diamondsuit$	$3_0 4_0 6_0$ $\clubsuit$	$3_2 0_2 1_2$ $\clubsuit$
$2_0 3_1 4_2$ $\diamondsuit$	$6_0 0_1 7_2$ $\clubsuit$	$4_2 1_2 2_2$ $\clubsuit$	$4_1 5_1 7_1$ $\clubsuit$	$5_1 6_1 0_1$ $\diamondsuit$	$4_0 5_0 7_0$ $\clubsuit$
$7_0 7_1 7_2$ $\diamondsuit$	$0_0 2_1 1_2$ $\clubsuit$	$0_2 5_2 6_2$ $\diamondsuit$	$6_0 7_0 1_0$ $\clubsuit$	$6_2 3_2 4_2$ $\clubsuit$	$6_1 7_1 1_1$ $\clubsuit$
$5_0 1_1 3_2$ $\clubsuit$	$1_0 3_1 2_2$ $\clubsuit$	$7_1 0_1 2_1$ $\clubsuit$	$5_2 2_2 3_2$ $\diamondsuit$	$7_0 0_0 2_0$ $\clubsuit$	$7_2 4_2 5_2$ $\clubsuit$
$6_0 5_1 2_2$ $\clubsuit$	$2_0 4_1 3_2$ $\diamondsuit$	$3_0 7_0 \infty_0$ $\clubsuit$	$1_2 6_2 7_2$ $\diamondsuit$	$5_2 1_2 \infty_2$ $\clubsuit$	$4_1 0_1 \infty_1$ $\clubsuit$
$1_0 4_1 0_2$ $\clubsuit$	$3_0 5_1 4_2$ $\diamondsuit$	$1_1 5_1 \infty_1$ $\clubsuit$	$4_0 0_0 \infty_0$ $\clubsuit$	$1_1 2_1 4_1$ $\diamondsuit$	$6_2 2_2 \infty_2$ $\clubsuit$
$3_0 0_1 6_2$ $\clubsuit$	$\infty_0 \infty_1 \infty_2$ $\diamondsuit$	$3_2 7_2 \infty_2$ $\clubsuit$	$2_1 6_1 \infty_1$ $\clubsuit$	$5_0 1_0 \infty_0$ $\clubsuit$	$0_0 1_0 3_0$ $\diamondsuit$
$4_0 1_2 \infty_1$ $\diamondsuit$	$7_0 1_1 0_2$ $\diamondsuit$	$5_0 6_0 0_0$ $\diamondsuit$	$4_2 0_2 \infty_2$ $\clubsuit$	$3_1 7_1 \infty_1$ $\clubsuit$	$6_0 2_0 \infty_0$ $\clubsuit$

FIGURE 2. A 2-$*$colorable special $\mathrm{GBTD}_1(3, 9)$ $(X, \mathcal{A})$, where $X = (\mathbb{Z}_8 \times \mathbb{Z}_3) \cup \{\infty_0, \infty_1, \infty_2\}$ and colors $\{\clubsuit, \diamondsuit\}$. The cell $(1, 5)$, occupied by the block $7_0 0_1 1_2$, is special. For succinctness, a set $\{x, y, z\}$ is written xyz.

defined for RBIBDs. Hence, we do not need a smaller GBTD to seed the recursion in Proposition 5.1. We make use of this fact to yield a special $\mathrm{GBTD}_1(3, 15)$ in Lemma 7.1.

4.2. Incomplete GBTPs

Incomplete designs are ubiquitous in combinatorial design theory and crucial in "filling in the holes" constructions described in Section 5.

Suppose that $(X, \mathcal{A})$ is a (v, K, λ)-packing. Let $W \subset X$ with $|W| = w$. Furthermore, we call $(X, W, \mathcal{A})$ as an *incomplete resolvable packing*, denoted by $\mathrm{IRP}(v, K, \lambda; w)$, if it satisfies the following conditions:

(i) any pair of points from W occurs in no blocks of $\mathcal{A}$,
(ii) the blocks in $\mathcal{A}$ can be partitioned into parallel classes and *partial parallel classes* $X \setminus W$.

Definition 4.5. Let $(X, W, \mathcal{A})$ be an IRP$(v, K, \lambda; w)$. Then $(X, W, \mathcal{A})$ is called an incomplete GBTP (IGBTP) if the blocks of $\mathcal{A}$ are arranged into an $m \times n$ array A, with rows and columns indexed by R and C, respectively, satisfying the following conditions:

(i) there exist a $P \subset R$ with $|P| = m'$ and a $Q \subset C$ with $|Q| = n'$ such that the cell (r, c) is empty if $r \in P$ and $c \in Q$;
(ii) for any row $r \in P$, every point in $X \setminus W$ is contained in either $\lceil n/m \rceil$ or $\lfloor n/m \rfloor$ cells and the points in W do not appear; for any row $r \in R \setminus P$, every point in X is contained in either $\lceil n/m \rceil$ or $\lfloor n/m \rfloor$ cells;
(iii) the blocks in any column $c \in Q$ form a partial parallel class of $X \setminus W$ and the blocks in any column $c \in C \setminus Q$ forms a parallel class of X.

Denote such an IGBTP by IGBTP$_\lambda(K, v, m \times n; w, m' \times n')$.

Example 4.3. An IGBTP$_1(\{2, 3^*\}, 29, 14 \times 25; 9, 4 \times 5)$ is given in Fig. 3.

Consider an IGBTP$_1(\{k\}, km, m \times \frac{km-1}{k-1}; k, 1 \times 1)$. Then its corresponding array has one empty cell and we fill this cell with the block W to obtain a GBTD$_1(k, m)$. A GBTD$_1(k, m)$ obtained in this way is called a *special* GBTD$_1(k, m)$ and the cell occupied by W is said to be *special*.

Example 4.4. The GBTD$_1(3, 9)$ in Fig. 2 is a special GBTD$_1(3, 9)$ with special cell $(1, 5)$.

A few more classes of auxiliary designs are also required.

4.3. Group Divisible Designs and Transversal Designs

Definition 4.6. Let $(X, \mathcal{A})$ be a set system and let $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ be a partition of X into subsets, called groups. The triple $(X, \mathcal{G}, \mathcal{A})$ is a group divisible design (GDD) when every 2-subset of X not contained in a group appears in exactly one block, and $|A \cap G| \leq 1$ for $A \in \mathcal{A}$ and $G \in \mathcal{G}$.

We denote a GDD $(X, \mathcal{G}, \mathcal{A})$ by K-GDD if $(X, \mathcal{A})$ is K-uniform. The *type* of a GDD $(X, \mathcal{G}, \mathcal{A})$ is the multiset $\langle |G| : G \in \mathcal{G} \rangle$. For convenience, the exponential notation is used to describe the type of a GDD: a GDD of type $g_1^{t_1} g_2^{t_2} \ldots g_s^{t_s}$ is a GDD with exactly t_i groups of size g_i, $i \in [s]$.

Definition 4.7. A transversal design TD(k, n) is a $\{k\}$-GDD of type n^k.

The following result on the existence of transversal designs (see [1]) is sometimes used without explicit reference throughout this paper.

Theorem 4.8. *Let* TD(k) *denote the set of positive integers n such that there exists a* TD(k, n). *Then, we have*

(i) TD$(4) \supseteq \mathbb{Z}_{>0} \setminus \{2, 6\}$,
(ii) TD$(5) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 6, 10\}$,
(iii) TD$(6) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 6, 10, 14, 18, 22\}$,
(iv) TD$(7) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 5, 6, 10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$,
(v) TD$(k) \supseteq \{q : q \geq k - 1$ *is a prime power*$\}$.

A	B

where A is the array

–	–	–	–	–	2,13	3,14	4,15	5,16	6,17	7,18	8,19	9,0
–	–	–	–	–	12,16	13,17	14,18	15,19	16,0	17,1	18,2	19,3
–	–	–	–	–	15,18	16,19	17,0	18,1	19,2	0,3	1,4	2,5
–	–	–	–	–	1,3	2,4	3,5	4,6	5,7	6,8	7,9	8,10
0,10	2,7	12,17	4,16	14,6	4,5,11	$i,18$	$h,1$	$g,12$	$f,18$	$e,13$	$d,16$	$c,13$
1,11	3,8	13,18	5,17	15,7	$a,0$	5,6,12	$i,19$	$h,2$	$g,13$	$f,19$	$e,14$	$d,17$
2,12	4,9	14,19	6,18	16,8	$b,7$	$a,1$	6,7,13	$i,0$	$h,3$	$g,14$	$f,0$	$e,15$
3,13	5,10	15,0	7,19	17,9	$c,6$	$b,8$	$a,2$	7,8,14	$i,1$	$h,4$	$g,15$	$f,1$
4,14	6,11	16,1	8,0	18,10	$d,10$	$c,7$	$b,9$	$a,3$	8,9,15	$i,2$	$h,5$	$g,16$
5,15	7,12	17,2	9,1	19,11	$e,8$	$d,11$	$c,8$	$b,10$	$a,4$	9,10,16	$i,3$	$h,6$
6,16	8,13	18,3	10,2	0,12	$f,14$	$e,9$	$d,12$	$c,9$	$b,11$	$a,5$	10,11,17	$i,4$
7,17	9,14	19,4	11,3	1,13	$g,9$	$f,15$	$e,10$	$d,13$	$c,10$	$b,12$	$a,6$	11,12,18
8,18	10,15	0,5	12,4	2,14	$h,19$	$g,10$	$f,16$	$e,11$	$d,14$	$c,11$	$b,13$	$a,7$
9,19	11,16	1,6	13,5	3,15	$i,17$	$h,0$	$g,11$	$f,17$	$e,12$	$d,15$	$c,12$	$b,14$

where B is the array

10,1	11,2	12,3	13,4	14,5	15,6	16,7	17,8	18,9	19,10	0,11	1,12
0,4	1,5	2,6	3,7	4,8	5,9	6,10	7,11	8,12	9,13	10,14	11,15
3,6	4,7	5,8	6,9	7,10	8,11	9,12	10,13	11,14	12,15	13,16	14,17
9,11	10,12	11,13	12,14	13,15	14,16	15,17	16,18	17,19	18,0	19,1	0,2
$b,15$	$a,9$	14,15,1	$i,8$	$h,11$	$g,2$	$f,8$	$e,3$	$d,6$	$c,3$	$b,5$	$a,19$
$c,14$	$b,16$	$a,10$	15,16,2	$i,9$	$h,12$	$g,3$	$f,9$	$e,4$	$d,7$	$c,4$	$b,6$
$d,18$	$c,15$	$b,17$	$a,11$	16,17,3	$i,10$	$h,13$	$g,4$	$f,10$	$e,5$	$d,8$	$c,5$
$e,16$	$d,19$	$c,16$	$b,18$	$a,12$	17,18,4	$i,11$	$h,14$	$g,5$	$f,11$	$e,6$	$d,9$
$f,2$	$e,17$	$d,0$	$c,17$	$b,19$	$a,13$	18,19,5	$i,12$	$h,15$	$g,6$	$f,12$	$e,7$
$g,17$	$f,3$	$e,18$	$d,1$	$c,18$	$b,0$	$a,14$	19,0,6	$i,13$	$h,16$	$g,7$	$f,13$
$h,7$	$g,18$	$f,4$	$e,19$	$d,2$	$c,19$	$b,1$	$a,15$	0,1,7	$i,14$	$h,17$	$g,8$
$i,5$	$h,8$	$g,19$	$f,5$	$e,0$	$d,3$	$c,0$	$b,2$	$a,16$	1,2,8	$i,15$	$h,18$
12,13,19	$i,6$	$h,9$	$g,0$	$f,6$	$e,1$	$d,4$	$c,1$	$b,3$	$a,17$	2,3,9	$i,16$
$a,8$	13,14,0	$i,7$	$h,10$	$g,1$	$f,7$	$e,2$	$d,5$	$c,2$	$b,4$	$a,18$	3,4,10

FIGURE 3. An $\text{IGBTP}_1(\{2,3\}, 29, 14 \times 25; 9, 4 \times 5)$ $(X, \mathcal{A})$, where $X = \mathbb{Z}_{20} \cup \{a, b, c, d, e, f, g, h, i\}$ and $W = \{a, b, c, d, e, f, g, h, i\}$. For succinctness, a block $\{x, y, z\}$ is written x, y, z.

