RELIABLE COMMUNICATIONS OVER POWER LINES THROUGH CODED MODULATION SCHEMES

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> Japanese mountain potatoes known as taros are rough and dirty when harvested, but when they are placed in a basin of running water together and rolled against each other, the skin peels away, leaving the potatoes clean and ready for cooking. Similarly, the only way for us to hone and polish our character is through our interactions with others.
>
> — Daisaku Ikeda

This dissertation is the fruit of the countless maddening arguments, lively discussions and meaningful interactions with many individuals, whom I had the great fortune to come in contact with.

Firstly, I am greatly indebted to my supervisor Yeow Meng Chee. Under his strict guidance and insightful advice, I have not only grown academically, but also gained strength in character. His thirst for precision and perfection have pushed me to make vast improvements in my technical writing, presentation skills and critical thinking. I am also deeply appreciative of the many opportunities that he has presented during this period. From technical conferences to informal meetings with academic giants, I have benefitted immensely from each experience.

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To my little girl, Ern Yee, if you ever read this dissertation, Daddy did not earn his doctoral degree through his efforts alone. He had the assistance of many comrades and mentors and Daddy hopes that you will meet similar friends in your life. :)

CREDITS

Most of the material in this thesis has appeared in conference proceedings or journal articles. The material in Chapter 2 is joint work with Yeow Meng Chee, Punarbasu Purkayasatha and Chengmin Wang and appear in [15,16]. Parts of Chapter 3 and Chapter 5 are joint work with Yeow Meng Chee and Punarbasu Purkayasatha and appear in [13] and [14] respectively. The construction in Chapter 4 is joint work with Yeow Meng Chee, Alan Ling and Chengmin Wang and has been presented in part at [12]. I am deeply indebted to Yeow Meng, Alan, Punarbasu and Chengmin for the most fruitful collaborations.

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ABSTRACT

Single-tone frequency shift keying (FSK) modulation with permutation codes has been found to be useful in establishing reliable communications over power lines. This dissertation is devoted to the study of generalizations to this coded modulation scheme.

In the first part of this dissertation, we extend this coded modulation scheme based on permutation codes to general block codes and establish the conditions for correct decoding over a power line communications (PLC) channel. In the process, we introduce a new parameter to measure the performance against narrowband noise. As a result, we define a new class of codes, namely, equitable symbol weight codes, which are optimal with respect to this measure. Simulation results validating the relevance of this new parameter are given.

Hence, we investigate the possible sizes of equitable symbol weight codes. Using an Elias-type bound, we determine the asymptotic size of equitable symbol weight codes under certain conditions. Using both classical coding and computational methods, we also tabulate the possible lower and upper bounds of an optimal equitable code for certain parameters.

However, the exact size of optimal equitable symbol codes is only known in a limited number of instances. Generalizing a class of combinatorial objects called generalized balanced tournament designs introduced by Lamken and Vanstone (1989), we define a class of combinatorial objects called generalized balanced tournament packings and establish a connection to equitable symbol weight codes. As a result, we construct new infinite families of optimal equitable symbol weight codes whose narrowband noise error-correcting capability to code length ratios are bounded away for zero.

Unfortunately, this general coded modulation scheme usually requires the use of a codebook and does not have an efficient decoding algorithm. Hence, we propose the use of multitone FSK and codes defined over binary matrices in the final part of this dissertation.

Adopting techniques from classical concatenation, we construct infinite families of efficiently decodable matrix codes with rates and relative distances bounded away from zero. Simulation results demonstrating the merits of multitone FSK modulation scheme are also given.

CONTENTS

1. INTRODUCTION

Power line communications (PLC) is a technology that enables the transmission of data over electric power lines. It was started in the 1910's for voice communication [69], and used in the 1950's in the form of ripple control for load and tariff management in power distribution. With the emergence of the Internet in the 1990's, research into broadband PLC gathered pace as a promising technology for Internet access and local area networking, since the electrical grid infrastructure provides "last mile" connectivity to premises and capillarity within premises. Recently, there has been a renewed interest in high-speed narrowband PLC due to applications in sustainable energy strategies, specifically in smart grids (see [30, 40, 53, 85]).

However, as power lines are not originally designed for information transmission, they present a difficult communications environment with the presence of various types of noise that include additive white Gaussian noise, fading, permanent narrowband noise, and impulse noise. Establishing reliable communication thus remains a challenging problem and a variety of coding and modulation strategies have been proposed [4, 5, 65, 80].

In particular, this dissertation studies generalizations to a coded modulation scheme proposed by Vinck [80] that utilizes single-tone (multiple) frequency shift keying (FSK) and permutation codes as ingredients.

1.1 Vinck's Coded Modulation Scheme and its Generalizations

As there are some inconsistent uses of various notions of FSK in the literature, we clarify our terminology before going further. In general FSK systems, each symbol is signaled by an element or a combination of elements from an alphabet of orthogonal sinusoidal waveforms (*tones*) tuned to different specific frequencies. FSK schemes can be either *single-tone* or *multitone*.

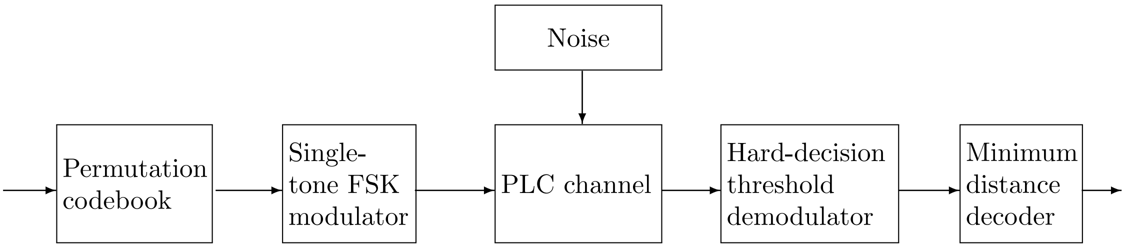
(i) *Single-tone* FSK is an FSK scheme where each symbol is signaled by a single tone.

(ii) *Multitone* FSK is an FSK scheme where each symbol is signaled by a combination of (one or more) tones.

Vinck's scheme is based on single-tone FSK, where channel state information is assumed to be unknown to the receiver and a hard-decision threshold demodulator is used. In this scheme, narrowband noise results in the unwanted appearance of a certain frequency (or information symbol) over a prolonged period, while impulse noise results in the unwanted appearance of all frequencies at certain time instances (see Figure 1.1). To determine the sequence of frequencies for transmission, Vinck proposed the use of a codebook that consists only of permutation words. Using a minimum distance decoder, Vinck showed that a permutation code of minimum (Hamming) distance d with single-tone FSK modulation is able to correct up to $d-1$ errors due to narrowband and impulse noise.

Many generalizations has since been made to Vinck's coded modulation scheme. Constant composition codes (see [10,18,21,24–27,38,42,56]), frequency permutation arrays (see [42,43]), and injection codes (see [28]) have been considered as possible replacements for permutation codes in PLC. Versfeld *et al.* [78,79] later introduced the notion of 'same-symbol weight' (henceforth, termed as *symbol weight*) of a code as a measure of the capability of a code in dealing with narrowband noise. They also showed empirically that low symbol weight cosets of Reed-Solomon codes outperform normal Reed-Solomon codes in the presence of narrowband noise and additive white Gaussian noise.

Unfortunately, symbol weight alone is not sufficient to capture the performance of a code in dealing with permanent narrowband noise and this motivates our study of coded modulation schemes that use general block codes as ingredients. Extending Vinck's analysis for permutation codes to general block codes, we introduce an additional new parameter that more precisely captures a code's performance against permanent narrowband noise. This parameter in turn motivates the study of a class of codes, called *equitable symbol weight codes*. Not surprisingly, the class of equitable symbol weight codes includes the classes of permutation codes, injection codes and frequency permutation arrays.



A codeword $u = (3, 4, 1, 2)$ is transmitted as:

time instances

frequencies	1	2	3	4
1			*	
2				*
3	*			
4		*		

Narrowband noise at frequency (or information symbol) 2 results in the following demodulator output:

time instances

frequencies	1	2	3	4
1			*	
2	*	*	*	*
3	*			
4		*		

Impulse noise at time instance 2 results in the following demodulator output:

time instances

frequencies	1	2	3	4
1		*	*	
2		*		*
3	*	*		
4		*		

Fig. 1.1: Vinck's coded modulation scheme and the noise arising from a PLC channel

Using classical coding and computational methods, we provide estimates on both asymptotic and finite sizes of equitable symbol weight codes. In particular, we determine the asymptotic sizes of equitable symbol weight codes and also provide tables estimating the optimal sizes for certain parameters. Unfortunately, the exact size of optimal equitable symbol codes is only known in a limited number of instances. Interestingly, a class of combinatorial objects called *generalized balanced tournament designs* (introduced by Lamken and Vanstone [46]) can be used to construct optimal equitable symbol weight codes. Generalizing their definitions, we then construct new infinite families of equitable symbol weight codes.

However, while this more general coded modulation scheme using block codes gives better flexibility and performance, it involves the use of codebooks that require large storage and do not have efficient decoding algorithms. Coded modulation schemes with low decoding complexity are possible if the size of the code is small enough so that exhaustive search can be performed, or if the codes have sufficient structure such that efficient decoding algorithms can be implemented. Some families of codes with low decoding complexity are given by:

(i) distance preserving maps from the Hamming space to the permutation space (see [8, 9, 19, 52, 75]),

(ii) permutation trellis codes [31],

(iii) permutation group codes (see [3]), and

(iv) cosets of Reed-Solomon codes with low symbol weight (see [78, 79]).

However, the lengths of the families of codes are constrained by the number of frequencies, which is at least as large the alphabet size of the code. In addition, the first three families of codes do not simultaneously achieve positive relative distance and positive rate, with increasing code length. This provides the impetus to determine code families that can be used to combat permanent narrowband noise in PLC and with the following (simultaneous) properties:

(i) positive relative distance,

(ii) positive rate,

(iii) have efficient decoding algorithms, and

(iv) without restriction that the length of the codes is at most the size of the alphabet.

This prompts the next modification to Vinck's coded modulation scheme. Instead of a single-tone FSK, we propose the use of multitone FSK modulation with binary matrix codes. With classical concatenation techniques, we establish an infinite family of *efficiently decodable* codes whose rate and relative distance are bounded away from zero, and uses a logarithmic number of frequencies in the length of the code.

1.2 Structure and Contributions of Thesis

This dissertation is organized as follows. Contributions of the thesis are indicated in *italics*.

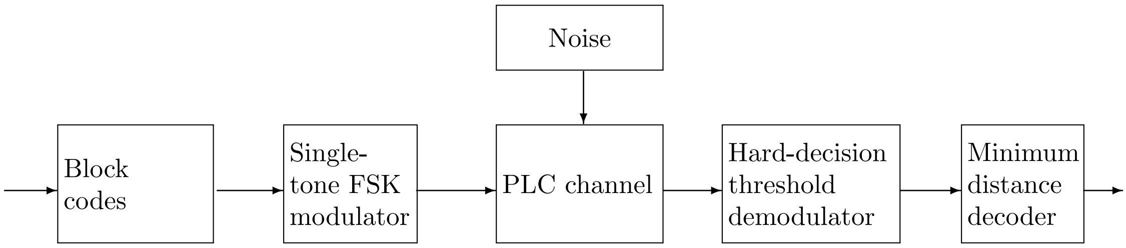
Chapter 2 introduces a generalization of Vinck's coded modulation scheme and describes the effects of noise over a PLC channel. We extend Vinck's analysis to general codes and *derive the conditions where correct decoding occurs*. A *new parameter is introduced to measure the performance against narrowband noise* and this parameter is related to symbol equity, the uniformity of frequencies of symbols in each codeword. Codes designed taking into account this new parameter, or equitable symbol weight codes, are *shown to perform better than general ones*.

This leads to the investigation of the possible sizes of equitable symbol weight codes in Chapter 3. Using an Elias-type bound, we *determine the asymptotic sizes of optimal equitable symbol weight codes under certain conditions*. Using both classical coding and computation methods, we provide *tables listing the possible upper and lower bounds for the size of an optimal equitable code.*

Chapter 4 examines a construction of optimal equitable symbol weight codes. Specifically, we *define a class of combinatorial objects called generalized balanced tournament packings* and employ tools from combinatorial design theory to *determine the existence of such objects*. As a result, we *construct new infinite families of optimal equitable symbol weight codes*.

The next generalization to Vinck's scheme is given in chapter 5. We *outline a coded modulation scheme* that uses codes defined over binary matrices and multitone FSK as ingredients. Using classical concatenation techniques, we *establish infinite families of efficiently*

Using general block codes for transmission.



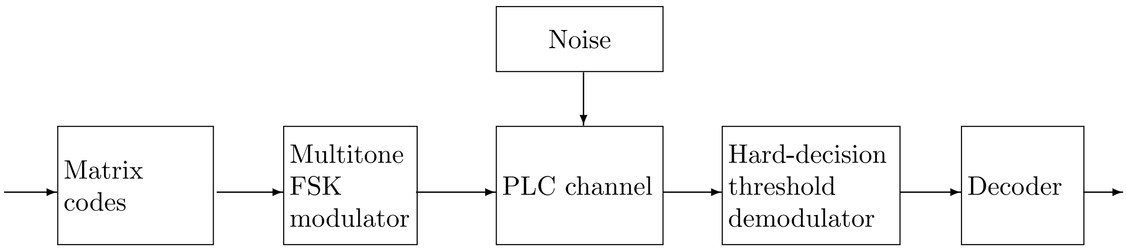
Codeword $(3, 4, 1, 2, 1, 2)$ is transmitted as:

time instances

frequencies	1	2	3	4	5	6
1			*		*	
2				*		*
3	*					
4		*				

.

Transmitting a combination of frequencies at each time instance.



A binary matrix $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$ is transmitted as:

time instances

frequencies	1	2	3	4
1			*	*
2	*	*		
3	*		*	
4		*		*

.

Fig. 1.2: Generalizations to Vinck's coded modulation scheme

decodable codes whose rate and relative distance are bounded away from zero. Simulation results demonstrating the merits of multitone FSK modulation scheme are also given.

1.3 Notation and Coding Preliminaries

We introduce notation that is used throughout this dissertation. We denote the set of integers and positive integers by $\mathbb{Z}$ and $\mathbb{Z}_{>0}$ respectively.

For integers $m, n \in \mathbb{Z}$ with $m \leq n$, the set $\{m, m+1, \ldots, n\}$ is denoted by $[m, n]$. For a positive integer n, the set $[1, n]$ is written as $[n]$. Let $\mathbb{Z}_n$ denote the ring of integers modulo n and $\mathbb{F}_q$ denote the finite field with q elements, where q is a prime power.

The cardinality of a finite set X is given by $|X|$. For $k \leq |X|$, denote the collection of all k-subsets by $\binom{X}{k}$ and denote the collection of all subsets of X, or the *power set* of X, by 2^X.

Let T be an index set and Σ be a set of *symbols*. We denote *a sequence or a vector with index set* T by $(u_t : t \in T, u_t \in \Sigma)$. In contrast, we denote a *multiset* by angled brackets, that is, $\langle u_t : t \in T\rangle$. For the latter, when more convenient, the exponential notation $\langle u_1^{t_1} u_2^{t_2} \cdots u_n^{t_n}\rangle$ is used to describe a multiset with exactly t_i elements u_i, $i \in [n]$.

When $|\Sigma| = q$, a *q-ary code* $\mathcal{C}$ of *length* n over the *alphabet* Σ is a subset of Σ^n. Elements of $\mathcal{C}$ are called *codewords*. The *size* of $\mathcal{C}$ is the number of codewords in $\mathcal{C}$. For $i \in [n]$, the ith *coordinate* of a codeword u is denoted by u_i.

Given two codewords $\mathsf{u}, \mathsf{v} \in \Sigma^n$, the *Hamming distance* between u and v, denoted by $d(\mathsf{u}, \mathsf{v})$, is given by the number of differing coordinates, or

$$d(\mathsf{u}, \mathsf{v}) := |\{i \in [n] : \mathsf{u}_i \neq \mathsf{v}_i\}|. \tag{1.1}$$

The (minimum) distance of a code $\mathcal{C}$ is then given by $\min\{d(\mathsf{u}, \mathsf{v}) : \mathsf{u}, \mathsf{v} \in \mathcal{C}, \mathsf{u} \neq \mathsf{v}\}$. An $(n, d)_q$-code then denotes a q-ary code of length n and distance d and $A_q(n, d)$ is defined to be the maximum size of an $(n, d)_q$-code. An $(n, d)_q$-code of size $A_q(n, d)$ is said to be *optimal*.