Definition 4.9. A doubly resolvable TD(k, n), denoted by DRTD(k, n), is a TD(k, n) whose blocks can be arranged in an $n \times n$ array such that each point appears exactly once in each row and once in each column.

The following proposition describes the relationship between DRTDs and TDs.

Proposition 4.10 (Folklore, see [1, Theorem 3.18] and [10]). *There exists a TD$(k+2, n)$ if and only if there exists a DRTD(k, n).*

Corollary 4.11. *A DRTD$(3, n)$ exists for all $n \geq 4$ and $n \notin \{6, 10\}$.*

Proof. A TD$(5, n)$ exists if $n \geq 4$ and $n \notin \{6, 10\}$ by Theorem 4.8. □

4.4. Frame GBTD

Let $(X, \mathcal{G}, \mathcal{A})$ be a $\{k\}$-GDD with $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ and $|G_i| \equiv 0 \bmod k(k-1)$ for all $i \in [s]$. Let $R = \frac{1}{k}\sum_{i=1}^{s}|G_i|$ and $C = \frac{1}{k-1}\sum_{i=1}^{s}|G_i|$. Suppose there exists a partition $[R] = \bigsqcup_{i=1}^{s} R_i$ and a partition $[C] = \bigsqcup_{i=1}^{s} C_i$ such that for each $i \in [s]$, we have $|R_i| = |G_i|/k$ and $|C_i| = |G_i|/(k-1)$.

We say that $(X, \mathcal{G}, \mathcal{A})$ is a *frame GBTD* (FrGBTD) if its blocks can be arranged in an $R \times C$ array such that the following conditions hold:

(i) the cell (r, c) is empty when $(r, c) \in R_i \times C_i$ for $i \in [s]$,
(ii) for any row $r \in R_i$, each point in $X \setminus G_i$ appears either once or twice and the points in G_i do not appear,
(iii) for any column $c \in C_i$, each point in $X \setminus G_i$ appears exactly once.

Denote this FrGBTD by FrGBTD(k, T), where $T = \langle |G_i| : i \in [s]\rangle$.

Example 4.5. An FrGBTD$(3, 6^6)$ is given in Fig. 4.

5. RECURSIVE CONSTRUCTIONS

In this section, we describe the necessary recursive constructions. We note that these are straightforward adaptions of methods in previous work [11, 21, 22, 33]. Here, we state the propositions without proof and the interested reader may refer to [6] for detailed proofs.

5.1. Recursive Constructions for GBTPs

First, for block size 3, we have the following tripling construction for GBTDs. This is an adaption of k-tupling construction for the case of GBTDs with index $k-1$ [22, Theorem 3.1] and the doubling construction for balanced tournament designs [27].

Proposition 5.1 (Tripling construction)**.** *Suppose there exists a 3-$*$colorable RBIBD$(m, 3, 1)$ and a DRTD$(3, m)$. Then there exists a 2-$*$colorable GBTD$_1(3, m)$. Suppose further that the RBIBD$(m, 3, 1)$ is 3-$*$colorable with property Π. Then the GBTD$_1(3, m)$ is a special GBTD$_1(3, m)$.*

Corollary 5.2. *Let $m > 3$ and suppose an RBIBD$(m, 3, 1)$ that is 3-$*$colorable with property Π exists. Then there exists a special GBTD$_1(3, 3^k m)$, for all $k \geq 0$.*

Proof. First note that $m \equiv 3 \bmod 6$ since this is a necessary condition for the existence of an RBIBD$(m, 3, 1)$. Hence, there exists a DRTD$(3, m)$ by Corollary 4.11. By Proposition 5.1, there exists a 2-$*$colorable special GBTD$_1(3, m)$, which may be regarded as an RBIBD$(3m, 3, 1)$ that is 3-$*$colorable with property Π. The corollary then follows by induction. □

The following propositions are simple generalizations of the standard "filling in the hole" construction to construct GBTPs or GBTDs using IGBTPs and FrGBTDs.

A	B

where A is the array

–	–	–	$4_0 1_0 7_0$	$4_1 1_1 7_1$	$4_2 1_2 7_2$	$6_0 3_0 9_0$	$6_1 3_1 9_1$	$6_2 3_2 9_2$
–	–	–	$6_0 7_2 8_0$	$6_1 7_0 8_1$	$6_2 7_1 8_2$	$8_0 9_2 0_0$	$8_1 9_0 0_1$	$8_2 9_1 0_2$
$2_0 8_1 1_0$	$2_1 8_2 1_1$	$2_2 8_0 1_2$	–	–	–	$4_1 8_2 \infty_4$	$4_2 8_0 \infty_3$	$4_0 8_1 \infty_5$
$6_2 7_2 3_1$	$6_0 7_0 3_2$	$6_1 7_1 3_0$	–	–	–	$9_1 1_2 \infty_5$	$9_2 1_0 \infty_4$	$9_0 1_1 \infty_3$
$4_0 0_1 3_0$	$4_1 0_2 3_1$	$4_2 0_0 3_2$	$1_1 3_0 9_1$	$1_2 3_1 9_2$	$1_0 3_2 9_0$	–	–	–
$8_2 9_2 5_1$	$8_0 9_0 5_2$	$8_1 9_1 5_0$	$4_2 6_1 8_2$	$4_0 6_2 8_0$	$4_1 6_0 8_1$	–	–	–
$6_0 2_1 5_0$	$6_1 2_2 5_1$	$6_2 2_0 5_2$	$8_1 1_2 \infty_0$	$8_2 1_0 \infty_1$	$8_0 1_1 \infty_2$	$3_1 5_0 1_1$	$3_2 5_1 1_2$	$3_0 5_2 1_0$
$0_2 1_2 7_1$	$0_0 1_0 7_2$	$0_1 1_1 7_0$	$2_0 3_1 \infty_1$	$2_1 3_2 \infty_2$	$2_2 3_0 \infty_0$	$6_2 8_1 0_2$	$6_0 8_2 0_0$	$6_1 8_0 0_1$
$8_0 4_1 7_0$	$8_1 4_2 7_1$	$8_2 4_0 7_2$	$2_2 9_0 \infty_2$	$2_0 9_1 \infty_0$	$2_1 9_2 \infty_1$	$0_1 3_2 \infty_0$	$0_2 3_0 \infty_1$	$0_0 3_1 \infty_2$
$2_2 3_2 9_1$	$2_0 3_0 9_2$	$2_1 3_1 9_0$	$3_2 4_1 \infty_3$	$3_0 4_2 \infty_5$	$3_1 4_0 \infty_4$	$4_0 5_1 \infty_1$	$4_1 5_2 \infty_2$	$4_2 5_0 \infty_0$
$0_0 6_1 9_0$	$0_1 6_2 9_1$	$0_2 6_0 9_2$	$2_1 6_2 \infty_4$	$2_2 6_0 \infty_3$	$2_0 6_1 \infty_5$	$4_2 1_0 \infty_2$	$4_0 1_1 \infty_0$	$4_1 1_2 \infty_1$
$4_2 5_2 1_1$	$4_0 5_0 1_2$	$4_1 5_1 1_0$	$7_1 9_2 \infty_5$	$7_2 9_0 \infty_4$	$7_0 9_1 \infty_3$	$5_2 6_1 \infty_3$	$5_0 6_2 \infty_5$	$5_1 6_0 \infty_4$

where B is the array

$8_0 5_0 1_0$	$8_1 5_1 1_1$	$8_2 5_2 1_2$	$0_0 7_0 3_0$	$0_1 7_1 3_1$	$0_2 7_2 3_2$	$2_0 9_0 5_0$	$2_1 9_1 5_1$	$2_2 9_2 5_2$
$0_0 1_2 2_0$	$0_1 1_0 2_1$	$0_2 1_1 2_2$	$2_0 3_2 4_0$	$2_1 3_0 4_1$	$2_2 3_1 4_2$	$4_0 5_2 6_0$	$4_1 5_0 6_1$	$4_2 5_1 6_2$
$6_2 3_0 \infty_2$	$6_0 3_1 \infty_0$	$6_1 3_2 \infty_1$	$4_1 7_2 \infty_0$	$4_2 7_0 \infty_1$	$4_0 7_1 \infty_2$	$9_1 1_0 7_1$	$9_2 1_1 7_2$	$9_0 1_2 7_0$
$7_2 8_1 \infty_3$	$7_0 8_2 \infty_5$	$7_1 8_0 \infty_4$	$8_0 9_1 \infty_1$	$8_1 9_2 \infty_2$	$8_2 9_0 \infty_0$	$2_2 4_1 6_2$	$2_0 4_2 6_0$	$2_1 4_0 6_1$
$6_1 0_2 \infty_4$	$6_2 0_0 \infty_3$	$6_0 0_1 \infty_5$	$8_2 5_0 \infty_2$	$8_0 5_1 \infty_0$	$8_1 5_2 \infty_1$	$6_1 9_2 \infty_0$	$6_2 9_0 \infty_1$	$6_0 9_1 \infty_2$
$1_1 3_2 \infty_5$	$1_2 3_0 \infty_4$	$1_0 3_1 \infty_3$	$9_2 0_1 \infty_3$	$9_0 0_2 \infty_5$	$9_1 0_0 \infty_4$	$0_0 1_1 \infty_1$	$0_1 1_2 \infty_2$	$0_2 1_0 \infty_0$
–	–	–	$8_1 2_2 \infty_4$	$8_2 2_0 \infty_3$	$8_0 2_1 \infty_5$	$0_2 7_0 \infty_2$	$0_0 7_1 \infty_0$	$0_1 7_2 \infty_1$
–	–	–	$3_1 5_2 \infty_5$	$3_2 5_0 \infty_4$	$3_0 5_1 \infty_3$	$1_2 2_1 \infty_3$	$1_0 2_2 \infty_5$	$1_1 2_0 \infty_4$
$5_1 7_0 3_1$	$5_2 7_1 3_2$	$5_0 7_2 3_0$	–	–	–	$0_1 4_2 \infty_4$	$0_2 4_0 \infty_3$	$0_0 4_1 \infty_5$
$8_2 0_1 2_2$	$8_0 0_2 2_0$	$8_1 0_0 2_1$	–	–	–	$5_1 7_2 \infty_5$	$5_2 7_0 \infty_4$	$5_0 7_1 \infty_3$
$2_1 5_2 \infty_0$	$2_2 5_0 \infty_1$	$2_0 5_1 \infty_2$	$7_1 9_0 5_1$	$7_2 9_1 5_2$	$7_0 9_2 5_0$	–	–	–
$6_0 7_1 \infty_1$	$6_1 7_2 \infty_2$	$6_2 7_0 \infty_0$	$0_2 2_1 4_2$	$0_0 2_2 4_0$	$0_1 2_0 4_1$	–	–	–

FIGURE 4. An $\text{FrGBTD}_1(3, 6^6)$ $(X, \mathcal{G}, \mathcal{A})$, where $X = (\mathbb{Z}_{10} \times \mathbb{Z}_3) \cup \{\infty_i : i \in \mathbb{Z}_6\}$ and $\mathcal{G} = \{\{t_0, t_1, t_2, (5+t)_0, (5+t)_1, (5+t)_2\} : t \in \mathbb{Z}_5\} \cup \{\infty_i : i \in \mathbb{Z}_6\}$. For succinctness, a set $\{x, y, z\}$ is written xyz.