A central problem in coding theory is to determine $A_q(n, d)$ given parameters n, d and

q. We conclude this section with some classical bounds for $A_q(n,d)$ (see [57] for example).

Theorem 1.3.1 (Singleton Bound)**.** *Let* $1 \leq d \leq n$*. Then*

$$A_q(n,d) \leq q^{n-d+1} \ .$$

Theorem 1.3.2 (Gilbert-Varshamov (GV) and Hamming Bound)**.** *Let* $1 \leq d \leq n$*. Then*

$$\frac{q^n}{V_q(n,d-1)} \leq A_q(n,d) \leq \frac{q^n}{V_q(n,\lfloor (d-1)/2 \rfloor)} \ ,$$

where

$$V_q(n,d) = \sum_{i=0}^{d} \binom{n}{i} (q-1)^i.$$

Theorem 1.3.3 (Plotkin Bound)**.** *Suppose* $qd - (q-1)n > 0$*. Then*

$$A_q(n,d) \leq \frac{qd}{qd-(q-1)n} \ .$$

2. IMPORTANCE OF SYMBOL EQUITY IN A CODED MODULATION SCHEME

This chapter extends the analysis of Vinck's coded modulation scheme based on permutation codes (see [80],[2, Subsection 5.2.4]) to general block codes. In particular, we outline a noise model for PLC and derive the criterion under which correct decoding can be performed. In the process, we define a new parameter that captures how well a code can perform under narrowband noise and show that equitable symbol weight codes are optimal with respect to this new parameter. At the end of the chapter, we present some simulation results to compare the performance of equitable symbol weight codes with other block codes previously studied in the literature. This chapter has been presented in part at the IEEE International Symposium on Information Theory, 2012 [15] and appears in Chee *et al.* [16].

2.1 Preliminaries

Recall that Σ is a set of q *symbols* and a *q-ary code* $\mathcal{C}$ of *length* n over the *alphabet* Σ is a subset of Σ^n. Elements of $\mathcal{C}$ are called *codewords*. For $i \in [n]$, the ith *coordinate* of a codeword u is denoted by u_i.

2.1.1 Symbol Weight

Let $\mathsf{u} \in \Sigma^n$. For $\sigma \in \Sigma$, let $w_\sigma(\mathsf{u})$ be the number of times the symbol σ appears among the coordinates of u, that is,

$$w_\sigma(\mathsf{u}) := |\{i \in [n] : \mathsf{u}_i = \sigma\}|.$$

The *symbol weight* of u is the maximum frequency of any symbol in u. That is,

$$\mathrm{swt}(\mathsf{u}) := \max_{\sigma \in \Sigma} w_\sigma(\mathsf{u}).$$

A code has *bounded symbol weight* r if the maximum symbol weight of all its codewords is r. A code $\mathcal{C}$ has *constant symbol weight* r if all its codewords have symbol weight exactly r. For any $\mathsf{u} \in \Sigma^n$, observe that $\mathrm{swt}(\mathsf{u}) \geq \lceil n/q \rceil$. A code has *minimum symbol weight* if it has constant symbol weight $\lceil n/q \rceil$.

A codeword $\mathsf{u} \in \Sigma^n$ is said to have *equitable symbol weight* if $w_\sigma(\mathsf{u}) \in \{\lfloor n/q \rfloor, \lceil n/q \rceil\}$ for all $\sigma \in \Sigma$. In other words, if $r = \lceil n/q \rceil$, then every symbol appears r or $r-1$ times in u. If all the codewords of $\mathcal{C}$ have equitable symbol weight, then the code $\mathcal{C}$ is called an *equitable symbol weight code*. Every equitable symbol weight code is hence a minimum symbol weight code.

Recall that a q-ary code of length n and distance d is called an $(n, d)_q$-code. Similarly, a q-ary code of length n having bounded symbol weight r and distance d is called an $(n, d, r)_q$-*symbol weight code*, while a q-ary equitable symbol weight code of length n and distance d is called an $(n, d)_q$-*equitable symbol weight code*.

2.1.2 Composition and Partition

The *composition* of $\mathsf{u} \in \Sigma^n$ is the sequence $(w_\sigma(\mathsf{u}) : \sigma \in \Sigma)$, while the *partition* of u is the multiset $\langle w_\sigma(\mathsf{u}) : \sigma \in \Sigma \rangle$. Fix a multiset of nonnegative numbers $\langle c_\sigma : \sigma \in \Sigma \rangle$ such that $\sum_{\sigma \in \Sigma} c_\sigma = n$. A code $\mathcal{C}$ is a *constant composition code with composition* $(c_\sigma : \sigma \in \Sigma)$ if all words in $\mathcal{C}$ have composition $(c_\sigma : \sigma \in \Sigma)$. Similarly, a code $\mathcal{C}$ is a *constant partition code with partition* $\langle c_\sigma : \sigma \in \Sigma \rangle$ if all words in $\mathcal{C}$ have partition $\langle c_\sigma : \sigma \in \Sigma \rangle$.

Clearly, a constant composition code is necessarily a constant partition code. The following example demonstrates that the converse is not true.

Example 2.1.1. The code $\{(1, 2, 3), (2, 3, 4), (3, 4, 1), (4, 1, 2)\}$ is a constant partition code with partition $\langle 1^3 0 \rangle$, since in each code word three symbols appear once each, and one symbol does not appear. However, the words have different compositions.

Furthermore, we show that an equitable symbol weight code is necessarily a constant partition code with minimum symbol weight. This follows from the next lemma that states that for any codeword $\mathsf{u} \in \Sigma^n$ having equitable symbol weight, the number of symbols occurring with frequency $\lceil n/q \rceil$ in u is uniquely determined. Hence, the frequencies of symbols in an equitable symbol weight codeword are as uniformly distributed as possible and the partition of the codeword is fixed.

Lemma 2.1.2. *Let $\mathsf{u} \in \Sigma^n$, $r = \lceil n/q \rceil$, and $t = qr - n$. If u has equitable symbol weight, then u has partition $\langle r^{q-t}(r-1)^t \rangle$*

Proof. Let $x = |\{\sigma \in \Sigma : w_\sigma(\mathsf{u}) = r\}|$ and $y = |\{\sigma \in \Sigma : w_\sigma(\mathsf{u}) = r-1\}|$. Then the following equations hold:

$$x + y = q,$$
$$rx + (r-1)y = n.$$

Solving this set of equations gives the lemma. ∎

Using the above notation, we observe that equitable symbol weight codes are generalizations of certain classes of codes which have been studied in PLC applications. For example, if $q|n$, then an equitable symbol weight code has constant partition $\langle (n/q)^q \rangle$, which is known as a *frequency permutation array* (FPA). If $n \leq q$ then an equitable symbol weight code has constant partition $\langle 1^n 0^{q-n} \rangle$, which is called an *injection code*. Finally, if $n = q$, then all definitions coincide to give the definition of a *permutation code*. We exhibit the (inclusion) relationships between these classes of codes in Figure 2.1.

2.2 Correcting Noise with Single-tone FSK Modulation

In coded modulation for power line communications [80], a q-ary code of length n is used, whose symbols are modulated using q-ary single-tone FSK. The receiver demodulates the received signal using an envelope detector to obtain an output, which is then decoded by a decoder.

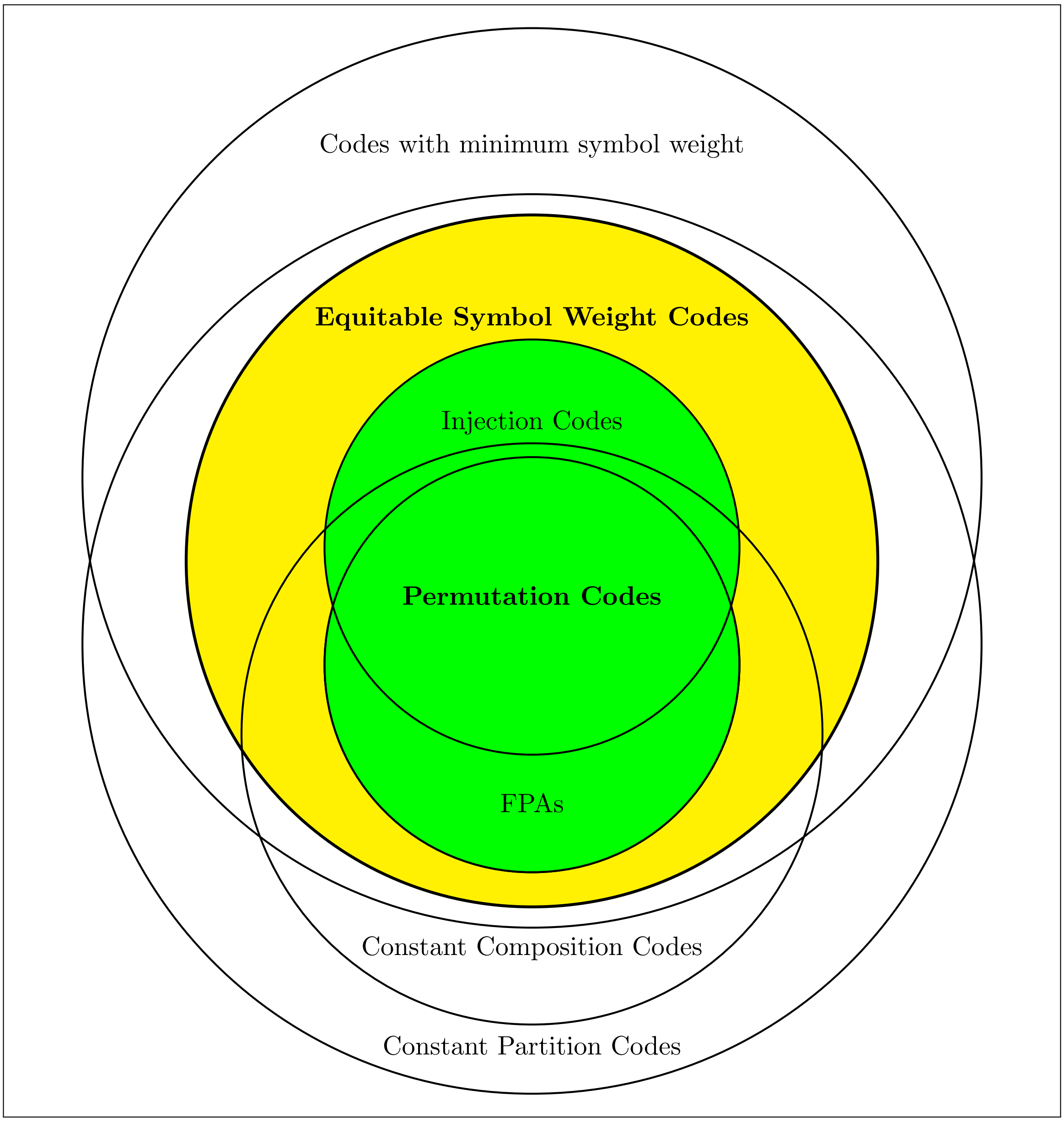


Fig. 2.1: Generalizations of Permutation Codes

Four detector/decoder combinations are possible: classical, modified classical, hard-decision threshold, and soft-decision threshold (see [2] for details). A soft-decision threshold detector/decoder requires exact channel state knowledge and is therefore not useful if we do not have channel state knowledge. Henceforth, we consider the hard-decision threshold detector/decoder, since it contains more information about the received signal compared to the classical and modified classical ones. We remark that in the case of the hard-decision threshold detector/decoder, the decoder used is a *minimum distance decoder*.

Let $\mathcal{C}$ be an $(n, d)_q$-code over alphabet Σ, and let $\mathsf{u} = (\mathsf{u}_1, \ldots, \mathsf{u}_n)$ be a codeword transmitted over the PLC channel where the symbol u_i is transmitted at discrete time instance i for $i \in [n]$. The received signal (which may contain errors caused by noise) is demodulated to give an output $\mathsf{v} = (\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n)$ in which each v_i is a subset of Σ. The errors that arise from the different types of noise in the channel (see [2, pp. 222–223]) have the following effects on the output of the detector.

1. Narrowband noise at a particular frequency introduces a symbol at several consecutive discrete time instances of the transmitted signals. The narrowband noise affects only a part of the transmission that occurs at discrete time instances from $i = 1$ to $i = n$. Hence, narrowband noise of duration l affects up to l consecutive positions in the discrete time instances from $i = 1$ to $i = n$, depending on whether the noise started prior to or during the current transmission. Narrowband noise may be present simultaneously at multiple frequencies corresponding to different symbols.

   Let $1 \le e \le q$ and $l \in \mathbb{Z}_{>0}$. If *e narrowband noise errors of duration l* occur, then there is a set $\Gamma \in \binom{\Sigma}{e}$ of e symbols and e corresponding starting instances $\{i_\sigma \le n : \sigma \in \Gamma\}$ such that for $\sigma \in \Gamma$,

   $$\sigma \in \mathsf{v}_i \text{ for } \max\{1, i_\sigma\} \le i \le \min\{i_\sigma + l - 1, n\}.$$

   We write the condition $\max\{1, i_\sigma\} \le i \le \min\{i_\sigma + l - 1, n\}$ compactly as the condition $i \in [i_\sigma, i_\sigma + l - 1] \cap [n]$.

2. A signal fading error results in the absence of a symbol in the received signal. Let

$1 \le e \le q$. If e *signal fading errors* occur, then there are e symbols, none of which appears in any v_i, that is, $(\bigcup_{i=1}^{n} \mathsf{v}_i) \cap \Gamma = \varnothing$ for some $\Gamma \in \binom{\Sigma}{e}$.

3. Impulse noise results in the entire set of symbols being received at a certain discrete time instance. Let $1 \le e \le n$. If e *impulse noise errors* occur, then there is a set $\Pi \in \binom{[n]}{e}$ of e positions such that $\mathsf{v}_i = \Sigma$ for all $i \in \Pi$.

4. An insertion error results in an unwanted symbol in the received signal. Let $1 \le e \le n(q-1)$. If e *insertion errors* occur, then there is a set $\Omega \in \binom{[n]\times\Sigma}{e}$ such that for each $(i, \sigma) \in \Omega$, v_i contains σ and $\sigma \ne \mathsf{u}_i$.

5. A deletion error results in the absence of a transmitted symbol in the received signal. Let $1 \le e \le n$. If e *deletion errors* occur, then there is a set $\Pi \in \binom{[n]}{e}$ of e positions such that v_i does not contain u_i for all $i \in \Pi$.

Both insertion and deletion errors are due to *background noise*. This definition of insertion and deletion error is different from the errors that arise in an "insertion-deletion channel" (see [50]).

Example 2.2.1. Suppose $\mathsf{u} = (1, 2, 3, 4)$.

1. Narrowband noise can start prior to or during the transmission of u. Narrowband noise error of duration four at symbol 1 starting at discrete time instance $i = -1$ results in detector output $\mathsf{v} = (\{1\}, \{1, 2\}, \{3\}, \{4\})$, while the same narrowband noise error starting at discrete time instance $i = 3$ results in detector output $\mathsf{v} = (\{1\}, \{2\}, \{1, 3\}, \{1, 4\})$.

2. The same detector output can arise from different combinations of error types. A signal fading error of symbol 1 and a deletion error at position 1 would each result in the same detector output of $\mathsf{v} = (\varnothing, \{2\}, \{3\}, \{4\})$.