Proposition 5.3 (IGBTP construction for GBTP). *If an* $IGBTP_\lambda(K, v, m \times n; w, m' \times n')$ *and a* $GBTP_\lambda(K, w, m' \times n')$ *exists, then a* $GBTP_\lambda(K, v, m \times n)$ *exists.*

FrGBTD is a useful tool to construct larger GBTPs from smaller ones.

Proposition 5.4 (FrGBTD construction for GBTP). *Let* $k \in K$. *Suppose there exists an* FrGBTD$(k, T)(X, \mathcal{G}, \mathcal{A})$, *where* $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$, *and let* $r_i = |G_i|/k$ *and* $c_i = |G_i|/(k-1)$, *for* $i \in [s]$. *If there exists an* $IGBTP_1(K, |G_i| + w, (r_i + m) \times (c_i + n); w, m \times n)$ *for all* $i \in [s]$, *then there exists an* $IGBTP_1(K, \sum_{i=1}^{s} |G_i| + w, (\sum_{i=1}^{s} r_i +$

$m) \times (\sum_{i=1}^{s} c_i + n); w, m \times n)$. Furthermore, if a GBTP$_1(K, w, m \times n)$ exists, then a GBTP$_1(K, \sum_{i=1}^{s} |G_i| + w, (\sum_{i=1}^{s} r_i + m) \times (\sum_{i=1}^{s} c_i + n))$ exists.

Since a GBTD is an instance of GBTP, we have the following recursive construction for GBTDs.

Corollary 5.5 (FrGBTD construction for GBTD). *Suppose an* FrGBTD(k, T) *exists with groups $\{G_1, G_2, \ldots, G_s\}$. Let $g_i = |G_i|/k$, for $i \in [s]$. If there exists a special GBTD$_1(k, g_i + 1)$ for all $i \in [s]$, then there exists a special GBTD$_1(k, \sum_{i=1}^{s} g_i + 1)$.*

When the groups are of the same size, we have the following corollary.

Corollary 5.6. *If there exists an* FrGBTD$(3, (3g)^t)$ *and a special* GBTD$_1(3, g + 1)$, *then there exists a special* GBTD$_1(3, gt + 1)$.

For Proposition 5.3 and Corollary 5.5 to be useful, we require large classes of FrGBTDs. We give three recursive constructions for FrGBTDs next.

5.2. Recursive Constructions for FrGBTDs

We adapt the standard direct product construction.

Proposition 5.7 (Inflation). *Suppose an* FrGBTD(k, T) *and a* DRTD(k, n) *exists. Then there exists an* FrGBTD(k, nT).

Wilson's fundamental construction for GDDs [32] can also be modified to construct FrGBTDs.

Proposition 5.8 (Fundamental construction). *Suppose there exists a (master) GDD $(X, \mathcal{G}, \mathcal{A})$ of type T and let $w : X \to \mathbb{Z}_{\geq 0}$ be a weight function. If for each $A \in \mathcal{A}$, there exists an (ingredient)* FrGBTD$(k, \langle w(a) : a \in A\rangle)$, *then there exists an* FrGBTD$(k, \langle \sum_{x \in G} w(x) : G \in \mathcal{G}\rangle)$.

Proposition 5.8 admits the following specialization.

Proposition 5.9 FrGBTD from truncated TD. *Let $s > 0$. Suppose there exists a TD$(u + s, m)$, and $g_1, g_2, \ldots, g_s$ are non-negative integers at most m. If there exists an* FrGBTD(k, g^t) *for each $t \in \{u, u + 1, \ldots, u + s\}$, then there exists an* FrGBTD(k, T), *where $T = (g \cdot m)^u (g \cdot g_1)(g \cdot g_2) \cdots (g \cdot g_s)$.*

6. DIRECT CONSTRUCTIONS

This section constructs some small GBTDs and FrGBTDs that are required to seed the recursive constructions given in Section 5. Our main tools are *starters* and the *method of differences* .

Starter–adder constructions are ubiquitous in the constructions for GBTDs with index $k - 1$, associated frames, and other types of similar designs (see, e.g., [9, 11, 21, 22, 33]). Unlike previous work and due to the lack of symmetry in our arrays, we fix the positions of the starters in our arrays and "develop" the blocks in a variety of "directions" (see Figs. 5–7). This removes the use of adders and surprisingly a careful analysis of the starter conditions allows a prime power construction that is given in Proposition 6.3.

$$\begin{array}{|c|c|}\hline \mathsf{A} & \mathsf{B} \\ \hline \end{array}$$

where A is the array

$$\begin{array}{|ccccc|}\hline A_0 & A_{-\alpha_1}+\alpha_1 & A_{-\alpha_2}+\alpha_2 & \cdots & A_{-\alpha_{m-1}}+\alpha_{m-1} \\ A_{\alpha_1} & A_0+\alpha_1 & A_{\alpha_1-\alpha_2}+\alpha_2 & \cdots & A_{\alpha_1-\alpha_{m-1}}+\alpha_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{\alpha_{m-1}} & A_{\alpha_{m-1}-\alpha_1}+\alpha_1 & A_{\alpha_{m-1}-\alpha_2}+\alpha_2 & \cdots & A_0+\alpha_{m-1} \\ \hline \end{array}$$

and B is the array

$$\begin{array}{|cccc|}\hline B_1 & B_2 & \cdots & B_{(m-1)/(k-1)} \\ B_1+\alpha_1 & B_2+\alpha_1 & \cdots & B_{(m-1)/(k-1)}+\alpha_1 \\ \vdots & \vdots & \ddots & \vdots \\ B_1+\alpha_{m-1} & B_2+\alpha_{m-1} & \cdots & B_{(m-1)/(k-1)}+\alpha_{m-1} \\ \hline \end{array}.$$

FIGURE 5. A $\text{GBTD}_1(k, m)$ from $(\Gamma \times [k])$-GBTD-starter $\mathcal{S} = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}$, where $\Gamma = \{0, \alpha_1, \ldots, \alpha_{m-1}\}$ and $T = [(m-1)/(k-1)]$.

First, we recall certain concepts with regards to the method of differences. Let Γ be an additive abelian group and let n be a positive integer. For a set system $(\Gamma, \mathcal{S})$, the *difference list* of $\mathcal{S}$ is the multiset

$$\Delta\mathcal{S} = \langle x - y : x, y \in A, x \neq y, \text{and } A \in \mathcal{S}\rangle.$$

For a set system $(\Gamma \times [n], \mathcal{S})$ and $i, j \in [n]$, the multiset

$$\Delta_{ij}\mathcal{S} = \langle x - y : x_i, y_j \in A, x_i \neq y_j, \text{and } A \in \mathcal{S}\rangle$$

is called a list of *pure differences* when $i = j$, and called a list of *mixed differences* when $i \neq j$.

6.1. Direct Constructions for GBTDs

Definition 6.1 (Starter for GBTD)**.** Let m be an odd positive integer, Γ be an additive abelian group of size m. Let T be an index set of size $(m-1)/2$. Let $(\Gamma \times [3], \mathcal{S})$ be a $\{3\}$-uniform set system of size $(3m-1)/2$, where

$$\mathcal{S} = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}.$$

$\mathcal{S}$ is called a $(\Gamma \times [3])$-GBTD-starter if the following conditions hold:

(i) $\Delta_{ii}\mathcal{S} = \Gamma \setminus \{0\}$, for $i \in [3]$,
(ii) $\Delta_{ij}\mathcal{S} = \Gamma$, for $i, j \in [3]$, $i \neq j$,
(iii) $\cup_{\alpha\in\Gamma} A_\alpha = \Gamma \times [3]$,

$$\begin{array}{|c|c|c|}\hline \mathsf{W} & \mathsf{B} & \mathsf{B}+0_1 \\ \hline \mathsf{A} & \mathsf{C} & \mathsf{C}+0_1 \\ \hline \end{array}$$

where W is a $(w-1)/2 \times (w-4)$ empty array, A is an $m \times (w-4)$ array,

$$\left|\begin{array}{c|ccccccc} \{0_0, 0_1\} & A_1 & A_1+0_1 & A_2 & A_2+0_1 & \cdots & A_{(w-5)/2} & A_{(w-5)/2}+0_1 \\ \{1_0, 1_1\} & A_1+1_0 & A_1+1_1 & A_2+1_0 & A_2+1_1 & \cdots & A_{(w-5)/2}+1_0 & A_{(w-5)/2}+1_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \{(m-1)_0, (m-1)_1\} & A_1-1_0 & A_1-1_1 & A_2-1_0 & A_2-1_1 & \cdots & A_{(w-5)/2}-1_0 & A_{(w-5)/2}-1_1 \end{array}\right|,$$

B and C are the following $(w-1)/2 \times m$ and $m \times m$ arrays,

$$\left|\begin{array}{cccc} B_1 & B_1+1_0 & \cdots & B_1-1_0 \\ B_2 & B_1+1_0 & \cdots & B_1-1_0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{(w-1)/2} & B_{(w-1)/2}+1_0 & \cdots & B_{(w-1)/2}-1_0 \end{array}\right|, \quad \left|\begin{array}{cccc} C_0 & C_{m-1}+1_0 & \cdots & C_1-1_0 \\ C_1 & C_0+1_0 & \cdots & C_2-1_0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{m-1} & C_{m-2}+1_0 & \cdots & C_0-1_0 \end{array}\right|.$$

FIGURE 6. An IGBTP$_1$({2, 3*}, $2m + w$, $(m + (w-1)/2) \times (2m + w - 4)$; w, $(w-1)/2 \times (w-4)$) from a $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$-GBTP-starter.

$$\begin{array}{|c|c|c|c|c|}\hline \mathsf{W} & \mathsf{B} & \mathsf{B}+0_1 & \mathsf{B}+0_2 & \mathsf{B}+0_3 \\ \hline \mathsf{A} & \mathsf{C} & \mathsf{D}+0_1 & \mathsf{C}+0_2 & \mathsf{D}+0_3 \\ \cline{2-5} & \mathsf{D} & \mathsf{C}+0_1 & \mathsf{D}+0_2 & \mathsf{C}+0_3 \\ \hline \end{array}$$

where W is a 4×5 empty array, A is a $2m \times 5$ array,

$$\left|\begin{array}{ccc|cc} \{0_0, 0_1\} & \{x_0, x_2\} & \{y_0, y_3\} & A & A+0_2 \\ \{1_0, 1_1\} & \{(x+1)_0, x_2\} & \{(y+1)_0, (y+1)_3\} & A+1_0 & A+1_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \{(m-1)_0, (m-1)_1\} & \{(x-1)_0, x_2\} & \{(y-1)_0, (y-1)_3\} & A+(m-1)_0 & A+(m-1)_2 \\ \hline \{0_2, 0_3\} & \{x_1, x_3\} & \{y_1, y_2\} & A+0_1 & A+0_3 \\ \{1_2, 1_3\} & \{(x+1)_1, x_3\} & \{(y+1)_1, (y+1)_2\} & A+1_1 & A+1_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \{(m-1)_2, (m-1)_3\} & \{(x-1)_1, x_3\} & \{(y-1)_1, (y-1)_2\} & A+(m-1)_1 & A+(m-1)_3 \end{array}\right|,$$

B, C and D are the following $4 \times m$, $m \times m$ and $m \times m$ arrays respectively,

$$\left|\begin{array}{cccc} B_1 & B_1+1_0 & \cdots & B_1-1_0 \\ B_2 & B_2+1_0 & \cdots & B_2-1_0 \\ B_3 & B_3+1_0 & \cdots & B_3-1_0 \\ B_4 & B_4+1_0 & \cdots & B_4-1_0 \end{array}\right|, \quad \left|\begin{array}{cccc} C_0 & C_{m-1}+1_0 & \cdots & C_1-1_0 \\ C_1 & C_0+1_0 & \cdots & C_2-1_0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{m-1} & C_{m-2}+1_0 & \cdots & C_0-1_0 \end{array}\right|, \quad \left|\begin{array}{cccc} D_0 & D_{m-1}+1_0 & \cdots & D_1-1_0 \\ D_1 & D_0+1_0 & \cdots & D_2-1_0 \\ \vdots & \vdots & \ddots & \vdots \\ D_{m-1} & D_{m-2}+1_0 & \cdots & D_0-1_0 \end{array}\right|.$$

FIGURE 7. An IGBTP$_1$({2, 3*}, $4m + 9$, $(2m + 4) \times (4m + 5)$; 9, 4×5) from a $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)$-GBTP-starter.

(iv) $\{j : \alpha_j \in B_t \text{ for some } \alpha \in \Gamma\} = [3]$, for $t \in T$,
(v) each element in $\Gamma \times [3]$ appears either once or twice in the multiset

$$R = \left(\bigcup_{\alpha\in\Gamma} A_\alpha - \alpha\right) \cup \left(\bigcup_{t\in T} B_t\right).$$

Furthermore, $\mathcal{S}$ is said to be special if
(vi) each element in A_0 appears exactly once in R.
Also, $\mathcal{S}$ is said to be 3-$*$colorable with property Π if each of the blocks in

$$\{A_\alpha - \alpha : \alpha \in \Gamma\} \text{ and } \{B_t : t \in T\},$$

can be colored with one of three colors so that
(vii) blocks of the same color are pairwise disjoint,
(viii) for each color c, there exists a point (a witness for c) that is not contained in any block assigned color c.

Proposition 6.2. *If a* $(\Gamma \times [k])$*-GBTD-starter exists, then a* $\mathrm{GBTD}_1(k, m)$ *exists. Similarly, if there exists a special* $(\Gamma \times [3])$*-GBTD-starter, then there exists a special GBTD*$_1(3, m)$*; and if there exists a* 3-$*$*colorable* $(\Gamma \times [3])$*-GBTD-starter with property* Π*, then there exists a* 3-$*$*colorable GBTD*$_1(3, m)$ *with property* Π.

Proof. Let $X = \Gamma \times [k]$, and suppose $\mathcal{S} = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}$ is an $(\Gamma \times [k])$-GBTD-starter. Let

$$\mathcal{A} = \bigcup_{A\in\mathcal{S}} \{A + \alpha : \alpha \in \Gamma\}.$$

Then $(X, \mathcal{A})$ is a BIBD$(km, k, 1)$, whose blocks can be arranged in an $m \times \frac{(km-1)}{k-1}$ array, whose rows and columns are indexed by Γ and $\Gamma \cup T$, respectively, as follows:

- for $\alpha, \beta \in \Gamma$, the block $A_\alpha + \beta$ is placed in cell $(\alpha + \beta, \beta)$, and
- for $t \in T$ and $\alpha \in \Gamma$, the block $B_t + \alpha$ is placed in cell (α, t).

Figure 5 depicts the placement of blocks in the array.

For $\beta \in \Gamma$, the set of blocks occupying column β is $\{A_\alpha + \beta : \alpha \in \Gamma\}$, which form a resolution class by condition (iii) of Definition 6.1. Similarly, for $t \in T$, the set of blocks occupying column t is $\{B_t + \alpha : \alpha \in \Gamma\}$, which form a resolution class by condition (iv) in Definition 6.1.

The set of blocks occupying row 0 is given by R, and by condition (v) of Definition 6.1, each point in X appears either once or twice in row 0. Since the blocks occupying row α ($\alpha \in \Gamma$) are exactly the translates of the blocks in R by α, every point in X also appears either once or twice in row α.

Suppose $\mathcal{S} = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}$ is a special $(\Gamma \times [3])$-GBTD-starter. Then condition (vi) of Definition 6.1 ensures that the cell (0, 0) is special.

On the other hand, if $\mathcal{S}$ be a 3-$*$colorable $(\Gamma \times [3])$-GBTD-starter and let

$$c_i \text{ be the color assigned to } \begin{cases} A_i - i, & \text{if } i \in \Gamma, \\ B_i, & \text{otherwise.} \end{cases}$$

For $\alpha, \beta \in \Gamma$ and $t \in T$, assign the block $A_\alpha + \beta$ color c_α and the block $B_t + \beta$ color c_t. Then conditions (vii) and (viii) of Definition 6.1 ensure that the $\mathrm{GBTD}_1(3, m)$ is 3-$*$colorable with property Π. $\square$

Proposition 6.3. *Let* $q \equiv 1 \bmod 6$. *Then there exists a special* $(\mathbb{F}_q \times [3])$*-GBTD-starter that is* 3-$*$*colorable with property* Π.

Proof. Let $s = (q-1)/6$ and ω be a primitive element of $\mathbb{F}_q$. Consider $\gamma \in \mathbb{F}_q$ that satisfies the following conditions (note that ω^{2s} has order three):

(A) $\gamma \notin \{0, -1, -\omega^{2s}, -\omega^{4s}\}$;
(B) $\gamma \notin \{\frac{\omega^{2is} - \omega^{t+2js}}{\omega^{t-1}} : i \neq j \in [3], t \in [s-1]\}$.

The existence of γ is guaranteed since the cardinality of the union of sets in (A) and (B) is at most $4 + 6(s-1) < 6s + 1 = q$.

Define Λ to be $\{-\gamma\omega^{t-1+2(j-1)s} : t \in [s], j \in [3]\}$ and construct the following $q + 3s = (3q-1)/2$ blocks. For $\alpha \in \mathbb{F}_q$, let

$$A_\alpha = \begin{cases} \left\{\left(\omega^{t-1+2(j-1)s}\right)_i : j \in [3]\right\}, & \text{if } \alpha = -\gamma\omega^{t-1+2(i-1)s} \text{ where } t \in [s], i \in [3], \\ \left\{\left(-\frac{\alpha}{\gamma}\omega^{2(i-1)s}\right)_i : i \in [3])\right\}, & \text{otherwise.} \end{cases}$$

For $(t, j) \in [s] \times [3]$, let

$$B_{(t,j)} = \{(\omega^{t-1+2(j-1)s}(\omega^{2(i-1)s} + \gamma))_i : i \in [3]\}.$$

Let $\mathcal{S} = \{A_\alpha : \alpha \in \mathbb{F}_q\} \cup \{B_{(t,j)} : (t, j) \in [s] \times [3]\}$ and we claim that $\mathcal{S}$ is the desired starter.

Define

$$\mathcal{D} = \{\{\omega^{t-1+2(j-1)s} : j \in [3]\} : t \in [s]\},$$

and Wilson [31] showed that the blocks in $\mathcal{D}$ are mutually disjoint and $\Delta\mathcal{D} = \mathbb{F}_q \setminus \{0\}$.

Hence, for condition (i) of Definition 6.1, we check for $i \in [3]$,

$$\begin{aligned} \Delta_{ii}\mathcal{S} &= \Delta_{ii}\{A_\alpha : \alpha = -\gamma\omega^{t-1+2(i-1)s}, t \in [s], i \in [3]\} \\ &= \Delta\mathcal{D} = \mathbb{F}_q \setminus \{0\}. \end{aligned}$$

For condition (ii), we verify for $i \neq i' \in [3]$,

$$\begin{aligned} \Delta_{ii'}\mathcal{S} &= \bigcup_{\alpha \notin \Lambda} \left(-\frac{\alpha}{\gamma}(\omega^{2(i-1)s} - \omega^{2(i'-1)s})\right) \cup \bigcup_{(t,j)\in[s]\times[3]} \omega^{t-1+2(j-1)s}(\omega^{2(i-1)s} - \omega^{2(i'-1)s}) \\ &= (\omega^{2(i-1)s} - \omega^{2(i'-1)s}) \left(\bigcup_{\alpha \notin \Lambda} -\frac{\alpha}{\gamma} \cup \bigcup_{(t,j)\in[s]\times[3]} \omega^{t-1+2(j-1)s} \right) \\ &= (\omega^{2(i-1)s} - \omega^{2(i'-1)s})\mathbb{F}_q = \mathbb{F}_q. \end{aligned}$$

For condition (iii) of Definition 6.1, since the number of points in $\bigcup_{\alpha\in\mathbb{F}_q} A_\alpha$ is kq, it suffices to check that each point $\beta_i \in \mathbb{F}_q \times [k]$ belongs to some block A_α. Indeed, if $\beta/\omega^{2(i-1)s} = \omega^{(t-1)+2(j-1)s}$ for some $(t, j) \in [s] \times [3]$, then let $\alpha = -\gamma\omega^{t-1+2(i-1)s}$

and so, $\beta_i = \left(\omega^{t-1+2(i+j-2)s}\right)_i$ belongs to A_α. Otherwise, $-\gamma\beta/\omega^{2(i-1)s} \notin \Lambda$. Let $\alpha = -\gamma\beta/\omega^{2(i-1)s}$ and $\beta_i \in A_\alpha$ as desired.

Condition (iv) of Definition 6.1 is clearly true from the definition of $B_{(t,j)}$. We establish condition (v) of Definition 6.1 through the following claims:

Claim 6.1. *The blocks in* $\bigcup_{\alpha\notin\Lambda}(A_\alpha - \alpha) \cup \bigcup_{(t,j)\in[s]\times[3]} B_{(t,j)}$ *form a resolution class.*

As above, it suffices to check that each point $\beta_i \in \mathbb{F}_q \times [3]$ belongs to some block in $\bigcup_{\alpha\notin\Lambda}(A_\alpha - \alpha) \cup \bigcup_{(t,j)\in[s]\times[k]} B_{(t,j)}$ as the total number of points is kq.

Indeed, if $\beta/(\omega^{2(i-1)s} + \gamma) = \omega^{t-1+2(j-1)s}$ for some $(t, j) \in [s] \times [k]$, then $\beta_i \in B_{(t,j)}$. Otherwise, $-\gamma\beta/(\omega^{2(i-1)s} + \gamma) \notin \Lambda$. Let $\alpha = -\gamma\beta/(\omega^{2(i-1)s} + \gamma)$ (note that α is well defined by Condition (A)) and $\beta_i \in A_\alpha - \alpha$.

Claim 6.5. *Each point in* $\mathbb{F}_q \times [k]$ *appears at most once in* $\bigcup_{\alpha\in\Lambda}(A_\alpha - \alpha)$.