For a codeword $\mathsf{u} \in \Sigma^n$ and an output $\mathsf{v} \in \left(2^{\Sigma}\right)^n$, define

$$d(\mathsf{u}, \mathsf{v}) := |\{i : \mathsf{u}_i \notin \mathsf{v}_i\}|.$$

Note that in this context, we identify $\mathsf{c} \in \Sigma^n$ with $(\{\mathsf{c}_1\}, \{\mathsf{c}_2\}, \ldots, \{\mathsf{c}_n\}) \in \left(2^{\Sigma}\right)^n$, so that $d(\mathsf{u}, \mathsf{c})$ coincides with the definition of Hamming distance given by (1.1). We also extend the definition of distance so that for $\mathcal{C} \subseteq \Sigma^n$, we have $d(\mathcal{C}, \mathsf{v}) = \min_{\mathsf{u} \in \mathcal{C}} d(\mathsf{u}, \mathsf{v})$. Given $\mathsf{v} \in (2^{\Sigma})^n$, a minimum distance decoder (for a code $\mathcal{C}$) outputs a codeword $\mathsf{u} \in \mathcal{C}$ which has the smallest distance to v, that is, a minimum distance decoder returns an element of

$$\arg\min_{\mathsf{u} \in \mathcal{C}} d(\mathsf{u}, \mathsf{v}) := \{\mathsf{u} \in \mathcal{C} : d(\mathsf{u}, \mathsf{v}) \leq d(\mathsf{u}', \mathsf{v}) \ \forall \mathsf{u}' \in \mathcal{C}\}. \tag{2.1}$$

In the following, we study the conditions under which a minimum distance decoder outputs the correct codeword, that is, when $\arg\min_{\mathsf{u} \in \mathcal{C}} d(\mathsf{u}, \mathsf{v}) = \{\mathsf{u}\}$. This is equivalent to saying that the decoder correctly outputs u if and only if $d(\mathcal{C} \setminus \{\mathsf{u}\}, \mathsf{v}) > d(\mathsf{u}, \mathsf{v})$.

Let $d' = d(\mathcal{C} \setminus \{\mathsf{u}\}, \mathsf{u})$. Since $\mathcal{C}$ has distance d, we have $d' \geq d$. Observe the following:

- Let $1 \leq e \leq n$. If e impulse noise errors occur, then in e coordinates all the symbols occur. Therefore, those e coordinates do not contribute to the distance between v and any codeword. Hence, we get

$$d(\mathsf{u}, \mathsf{v}) = 0 \quad \text{and} \quad d(\mathcal{C} \setminus \{\mathsf{u}\}, \mathsf{v}) \geq d' - e.$$

- Let $1 \leq e \leq n(q-1)$. If e insertion errors occur, then there are at most e coordinates which do not contribute to the distance between v and some codeword in the code. Hence, we get

$$d(\mathsf{u}, \mathsf{v}) = 0 \quad \text{and} \quad d(\mathcal{C} \setminus \{\mathsf{u}\}, \mathsf{v}) \geq d' - e.$$

- Let $1 \leq e \leq n$. If e deletion errors occur, then there are exactly e coordinates where the transmitted codeword u differs from v. Any other codeword still differs from v in at least d' coordinates. Therefore, we get

$$d(\mathsf{u}, \mathsf{v}) = e \quad \text{and} \quad d(\mathcal{C} \setminus \{\mathsf{u}\}, \mathsf{v}) \geq d'.$$

For errors due to narrowband noise we introduce a quantity that measures how many coordinates of any codeword in the code are affected by the noise. Specifically, a narrowband noise at the frequency corresponding to symbol σ can affect up to n coordinates in a codeword, depending on the number of times the symbol σ appears in the codeword. If narrowband noise is present in the set of symbols $\Gamma \subseteq \Sigma$, then the maximum number of entries of any codeword c that can be affected by the noise is $\sum_{\sigma\in\Gamma} w_\sigma(\mathsf{c})$. Therefore, we define

$$E(e;\mathcal{C}) := \max_{\mathsf{c}\in\mathcal{C},\, \Gamma\in\binom{\Sigma}{e}} \sum_{\sigma\in\Gamma} w_\sigma(\mathsf{c}). \tag{2.2}$$

The expression $E(e;\mathcal{C})$ measures the maximum number of coordinates, over all codewords in $\mathcal{C}$ that are affected by e narrowband noise. Equation (2.2) assumes that the duration of the narrowband noise is at least n and that it is present in all the coordinates of the codeword transmitted. In general, a narrowband noise error of duration l at symbol σ may not be present for the full duration of the codeword. In Subsection 2.2.1 we show that it suffices to consider narrowband noise of duration n since it measures the maximum effect of narrowband noise on the codewords.

Recall that $d' = d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{u})$. From the definition of $E(e;\mathcal{C})$, it is clear that the distance between any codeword, other than the transmitted codeword u, and the output v decreases by $E(e;\mathcal{C})$. Similarly, in the presence of a fading error the distance between u and v increases by at most $E(e;\mathcal{C})$. Therefore we get the two inequalities below.

- Let $1 \le e \le q$. If e narrowband noise errors occur, then

$$d(\mathsf{u},\mathsf{v}) = 0 \quad \text{and} \quad d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \ge d' - E(e;\mathcal{C}).$$

- Let $1 \le e \le q$. If e signal fading errors occur, then

$$d(\mathsf{u},\mathsf{v}) \le E(e;\mathcal{C}) \quad \text{and} \quad d(\mathcal{C}\setminus\{\mathsf{u}\},\mathsf{v}) \ge d'.$$

Hence, if we denote by e_{N}, e_{F}, e_{IMP}, e_{INS}, and e_{DEL} the number of errors due to narrowband noise, signal fading, impulse noise, insertion, and deletion, respectively, we have

$$d(\mathsf{u},\mathsf{v}) \leq e_{\mathsf{DEL}} + E(e_{\mathsf{F}};\mathcal{C}),$$
$$d(\mathcal{C} \setminus \{\mathsf{u}\},\mathsf{v}) \geq d' - e_{\mathsf{IMP}} - e_{\mathsf{INS}} - E(e_{\mathsf{N}};\mathcal{C}).$$

Hence,

$$\begin{aligned} d(\mathsf{u},\mathsf{v}) - d(\mathcal{C} \setminus \{\mathsf{u}\},\mathsf{v}) &\leq (e_{\mathsf{DEL}} + E(e_{\mathsf{F}};\mathcal{C})) - (d' - e_{\mathsf{IMP}} - e_{\mathsf{INS}} - E(e_{\mathsf{N}};\mathcal{C})) \\ &= e_{\mathsf{DEL}} + e_{\mathsf{IMP}} + e_{\mathsf{INS}} + E(e_{\mathsf{F}};,\mathcal{C}) + E(e_{\mathsf{N}};\mathcal{C}) - d'. \end{aligned} \tag{2.3}$$

Under the condition

$$e_{\mathsf{DEL}} + e_{\mathsf{IMP}} + e_{\mathsf{INS}} + E(e_{\mathsf{F}};\mathcal{C}) + E(e_{\mathsf{N}};\mathcal{C}) < d,$$

the inequality (2.3) reduces to $d(\mathsf{u},\mathsf{v}) < d(\mathcal{C} \setminus \{\mathsf{u}\},\mathsf{v})$, which implies correct decoding.

On the other hand, if

$$e_{\mathsf{DEL}} + e_{\mathsf{IMP}} + e_{\mathsf{INS}} + E(e_{\mathsf{F}};\mathcal{C}) + E(e_{\mathsf{N}};\mathcal{C}) \geq d,$$

say $e_{\mathsf{IMP}} = d$, and $\mathsf{u},\mathsf{w} \in \mathcal{C}$ is such that $d(\mathsf{u},\mathsf{w}) = d$ (since $\mathcal{C}$ has distance d, the codewords u,w must exist), then $d' = d(\mathcal{C} \setminus \{\mathsf{u}\},\mathsf{u}) = d$, and we have $d(\mathsf{u},\mathsf{v}) - d(\mathcal{C} \setminus \{\mathsf{u}\},\mathsf{v}) \leq d - d' = 0$. In this case, the correctness of the decoder output cannot be guaranteed. We therefore have the following theorem.

Theorem 2.2.2. *Let $\mathcal{C}$ be an $(n,d)_q$-code and $0 \leq e_{\mathsf{DEL}}, e_{\mathsf{IMP}}, e_{\mathsf{INS}} \leq n$, $0 \leq e_{\mathsf{N}}, e_{\mathsf{F}} \leq q$. Then $\mathcal{C}$ is able to correct e_{N} narrowband noise errors, e_{F} signal fading errors, e_{IMP} impulse noise errors, e_{INS} insertion errors, and e_{DEL} deletion errors if and only if*

$$e_{\mathsf{DEL}} + e_{\mathsf{IMP}} + e_{\mathsf{INS}} + E(e_{\mathsf{F}};\mathcal{C}) + E(e_{\mathsf{N}};\mathcal{C}) < d.$$

Therefore, the parameters n, q, d, and r (symbol weight) of a code are insufficient to characterize the total error-correcting capability of a code in a PLC system using single-tone

FSK, since $E(\mathcal{C})$ cannot be specified by n, q, d, and r alone. We now introduce an additional new parameter that together with n, q, and d, more precisely captures the error-correcting capability of a code for PLC using single-tone FSK.

Definition 2.2.3. Let $\mathcal{C}$ be a code of distance d. The *narrowband noise error-correcting capability* of $\mathcal{C}$ is

$$c(\mathcal{C}) = \min\{e : E(e;\mathcal{C}) \geq d\}.$$

From Theorem 2.2.2 we infer that a code $\mathcal{C}$ can correct up to $c(\mathcal{C})-1$ narrowband noise errors. In general, the minimum value of $c(\mathcal{C})$ is about d/r if all the symbols occur exactly r times, and the maximum value of $c(\mathcal{C})$ is at most d if all the symbols appear once. Therefore, for a code $\mathcal{C}$ with bounded symbol weight r, we have $\lceil d/r \rceil \leq c(\mathcal{C}) \leq \min\{d, q\}$. However, the gap between the upper and lower bounds can be large. Furthermore, the lower bound can be attained, giving codes of low resilience against narrowband noise, as is shown in the following example.

Example 2.2.4. The code

$$\mathcal{C} = \{(\underbrace{1,\ldots,1}_{r \text{ times}}, 2, 3, 4, \ldots, q), (\underbrace{2,\ldots,2}_{r \text{ times}}, 1, 3, 4, \ldots, q)\}$$

is a $(q+r-1, r+1, r)_q$-symbol weight code with narrowband noise error-correcting capability $c(\mathcal{C}) = \lceil d/r \rceil = 2$.

In the next section, we provide a tight upper bound for $c(\mathcal{C})$ and demonstrate that equitable symbol weight codes attain this upper bound.

2.2.1 Narrowband noise of different durations and $E(\mathcal{C})$

In this subsection we show that it suffices to consider narrowband noise of length n instead of smaller lengths since it measures the maximum effect of narrowband noise on the codewords. Consequently, we justify the definition of $E(\mathcal{C})$ given by (2.2).

Given n and for an integer $i_\sigma \leq n$, we can write $\big\{i : \max\{1, i_\sigma\} \leq i \leq \min\{i_\sigma + l - 1, n\}\big\}$ as $[i_\sigma, i_\sigma + l - 1] \cap [n]$. For errors due to narrowband noise, we define the following quantity for $\Gamma \subset \Sigma$, $l \in \mathbb{Z}_{>0}$, $\mathsf{c} \in \mathcal{C}$,

$$E(\Gamma; l, \mathsf{c}) = \max_{i_\sigma \leq n \colon \sigma \in \Gamma} \left|\{i : i \in [i_\sigma, i_\sigma + l - 1] \cap [n], \mathsf{c}_i = \sigma\}\right|.$$

The quantity $E(\Gamma; l, \mathsf{c})$ measures the maximum number of coordinates in c that can be affected by narrowband noise of duration l at symbols in Γ.

Let $L \subset \mathbb{Z}_{>0}$. We consider the following quantity as a function in e, $E(L, \mathcal{C}) : [q] \to [n]$,

$$E(e; L, \mathcal{C}) = \max_{l \in L,\ \Gamma \in \binom{\Sigma}{e},\ \mathsf{c} \in \mathcal{C}} E(\Gamma; l, \mathsf{c}).$$

Then $E(e; L, \mathcal{C})$ measures the maximum number of coordinates, over all codewords in $\mathcal{C}$, that can be affected by e narrowband noise of duration $l \in L$. The following lemma states that it suffices to consider the maximum duration when determining the performance of a code in a PLC.

Lemma 2.2.5. *Let $\mathcal{C}$ be a q-ary code of length n. Consider $L \subset \mathbb{Z}_{>0}$ and define $n' = \min\{n, \max L\}$. Then*

$$E(L, \mathcal{C}) = E(\{n'\}, \mathcal{C}).$$

Proof. Let $l' = \max L$ and fix $l \in L$ and $e \in [q]$.

Observe that since $[i, i + l - 1] \subseteq [i, i + l' - 1]$ for $i \leq n$,

$$E(\Gamma; l, \mathsf{c}) \leq E(\Gamma; l', \mathsf{c}) \text{ for } \mathsf{c} \in \mathcal{C},\ \Gamma \subset \Sigma.$$

Hence, $E(e; \{l\}, \mathcal{C}) \leq E(e; \{l'\}, \mathcal{C})$ and so, $E(e; L, \mathcal{C}) \leq E(e; \{l'\}, \mathcal{C})$.

In addition, since $[i, i + l - 1] \cap [n] \subseteq [n]$ for $i \leq n$,

$$E(\Gamma; l, \mathsf{c}) \leq E(\Gamma; n, \mathsf{c}) \text{ for } \mathsf{c} \in \mathcal{C},\ \Gamma \subset \Sigma.$$

Similar argument shows that $E(e; L, \mathcal{C}) \leq E(e; \{n\}, \mathcal{C})$. Since $l' \in L$, we have $E(e; L, \mathcal{C}) \geq E(e; \{l'\}, \mathcal{C})$ and the lemma follows. ∎

The following is now immediate.

Corollary 2.2.6. *Let $\mathcal{C}$ be a q-ary code of length n. For $L \subset \mathbb{Z}_{>0}$,*

$$E(e; L, \mathcal{C}) \leq E(e; \{n\}, \mathcal{C}) \text{ for all } e \in [q].$$

Therefore, $E(L, \mathcal{C})$, which measures the maximum effect of narrowband noise on codewords, is maximized when $L = \{n\}$. Therefore, we assume that only narrowband noise of duration n occurs and write $E(e; \mathcal{C})$ and $E(\mathcal{C})$ in lieu of $E(e; \{n\}, \mathcal{C})$ and $E(\{n\}, \mathcal{C})$, respectively.

2.3 Equitable Symbol Weight Codes and $E(\mathcal{C})$

In this section, we demonstrate the optimality of equitable symbol weight codes with respect to parameter $E(\mathcal{C})$. First, we make certain observations on the parameter $E(\mathcal{C})$.

2.3.1 Relation with Symbol Weight and Partition

Symbol weight provides an estimate for $E(\mathcal{C})$. Specifically, if $\mathcal{C}$ is a code of length n with bounded symbol weight r, then $E(1; \mathcal{C}) = r$, and for $e > 1$ the minimum possible value is $r + e - 1$ if any other symbol occurs exactly once. Therefore, $E(e; \mathcal{C}) \geq \min\{n, r + e - 1\}$.

On the other hand, if $\mathcal{C}$ is a constant partition code with partition $\langle c_\sigma : \sigma \in \Sigma \rangle$, $E(\mathcal{C})$ can be determined precisely. Assume $\Sigma = [q]$ and $c_1 \geq c_2 \geq \cdots \geq c_q$, then $E(e; \mathcal{C})$ is the sum of e largest symbol weights in any codeword, that is,

$$E(e; \mathcal{C}) = \sum_{i=1}^{e} c_i \text{ for all } e \in [q].$$

Further, suppose that $\mathcal{C}$ is an equitable symbol weight code. Then from Lemma 2.1.2, $\mathcal{C}$ has constant partition $\langle r^{q-t}(r-1)^t \rangle$, where $r = \lceil n/q \rceil$ and $t = qr - n$. Hence,

$$E(e; \mathcal{C}) = \begin{cases} re, & \text{if } e \leq q - t, \\ r(q-t) + (e-q+t)(r-1), & \text{if } q - t < e \leq q. \end{cases}$$

2.3.2 Importance of Symbol Equity

For narrowband noise error-correcting capability $c(\mathcal{C})$ to be large, the parameter $E(\mathcal{C})$ must grow slowly as a function of e. We seek codes $\mathcal{C}$ for which $E(\mathcal{C})$ grows as slowly as possible. In this subsection we show that the minimum growth of $E(\mathcal{C})$ is achieved when the maximum symbol weight in any codeword of the code is at most $\lceil n/q \rceil$, that is, the symbols are equitably distributed in any codeword. Fix n, q, and let $\mathcal{F}_{n,q}$ be the (finite) family of functions

$$\mathcal{F}_{n,q} = \{E(\mathcal{C}) : \mathcal{C} \text{ is a } q\text{-ary code of length } n\}.$$

If $f \in \mathcal{F}_{n,q}$, then f is a monotone increasing function with $f(q) = n$. We say that $f \prec g$ if

$$\text{there exists } e' \in [q] \text{ with } f(e) = g(e) \text{ for } e \leq e' - 1, \text{ and } f(e') < g(e'). \tag{2.4}$$

Define the total order $\preceq$ on $\mathcal{F}_{n,q}$ so that $f \preceq g$ if either $f(e) = g(e)$ for all $e \in [q]$ or $f \prec g$.