Note that the blocks are of the form

$$\left\{\left(\omega^{t-1+2(j-1)s} + \gamma\omega^{t-1+2(i-1)s}\right)_i : j \in [3]\right\}$$

for $(t, i) \in [s] \times [3]$. Suppose otherwise that a point appears twice. That is, there exist $j, j' \in [3]$, $(t, i), (t', i) \in [s] \times [3]$ with $t > t'$ such that

$$\omega^{t-1+2(j-1)s} + \gamma\omega^{t-1+2(i-1)s} = \omega^{t'-1+2(j'-1)s} + \gamma\omega^{t'-1+2(i-1)s}.$$

Hence,

$$\gamma = \frac{\omega^{2(j'-i)s} - \omega^{2(j-i)s+(t-t')}}{\omega^{t-t'} - 1}.$$

Since $t \neq t'$, we have $t - t' \in [s-1]$. If $j \neq j'$, this contradicts Condition (B). Otherwise $j = j'$ implies $\gamma = -\omega^{2(j-i)s}$ contradicting (A).

Next, observe that $A_0 = \{(0, i) : i \in [3]\}$. By Claim 6.1, to establish condition (vi) of Definition 6.1, it suffices to show that $0_i \notin A_\alpha - \alpha$ for $\alpha \in \Lambda$ and $i \in [3]$. Suppose otherwise. Then there exists $(t, j) \in [s] \times [3]$ and $i \in [3]$ such that

$$(\omega^{(j-1)s} + \gamma)\omega^{t+(i-1)s} = 0,$$

contradicting (A).

Finally, we exhibit that $\mathcal{S}$ is 3-$*$colorable with property Π by assigning the block A_0 color $\clubsuit$, the blocks $A_\alpha - \alpha$ for $\alpha \notin \Lambda$ and B_t for $t \in T$ color $\heartsuit$, and the blocks $A_\alpha - \alpha$ for $\alpha \in \Lambda$ color $\diamondsuit$. Then this assignment satisfies condition (vii) of Definition 6.1. In addition, 0_1 is a witness for both $\heartsuit$ and $\diamondsuit$ and α_1 is a witness for $\clubsuit$ for some $\alpha \neq 0$, satisfying condition (viii) of Definition 6.1. $\square$

Corollary 6.4. *Let* $q \equiv 1 \bmod 6$. *Then a* 3-$*$*colorable* $\mathrm{GBTD}_1(3, m)$ *with property* Π *exists.*

Proof. This follows from Propositions 6.2 and 6.3. $\square$

Corollary 6.5. *A special* $\mathrm{GBTD}_1(3, m)$ *exists for* $m \in \{1, 17, 29, 35, 47, 53, 55\}$*, a* 3-$*$*colorable special* $\mathrm{GBTD}_1(3, m)$ *with property* Π *for* $m \in \{9, 11, 23\}$ *and a* 3-$*$*colorable* RBIBD(15, 3, 1) *with property* Π.

Proof. A special $\text{GBTD}_1(3, 1)$ exists trivially. In addition, a 3-$*$colorable special $\text{GBTD}_1(3, 9)$ with property Π is given by Example 4.4, and a 3-$*$colorable RBIBD(15, 3, 1) with property Π is given by Example 4.1.

For $m \in \{11, 17, 23, 29, 35, 47, 53, 55\}$, apply Proposition 6.2 with special $(\mathbb{Z}_m \times [3])$-GBTD-starters and 3-$*$colorable special $(\mathbb{Z}_m \times [3])$-GBTD-starters with property Π given in [5]. □

6.2. Direct Constructions for an $\text{IGBTP}_1(\{2, 3^*\}, 2m + w, (m + (w-1)/2) \times (2m + w - 4); w, (w-1)/2 \times (w-4))$

As with GBTDs, we use a set of starters to construct GBTPs. To construct these starters, we need the notion of *infinite elements* and *intransitive starters*.

Given an abelian group Γ, we augment the point set with *infinite* elements, denoted by ∞_i, where i belongs to some index set I. The infinite elements are fixed under addition by elements in Γ. That is, $\infty_i + \gamma = \infty_i$ for $\gamma \in \Gamma$. Let w be a positive integer and $W_w = \{\infty_i : i \in [w]\}$. So, given a block $A \subset \Gamma \cup W_w$ and $\gamma \in \Gamma$, $A + \gamma = \{a + \gamma : a \in A \setminus W_w\} \cup (A \cap W_w)$.

We also extend the definition of difference lists. For a set system $(\Gamma \cup W_w, \mathcal{S})$, then the difference list of $\mathcal{S}$ is given by the multiset

$$\Delta\mathcal{S} = \langle x - y : x, y \in A \setminus W_w, x \neq y, A \in \mathcal{S}\rangle.$$

Definition 6.6. Let m be an odd integer with $m \geq 11$. Let $(\mathbb{Z}_m \times \mathbb{Z}_2 \cup W_w, \mathcal{S})$ be a $\{2, 3\}$-uniform set system of size $w - 3 + m$, where

$$\mathcal{S} = \{A_i : i \in [(w-5)/2]\} \cup \{B_i : i \in [(w-1)/2]\} \cup \{C_i : i \in \mathbb{Z}_m\}$$

satisfying $|A_i| = 2$ for $i \in [(w-5)/2]$, $|B_i| = 2$ for $i \in [(w-1)/2]$, $|C_0| = 3$, and $|C_i| = 2$ for $i \in \mathbb{Z}_m \setminus \{0\}$.

$\mathcal{S}$ is called a $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$-IGBTP-starter if the following conditions hold:

(i) $\Delta\mathcal{S} = \mathbb{Z}_m \times \mathbb{Z}_2 \setminus \{0_0, 0_1\}$,
(ii) $\{j : a_j \in A_i\} = \mathbb{Z}_2$ for $i \in [(w-5)/2]$,
(iii) $\{B_i : i \in [(w-1)/2]\} \cup \{C_j : j \in \mathbb{Z}_m\} = (\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$,
(iv) $|C_i \cap W_w| \leq 1$ for $i \in \mathbb{Z}_m$,
(v) each element in $(\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$ appears either once or twice in the multiset

$$R = \{0_0, 0_1\} \cup \left(\bigcup_{\substack{i \in [(w-5)/2] \\ j \in \mathbb{Z}_2}} A_i + 0_j \right) \cup \left(\bigcup_{i_j \in \mathbb{Z}_m \times \mathbb{Z}_2} C_i - i_j \right).$$

Proposition 6.7. *Suppose there exists a $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$-IGBTP-starter. Then there exists an $\text{IGBTP}_1(\{2, 3^*\}, 2m + w, (m + (w-1)/2) \times (2m + w - 4); w, (w-1)/2 \times (w-4))$.*

Proof. Let

$$X = \mathbb{Z}_m \times \mathbb{Z}_2 \cup W_w,$$
$$\mathcal{A} = \{S + j : S \in \mathcal{S} \text{ and } j \in \mathbb{Z}_m \times \mathbb{Z}_2\} \cup \{\{i_0, i_1\} : i \in \mathbb{Z}_m\}.$$

Then $(X, W_w, \mathcal{A})$ is an IRP$(2m + w, K, 1; w)$, whose blocks can be arranged in an $(m + (w - 1)/2) \times (2m + w - 4)$ array as in Fig. 7. We index the rows by $[(w - 1)/2] \cup \mathbb{Z}_m$ and the columns by $[w - 4] \cup (\mathbb{Z}_m \times \mathbb{Z}_2)$.

First, check that the cell (r, c) is empty for $(r, c) \in [(w - 1)/2] \times [w - 4]$.

For $j \in [w - 4]$, the set of blocks occupying column j is $\mathbb{Z}_m \times \mathbb{Z}_2$ by condition (ii) of Definition 6.6. For $j \in \mathbb{Z}_m \times \mathbb{Z}_2$, first observe that the set of the blocks occupying the column 0_0 by condition (iii) of Definition 6.6 is $(\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$. Since the blocks of column j are translates (by j) of the blocks in column 0_0, the union of the blocks in column j is also $(\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$.

For $i \in [(w - 1)/2]$, each element in $\mathbb{Z}_m \times \mathbb{Z}_2$ appears exactly twice in row i by construction. For $i \in \mathbb{Z}_m$, let R_i denote the multiset containing all the points appearing in the blocks of row i. Then $R_0 = R$ and $R_i = R_0 + i_0$, for all $i \in \mathbb{Z}_m$. Hence, it suffices each element in X appears either once or twice in R, which follows immediately from conditions (v) in Definition 6.6. □

Definition 6.8. Let m be an odd integer with $m \geq 11$. Let $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9, \mathcal{S})$ be a $\{1, 2, 3\}$-uniform set system of size $7 + 2m$, where

$$\mathcal{S} = \{x_0\} \cup \{y_0\} \cup A \cup \{B_i : i \in [4]\} \cup \{C_i : i \in \mathbb{Z}_m\} \cup \{D_i : i \in \mathbb{Z}_m\}$$

satisfying $|A| = 2$, $|B_i| = 2$ for $i \in [4]$, $|C_0| = 3$, $|C_i| = 2$ for $i \in \mathbb{Z}_m \setminus \{0\}$, and $|D_i| = 2$ for $i \in \mathbb{Z}_m$.

$\mathcal{S}$ is called a $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)$-IGBTP-starter if the following conditions hold:

(i) $\Delta\mathcal{S} = (\mathbb{Z}_m \times \mathbb{Z}_4) \setminus \{0_0, 0_1, 0_2, 0_3\}$,
(ii) $\{j : a_j \in A\} = \{0, 2\}$,
(iii) $\{B_i : i \in [(w - 1)/2]\} \cup \{C_i : i \in \mathbb{Z}_m\} \cup \{D_i : i \in \mathbb{Z}_m\} = (\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$,
(iv) $|C_i \cap W_9| \leq 1$ and $|D_i \cap W_9| \leq 1$ for $i \in \mathbb{Z}_m$,
(v) each element in $(\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$ appears either once or twice in the multisets

$$\begin{aligned} R_\circ = {} & \{0_0, 0_1, x_0, x_2, y_0, y_3\} \cup A \cup A \\ & + 0_2 \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{0,2\}} C_i - i_j \right) \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{1,3\}} D_i - i_j \right), \\ R_\bullet = {} & \{0_2, 0_3, x_1, x_3, y_1, y_2\} \cup A \\ & + 0_1 \cup A + 0_3 \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{1,3\}} C_i - i_j \right) \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{0,2\}} D_i - i_j \right). \end{aligned}$$

Proposition 6.9. *Suppose there exists a $(\mathbb{Z}_m \times \mathbb{Z}_4 \cup W_9)$-IGBTP-starter. Then there exists an IGBTP$_1(\{2, 3^*\}, 4m + 9, (2m + 4) \times (4m + 5); 9, 4 \times 5)$.*

Proof. Let

$$X = (\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9,$$
$$\mathcal{A} = \{S + j : S \in \mathcal{S}, |S| \neq 1, j \in \mathbb{Z}_m \times \mathbb{Z}_2\} \cup \{\{i_0, i_1\} : i \in \mathbb{Z}_m\} \cup \{\{i_2, i_3\} : i \in \mathbb{Z}_m\}$$
$$\cup \{\{(x+i)_0, (x+i)_2\} : i \in \mathbb{Z}_m\} \cup \{\{(x+i)_1, (x+i)_3\} : i \in \mathbb{Z}_m\}$$
$$\cup \{\{(y+i)_0, (y+i)_3\} : i \in \mathbb{Z}_m\} \cup \{\{(y+i)_1, (y+i)_2\} : i \in \mathbb{Z}_m\}.$$

Then $(X, W_9, \mathcal{A})$ is an IRP$(4m + 9, K, 1; 9)$, whose blocks can be arranged in a $(2m + 4) \times (4m + 5)$ array as in Fig. 6. We index the rows by $[4] \cup (\mathbb{Z}_m \times \{\circ, \bullet\})$ and the columns by $[5] \cup (\mathbb{Z}_m \times \mathbb{Z}_4)$.