The following proposition states that the total order $\preceq$, in some sense, orders codes of same length and alphabet size in accordance to their capabilities in a PLC system.

Proposition 2.3.1. *Let $\mathcal{C}$ and $\mathcal{C}'$ be $(n,d)_q$-codes. Suppose $E(\mathcal{C}) \prec E(\mathcal{C}')$ with e' satisfying equation (2.4). If $E(e';\mathcal{C}) < d$, then there exists a set of errors that $\mathcal{C}$ is able to correct but $\mathcal{C}'$ is unable to correct.*

Proof. Consider e' narrowband noise errors of duration n and $d - E(e';\mathcal{C}) - 1$ impulse errors. Then $E(e';\mathcal{C}) + (d - E(e';\mathcal{C}) - 1) < d$, but $E(e';\mathcal{C}') + (d - E(e';\mathcal{C}) - 1) \geq d$. The proposition then follows from Theorem 2.2.2. ∎

Hence we seek the least element in $\mathcal{F}_{n,q}$ with respect to the total order $\preceq$.

Proposition 2.3.2. *Let $f^*_{n,q} : [q] \to [n]$ be defined by*

$$f^*_{n,q}(e) = \begin{cases} re, & \text{if } 1 \leq e \leq q - t, \\ r(q-t) + (e - q + t)(r - 1), & \text{otherwise,} \end{cases}$$

where $r = \lceil n/q \rceil$ and $t = qr - n$. Then $f^*_{n,q}$ is the unique least element in $\mathcal{F}_{n,q}$ with respect *to the total order* $\preceq$.

Proof. Since $\preceq$ is total, it suffices to establish that $f^*_{n,q} \preceq f$ for all $f \in \mathcal{F}_{n,q}$, and that $f^*_{n,q} \in \mathcal{F}_{n,q}$.

Let $f = E(\mathcal{C}) \in \mathcal{F}_{n,q}$, where $\mathcal{C}$ is a q-ary code of length n over the alphabet $[q]$. Let $\mathsf{u} \in \mathcal{C}$. By permuting symbols if necessary, we may assume that $w_1(\mathsf{u}) \geq w_2(\mathsf{u}) \geq \cdots \geq w_q(\mathsf{u})$. We show that for all $e \in [q]$,

$$\sum_{i=1}^{e} w_i(\mathsf{u}) \geq f^*_{n,q}(e). \tag{2.5}$$

Suppose on the contrary that $\sum_{i=1}^{e} w_i(\mathsf{u}) < f^*_{n,q}(e)$ for some $e \in [q]$. If $e \leq q-t$, then we have $\sum_{i=1}^{e} w_i(\mathsf{u}) < re$ and $r-1 \geq w_e(\mathsf{u}) \geq w_j(\mathsf{u})$ for $j \geq e+1$. Hence,

$$n = \sum_{i=1}^{q} w_i(\mathsf{u}) < re + (q-e)(r-1) = qr - q + e \leq qr - t = n,$$

a contradiction.

Similarly, when $e > q-t$, we have $\sum_{i=1}^{e} w_i(\mathsf{u}) < r(q-t) + (e-q+t)(r-1)$ and $r-1 \geq w_e(\mathsf{u}) \geq w_j(\mathsf{u})$ for $j \geq e+1$. Hence,

$$\begin{aligned} n &= \sum_{i=1}^{q} w_i(\mathsf{u}) \\ &< r(q-t) + (e-q+t)(r-1) + (q-e)(r-1) \\ &= qr - t = n, \end{aligned}$$

also a contradiction. Hence, (2.5) holds. This then implies $E(e;\mathcal{C}) \geq f^*_{n,q}(e)$ for all $e \in [q]$, and consequently $f \succeq f^*_{n,q}$.

The proposition then follows by noting that $f^*_{n,q} \in \mathcal{F}_{n,q}$, since $E(\mathcal{C}) = f^*_{n,q}$ when $\mathcal{C}$ is a q-ary equitable symbol weight code of length n. ■

Corollary 2.3.3. *A q-ary code $\mathcal{C}$ of length n is equitable symbol weight if and only if its parameter $E(\mathcal{C})$ is given by $f^*_{n,q}$.*

Proof. If $\mathcal{C}$ is a q-ary equitable symbol weight code of length n, we have already determined that $E(\mathcal{C}) = f^*_{n,q}$. Hence, it only remains to show that $E(\mathcal{C}) = f^*_{n,q}$ implies $\mathcal{C}$ is a q-ary equitable symbol weight code of length n. Let $\mathsf{u} \in \mathcal{C}$ and we follow the notation in the proof of Proposition 2.3.2. Equality holds in (2.5) if and only if $w_i(\mathsf{u}) = r$ for $1 \leq i \leq q-t$ and $w_i(\mathsf{u}) = r-1$, otherwise. That is, u has equitable symbol weight. Hence, $\mathcal{C}$ is an equitable symbol weight code. ■

It follows that an equitable symbol weight code $\mathcal{C}$ gives $E(\mathcal{C})$ of the slowest growth rate. From Proposition 2.3.1, this is the desired condition for correcting as many narrowband noise and signal fading errors as possible.

We end this section with a tight upper bound on $c(\mathcal{C})$.

Corollary 2.3.4. *Let $\mathcal{C}$ be an $(n,d)_q$-code. Then*

$$c(\mathcal{C}) \leq \min\{e : f^*_{n,q}(e) \geq d\},$$

and equality is achieved when $\mathcal{C}$ is an equitable symbol weight code.

Proof. Let $c' = \min\{e : f^*_{n,q}(e) \geq d\}$. Observe that

$$E(c';\mathcal{C}) \geq f^*_{n,q}(c') \geq d.$$

Hence, by minimality of $c(\mathcal{C})$, we have $c(\mathcal{C}) \leq c'$. The second part of the statement follows from Corollary 2.3.3. ■

The results in this section establish that an equitable symbol weight code has the best narrowband noise error-correcting capability, among codes of the same distance and symbol weight.

2.4 Simulation Results

In this section, we study the performance of equitable symbol weight codes in a simulated setup. The setup is as follows. We transmit with a code of length n over alphabet Σ. Let $0 < p < 1$ and $L = \{bn : b \in [10]\}$. We simulate a PLC channel with the following characteristics:

1. for each $\sigma \in \Sigma$, narrowband noise error[1] of duration $l \in L$ occurs at symbol σ with probability p,

2. for each $\sigma \in \Sigma$, a signal fading error occurs at symbol σ with probability 0.05,

3. for each $i \in [n]$, an impulse noise error occurs at coordinate i with probability 0.05, and

4. for each $(\sigma, i) \in \Sigma \times [n]$, an insertion/deletion error occurs at symbol σ and coordinate i with probability 0.05.

These errors occur independently.

We choose random codewords (with repetition) from each code to transmit through the simulated PLC channel. At the receiver, we decode the detector output v to the codeword u' using the minimum distance decoder defined in equation (2.1). The number of symbols in error is then $d(\mathsf{u}', \mathsf{u})$ and the *symbol error rate* is the ratio of the total number of symbols in error to the total number of symbols transmitted.

We remark that the choice of the code lengths in our simulations is similar to code lengths studied in prior work in this area [63,78,79]. While in theory the minimum distance decoder described by (2.1) works for all code lengths, this decoding algorithm becomes inefficient when the size of the code is big. We address the issue of efficient decodability in Chapter 5.

Decoding with narrowband noise detection. Versfeld *et al.* [78,79] introduced a method to detect narrowband noise in order to enhance the error correction capability of the detector introduced in Section 2.2, when used with bounded distance decoding. Based on the energy metrics obtained at each time slot for each frequency, they first determine the presence of narrowband interference and if so, the metrics of the corresponding frequency are set to zero. Depending on the detector/decoder combination, a signal is sent to the decoder. Specifically, consider narrowband noise detection with the use of an (n, d, r)-symbol weight code. If the number of discrete time instances in which a particular symbol appears, exceeds $\lfloor (n + r)/2 \rfloor$, the particular symbol is removed from the coordinates in which it occurs. We describe an algorithm to detect and remove narrowband noise in Figure 2.2.

[1] The choice of L is similar to that of the narrowband noise model in the setup of Versfeld *et al.* [78,79].

Input: Detector Output, $\mathsf{v} \in (2^{\Sigma})^n$
Output: Modified $\mathsf{v} \in (2^{\Sigma})^n$

$\tau \leftarrow \lfloor (n+r)/2 \rfloor$;
for $\sigma \in \Sigma$ **do**
  if $|\{i : \sigma \in \mathsf{v}_i\}| > \tau$ **then**
    for $i \in [n]$ **do**
      $\mathsf{v}_i \leftarrow \mathsf{v}_i \setminus \{\sigma\}$
    end
  end
end

Fig. 2.2: Narrowband noise detection with a (n, d, r)-symbol weight code

Tab. 2.1: Comparison of Equitable Symbol Weight Codes and Minimum Symbol Weight Codes

$(n, d, r)_q$-symbol weight code	Distance	Narrowband noise error-correcting capability	Symbol weight	Size	Remarks
$\mathrm{ESW}(25, 24, 2)_{17}$	24	16	2	51	equitable symbol weight
$\mathrm{MSW}(25, 24, 2)_{17}$	24	12	2	51	minimum symbol weight
$\mathrm{ESW}(11, 6, 2)_{10}$	6	5	2	1000	equitable symbol weight
$\mathrm{MSW}(11, 6, 2)_{10}$	6	3	2	1000	minimum symbol weight

2.4.1 Minimum Symbol Weight Codes

We exhibit the difference in performance between equitable symbol weight codes and (non-equitable) minimum symbol weight codes. Specifically, we consider the codes of various lengths and relative distances in Table 2.1.

The results of the simulation are displayed in Fig. 2.3. We detect the presence of narrowband noise[2] using the algorithm given in Fig. 2.2 in the simulations denoted by solid lines and labeled by "(NB)". The similarly colored dashed lines denote simulations without narrowband noise detection. From the results, observe that $\mathrm{ESW}(25, 24, 2)_{17}$ and $\mathrm{ESW}(11, 6, 2)_{10}$ achieve lower symbol error rates compared to $\mathrm{MSW}(25, 24, 2)_{17}$ and $\mathrm{MSW}(11, 6, 2)_{10}$, respectively.

[2] As discussed in Section 2.2, after narrowband noise detection, the multivalued output is given directly to a minimum distance decoder. This deviation from the setup by Versfeld *et al.* (where envelope detection and Viterbi threshold ratio test is applied prior to decoding) means that the results are independent of the choice of demodulation rule.

2.4.2 Cosets and Subcodes of Reed-Solomon Codes

Versfeld *et al.* [78, 79] showed empirically that using narrowband detection, low symbol weight cosets of Reed-Solomon codes outperform normal Reed-Solomon codes in the presence of narrowband noise and additive white Gaussian noise. We continue this investigation and observe the difference in performance between equitable symbol weight codes and low symbol weight cosets of Reed-Solomon codes. In addition, we consider subcodes of Reed-Solomon codes with low symbol weight.

Tab. 2.2: Comparison of Equitable Symbol Weight Codes and Low Symbol Weight Cosets and Subcodes of Reed-Solomon Codes

$(n, d, r)_q$-symbol weight code	Distance	Narrowband noise error-correcting capability	Symbol weight	Size	Remarks
$\mathrm{ESW}(7, 5, 1)_8$	5	5	1	336	equitable symbol weight
$\mathrm{RSS}(7, 5, 2)_8$	5	3	2	336	subcode of Reed-Solomon code
$\mathrm{RSC}(7, 6, 2)_8$	6	3	2	64	coset of Reed-Solomon code
$\mathrm{ESW}(7, 2, 1)_8$	2	2	1	20160	equitable symbol weight
$\mathrm{RSS}(7, 3, 2)_8$	3	2	2	20160	subcode of Reed-Solomon code
$\mathrm{RSC}(7, 4, 4)_8$	4	1	4	4096	coset of Reed-Solomon code
$\mathrm{ESW}(15, 11, 1)_{16}$	11	11	1	21120	equitable symbol weight
$\mathrm{RSS}(15, 12, 3)_{16}$	12	4	3	21120	subcode of Reed-Solomon code
$\mathrm{RSC}(15, 13, 3)_{16}$	13	5	3	4096	coset of Reed-Solomon code

Specifically, we consider the codes in Table 2.2. See [78, 79] for the construction of Reed-Solomon coset codes, denoted by RSC. The codes denoted by RSS are subcodes of Reed-Solomon codes. They are obtained by expurgation of a Reed-Solomon code and retaining only the codewords with low symbol weight.

We note that it is not possible for equitable symbol weight codes and Reed-Solomon coset codes of the same minimum distance and length over the same alphabet to be of the same size. Therefore, for each Reed-Solomon coset codes, we make comparisons with an equitable symbol weight code of a larger size, albeit with a smaller distance. However, these equitable symbol weight codes have larger narrowband noise error-correcting capabilities. In addition, we make comparisons with subcodes of Reed-Solomon codes with parameters as close as possible to the corresponding equitable symbol weight codes. In particular, we ensure that the subcodes and the equitable symbol weight codes have the same size.

The results of the simulation are displayed in Fig. 2.4, where we adopt similar conventions as in Fig. 2.3, and we make the following observations.

(i) While narrowband noise detection in general improves the performance of codes in PLC, it has negligible effect on the performance of equitable symbol weight codes. A natural question is if there is another parameter that measures this improvement and if this parameter is related to symbol equity.

(ii) Equitable symbol weight codes show larger improvement over Reed-Solomon coset codes at higher narrowband noise probabilities. This reflects the relevance of narrowband noise error-correcting capabilities as a measure of performance when the effects of narrowband interference are significant. In contrast, when the effects of narrowband interference are negligible, the classical Hamming distance parameter provides a better measure of performance.

2.5 Concluding Remarks

We introduce a new code parameter that captures the error-correcting capability of a code with respect to narrowband noise. Equitable symbol weight codes are shown to be optimal with respect to this parameter when code length, alphabet size and distance are fixed. This makes equitable symbol weight codes a viable option to handle narrowband noise in a PLC channel and we study their sizes and some constructions in Chapter 3 and 4. However, our analysis is based on a minimum distance decoder and this algorithm becomes inefficient when the size of the code is big. We address the issue of efficient decodability in Chapter 5.

Next, we remark that the notion of symbol equity used in this chapter differs from the notion of symbol equity that is used in Swart and Ferreira [74]. In that work, the authors consider the code-matrix of the code (the matrix whose rows consist of all the codewords), and show that an equal distribution of symbols in each column of the code-matrix results in the maximum possible separation between all the codewords. This notion of symbol equity also appears in the computation of the Plotkin bound on codes. In contrast, the symbol equity discussed here considers the distribution of symbols in every codeword of the code, that is, in every row of the code-matrix.