First, check that the cell (r, c) is empty for $(r, c) \in [4] \times [5]$.

For $j \in [5]$, the set of blocks occupying column j is $\mathbb{Z}_m \times \mathbb{Z}_4$ by condition (ii) of Definition 6.8. For $j \in \mathbb{Z}_m \times \mathbb{Z}_4$, first observe that the set of the blocks occupying the column 0_0 by condition (iii) of Definition 6.8 is $(\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$. Since the blocks of column j are translates (by j) of the blocks in column 0_0, the union of the blocks in column j is also $(\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$.

For $i \in [4]$, each element in $\mathbb{Z}_m \times \mathbb{Z}_4$ appears exactly twice in row i by construction. For $(i, *) \in \mathbb{Z}_m \times \{\circ, \bullet\}$, let $R_{(i,*)}$ denote the multiset containing all the points appearing in the blocks of row $(i, *)$. Then $R_{(0,*)} = R_*$ and $R_{(i,*)} = R_{(0,*)} + i_0$, for all $i \in \mathbb{Z}_m$. Hence, it suffices each element in X appears either once or twice in R_*, which follows immediately from conditions (v) in Definition 6.8. □

Corollary 6.10. *An IGBTP*$_1(\{2, 3^*\}, 2m + 9, (m + 4) \times (2m + 5); 9, 4 \times 5)$ *exists for* $m \in \{s : 10 \leq s \leq 45\} \cup \{47, 49, 53, 57, 77\} \setminus \{16, 20, 24, 28, 36, 40, 44\}$*, and an IGBTP*$_1(\{2, 3^*\}, 2m + 11, (m + 5) \times (2m + 7); 11, 5 \times 7)$ *exists for* $m \in \{15, 19, 23, 27, 31, 35, 45, 49\}$.

Proof. The required $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_9)$-IGBTP-starter for $m \in \{s : 11 \leq s \leq 49, s \text{ odd}\} \cup \{53, 57, 77\}$ and $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)$-IGBTP-starter for $m \in \{s : 5 \leq s \leq 21, s \text{ odd}\}$ is given in [5] and we apply Propositions 6.7 and 6.9 to obtain the corresponding IGBTP.

Similarly, to construct an IGBTP$_1(\{2, 3^*\}, 2m + 11, (m + 5) \times (2m + 7); 11, 5 \times 7)$ for $m \in \{15, 19, 23, 27, 31, 35, 45, 49\}$, we apply Proposition 6.7 to $(\mathbb{Z}_m \times \mathbb{Z}_2 \cup W_{11})$-IGBTP-starters listed in [5].

It remains to construct an IGBTP$_1(\{2, 3^*\}, 33, 16 \times 29; 9, 4 \times 5)$. Consider $((\mathbb{Z}_3 \times \mathbb{Z}_8) \cup W_9, \mathcal{S})$, a $\{2, 3\}$-uniform set system of size 36, where $\mathcal{S}$ comprise the blocks below:

$$\begin{array}{llll}
A_1 = \{1_0, 1_2\} & A_2 = \{1_1, 1_5\} & A_3 = \{0_0, 0_4\} & A_4 = \{1_3, 1_6\} \\
A_5 = \{0_3, 0_5\} & A_6 = \{1_1, 1_3\} & A_7 = \{1_4, 1_7\} & A_8 = \{0_1, 0_6\} \\
A_9 = \{0_0, 0_5\} & A_{10} = \{0_2, 0_4\} & A_{11} = \{1_4, 1_6\} & A_{12} = \{1_0, 1_3\} \\
A_{13} = \{0_2, 0_5\} & A_{14} = \{1_2, 1_7\} & A_{15} = \{0_1, 0_7\} & A_{16} = \{1_5, 1_7\} \\
A_{17} = \{0_2, 0_6\} & A_{18} = \{0_3, 0_7\} & A_{19} = \{1_1, 1_4\} & A_{20} = \{1_0, 1_6\} \\
B_1 = \{0_0, 0_1\} & B_2 = \{0_5, 1_5\} & B_3 = \{1_1, 2_4\} & B_4 = \{0_7, 1_3\} \\
C_0^1 = \{1_0, 2_1, 2_6\} & C_1^1 = \{1_0, 2_1\} & C_2^1 = \{1_0, 2_1\} & \\
C_0^2 = \{0_2, \infty_1\} & C_1^2 = \{0_4, \infty_2\} & C_2^2 = \{1_2, \infty_3\} & \\
C_0^3 = \{2_0, \infty_4\} & C_1^3 = \{2_3, \infty_5\} & C_2^3 = \{1_6, \infty_6\} & \\
C_0^4 = \{2_7, \infty_7\} & C_1^4 = \{2_2, \infty_8\} & C_2^4 = \{2_5, \infty_9\}. &
\end{array}$$

Let

$$X = (\mathbb{Z}_3 \times \mathbb{Z}_8) \cup W$$
$$\mathcal{A} = \{S + j : S \in \mathcal{S}, j \in \mathbb{Z}_3 \times \mathbb{Z}_8\}.$$

Then $(X, W, \mathcal{A})$ is an IRP(33, $\{2, 3^*\}$, 1; 9), whose blocks can be arranged in a 16×29 array as in Fig. 8. It can be readily verified that this arrangement results in an IGBTP$_1$($\{2, 3^*\}$, 33, 16×29; 9, 4×5). □

6.3. Direct Constructions for FrGBTDs

Lemma 6.11. *There exists an FrGBTD*$(2, 2^t)$ *for* $t \in \{4, 5\}$.

Proof. The desired FrGBTDs are given in Figs. 9 and 10. □

Definition 6.12. Let t be a positive integer, and let $I = [t-1] \times [2]$. Let $(\mathbb{Z}_{3t} \times [2], \mathcal{S})$ be a 3-uniform set system of size $2(t-1)$, where $\mathcal{S} = \{A_i : i \in I\}$. $\mathcal{S}$ is called a $(\mathbb{Z}_{3t} \times [2])$-FrGBTD-starter if the following conditions hold:

(i) $\Delta_{ij}\mathcal{S} = \mathbb{Z}_{3t} \setminus \{0, t, 2t\}$ for $i, j \in [2]$,
(ii) $\cup_{i \in I} A_i = (\mathbb{Z}_{3t} \setminus \{0, t, 2t\}) \times [2]$,
(iii) for $j \in [2]$, each element in $(\mathbb{Z}_t \setminus \{0\}) \times [2]$ appears either once or twice in the multiset

$$R_j = \bigcup_{i=1}^{t-1} A_{(i,j)} - i \bmod t,$$

(4) $r \in (\mathbb{Z}_t \setminus \{0\}) \times [2]$ for each $r \in R_1 \cup R_2$.

Proposition 6.13. *If a* $(\mathbb{Z}_{3t} \times [2], 6^t)$*-FrGBTD-starter exists, then an* FrGBTD$(3, 6^t)$ *exists.*

Proof. Let

$$X = \mathbb{Z}_{3t} \times [2],$$
$$\mathcal{G} = \{G_i = \{i_1, (t+i)_1, (2t+i)_1, i_2, (t+i)_2, (2t+i)_2\} : i \in \mathbb{Z}_t\},$$
$$\mathcal{A} = \{A_i + j : i \in I \text{and} j \in \mathbb{Z}_{3t}\}.$$

Then $(X, \mathcal{G}, \mathcal{A})$ is a $\{3\}$-GDD of type 6^t, whose blocks can be arranged in a $2t \times 3t$ array, with rows and columns indexed by $\mathbb{Z}_t \times [2]$ and $\mathbb{Z}_{3t}$, respectively, as follows: the block $A_{(i,j)} + k$ is placed in cell $((i+k, j), k)$.

The set of blocks occupying column zero are $\{A_i : i \in I\}$ and by condition (ii) of Definition 6.12, $\bigcup_{i \in I} A_i = X \setminus G_0$. For other $j \in \mathbb{Z}_{3t}$, observe that the blocks occupying column j are translates (by j) of the blocks in column zero, and hence the union of the blocks in column j is $X \setminus G_{j'}$, where $j' \equiv j \bmod t$.

For $(i, j) \in \mathbb{Z}_t \times [2]$, let $R_{(i,j)}$ denote the multiset containing all the points appearing in the blocks of row (i, j). Then $R_{(i,j)} = R_{(0,j)} + i$, for all $i \in \mathbb{Z}_t$. Hence, it suffices to check that each element of $X \setminus G_0$ appears either once or twice in $R_{(0,j)}$ and the elements of $R_{(0,j)}$ belong to $X \setminus G_0$ for $j \in [2]$. This, however, follows immediately from conditions (iii) and (iv) in Definition 6.12, since $R_{(0,j)} = R_j \cup (R_j + t) \cup (R_j + 2t)$ for $j \in [2]$. □

W	B	B $+ 0_1$	B $+ 0_2$	B $+ 0_3$	B $+ 0_4$	B $+ 0_5$	B $+ 0_6$	B $+ 0_7$
A	C_1	$\mathsf{C}_4 + 0_1$	$\mathsf{C}_3 + 0_2$	$\mathsf{C}_2 + 0_3$	$\mathsf{C}_1 + 0_4$	$\mathsf{C}_4 + 0_5$	$\mathsf{C}_3 + 0_6$	$\mathsf{C}_2 + 0_7$
	C_2	$\mathsf{C}_1 + 0_1$	$\mathsf{C}_4 + 0_2$	$\mathsf{C}_3 + 0_3$	$\mathsf{C}_2 + 0_4$	$\mathsf{C}_1 + 0_5$	$\mathsf{C}_4 + 0_6$	$\mathsf{C}_3 + 0_7$
	C_3	$\mathsf{C}_2 + 0_1$	$\mathsf{C}_1 + 0_2$	$\mathsf{C}_4 + 0_3$	$\mathsf{C}_3 + 0_4$	$\mathsf{C}_2 + 0_5$	$\mathsf{C}_1 + 0_6$	$\mathsf{C}_4 + 0_7$
	C_4	$\mathsf{C}_3 + 0_1$	$\mathsf{C}_2 + 0_2$	$\mathsf{C}_1 + 0_3$	$\mathsf{C}_4 + 0_4$	$\mathsf{C}_3 + 0_5$	$\mathsf{C}_2 + 0_6$	$\mathsf{C}_1 + 0_7$

where W is a 4×5 empty array, A is a 12×5 array,

$$\begin{array}{|ccccc|}
\hline
A_1 & A_2 & A_3 & A_4 & A_5 \\
A_1 + 1_0 & A_2 + 1_0 & A_3 + 1_0 & A_4 + 1_0 & A_5 + 1_0 \\
A_1 + 2_0 & A_2 + 2_0 & A_3 + 2_0 & A_4 + 2_0 & A_5 + 2_0 \\
A_6 & A_7 & A_8 & A_9 & A_{10} \\
A_6 + 1_0 & A_7 + 1_0 & A_8 + 1_0 & A_9 + 1_0 & A_{10} + 1_0 \\
A_6 + 2_0 & A_7 + 2_0 & A_8 + 2_0 & A_9 + 2_0 & A_{10} + 2_0 \\
A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\
A_{11} + 1_0 & A_{12} + 1_0 & A_{13} + 1_0 & A_{14} + 1_0 & A_{15} + 1_0 \\
A_{11} + 2_0 & A_{12} + 2_0 & A_{13} + 2_0 & A_{14} + 2_0 & A_{15} + 2_0 \\
A_{16} & A_{17} & A_{18} & A_{19} & A_{20} \\
A_{16} + 1_0 & A_{17} + 1_0 & A_{18} + 1_0 & A_{19} + 1_0 & A_{20} + 1_0 \\
A_{16} + 2_0 & A_{17} + 2_0 & A_{18} + 2_0 & A_{19} + 2_0 & A_{20} + 2_0 \\
\hline
\end{array},$$

B is a 4×3 array,

$$\begin{array}{|ccc|}
\hline
B_1 & B_1 + 1_0 & B_1 + 2_0 \\
B_2 & B_2 + 1_0 & B_2 + 2_0 \\
B_3 & B_3 + 1_0 & B_3 + 2_0 \\
B_4 & B_4 + 1_0 & B_4 + 2_0 \\
\hline
\end{array},$$

C_i for $i \in [4]$ is a 3×3 array,

$$\begin{array}{|ccc|}
\hline
C_0^i & C_2^i + 1_0 & C_1^i + 2_0 \\
C_1^i & C_0^i + 1_0 & C_2^i + 2_0 \\
C_2^i & C_1^i + 1_0 & C_0^i + 2_0 \\
\hline
\end{array}.$$

FIGURE 8. An IGBTP$_1$($\{2, 3^*\}$, 33, 16×29; 9, 4×5).

—	—	{2,7}	{6,3}	{7,1}	{3,5}	{5,6}	{1,2}
{2,3}	{6,7}	—	—	{3,0}	{7,4}	{0,2}	{4,6}
{5,7}	{1,3}	{3,4}	{7,0}	—	—	{4,1}	{0,5}
{1,6}	{5,2}	{6,0}	{2,4}	{4,5}	{0,1}	—	—

FIGURE 9. An FrGBTD$_1$($2, 2^4$) $(X, \mathcal{G}, \mathcal{A})$, where $X = \mathbb{Z}_8$ and $\mathcal{G} = \{\{i, 4+i\} : i \in \mathbb{Z}_4\}$.

—	—	{7,9}	{2,4}	{3,4}	{8,9}	{6,2}	{1,7}	{1,8}	{6,3}
{7,4}	{2,9}	—	—	{8,0}	{3,5}	{4,5}	{9,0}	{7,3}	{2,8}
{3,9}	{8,4}	{8,5}	{3,0}	—	—	{9,1}	{4,6}	{5,6}	{0,1}
{1,2}	{6,7}	{4,0}	{9,5}	{9,6}	{4,1}	—	—	{0,2}	{5,7}
{6,8}	{1,3}	{2,3}	{7,8}	{5,1}	{0,6}	{0,7}	{5,2}	—	—

FIGURE 10. An $\mathrm{FrGBTD}_1(2, 2^5)$ $(X, \mathcal{G}, \mathcal{A})$, where $X = \mathbb{Z}_{10}$ and $\mathcal{G} = \{\{i, 5+i\} : i \in \mathbb{Z}_5\}$.

Corollary 6.14. *There exist an* $\mathrm{FrGBTD}(3, 6^t)$ *for all* $t \in \{5, 6, 7, 8\}$*, an* $\mathrm{FrGBTD}(3, 24^t)$ *for all* $t \in \{5, 8\}$ *and an* $\mathrm{FrGBTD}(3, 30^t)$ *for all* $t \in \{5, 7\}$.

Proof. An $\mathrm{FrGBTD}_1(3, 6^6)$ is given by Example 4.5. An $\mathrm{FrGBTD}(3, 6^t)$ for $t \in \{5, 7\}$ exists by applying Proposition 6.13 with FrGBTD-starters given in [5].

The existence of an $\mathrm{FrGBTD}(3, 24^t)$, $t \in \{5, 8\}$ follows by applying Proposition 5.7 with an $\mathrm{FrGBTD}(3, 6^t)$ (constructed in this proof) and a DRTD(3, 4), whose existence is provided by Corollary 4.11. The existence of an $\mathrm{FrGBTD}(3, 30^t)$, $t \in \{5, 7\}$ follows by applying Proposition 5.7 similarly.

To prove the existence of an $\mathrm{FrGBTD}(3, 6^8)$, consider $(\mathbb{Z}_{48}, \mathcal{S})$, a $\{3\}$-uniform set system of size 7, where $\mathcal{S}$ comprise the blocks below:

$$\begin{array}{llll} A_1 = \{2, 3, 5\} & A_2 = \{4, 14, 31\} & A_3 = \{9, 22, 45\} & A_4 = \{15, 34, 43\} \\ A_5 = \{20, 35, 42\} & A_6 = \{13, 17, 47\} & A_7 = \{1, 6, 12\}. & \end{array}$$

Observe that $\mathcal{S}$ satisfies the following conditions:

(i) $\Delta\mathcal{S} = \mathbb{Z}_{48} \setminus \{0, 8, 16, 24, 32, 40\}$,
(ii) $\cup_{i\in[7]} A_i \bmod 24 = \mathbb{Z}_{24} \setminus \{0, 8, 16\}$,
(iii) each element in $\mathbb{Z}_{16} \setminus \{0, 8\}$ appears either once or twice in the multiset

$$R = \bigcup_{i\in[7]} A_i - i \bmod 16,$$

(iv) $r \in \mathbb{Z}_{16} \setminus \{0, 8\}$ for each $r \in R$.

Further, let

$$\begin{aligned} X &= \mathbb{Z}_{48}, \\ \mathcal{G} &= \{\{i + 8k : k \in \mathbb{Z}_6\} : i \in \mathbb{Z}_8\}, \\ \mathcal{A} &= \{A_i + j : i \in [7] \text{and} j \in \mathbb{Z}_{48}\}. \end{aligned}$$

Then $(X, \mathcal{G}, \mathcal{A})$ is a $\{3\}$-GDD of type 6^8, whose blocks can be arranged in a 16×24 array, with rows and columns are indexed by $\mathbb{Z}_{16}$ and $\mathbb{Z}_{24}$, respectively, as follows: the block $A_i + j$ is placed in cell $(i + j, j)$. This array can be verified to be an $\mathrm{FrGBTD}(3, 6^8)$. □

TABLE II. Existence of special $\mathrm{GBTD}_1(3, m)$.

Authority	m
Corollary 6.5	9, 11, 17, 23, 29, 35, 47, 53, 55
Lemma 7.1	7, 13, 15, 19, 21, 25, 27, 31, 33, 37, 39, 43, 45, 49, 57, 61, 63, 67, 69, 73, 75
Corollary 5.6 with (g, t) in $\{(8, 5), (5, 10), (8, 8), (7, 10)\}$	41, 51, 65, 71
Lemma 7.2 with $n = 5$, $g_1 = 4$	59
Lemma 7.2 with $n = 7$, $g_1, g_2 \in \{0\} \cup \{t : 3 \leq t \leq 7\}$	$\{s : 77 \leq s \leq 95, s \text{ odd}\}$

7. EXISTENCE OF GBTDS AND GBTPs

We apply the recursive constructions in Section 5 using the small designs constructed directly in Section 6 to completely settle the existence of $\mathrm{GBTD}_1(3, m)$ and $\mathrm{GBTP}_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$.

7.1. Existence of $\mathrm{GBTD}_1(3, m)$

Lemma 7.1. *There exists a special* $\mathrm{GBTD}_1(3, 3^r q)$ *for all* $r \geq 0$ *and* $q \in Q$*, where* $Q = \{q : q \equiv 1 \bmod 6$ *is a prime power*$\} \cup \{5, 9, 11, 23\}$*, except when* $(r, q) = (0, 5)$.

Proof. Existence of a special $\mathrm{GBTD}_1(3, q)$ for all $q \in Q \setminus \{5\}$ is provided by Corollaries 6.4 and 6.5. These GBTDs are all 3-$*$colorable with property Π. The lemma then follows by considering these GBTDs as RBIBDs and applying Corollary 5.2. □

Lemma 7.2. *Let* $s \in [2]$ *and suppose there exists a* $\mathrm{TD}(5 + s, n)$*. If* $0 \leq g_i \leq n$, $i \in [s]$ *and that there exists a special* $\mathrm{GBTD}_1(3, m)$ *for all* $m \in \{2n + 1\} \cup \{2g_i + 1 : i \in [s]\}$*, then there exists a special* $\mathrm{GBTD}_1(3, 10n + 1 + 2\sum_{i=1}^{s} g_i)$.

Proof. By Corollary 6.14, there exists an $\mathrm{FrGBTD}(3, 6^t)$ for all $t \in \{5, 6, 7\}$. By Proposition 5.9, there exists an $\mathrm{FrGBTD}(3, (6n)^5(6g_1) \cdots (6g_s))$. Now apply Corollary 5.5 to obtain a special $\mathrm{GBTD}_1(3, 10n + 1 + 2\sum_{i=1}^{s} g_i)$. □

Lemma 7.3. *A special* $\mathrm{GBTD}_1(3, m)$ *exists for odd* $m \geq 7$.

Proof. First, a special $\mathrm{GBTD}_1(3, m)$ can be constructed for odd m, $7 \leq m \leq 95$. Details are provided in Table II.

We then prove the lemma by induction on $m \geq 97$.

Let $E = \{t : t \geq 9\} \setminus \{10, 14, 15, 18, 20, 22, 26, 30, 34, 38, 46, 60\}$. By Theorem 4.8, a $\mathrm{TD}(7, n)$ exists for any $n \in E$. If there exists a special $\mathrm{GBTD}_1(3, m')$ for odd m', $7 \leq m' \leq 2n + 1$, then apply Lemma 7.2 with $3 \leq g_1, g_2 \leq n$ to obtain a special $\mathrm{GBTD}_1(3, m)$ for odd m, $10n + 7 \leq m \leq 14n + 1$.

Hence, take $n = 9$ to obtain a special $\mathrm{GBTD}_1(3, 97)$.

Suppose there exists a $\mathrm{GBTD}_1(3, m')$ for all odd $m' < m$. Then there exists $n \in E$ with $10n + 7 \leq m \leq 14n + 1$. Suppose otherwise. Then there exists $n_1 \in E$ such that $14n_1 + 1 < 10n_2 + 7$ for all $n_2 > n_1$ and $n_2 \in E$. This, together with the fact that $n_1 \geq 9$,

TABLE III. Existence of IGBTP$_1$({2, 3*}, 2m + 9, (m + 4) × (2m + 5); 4 × 5).

Authority	m
Corollary 6.10	$\{s : 10 \leq s \leq 57\} \setminus \{16, 20, 24, 28, 32, 36, 40, 44, 48, 50, 52, 54, 55, 56\}$
Lemma 7.4 with $(n, g) \in \{(10, 0), (11, 0), (12, 0), (13, 0), (11, 10), (11, 11), (14, 0)\}$	40, 44, 48, 52, 54, 55, 56

implies that $n_2 - n_1 > 3$ for all $n_2 \in E$ and $n_2 > n_1$. However, a quick check on E gives a contradiction.