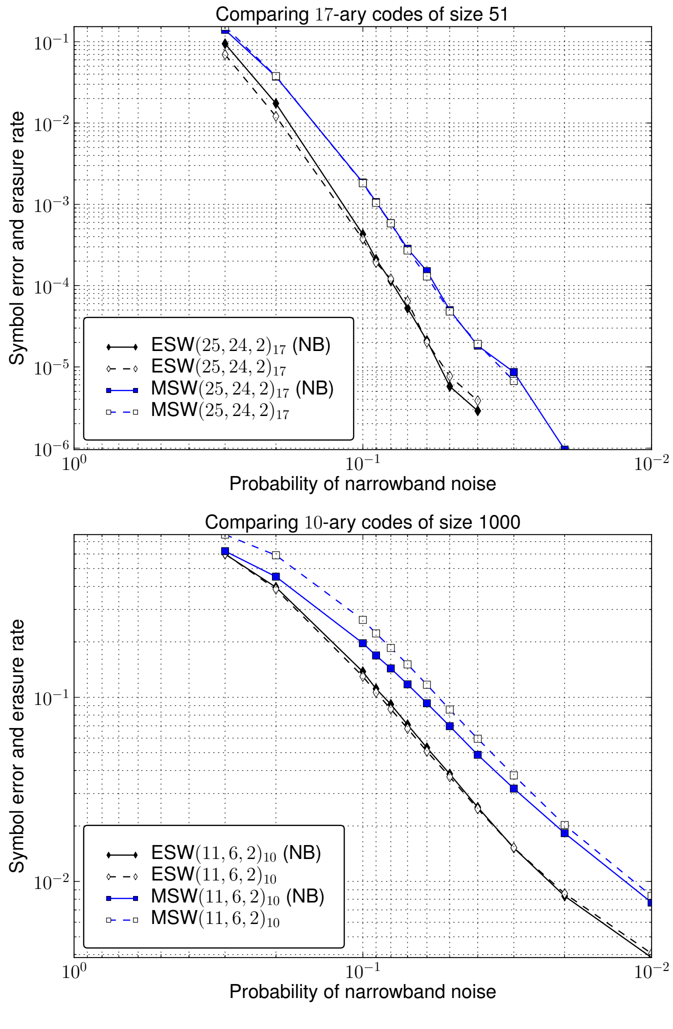


Fig. 2.3: Comparison of equitable and minimum symbol weight codes

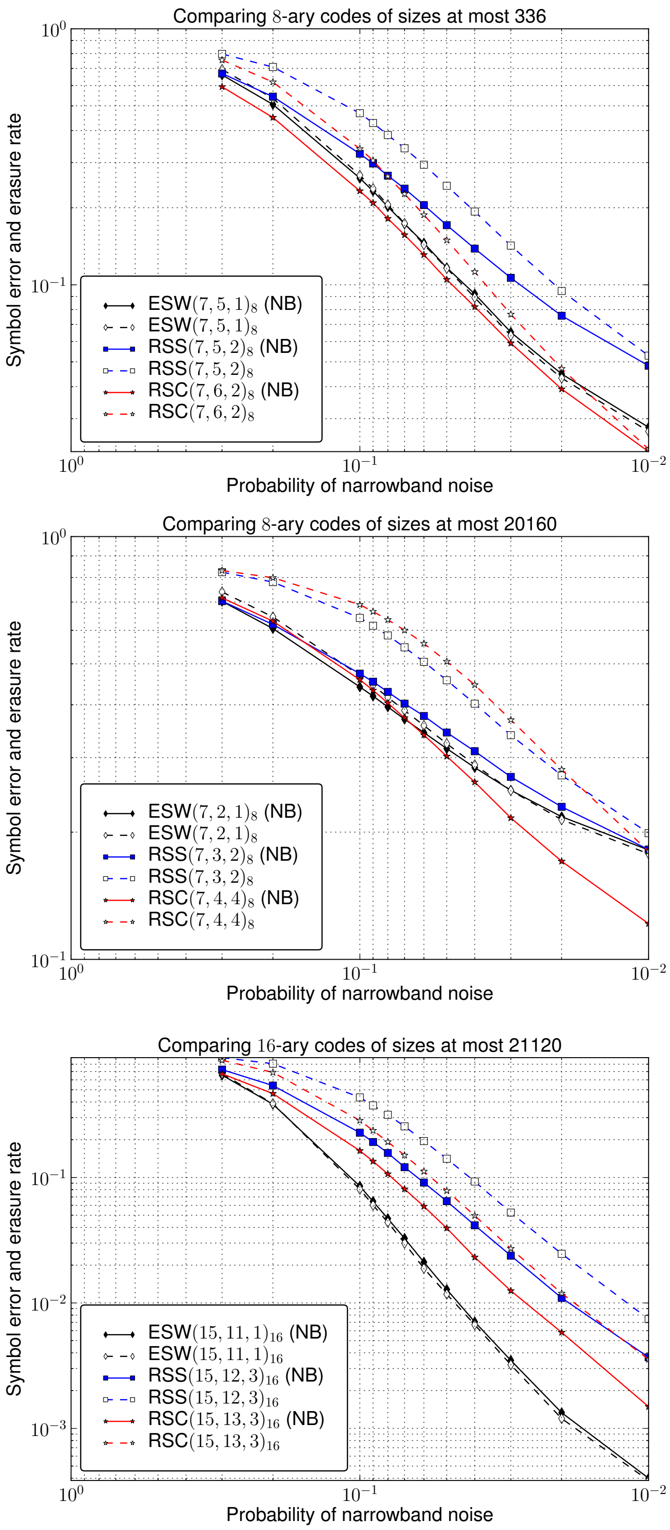


Fig. 2.4: Comparison of equitable symbol weight codes and low symbol weight cosets and subcodes of Reed-Solomon codes

Finally, the notion of symbol equity discussed in this chapter is also applicable to systems where crisscross types of errors are encountered [66].

3. ESTIMATES ON THE SIZES OF EQUITABLE SYMBOL WEIGHT CODES

Motivated by results in Chapter 2, we investigate the possible sizes of equitable symbol weight codes and in particular, provide lower and upper bounds on the sizes. Of significance, we determine the asymptotic size for equitable symbol weight codes under certain conditions and tabulate the estimated size of optimal equitable symbol weight codes for certain parameters.

Section 3.2 is presented part in Chee *et al.* [13], where we examine the asymptotic behavior of codes in the bounded symbol weight space. Interestingly, even though the equitable symbol weight space is a subset of the bounded symbol weight space, the asymptotic sizes of the two are approximately the same.

3.1 Preliminaries

Throughout this chapter, let Σ denote an alphabet of size q and n denote the code length. Let $\mathsf{u} \in \Sigma^n$. For $\sigma \in \Sigma$, recall that $w_\sigma(\mathsf{u})$ is the number of times the symbol σ appears among the coordinates of u.

A word u has symbol weight r if $r = \max_{\sigma\in\Sigma} w_\sigma(\mathsf{u})$. Denote the space of all codewords of length n with symbol weight at most r by $\mathrm{SW}(q, n, \le r)$. The size of $\mathrm{SW}(q, n, \le r)$ is hence given by

$$|\mathrm{SW}(q, n, \le r)| = \sum_{(r_1,r_2,\ldots,r_q)\in P(n,q,r)} \binom{n}{r_1, r_2, \ldots, r_q}, \tag{3.1}$$

where $P(n, q, r)$ denote the compositions of n into q parts where each part is bounded between 0 and r.

On the other hand, a word u has equitable symbol weight if $w_\sigma(\mathsf{u}) \in \{\lceil n/q\rceil, \lfloor n/q\rfloor\}$ for all $\sigma \in \Sigma$. Denote the space of all codewords with equitable symbol weight by $\mathrm{ESW}(q,n)$ and the size of the equitable symbol weight space can be derived easily. Let $r = \lceil n/q \rceil$ and $t = qr - n$. Then,

$$|\mathrm{ESW}(q,n)| = \binom{q}{t}\binom{n}{\underbrace{r,r,\ldots,r}_{q-t},\underbrace{r-1,r-1,\ldots,r-1}_{t}} = \binom{q}{t}\frac{n!}{(r!)^{q-t}(r-1)!^{t}} \tag{3.2}$$

Denote the maximum size of an $(n,d)_q$-equitable symbol weight code by $A_q^{ESW}(n,d)$. Any $(n,d)_q$-equitable symbol weight code of size $A_q^{ESW}(n,d)$ is said to be *optimal*.

3.1.1 Classical Bounds

We provide upper and lower bounds for $A_q^{ESW}(n,d)$ by considering familiar techniques in classical coding theory. While certain bounds are not expressed in simple forms and are computationally infeasible for large values, we use them in Section 3.3 to provide bounds for specific values of q, n and d.

Proposition 3.1.1 (Singleton-type bound)**.** *Let $1 \le d \le n$. Let $r = \lceil n/q \rceil$. Then*

$$A_q^{ESW}(n,d) \le |SW(q, n-d+1, \le r)|. \tag{3.3}$$

Proof. Let $\mathcal{C}$ be an $(n,d)_q$-equitable symbol weight code. Pick any $n-d+1$ coordinates of $[n]$, that is, $I \in \binom{[n]}{n-d+1}$, and consider the code $\mathcal{C}|_I := \{(\mathsf{u}_i) : i \in I\}$. Then $\mathcal{C}|_I$ consists of distinct words from $\mathrm{SW}(q, n-d+1, \le r)$. The inequality is then immediate. ∎

We remark that Proposition 3.1.1 generalizes previously known bounds for equitable symbol weight codes. Namely, when $n \le q$, the proposition reduces to the Singleton-type bound for injection codes given by Dukes [28, Theorem 1].

As with classical GV and Hamming bounds, we consider the volume of a Hamming ball in the equitable symbol weight space. In particular, fix any $\mathsf{u}_0 \in \mathrm{ESW}(q,n)$ and define the following,

$$V^{ESW}(q,n,d) := |\{\mathsf{u} \in \mathrm{ESW}(q,n) : d(\mathsf{u},\mathsf{u}_0) \le d\}|. \tag{3.4}$$

Note that the value of $V^{ESW}(q,n,d)$ is independent of the choice u_0. Hence, we obtain the following GV-type and Hamming-type bounds.

Proposition 3.1.2 (GV-type and Hamming-type bounds)**.** *Let* $1 \le d \le n$. *Then*

$$\frac{|\mathrm{ESW}(q,n)|}{V^{ESW}(q,n,d-1)} \le A_q^{ESW}(n,d) \le \frac{|\mathrm{ESW}(q,n)|}{V^{ESW}(q,n,\lfloor (d-1)/2 \rfloor)}. \tag{3.5}$$

To end this section, we remark that Luo *et al.* gave a nonrecursive Johnson-type bound for constant composition codes [56, Lemma 2] and Huczynska and Mullen observed that this bound reduces to the Plotkin bound for frequency permutation arrays [43]. For general equitable symbol weight codes, we remark the classical Plotkin bound suffices in the sense that there are families of optimal equitable symbol weight codes that meet the Plotkin bound. We construct these codes in Chapter 4.

3.2 Asymptotic Size of Equitable Symbol Weight Codes

In this section, we show that the asymptotic sizes of an optimal equitable symbol weight code and an optimal classical code with the same parameters are approximately the same, provided that the ratio q/n tends to a constant. Moreover, when q also grows with n, the asymptotic size of an optimal equitable symbol weight code can be precisely determined.

We remark that the condition that q grows at most proportional to n is a reasonable assumption. This is because in many applications the number of frequencies (or symbols) available for transmission is restricted as compared to the code length.

To establish this result, we require the following lemma that states that the asymptotic size of the equitable symbol weight space is approximately one. As the proof of the lemma is a technical application of Stirling's approximation, we defer the proof to the Subsection 3.2.1.

Lemma 3.2.1. *Let* $q/n \to \theta$ *as* $n \to \infty$. *Then*

$$\frac{1}{n}\log_q |\mathrm{ESW}(q,n)| = 1 - o(1).$$

With this simple lemma and an Elias-type argument, we establish the main result of this chapter.

Theorem 3.2.2. *Let $q/n \to \theta$ as $n \to \infty$ where $0 \le \theta < \infty$. Then*

$$\frac{1}{n}\log_q A_q^{ESW}(n,d) = \frac{1}{n}\log_q A_q(n,d) - o(1).$$

Furthermore, if $q \to \infty$ and $d/n \to \delta$ as $n \to \infty$,

$$\lim_{n\to\infty}\frac{1}{n}\log_q A_q^{ESW}(n,d) = 1-\delta.$$

Proof. Using an Elias-type argument, we have

$$A_q(n,d) \le \frac{q^n}{|\mathrm{ESW}(q,n)|}A_q^{ESW}(n,d).$$

Taking logarithms and applying Lemma 3.2.1, we have

$$\begin{aligned}\frac{1}{n}\log_q A_q(n,d) &\le 1 + \frac{1}{n}\log_q|\mathrm{ESW}(q,n)| + \frac{1}{n}\log_q A_q^{ESW}(n,d)\\ &\le \frac{1}{n}\log_q A_q^{ESW}(n,d) + o(1).\end{aligned}$$

On the other hand, we have the fact $A_q^{ESW}(n,d) \le A_q(n,d)$, and so,

$$\frac{1}{n}\log_q A_q(n,d) - o(1) \le \frac{1}{n}\log_q A_q^{ESW}(n,d) \le \frac{1}{n}\log_q A_q(n,d).$$

Suppose in addition $d/n \to \delta$ and $q \to \infty$. Then by GV bound,

$$\begin{aligned}\frac{1}{n}\log_q A_q^{ESW}(n,d) &\ge \frac{1}{n}\log_q A_q(n,d) - o(1)\\ &\ge 1 - h_q(\delta) - o(1).\end{aligned}$$

Since $h_q(x) \to x$ as $q \to \infty$, we have

$$\lim_{n\to\infty}\frac{1}{n}\log_q A_q^{ESW}(n,d) \ge 1-\delta.$$

On the other hand, by Singleton bound,

$$\frac{1}{n}\log_q A_q^{ESW}(n,d) \le \frac{1}{n}\log_q A_q(n,d) \le 1-\delta,$$

and this gives the last equality. ■

3.2.1 Asymptotic Size of ESW(q,n)

In this subsection, we establish Lemma 3.2.1. Recall that given n and q, we define $r = \lceil n/q \rceil$ and $t = qr - n$. In other words, we have $r = (n+t)/q$ and $0 \le t < q$. The size of ESW(q,n) is given by (3.2) and to estimate the size we make use of Stirling's approximation.

Theorem 3.2.3 (Stirling Approximation)**.** *Let $n > 0$. Then*

$$n! \ge n\ln n - n, \tag{3.6}$$

$$n! = n\ln n - n + \frac{1}{2}\ln n + O(1). \tag{3.7}$$

Let q be a function of n such that $q/n \to \theta$ when $n \to \infty$. Consider the following cases.

First, suppose $\theta > 0$. Hence, $q \to \infty$ as $n \to \infty$.

Observe that ESW$(q,n) \ge \frac{n!}{r^n}$ and applying Stirling's approximation,

$$\begin{aligned}
\frac{1}{n}\log_q |\mathrm{ESW}(q,n)| &\ge \log_q n - \log_q e - \log_q r \\
&= \log_q n - \log_q e - \log_q(n+t) + 1 \\
&= 1 - \log_q e - \log_q \frac{n+t}{n} \\
&\ge 1 - \log_q e - \log_q \frac{n+q}{n} \\
&= 1 - o(1),
\end{aligned}$$

where the last equality follow from the fact that $\log_q e \to 0$ and $\log_q(n+q)/n = \log_q(1+q/n) \to 0$ since $q/n \to \theta$ and $q \to \infty$.

Suppose $\theta = 0$. Then $r = (n+t)/q \to \infty$ as $n \to \infty$. Applying Stirling's approximation, we have

$$\begin{aligned}\frac{1}{n}\log_q|\mathrm{ESW}(q,n)| &= \frac{1}{n}\log_q\frac{(n!)r^t}{(r!)^q}\\ &\geq \log_q n - \log_q e + \frac{t}{n}\log_q r - \frac{qr}{n}\log_q r + \frac{qr}{n}\log_q e\\ &\qquad - \frac{q}{2n}\log_q r - \frac{q}{n}\log_q O(1).\end{aligned}$$

We make the following observations:

$$\begin{aligned}\frac{t}{n}\log_q r - \frac{qr}{n}\log_q r &= -\log_q r = 1 - \log_q(n+t),\\ -\log_q e + \frac{qr}{n}\log_q e &= \frac{t}{n}\log_q e \leq \frac{q}{n}\log_2 e = o(1),\\ \frac{q}{2n}\log_q r &= \frac{q}{2n}\log_q\frac{n+t}{q} \leq \frac{q}{2n}\log_2\frac{2n}{q} = o(1),\\ \frac{q}{n}\log_q O(1) &= o(1),\end{aligned}$$

Therefore,

$$\frac{1}{n}\log_q|\mathrm{ESW}(q,n)| \geq 1 - \log_q(n+t)/n - o(1) \geq 1 - \log_q(1+q/n) - o(1) = 1 - o(1),$$

since $1 + q/n \to 1$ as $n \to \infty$.

3.3 Sizes of Equitable Symbol Weight Codes for Specific Parameters

This section looks at the possible sizes of optimal equitable symbol weight codes for specific parameters. In particular, we provide a table of values of lower and upper bounds for $A_q^{ESW}(n,d)$ where $q \in \{3,4\}$ at the end of the chapter.

While there is extensive literature on constant composition codes (see [10, 18, 21, 24–27, 38, 42, 56]) and frequency permutation arrays (see [42, 43]), few results apply in the range of parameters in our tables. In fact, it turns out that computational methods yield better results for this set of parameters.

But first we provide a survey on known values of $A_q^{ESW}(n,d)$. The following facts on the size of optimal equitable symbol weight codes are trivial.

(i) Given q and n, $A_q^{ESW}(n,1) = \mathrm{ESW}(q,n)$.

(ii) Given q and r, $A_q^{ESW}(rq,2) = \mathrm{ESW}(q,rq) = (rq)!/(r!)^q$.

(iii) Given q and n, $A_q^{ESW}(n,n) = q$.

For other values of q, n and d, relatively little is known about optimal equitable symbol weight codes, other than those that correspond to permutation codes, injection codes and frequency permutation arrays. In Table 3.1, we provide a summary of the known infinite families of optimal equitable symbol weight codes. We observe that only six infinite families of optimal equitable symbol weight codes with code length greater than alphabet size are known. These have all been constructed by Ding and Yin [26], and Huczynska and Mullen [42] as frequency permutation arrays and they meet the Plotkin bound.

One drawback with the code parameters of these families is that the narrowband noise error-correcting capability (see Definition 2.2.3) to length ratio diminishes as its length grows. This is undesirable for narrowband noise correction for PLC and the following theorem provides infinite families of optimal equitable symbol weight codes with code lengths are larger than alphabet size and whose relative narrowband noise capability to length ratios tend to a positive constant as length grows.

Theorem 3.3.1. *The following holds.*

(i)

$$A_q^{ESW}(2q-1, 2q-2) = \begin{cases} 3, & q = 2, \\ 2q, & q \geq 3. \end{cases}$$

(ii)

$$A_q^{ESW}(3q-1, 3q-3) = \begin{cases} 4, & q = 2, \\ 3q, & q \geq 3. \end{cases}$$

(iii)

$$A_q^{ESW}(4q-1, 4q-4) = \begin{cases} 4q-1, & q = 2, 3, \\ 4q, & q \geq 4. \end{cases}$$

Tab. 3.1: Infinite families of optimal $(n, d)_q$ equitable symbol weight codes

$(n,d)_q$-equitable symbol weight code $\mathcal{C}$	$\lvert\mathcal{C}\rvert$	$c(\mathcal{C})$	$\lim\limits_{q\to\infty} c(\mathcal{C})/n$	Remarks
$(n,n)_q$ for $q \geq 2$	q	$\min\{n,q\}$	–	easy
$(3,2)_q$ for $q \geq 3$	$q(q-1)$	2	0	injection code [28]
$(4,2)_q$ for $q \geq 4$, $q \neq 7$	$q(q-1)(q-2)$	2	0	injection code [28]
$(n,1)_q$ for $n < q$	$q(q-1)\cdots(q-n+1)$	1	$1/n$	injection code, easy
$(qn,2)_q$ for $q \geq 2$	$(qn)!/(n!)^q$	2	0	frequency permutation array, easy
$(q,3)_q$ for $q \geq 3$	$q!/2$	3	0	permutation code, easy
$(q,q-1)_q$ for prime powers q	$q(q-1)$	$q-1$	1	injection code [22]
$(n,n-1)_q$ for q sufficiently large and $n \leq q$	$q(q-1)$	$n-1$	$1-1/n$	injection code [28]
$(q,q-2)_q$ for prime powers $q-1$	$q(q-1)(q-2)$	$q-2$	1	permutation code [33]
$\left(q(q+1),q^2\right)_q$ for prime powers q	q^2	q	0	frequency permutation array [25]
$\left(\frac{q(kq^2-1)}{k-1}, \frac{kq^2(q-1)}{k-1}\right)_q$ for prime powers q, $2 \leq k \leq 5$, $(k,q) \neq (5,9)$	kq^2	q	0	frequency permutation array[26]
$\left(\frac{\mu q^{s-t}(q^{2s-t}-1)}{q^t-1}, \frac{\mu q^{2s-t}(q^{s-t}-1)}{q^t-1}\right)_{q^{s-t}}$ for prime powers q, $1 \leq t < s$, $\mu = \prod_{i=1}^{t-1} \frac{q^{s-i}-1}{q^i-1}$	q^{2s-t}	q^{s-t}	0	frequency permutation array[26]
$(q^s(q^{2s+c}-1), q^{2s+c}(q^s-1))_{q^s}$, for prime powers q, and $s,c \geq 1$	q^{2s+c}	q^s	0	frequency permutation array[26]
$(\binom{kq}{k}, \frac{kq-k}{kq-1}\binom{kq}{k})_q$ for $q,k \geq 1$	kq	$q-1$	0	frequency permutation array[42]
$(2q^2-q, 2q^2-2q)_q$ for even q, $q \notin \{2,6\}$	$2q$	q	0	frequency permutation array[42]
$(2q-1,2q-2)_q$ for $q \geq 3$	$2q$	$q-1$	$1/2$	Theorem 3.3.1
$(3q-1,3q-3)_q$ for $q \geq 3$	$3q$	$q-1$	$1/3$	Theorem 3.3.1
$(4q-1,4q-4)_q$ for $q \geq 4$	$4q$	$q-1$	$1/4$	Theorem 3.3.1
$(5q-1,5q-5)_q$ for $q \geq 62$	$5q$	$q-1$	$1/5$	Theorem 3.3.1
$(q^2-1,q^2-q)_q$ for $q \geq 4$	q^2	$q-1$	0	Theorem 3.3.1
$\left(\frac{3q-1}{2}, \frac{3q-3}{2}\right)_q$ for $q \geq 7$ and q odd	$3q$	$q-1$	$2/3$	Theorem 3.3.1
$(2q-3,2q-4)_q$ for $q \geq 14$	$2q+1$	$q-2$	$1/2$	Theorem 3.3.1

(iv) If $q \geq 62$ *or* $q \in \{5-18, 30, 42, 46, 48-50, 54-57\}$,

$$A_q^{ESW}(5q-1, 5q-5) = 5q.$$

(v) If q is an odd prime power,

$$A_q^{ESW}(q^2-1, q^2-q) = q^2.$$

(vi)

$$A_q^{ESW}\left(\frac{3q-1}{2}, \frac{3q-3}{2}\right) = \begin{cases} 4q-6, & q = 3, 5, \\ 3q, & q \geq 7 \textit{ is odd.} \end{cases}$$

(vii)

$$A_q^{ESW}(2q-3, 2q-4) = \begin{cases} 6q-12, & q = 3, 4, \\ 14, & q = 5, 6, \\ 2q+1, & q \geq 7, \textit{except possibly } q \in \{12, 13\}. \end{cases}$$

The proof of Theorem 3.3.1 is based on the construction of equivalent combinatorial objects called *generalized balanced tournament packings* (GBTPs) and reveals an interesting interplay with combinatorial design theory. The construction of GBTPs forms the theme of Chapter 4. In particular, Theorem 3.3.1(i) to (v) follows from Proposition 4.1.2, Theorem 4.2.4 and Theorem 4.2.5. Theorem 3.3.1(vi) follows from Proposition 4.1.2, Theorem 4.2.4 and Lemma 4.6.3, while Theorem 3.3.1(vii) follows from Proposition 4.1.2, Theorem 4.2.6 and Lemma 4.6.6.

In the following subsections, we describe certain constructions of equitable symbol weight codes which help to establish estimates for $A_q^{ESW}(n, d)$.

3.3.1 Subcodes of Cosets of Linear Codes

Motivated by Theorem 3.2.2, we construct equitable symbol weight codes by looking at subcodes of cosets of classical linear codes.

Specifically, let $\mathcal{C}$ be a linear code over $\mathbb{F}_q$ of length n with distance d. Suppose k is the dimension of $\mathcal{C}$ and H is the parity check matrix for $\mathcal{C}$. For $\mathsf{s} \in \mathbb{F}_q^{n-k}$, if we consider the code

$$\mathcal{C}(\mathsf{s}) := \{\mathsf{u} \in \mathrm{ESW}(q,n) : \mathsf{u}\mathsf{H}^T = \mathsf{s}\},$$

then $\mathcal{C}(\mathsf{s})$ is an $(n,d)_q$-equitable symbol weight code. It then follows that

$$A_q^{ESW}(n,d) \geq \max_{\mathsf{s} \in \mathbb{F}_q^{n-k}} |\mathcal{C}(\mathsf{s})| \,. \tag{3.8}$$

In Table 3.2 and Table 3.3, we make use of (3.8) with known optimal linear codes (recorded at [39]) to establish certain lower bounds.

A notable class of optimal linear code is the Reed-Solomon codes. In particular, for $d \leq n \leq q$ and q is a prime power, there exists a linear $(n,d)_q$-code of dimension $n-d+1$. Since there are q^{d-1} cosets, an averaging argument shows that there exists a $(n,d)_q$ equitable symbol weight codes with size at least $q!/((q-n)!q^{d-1})$. Since $n \leq q$, these codes are in fact injection codes. Furthermore, observe that

$$\frac{q!}{(q-n)!q^{d-1}} = \frac{q!}{(q-n+d-1)!} \cdot \frac{(q-n+1)(q-n+2)\cdots(q-n+d-1)}{q^{d-1}}$$

and consider n as a function of q. If $n = o(q)$, then this construction shows that the Singleton-type bound given in Proposition 3.1.1 is asymptotically tight. In addition, as these codes are subcodes of cosets of Reed-Solomon codes, classical decoding algorithms[1] of the latter apply to these injection codes.

Proposition 3.3.2. *Let $n \leq q$ and q be a prime power. Then*

$$A_q^{ESW}(n,d) \geq \frac{q!}{(q-n)!q^{d-1}}.$$

Furthermore, when $n = o(q)$, then $A_q^{ESW}(n,d) = (1-o(1))q!/(q-n+d-1)!$, where the

[1] Versfeld *et al.* [78, 79] (see also Section 2.4) adapted classical bounded distance decoding methods to cosets of Reed-Solomon codes in the presence of narrowband interference. Their methods can be applied directly to the subcodes and decode up to the distance of the supercodes. Decoding up to the distance of the subcode is an area for future study.

asymptotics are in q.

Next, we observe that our construction can easily extended to codes that are nonlinear over a finite field, but are 'additive' over some abelian group.

Proposition 3.3.3. *Consider an abelian group Γ of order q. Let $\gamma \in \Gamma$ and define*

$$\mathcal{C}(q,n,\gamma) := \{\mathsf{u} \in \mathrm{ESW}(q,n) : \sum_{i=1}^{n} \mathsf{u}_i = \gamma\}.$$

Then $\mathcal{C}(q,n,\gamma)$ is a $(n,2)_q$-equitable symbol weight code. Hence,

$$A_q^{ESW}(n,2) \geq \max_{\gamma \in \Gamma} |\mathcal{C}(q,n,\gamma)|\,.$$

3.3.2 Computational Methods Based on Maximum Clique Problem

Given any coding metric, the problem of determining the size of an optimal code can be reduced to an instance of the maximum clique problem. More precisely, a *clique* of a graph is a set of mutually adjacent vertices and a *maximum clique* is a clique with the maximum number of vertices. Fix q, n, and d and define the graph $G(q,n,d)$ whose vertex set is $\mathrm{ESW}(q,n)$. The vertices $\mathsf{u},\mathsf{v} \in \mathrm{ESW}(q,n)$ are adjacent if the Hamming distance between u and v is at least d. Hence, it is not difficult to observe that the maximum clique of $G(q,n,d)$ corresponds to an optimal $(n,d)_q$-equitable symbol weight code.

Unfortunately, the maximum clique problem for a general graph is a difficult problem. Specifically, given M and an arbitrary graph, the problem of determining the existence of a clique of size M is $\mathcal{NP}$-hard. Despite this theoretic complexity for general graphs, many clique-finding algorithms and heuristics have been developed and shown experimentally to be effective for maximum clique problems of practical importance (see Pardolos and Xue [64] and Bomze *et al.* [7] for a survey).

Note that while the maximum clique problem for general graphs is $\mathcal{NP}$-hard, it remains an open problem to determine if the maximum clique problem for the family of graphs $G(q,n,d)$ is $\mathcal{NP}$-hard. However, the exact algorithms and heuristics for clique-finding are still applicable and we outline a few to determine certain exact values and lower bounds in

Table 3.2 and Table 3.3. We remark that there are many other computationally intensive search methods that may lead to tighter lower bounds.

In the following, let $G = (V, E)$ be a general graph.

Exact Algorithm - Branch-and-bound. A branch-and-bound algorithm for the maximum clique problem typically consists of a systematic enumeration (*branching* process) of all subsets of vertices or candidate solutions, where large sets of 'fruitless' candidates are discarded by using a bounding function on the candidate solutions (*bounding* or *pruning* process). Hence, the algorithm is exhaustive and always produces a maximum clique.

A typical branching algorithm for maximum clique orders the vertex set V and maintains a candidate solution set $K \subseteq V$, which is initialized to be the empty set. We recursively add and remove vertices (according to the order) to K and check if K indeed forms a clique. Clearly, when K is not a clique, we have a simple criterion to abandon all candidate solutions $K' \supseteq K$. This pruning process turns out to be effective in reducing the running time of the algorithm.

Various pruning methods [61, 71, 76] have been proposed and we use the algorithm `MaxCliqueDyn` implemented and proposed by Konc and Janežič [45]. The algorithm which builds on the work of Tomita *et al.* [76] is based on vertex coloring. Broadly speaking, `MaxCliqueDyn` in addition to K maintains another global set $K_{\max}$, which is the maximum clique currently found. Suppose v is the last added vertex to K and amongst the vertices yet to be considered, we look at $N(v)$, the set of vertices adjacent to v. The algorithm then colors the vertices in $N(v)$ such that adjacent vertices are of different color. If $N(v)$ can be colored with c colors, then from graph theory the size of a maximum clique in $N(v)$ is at most c. Hence, if $|K| + c < |K_{\max}|$, we are able to prune the search space.

While it is desirable to use as little colors as possible in the pruning step, finding an optimal coloring (with the least colors) is time-consuming and in fact $\mathcal{NP}$-hard. Hence, there exists a trade-off between the time needed for an approximate coloring and the resulting reduction in search space. `MaxCliqueDyn` in our experiments turns out to be suited for determining the maximum clique in $G(q, n, d)$.

Heuristic Search - Local Search Techniques. Unfortunately, running times of exact algorithms grow exponentially and for most instances, we resort to heuristic techniques to determine lower bounds for $A_q^{ESW}(n,d)$. Despite having no a priori guarantee on the size nor the running time, these heuristic techniques often produce better lower bounds than conventional methods in reasonable time. Such methods are broadly called *local search* and Honkala and Östergård [41] documents the success of local search methods in coding theory.

Hill-climbing is one simple variant of local search methods. In our constructions, we begin with an initial $(n,d)_q$-equitable symbol weight code $\mathcal{C}$. For each iteration, we randomly pick $\mathsf{u} \in \mathrm{ESW}(q,n)$ and we have three possibilities:

(I) if $d(\mathsf{u},\mathsf{v}) \geq d$ for all $\mathsf{v} \in \mathcal{C}$, then repeat the iteration with $\mathcal{C} \cup \{\mathsf{u}\}$;

(II) if $d(\mathsf{u},\mathsf{v}) < d$ but $d(\mathsf{u},\mathsf{v}) \geq d$ for all $\mathsf{v} \in \mathcal{C} \setminus \{\mathsf{v}_0\}$, we repeat the iteration with $\mathcal{C} \cup \{\mathsf{u}\} \setminus \{\mathsf{v}_0\}$;

(III) otherwise, we repeat the iteration with $\mathcal{C}$ unchanged.

The crucial feature of this algorithm is at Step (II) where we 'alter' the code and prevent the algorithm from being 'trapped at a local maximum'.

Heuristic Search - Partitioning Techniques. A heuristic method that is peculiar to coding theory is to transform a coding problem to an instance of the maximum clique problem defined over a graph with less vertices. More precisely, to determine $A_q^{ESW}(n,d)$, we partition $\mathrm{ESW}_q(n,d)$ into parts $C_1, C_2, \ldots, C_v$ such that C_i is an equitable symbol weight code with distance at least d for each $i \in [v]$. Consider the graph $G^*(q,n,d)$ whose vertices are $C_1, C_2, \ldots, C_v$. For distinct $i,j \in [v]$, we define C_i and C_j to be adjacent if the distance between u and v is at least d for all $\mathsf{u} \in C_i$ and $\mathsf{v} \in C_j$. Hence, a clique in $G^*(q,n,d)$ yields an $(n,d)_q$-equitable symbol weight code.