Since $n \in E$ and there exists a special GBTD$_1(3, m')$ for all $m' \leq 2n + 1 < 10n + 7 \leq m$ (induction hypothesis), there exists a special GBTD$_1(3, m)$ and induction is complete. □

Lemma 7.3 shows that a GBTD$_1(3, m)$ exists for all odd $m \neq 3, 5$. Theorem 2.3 (vi) now follows.

7.2. Existence of GBTP$_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$

Lemma 7.4. *Suppose there exists a TD(5, n). Suppose* $0 \leq g \leq n$ *and that there exists an IGBTP*$_1(\{2, 3^*\}, 2m + 9, (m + 4) \times (2m + 5); 9, 4 \times 5)$ *for* $m \in \{n, g\}$. *Then there exists an IGBTP*$_1(\{2, 3^*\}, 2M + 9, (M + 4) \times (2M + 5); 9, 4 \times 5)$*, where* $M = 4n + g$.

Proof. By Lemma 6.11, there exists an FrGBTD$(2, 2^t)$ for all $t \in \{4, 5\}$. By Proposition 5.9, there exists an FrGBTD$(2, (2n)^4(2g))$. Now apply Proposition 5.4 to obtain an IGBTP$_1(\{2, 3^*\}, 2M + 9, (M + 4) \times (2M + 5); 9, 4 \times 5)$. □

Lemma 7.5. *There exists an IGBTP*$_1(\{2, 3^*\}, 2m + 9, (m + 4) \times (2m + 5); 9, 4 \times 5)$ *for any* $m \geq 10$*, except possibly for* $m \in \{16, 20, 24, 28, 32, 36, 46, 50\}$.

Proof. Let $E = \{16, 20, 24, 28, 32, 36, 46, 50\}$. An IGBTP$_1(\{2, 3^*\}, 2m + 9, (m + 4) \times (2m + 5); 9, 4 \times 5)$ can be constructed for $10 \leq m \leq 57$ and $m \notin E \cup \{51\}$. Details are provided in Table III. When $m = 51$, consider a TD(5, 11) and delete four points from a block to form a {4, 5}-GDD of type $10^4 11$. Proposition 5.8 yields an FrGBTD$(2, 20^4 22)$ and hence, Proposition 5.4 yields an IGBTP$_1(\{2, 3^*\}, 2m + 9, (m + 4) \times (2m + 5); 9, 4 \times 5)$ with $m = 51$.

We then prove the lemma by induction on $m \geq 57$. Let $E' = \{4n + g : n \in E, 10 \leq g \leq 13\}$ and assume the lemma is true for $n < m$.

When $m \notin E'$, then write $m = 4n + g$ with $13 \leq n < m$, $n \notin E$ and $g \in \{10, 11, 12, 13\}$. Since a TD$(5, n)$ that exists by Theorem 4.8, applying Lemma 7.4 with the corresponding n and g, we obtain the desired IGBTP.

When $m \in E'$, we have two cases.

- If $m = 77$, the required IGBTP is given by Corollary 6.10.
- Otherwise, apply Lemma 7.4 with (n, g) taking values in {(15, 14), (15, 15), (19, 0), (18, 18), (19, 15), (23, 0), (19, 17), (22, 18), (22, 19), (27, 0), (22, 21), (25, 22), (25,

23), (31, 0), (25, 25), (29, 22), (29, 23), (35, 0), (29, 25), (31, 30), (31, 31), (39, 0), (33, 25), (39, 38), (39, 39), (49, 0), (40, 37), (42, 42), (43, 39), (43, 40), (43, 41)}.

This completes the induction. □

Lemma 7.6. *A* $GBTP_1(\{2, 3^*\}, 2m + 1, m \times (2m - 3))$ *exists for* $m \geq 4$*, except possibly for* $m \in \{12, 13\}$.

Proof. A $GBTP_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$ can be found via computer search for $4 \leq m \leq 11$. The GBTPs are listed in [5].

For $m \in \{20, 24, 28, 32, 36, 40, 50, 54\}$, set $M = m - 5$ and we apply Proposition 5.3 with the $GBTP_1(\{2, 3^*\}, 11, 5 \times 7)$ and the $IGBTP_1(\{2, 3^*\}, 2M + 11, (M + 5) \times (2M + 7); 11, 5 \times 7)$ constructed in Corollary 6.10.

Finally, for $m \geq 14$ and $m \notin \{20, 24, 28, 32, 36, 40, 50, 54\}$, set $M = m - 4$ and apply Proposition 5.3 with $GBTP_1(\{2, 3^*\}, 9, 4 \times 5)$ and the $IGBTP_1(\{2, 3^*\}, 2M + 9, (M + 4) \times (2M + 5); 9, 4 \times 5)$ constructed in Lemma 7.5. □

Lemma 7.6 shows that a $GBTP_1(\{2, 3^*\}, 2m + 1, m \times (2m - 3))$ exists for all $m \geq 4$, except possibly for $m \in \{12, 13\}$. Theorem 2.3 (vii) now follows.

8. CONCLUSION

In this paper, we establish infinite families of ESWCs, whose code lengths are greater than alphabet size and whose relative narrowband noise error-correcting capabilities tend to a positive constant as length grows. The construction method used is combinatorial and reveals interesting interplays with equivalent combinatorial designs called GBTPs. These have enabled us to borrow ideas from combinatorial design theory to construct ESWCs. In return, questions on ESWCs offer new problems to combinatorial design theory. We expect this symbiosis to deepen.

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REFERENCES

[1] R. J. R. Abel, C. J. Colbourn, and J. H. Dinitz, “Mutually orthogonal Latin squares (MOLS),” in The CRC Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz (Editors), CRC Press, Boca Raton, 2nd ed., 2007, pp. 160–193.

[2] E. Biglieri, Coding and modulation for a horrible channel, IEEE Commun Mag 41 (2003), 92–98.

[3] G. T. Bogdanova, A. E. Brouwer, S. N. Kapralov, and P. R. Östergård, Error-correcting codes over an alphabet of four elements, Des Codes Cryptogr 23 (2001), 333–342.

[4] G. T. Bogdanova, V. Zinoviev, and T. J. Todorov, On the construction of q-ary equidistant codes, Problems Inform Transmission 43 (2007), 280–302.

[5] Y. M. Chee, H. M. Kiah, A. C. H. Ling, and C. Wang, Addendum to optimal equitable symbol weight codes for power line communications, (2014), http://www.ifp.illinois.edu/hmkiah/optimalESWC.html .

[6] Y. M. Chee, H. M. Kiah, A. C. H. Ling, and C. Wang, Generalized balanced tournament packings and optimal equitable symbol weight codes for power line communications, (2013), http://arxiv.org/abs/1304.0278.

[7] Y. M. Chee, H. M. Kiah, A. C. H. Ling, and C. M. Wang, Optimal equitable symbol weight codes for power line communications. In Proceedings of the IEEE International Symposium on Information Theory, IEEE, Cambridge, MA, 2012, pp. 671–675.

[8] Y. M. Chee, H. M. Kiah, P. Purkayastha, and C. Wang, Importance of symbol equity in coded modulation for power line communications, IEEE Trans Commun 61 (2013), 4381–4390.

[9] Y. M. Chee, H. M. Kiah, and C. Wang, Generalized balanced tournament designs with block size four, Electron J Combin 20 (2013), P51 (electronic).

[10] C. J. Colbourn, T. Kløve, and A. C. H. Ling, Permutation arrays for powerline communication and mutually orthogonal Latin squares, IEEE Trans Inform Theory 50 (2004), 1289–1291.

[11] P. Dai, J. Wang, and J. Yin, Two series of equitable symbol weight codes meeting the Plotkin bound, Des Codes Cryptogr (2013).

[12] C. Ding and J. Yin, Combinatorial constructions of optimal constant-composition codes, IEEE Trans Inform Theory 51 (2005), 3671–3674.

[13] C. Ding and J. Yin, A construction of optimal constant composition codes, Des Codes Cryptogr 40 (2006), 157–165.

[14] P. J. Dukes, Coding with injections, Des Codes Cryptogr 65 (2012), 213–222.

[15] D. Dzung, I. Berganza, and A. Sendin, Evolution of powerline communications for smart distribution: from ripple control to OFDM. In Proceedings of the IEEE International Symposium on Power Line Communications and its Applications, IEEE, Udine, 2011, pp. 474–478.

[16] P. Frankl and M. Deza, On the maximum number of permutations with given maximal or minimal distance, J Comb Theory Ser A 22 (1977), 352–360.

[17] A. Haidine, B. Adebisi, A. Treytl, H. Pille, B. Honary, and A. Portnoy, High-speed narrow-band PLC in smart grid landscape—state-of-the-art. In Proceedings of the IEEE International Symposium on Power Line Communications and its Applications, IEEE, Udine, 2011, pp. 468–473.

[18] S. Huczynska, Equidistant frequency permutation arrays and related constant composition codes, Des Codes Cryptogr 54 (2010), 109–120.

[19] E. Lamken and S. Vanstone, Balanced tournament designs and related topics, Discrete Math 77 (1989), 159–176.

[20] E. R. Lamken, Generalized balanced tournament designs, Trans Amer Math Soc 318 (1990), 473–490.

[21] E. R. Lamken, Existence results for generalized balanced tournament designs with block size 3, Des Codes Cryptogr 3 (1992), 33–61.

[22] E. R. Lamken, Constructions for generalized balanced tournament designs, Discrete Math 131 (1994), 127–151.

[23] E. R. Lamken, The existence of doubly resolvable $(v, 3, 2)$-BIBDs, J Comb Theory Ser A 72 (1995), 50–76.

[24] E. R. Lamken, The existence of partitioned generalized balanced tournament designs with block size 3, Des Codes Cryptogr 11 (1997), 37–71.

[25] J. Liu, B. Zhao, L. Geng, Z. Yuan, and Y. Wang, Communication performance of broadband PLC technologies for smart grid. In Proceedings of the IEEE International Symposium on Power Line Communications and its Applications, IEEE, Udine, 2011, pp. 491–496.

[26] N. Pavlidou, A. J. H. Vinck, J. Yazdani, and B. Honary, Power line communications: state of the art and future trends, IEEE Commun Mag 41 (2003), 34–40.

[27] P. Schellenberg, G. Van Rees, and S. Vanstone, The existence of balanced tournament designs, Ars Combinatoria 3 (1977), 303–318.

[28] M. Schwartz, Carrier-wave telephony over power lines: Early history [history of communications], IEEE Commun Mag 47 (2009), 14–18.

[29] N. V. Semakov and V. A. Zinoviev, Equidistant q-ary codes with maximal distance and resolvable balanced incomplete block designs, Problemy Peredači Informacii 4 (1968), 3–10.

[30] A. J. H. Vinck, Coded modulation for power line communications, AEÜ - Int J Electron Commun 54 (2000), 45–49.

[31] R. M. Wilson, Cyclotomy and difference families in elementary abelian groups, J Number Theory 4 (1972), 17–47.

[32] R. M. Wilson, An existence theory for pairwise balanced designs. I. Composition theorems and morphisms, J Combin Theory Ser A 13 (1972), 220–245.

[33] J. Yin, J. Yan, and C. Wang, Generalized balanced tournament designs and related codes, Des Codes Cryptogr 46 (2008), 211–230.

[34] W. Zhang and L. Yang, SC-FDMA for uplink smart meter transmission over low voltage power lines. In Proceedings of the IEEE International Symposium on Power Line Communications and its Applications, IEEE, Udine, 2011, pp. 497–502.