Similar techniques have been employed in constructing binary codes [60, 62] and permutation codes [20, 73]. Unfortunately, unlike binary codes and permutation codes, we do not have an obvious partitioning of the equitable symbol weight space. In determining the values of Table 3.2 and Table 3.3, we consider partitioning $\mathrm{ESW}(q,n)$ (defined over $\mathbb{Z}_q$) via the equivalence relation $\mathsf{u} \sim \mathsf{v}$ if and only if $\mathsf{u} = \mathsf{v} + (c,c,\ldots,c)$ for some $c \in \mathbb{Z}_q$. In other words, each part is a code of size q and distance n. For convenience, we call such a code a partition code of type I. Then, we used either exact algorithms or heuristic techniques in the previous parts to determine the corresponding lower bounds.

3.3.3 Tables of Equitable Symbol Weight Codes

Finally, we provide estimates of the size of optimal equitable symbol weight codes based on upper/lower bounds and constructions given in this chapter. Here, we look at $q \in \{3, 4\}$ and look at length $n \geq q + 1$, as corresponding tables for injection codes and permutation codes are given by Dukes [28] and Chu *et al.* [20], Smith and Montemanni [73] respectively.

We explain the annotations. The rows marked by 'UB' provides the upper bounds and the bounds corresponding to the superscripts are as follow:

Superscript s: Singleton-type bound from Proposition 3.1.1.

Superscript h: Hamming-type bound from Proposition 3.1.2.

Superscript p: Plotkin bound from Theorem 1.3.3.

The rows marked by 'LB' provides the lower bounds and the bounds/constructions corresponding to the superscripts are as follow:

Superscript cc: A subcode of a coset of an optimal linear code (see (3.8)).

Superscript hc: A code resulting from hill-climbing heuristic.

Superscript pc: Partition code of type I.

Exact values of $A_q^{ESW}(n, d)$ are highlighted in **bold**. The superscripts explain how the values are derived.

Superscript t : Trivial.

Superscript g : Theorem 3.3.1.

Superscript m: Exhaustive search by `MaxCliqueDyn`.

3.4 Concluding Remarks

We determine the value of $\lim_{n\to\infty} \frac{1}{n} \log_q A_q^{ESW}(n, d)$ under the condition that q/n tends to a constant and explore certain constructions of equitable symbol weight codes. In addition, we provide a table listing possible upper and lower bounds for $A_q^{ESW}(n, d)$ when $q \in \{3, 4\}$.

Tab. 3.2: Table of possible values for $A_3^{ESW}(n,d)$ for $4 \le n \le 10$

$n\backslash d$		1	2	3	4	5	6	7	8	9	10
4	UB	$\mathbf{36}^t$	$\mathbf{12}^m$	$\mathbf{6}^g$	$\mathbf{3}^t$						
	LB	–	–	–	–						
5	UB	$\mathbf{90}^t$	$\mathbf{36}^m$	$\mathbf{15}^m$	$\mathbf{6}^g$	$\mathbf{3}^t$					
	LB	–	–	–	–	–					
6	UB	$\mathbf{90}^t$	$\mathbf{90}^t$	$\mathbf{30}^m$	$\mathbf{15}^m$	$\mathbf{3}^m$	$\mathbf{3}^t$				
	LB	–	–	–	–	–	–				
7	UB	$\mathbf{630}^t$	510^s	90^h	$\mathbf{30}^m$	$\mathbf{9}^m$	$\mathbf{3}^m$	$\mathbf{3}^t$			
	LB	–	210^{cc}	72^{hc}	–	–	–	–			
8	UB	$\mathbf{1680}^t$	1050^s	240^h	210^s	36^h	$\mathbf{9}^g$	$\mathbf{3}^m$	$\mathbf{3}^t$		
	LB	–	583^{hc}	141^{pc}	58^{hc}	24^{pc}	–	–	–		
9	UB	$\mathbf{1680}^t$	$\mathbf{1680}^t$	1050^s	510^s	60^h	60^h	$\mathbf{6}^m$	$\mathbf{3}^m$	$\mathbf{3}^t$	
	LB	–	–	312^{hc}	168^{pc}	36^{pc}	24^{pc}	–	–	–	
10	UB	$\mathbf{12600}^t$	11130^s	1400^h	1400^h	190^h	190^h	21^p	$\mathbf{6}^m$	$\mathbf{3}^m$	$\mathbf{3}^t$
	LB	–	4200^{cc}	874^{hc}	238^{hc}	75^{pc}	50^{cc}	12^{hc}	–	–	–

Tab. 3.3: Table of possible values for $A_4^{ESW}(n,d)$ for $4 \le n \le 9$

$n\backslash d$		1	2	3	4	5	6	7	8	9
5	UB	$\mathbf{240}^t$	204^s	34^h	$\mathbf{12}^g$	$\mathbf{4}^t$				
	LB	–	60^{cc}	28^{hc}	–	–				
6	UB	$\mathbf{1080}^t$	600^s	120^h	60^s	$\mathbf{9}^m$	$\mathbf{4}^t$			
	LB	–	360^{cc}	80^{pc}	36^{cc}	–	–			
7	UB	$\mathbf{2520}^t$	1440^s	360^h	204^s	51^h	$\mathbf{8}^g$	$\mathbf{4}^t$		
	LB	–	864^{hc}	216^{hc}	63^{hc}	18^{hc}	–	–		
8	UB	$\mathbf{2520}^t$	$\mathbf{2520}^t$	1440^s	600^s	100^h	60^s	$\mathbf{5}^m$	$\mathbf{4}^t$	
	LB	–	–	672^{hc}	148^{hc}	37^{hc}	15^{hc}	–	–	
9	UB	$\mathbf{30240}^t$	30240^h	3024^h	3024^h	397^h	252^s	28^p	$\mathbf{5}^m$	$\mathbf{4}^t$
	LB	–	7560^{cc}	1916^{hc}	427^{hc}	112^{hc}	36^{hc}	14^{hc}	–	–

We remark that the upper and lower bounds employed are simple generalizations of classical coding techniques and computational methods. More sophisticated tools such as linear programming and genetic algorithms can be used to derive tighter upper and lower bounds respectively. An in-depth study of these methods is part of future research.

4. GENERALIZED BALANCED TOURNAMENT PACKINGS AND OPTIMAL EQUITABLE SYMBOL WEIGHT CODES

This chapter is devoted to the construction of optimal equitable symbol weight codes. The construction implies the proof of Theorem 3.3.1 and is based on the construction of an equivalent class of combinatorial objects called generalized balanced tournament packings (GBTPs). These packings extend the concept of generalized balanced tournament designs (GBTDs) introduced by Lamken and Vanstone [46] and our methods reveal an interesting interplay between coding theory and combinatorial design theory.

In particular, we formally define GBTPs and establish the equivalence between GBTPs and equitable symbol weight codes. In Section 4.2, we establish two classes of GBTPs that correspond to optimal equitable symbol weight codes and subsequently settle the existence of these two classes of GBTPs in the rest of the chapter. This chapter has been presented in part at the IEEE International Symposium on Information Theory, 2012 [12]

4.1 Preliminaries

Throughout this chapter, let Σ denote an alphabet of size q and n denote the code length. Let $\mathsf{u} \in \Sigma^n$. For $\sigma \in \Sigma$, recall that $w_\sigma(\mathsf{u})$ is the number of times the symbol σ appears among the coordinates of u.

A codeword $\mathsf{u} \in \Sigma^n$ has *equitable symbol weight* if $w_\sigma(\mathsf{u}) \in \{\lfloor n/q \rfloor, \lceil n/q \rceil\}$ for any $\sigma \in \Sigma$. If all the codewords of $\mathcal{C}$ have equitable symbol weight, then the code $\mathcal{C}$ is called an *equitable symbol weight code*. Denote the maximum size of an $(n,d)_q$-equitable symbol weight code by $A_q^{ESW}(n,d)$. Any $(n,d)_q$-equitable symbol weight code of size $A_q^{ESW}(n,d)$ is said to be *optimal.*

Taken as a q-ary code of length n, an optimal $(n,d)_q$-equitable symbol weight code satisfies the generalized Plotkin bound [44, Ch.2, Theorem 2.82, Corollary 2.84, Theorem 2.86].

Theorem 4.1.1 (Generalized Plotkin Bound [44])**.** *If there is an $(n,d)_q$-code $\mathcal{C}$ of size M, then*

$$\binom{M}{2} d \leq n \sum_{i=0}^{q-2} \sum_{j=i+1}^{q-1} M_i M_j, \tag{4.1}$$

where $M_i = \lfloor (M+i)/q \rfloor$. If $M \equiv 0 \bmod q$ and $\binom{M}{2} d = n\binom{q}{2}(M/q)^2$, then $\mathcal{C}$ is optimal.

In particular, the codes constructed in Theorem 3.3.1 meet this generalized Plotkin bound with the exception of certain small values of n, q and d. For these small values (with the exception of $A_6^{ESW}(9,8)$), an exhaustive computer search established their respective values of $A_q^{ESW}(n,d)$. For $A_6^{ESW}(9,8)$, a $(9,8)_6$-equitable symbol weight code of size 14 was found via computer search. Since a $(9,8)_6$-equitable symbol weight code of size 15 cannot exist by the generalized Plotkin bound, it follows that $A_6^{ESW}(9,8) = 14$. We record the results of the computations in the following proposition and the corresponding optimal codes can be found at [11].

Proposition 4.1.2. *We have the following:*

$$\begin{array}{lll}
A_2^{ESW}(3,2) = 3 & A_2^{ESW}(5,3) = 4 & A_2^{ESW}(7,4) = 7 \\
A_3^{ESW}(3,2) = 6 & A_3^{ESW}(4,3) = 6 & A_3^{ESW}(11,8) = 11 \\
A_4^{ESW}(5,4) = 12 & A_5^{ESW}(7,6) = 14 & A_6^{ESW}(9,8) = 14.
\end{array}$$

The rest of the chapter establishes the remaining values in Theorem 3.3.1.

4.2 Generalized Balanced Tournament Packings

A *set system* is a pair $\mathfrak{S} = (X, \mathcal{A})$, where X is a finite set of *points* and $\mathcal{A} \subseteq 2^X$. Elements of $\mathcal{A}$ are called *blocks*. The *order* of $\mathfrak{S}$ is the number of points in X, and the *size* of $\mathfrak{S}$ is the number of blocks in $\mathcal{A}$. Let K be a set of nonnegative integers. The set system $(X, \mathcal{A})$ is said to be K-*uniform* if $|A| \in K$ for all $A \in \mathcal{A}$.

Let λ, v be positive integers and K be a set of nonnegative integers. A (v, K, λ)-*packing* is a K-uniform set system of order v such that every pair of distinct points is contained in at most λ blocks. A *parallel class* (or *resolution class*) of a packing is a subset of the blocks that partitions the set of points X. If the set of blocks can be partitioned into parallel classes, then the packing is *resolvable*, and denoted by $\mathrm{RP}(v, K, \lambda)$. An $\mathrm{RP}(v, K, \lambda)$ is called a *maximum resolvable packing*, denoted by $\mathrm{MRP}(v, K, \lambda)$, if it contains maximum possible number of parallel classes.

Furthermore, an $\mathrm{MRP}(v, \{k\}, \lambda)$ is called a *resolvable* $(v, \{k\}, \lambda)$-*balanced incomplete block design*, or $\mathrm{RBIBD}(v, k, \lambda)$ in short, if every pair of distinct points is contained in exactly λ blocks. A simple computation gives the size of an $\mathrm{RBIBD}(v, k, \lambda)$ to be $\frac{\lambda v(v-1)}{k(k-1)}$.

Definition 4.2.1. Let $(X, \mathcal{A})$ be an $\mathrm{RP}(v, K, \lambda)$ with n parallel classes. Then $(X, \mathcal{A})$ is called a *generalized balanced tournament packing* if the blocks of $\mathcal{A}$ are arranged into an $m \times n$ array satisfying the following conditions:

(i) every point in X is contained in exactly one cell of each column,

(ii) every point in X is contained in either $\lceil n/m \rceil$ or $\lfloor n/m \rfloor$ cells of each row.

We denote such a GBTP by $\mathrm{GBTP}_\lambda(K; v, m \times n)$.

Unless otherwise stated, the rows of a $\mathrm{GBTP}_\lambda(K; v, m \times n)$ are indexed by $[m]$ and the columns by $[n]$.

In a $\mathrm{GBTP}_\lambda(K; v, m \times n)$, given point x and column j, there is a unique row that contains the point x in column j. Hence, for each point $x \in X$ of a $\mathrm{GBTP}_\lambda(K; v, m \times n)$ $(X, \mathcal{A})$, we may correspond the codeword $\mathsf{c}(x) = (r_1, r_2, \ldots, r_n) \in [m]^n$, where r_j is the row in which point x appears in column j. It is obvious that $\mathcal{C} = \{\mathsf{c}(x) : x \in X\}$ is an m-ary code of length n over the alphabet $[m]$. We note that this correspondence is precisely the one used by Semakov and Zinoviev [70] to show the equivalence between *equidistant codes* and resolvable balanced incomplete block designs.

For distinct points $x, y \in X$, the distance between $\mathsf{c}(x)$ and $\mathsf{c}(y)$ is the number of columns for which x and y are not both contained in the same row. Since there are at most λ blocks containing both x and y, and that no two such blocks can occur in the same column of the $\text{GBTP}_\lambda(K; v, m \times n)$, the distance between $\mathsf{c}(x)$ and $\mathsf{c}(y)$ is at least $n - \lambda$.

Next, we determine $w_i(\mathsf{c}(x))$, for $x \in X$ and $i \in [m]$. From the construction of $\mathsf{c}(x)$, the number of times a symbol i appears in $\mathsf{c}(x)$ is the number of cells in row i that contains x. By the definition of a $\text{GBTP}_\lambda(K; v, m \times n)$, this number belongs to $\{\lfloor n/m \rfloor, \lceil n/m \rceil\}$. Hence, $\mathcal{C}$ is an equitable symbol weight code with size v.

Finally, this construction of an equitable symbol weight code from a generalized balanced tournament packing can easily be reversed. We record these observations as:

Theorem 4.2.2. *Let K be set of nonnegative integers. Then a $\text{GBTP}_\lambda(K; v, m \times n)$ exists if and only if an $(n, n-\lambda)_m$-equitable symbol weight code of size v exists.*

Example 4.2.3. Consider the $\text{GBTP}_1(\{2, 3\}; 6, 3 \times 4)$ below.

	$\{1,4\}$	$\{2,6\}$	$\{3,5\}$
$\{1,2,3\}$	$\{2,5\}$	$\{3,4\}$	$\{1,6\}$
$\{4,5,6\}$	$\{3,6\}$	$\{1,5\}$	$\{2,4\}$

Each point $x \in [6]$ gives a codeword $\mathsf{c}(x) = (r_1, r_2, \ldots, r_5)$, where r_j is the row in which point x appears in column j. Hence, we have

$$\begin{array}{lll} \mathsf{c}(1) = (2,1,3,2), & \mathsf{c}(2) = (2,2,1,3), & \mathsf{c}(3) = (2,3,2,1), \\ \mathsf{c}(4) = (3,1,2,3), & \mathsf{c}(5) = (3,2,3,1), & \mathsf{c}(6) = (3,3,1,2). \end{array}$$

Hence, $\{\mathsf{c}(1), \mathsf{c}(2), \mathsf{c}(3), \mathsf{c}(4), \mathsf{c}(5), \mathsf{c}(6)\}$ is a $(4,3)_3$-equitable symbol weight code of size six.

Theorem 4.2.2 set up the equivalence between GBTPs and equitable symbol weight codes. In general, a GBTP may not correspond to an optimal equitable symbol weight code. However, in the following, we look at specific K to derive families of optimal equitable symbol weight codes.

4.2.1 Optimal Equitable Symbol Weight Codes from Generalized Balanced Tournament Designs

A $\mathrm{GBTP}_\lambda\left(\{k\}; km, m \times \frac{\lambda(km-1)}{k-1}\right)$ is a *generalized balanced tournament design* (GBTD), denoted by $\mathrm{GBTD}_\lambda(k,m)$. In this case, we check that each pair of distinct points is contained in exactly λ blocks and every point is contained in either $\left\lceil \frac{\lambda(km-1)}{m(k-1)} \right\rceil$ or $\left\lfloor \frac{\lambda(km-1)}{m(k-1)} \right\rfloor$ cells of each row.

Applying Theorem 4.2.2, a $\left(\frac{\lambda(km-1)}{k-1}, \frac{\lambda k(m-1)}{k-1}\right)_m$-equitable symbol weight code of size km exists and the corresponding code is optimal by generalized Plotkin bound. So, we have the following.

Theorem 4.2.4. *A $GBTD_\lambda(k,m)$ exists if and only if an optimal $\left(\frac{\lambda(km-1)}{k-1}, \frac{\lambda k(m-1)}{k-1}\right)_m$-equitable symbol weight code of size km exists and attains the generalized Plotkin bound.*

We remark that our definition of a generalized balanced tournament design extends that of Lamken and Vanstone [46], which corresponds in our definition to the case when $\lambda = k-1$. The following summarizes the state-of-the-art results on the existence of $\mathrm{GBTD}_{k-1}(k,m)$.

Theorem 4.2.5 (Lamken [46–49], Yin *et al.* [84], Chee *et al.* [17], Dai *et al.* [23])**.** *The following holds.*

(i) A $GBTD_1(2,m)$ exists if and only if $m = 1$ or $m \geq 3$.

(ii) A $GBTD_2(3,m)$ exists if and only if $m = 1$ or $m \geq 3$.

(iii) A $GBTD_3(4,m)$ exists if and only if $m = 1$ or $m \geq 4$.

(iv) A $GBTD_4(5,m)$ exists if $m \geq 62$ or $m \in \{5-18, 30, 42, 46, 48-50, 54-57\}$.

(v) A $GBTD_{k-1}(k,k)$ exists if k is an odd prime power.

Theorem 3.3.1(i) to (v) is now an immediate consequence of Theorem 4.2.4, Theorem 4.2.5. and Proposition 4.1.2. The existence of $\mathrm{GBTD}_\lambda(k,m)$ when $\lambda \neq k-1$ has not been previously investigated. The smallest open case is when $k = 3$ and $\lambda = 1$, which is the case dealt with in this chapter.

It follows readily from the fact that a $\text{GBTD}_1(3, m)$ is also an $\text{RBIBD}(3m, 3, 1)$, that a necessary condition for a $\text{GBTD}_1(3, m)$ to exist is that m must be odd. We note from Proposition 4.1.2 that $A_3^{ESW}(4, 3) = 6$ and $A_5^{ESW}(7, 6) = 14$, which do not meet the Plotkin bound. Hence, the corresponding designs $\text{GBTD}_1(3, 3)$ and $\text{GBTD}_1(3, 5)$ do not exist by Theorem 4.2.4.

Hence, a $\text{GBTD}_1(3, m)$ can exist only if m is odd and $m \notin \{3, 5\}$. In Sections 4.3 to 4.6, we prove that this necessary condition is also sufficient for the existence of $\text{GBTD}_1(3, m)$. A direct consequence of this is Theorem 3.3.1(vi).

4.2.2 Optimal Equitable Symbol Weight Codes a class of GBTPs

Theorem 4.2.4 constructs optimal equitable symbol weight codes from GBTDs. In this subsection, we make slight variations to obtain another infinite family of optimal equitable symbol weight codes.

Consider a $\text{GBTP}_1(\{2, 3\}; v, m \times n)$. If there is exactly one block of size three in each resolution class, then we denote the GBTP by $\text{GBTP}_1(\{2, 3^*\}; v, m \times n)$. A simple computation then shows $v = 2m + 1$. Now we establish the following construction for optimal equitable symbol weight codes.

Theorem 4.2.6. *Let $m \geq 7$. If there exists a $GBTP_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$, then there exists an optimal $(2m - 3, 2m - 4)_m$-equitable symbol weight code of size $2m + 1$ which attains the generalized Plotkin bound.*

Proof. By Theorem 4.2.2, we have a $(2m-3, 2m-4)_m$-equitable symbol weight code of size $2m + 1$. It remains to verify its optimality.

Suppose otherwise that there exists a $(2m - 3, 2m - 4)_m$-equitable symbol weight code of size $2m + 2$. Consider (4.1) in Theorem 4.1.1. On the left hand side, we have

$$\binom{2m+2}{2} \cdot (2m - 4) = 4m^3 - 2m^2 - 10m - 4.$$

Since $\left\lfloor \frac{2m+2+i}{m} \right\rfloor = 2$ for $0 \leq i \leq m - 3$ and $\left\lfloor \frac{2m+2+(m-2)}{m} \right\rfloor = \left\lfloor \frac{2m+2+(m-1)}{m} \right\rfloor = 3$, the term on the right hand side is

$$
\begin{aligned}
(2m-3)\left(\left(\sum_{i=0}^{m-3} 4(m-3-i)+12\right)+9\right) &= (2m-3)(4m(m-2)-2(m-3)(m-2)+9)\\
&= 4m^3-2m^2-12m+9
\end{aligned}
$$

But for $m \geq 7$,

$$4m^3-2m^2-10m-4 > 4m^3-2m^2-12m+9,$$

contradicting (4.1). Hence, a $(2m-3, 2m-4)_m$-equitable symbol weight code of size $2m+2$ does not exist and the result follows. ■

In the rest of this chapter, we construct a $\text{GBTP}_1(\{2, 3^*\}; 2m+1, m \times (2m-3))$ for $m \geq 4$, except possibly $m \in \{12, 13\}$. This with Theorem 4.2.6 and Proposition 4.1.2 gives Theorem 3.3.1(vii).

4.3 Proof Strategy

For the rest of the chapter, we determine with finite possible exceptions the existence of $\text{GBTD}_1(3, m)$ and $\text{GBTP}_1(\{2, 3^*\}; 2m+1, m \times (2m-3))$. Our proof is technical and rather complex. This section outlines the general strategy used and introduces the required auxiliary designs.

As with most combinatorial designs, direct construction to settle their existence is often difficult. Instead, we develop a set of recursive constructions, building big designs from smaller ones. Direct methods are used to construct a large enough set of small designs on which the recursions can work to generate all larger designs. For our recursion techniques to work, the generalized balanced tournament packing must possess more structure than stipulated in its definition. First, we consider $\text{GBTD}_1(3, m)$s that are $*$*colorable* which are defined below.

$0_0 0_1 \infty$ ♣	$2_0 4_0 3_1$ ♣	$6_1 4_0 1_0$ ♢	$2_1 1_1 6_1$ ♣	$5_1 1_0 2_0$ ♡	$4_1 3_1 1_1$ ♡	$5_1 4_1 2_1$ ♢
$6_1 5_1 3_1$ ♣	$1_0 1_1 \infty$ ♣	$3_0 5_0 4_1$ ♣	$3_0 3_1 \infty$ ♢	$5_0 0_0 6_1$ ♢	$6_0 1_0 0_1$ ♢	$0_0 2_0 1_1$ ♡
$1_0 3_0 2_1$ ♣	$0_1 6_1 4_1$ ♣	$2_0 2_1 \infty$ ♢	$4_0 6_0 5_1$ ♣	$1_1 6_0 3_0$ ♢	$5_0 5_1 \infty$ ♡	$3_1 1_0 5_0$ ♢
$4_1 2_0 6_0$ ♣	$5_1 3_0 0_0$ ♣	$1_1 0_1 5_1$ ♢	$0_1 5_0 2_0$ ♡	$4_0 4_1 \infty$ ♢	$2_1 0_0 4_0$ ♡	$6_0 6_1 \infty$ ♡
$1_1 4_0 5_0$ ♣	$2_1 5_0 6_0$ ♢	$3_1 6_0 0_0$ ♣	$4_1 0_0 1_0$ ♢	$3_1 2_1 0_1$ ♡	$6_1 2_0 3_0$ ♣	$0_1 3_0 4_0$ ♢

Fig. 4.1: A 3-$*$colorable RBIBD$(15, 3, 1)$ $(X, \mathcal{A})$, where $X = (\mathbb{Z}_7 \times \mathbb{Z}_2) \cup \{\infty\}$. The set of colors used is $\{♣, ♢, ♡\}$. $(X, \mathcal{A})$ has property Π as 1_0 is a witness for ♣ and ∞ is a witnesses for both ♢ and ♡ in row 1. For succinctness, a block $\{x, y, z\}$ is written xyz

4.3.1 c-$*$colorable Generalized Balanced Tournament Designs

Definition 4.3.1. A *c-$*$colorable* RBIBD(v, k, λ) is an RBIBD(v, k, λ) with the property that its $\frac{\lambda v(v-1)}{k(k-1)}$ blocks can be arranged in a $\frac{v}{k} \times \frac{\lambda(v-1)}{k-1}$ array, and each block can be colored with one of c colors so that

(i) each point appears exactly once in each column, and

(ii) in each row, blocks of the same color are pairwise disjoint.

Definition 4.3.2. A GBTD$_\lambda(k, m)$ is *c-$*$colorable* if each of its blocks can be colored with one of c colors so that in each row, blocks of the same color are pairwise disjoint.

Definition 4.3.3. A c-$*$colorable RBIBD$(v, k, 1)$ is *c-$*$colorable with property* Π if there exists a row r such that for each color i, there exists a point (called a *witness* for i) that is not contained in any block in row r that is colored i.

A GBTD$_1(k, m)$ that is *c-$*$colorable with property* Π is similarly defined.

Example 4.3.4. The RBIBD$(15, 3, 1)$ in Fig. 4.1 is 3-$*$colorable with property Π.

Proposition 4.3.5. *If an RBIBD$(v, k, 1)$ is $(k-1)$-$*$colorable, then it is k-$*$colorable with property* Π.

Proof. Consider a $(k-1)$-$*$colorable RBIBD$(v, k, 1)$ with colors $c_1, c_2, \cdots, c_{k-1}$. There must exists a point, say x, that appears only once in the first row. Recolor the block that contains this point with color c_k. This new coloring shows that the RBIBD$(v, k, 1)$ is k-$*$colorable with property Π, since for the first row, the point x is a witness for the colors $c_1, c_2, \ldots, c_{k-1}$, while any point not in the block colored c_k is a witness for color c_k. ■

Example 4.3.6. The $\mathrm{GBTD}_1(3, 9)$ in Fig. 4.2 is 2-$*$colorable and is therefore 3-$*$colorable with property Π by Proposition 4.3.5.

4.3.2 Ingredient Generalized Balanced Tournament Packings

Suppose that $(X, \mathcal{A})$ is a (v, K, λ)-packing. Let $W \subset X$ with $|W| = w$. Furthermore, we call $(X, W, \mathcal{A})$ is an *ingredient resolvable packing*, denoted by $\mathrm{IRP}(v, K, \lambda; w)$, if it satisfies the following conditions:

(i) any pair of points from W occurs in no blocks of $\mathcal{A}$,

(ii) the blocks in $\mathcal{A}$ can be partitioned into parallel classes and *partial parallel classes* $X \setminus W$.

Definition 4.3.7. Let $(X, W, \mathcal{A})$ be an $\mathrm{IRP}(v, K, \lambda; w)$. Then $(X, W, \mathcal{A})$ is called an *ingredient generalized balanced tournament packing* (IGBTP) if the blocks of $\mathcal{A}$ are arranged into an $m \times n$ array A, with rows and columns indexed by R and C respectively, satisfying the following conditions:

(i) there exist a $P \subset R$ with $|P| = m'$ and a $Q \subset C$ with $|Q| = n'$ such that the cell (r, c) is empty if $r \in P$ and $c \in Q$;

(ii) for any row $r \in P$, every point in $X \setminus W$ is contained in either $\lceil n/m \rceil$ or $\lfloor n/m \rfloor$ cells and the points in W do not appear; for any row $r \in R \setminus P$, every point in X is contained in either $\lceil n/m \rceil$ or $\lfloor n/m \rfloor$ cells;

(iii) the blocks in any column $c \in Q$ form a partial parallel class of $X \setminus W$ and the blocks in any column $c \in C \setminus Q$ forms a parallel class of X.

Denote such an IGBTP by $\mathrm{IGBTP}_\lambda(K, v, m \times n; w, m' \times n')$.

Example 4.3.8. An $\mathrm{IGBTP}_1(\{2, 3^*\}, 29, 14 \times 25; 9, 4 \times 5)$ is given in Fig. 4.3.

Consider an $\mathrm{IGBTP}_1(\{k\}, km, m \times \frac{km-1}{k-1}; k, 1 \times 1)$. Then its corresponding array has one empty cell and we fill this cell with the block W to obtain a $\mathrm{GBTD}_1(k, m)$. A $\mathrm{GBTD}_1(k, m)$ obtained in this way is called a *special* $\mathrm{GBTD}_1(k, m)$ and the cell occupied by W is said to be *special*.

A	B

where A is the array

$1_0 7_1 \infty_2 \ \clubsuit$	$6_0 3_2 \infty_1 \ \clubsuit$	$0_0 4_1 6_2 \ \clubsuit$	$6_0 7_1 0_2 \ \diamondsuit$	$7_0 0_1 1_2 \ \clubsuit$	$5_0 5_1 5_2 \ \diamondsuit$	$1_1 4_2 \infty_0 \ \clubsuit$
$5_0 2_2 \infty_1 \ \diamondsuit$	$2_0 0_1 \infty_2 \ \clubsuit$	$7_0 4_2 \infty_1 \ \clubsuit$	$1_0 5_1 7_2 \ \clubsuit$	$4_0 4_1 4_2 \ \diamondsuit$	$0_0 1_1 2_2 \ \clubsuit$	$7_0 5_1 \infty_2 \ \diamondsuit$
$3_1 6_2 \infty_0 \ \clubsuit$	$4_1 7_2 \infty_0 \ \diamondsuit$	$3_0 1_1 \infty_2 \ \clubsuit$	$0_0 5_2 \infty_1 \ \clubsuit$	$2_0 6_1 0_2 \ \clubsuit$	$3_0 7_1 1_2 \ \diamondsuit$	$1_0 2_1 3_2 \ \clubsuit$
$3_0 4_1 5_2 \ \clubsuit$	$1_0 1_1 1_2 \ \diamondsuit$	$5_1 0_2 \infty_0 \ \clubsuit$	$4_0 2_1 \infty_2 \ \diamondsuit$	$5_0 3_1 \infty_2 \ \clubsuit$	$2_0 7_2 \infty_1 \ \clubsuit$	$4_0 0_1 2_2 \ \clubsuit$
$6_0 2_1 4_2 \ \diamondsuit$	$4_0 5_1 6_2 \ \clubsuit$	$5_0 6_1 7_2 \ \diamondsuit$	$6_1 1_2 \infty_0 \ \clubsuit$	$1_0 6_2 \infty_1 \ \diamondsuit$	$6_0 4_1 \infty_2 \ \clubsuit$	$3_0 0_2 \infty_1 \ \clubsuit$
$0_0 0_1 0_2 \ \diamondsuit$	$5_0 2_1 0_2 \ \clubsuit$	$4_0 7_1 3_2 \ \clubsuit$	$2_0 1_1 6_2 \ \clubsuit$	$7_1 2_2 \infty_0 \ \diamondsuit$	$1_0 6_1 4_2 \ \clubsuit$	$0_0 3_1 7_2 \ \clubsuit$
$7_0 6_1 3_2 \ \clubsuit$	$7_0 3_1 5_2 \ \diamondsuit$	$6_0 3_1 1_2 \ \clubsuit$	$5_0 0_1 4_2 \ \clubsuit$	$3_0 2_1 7_2 \ \clubsuit$	$0_1 3_2 \infty_0 \ \diamondsuit$	$2_0 7_1 5_2 \ \clubsuit$
$2_0 5_1 1_2 \ \clubsuit$	$0_0 7_1 4_2 \ \clubsuit$	$2_0 2_1 2_2 \ \diamondsuit$	$7_0 4_1 2_2 \ \clubsuit$	$6_0 1_1 5_2 \ \clubsuit$	$4_0 3_1 0_2 \ \clubsuit$	$6_0 6_1 6_2 \ \diamondsuit$
$4_0 1_1 7_2 \ \clubsuit$	$3_0 6_1 2_2 \ \clubsuit$	$1_0 0_1 5_2 \ \clubsuit$	$3_0 3_1 3_2 \ \diamondsuit$	$0_0 5_1 3_2 \ \clubsuit$	$7_0 2_1 6_2 \ \clubsuit$	$5_0 4_1 1_2 \ \clubsuit$

where B is the array

$0_0 6_1 \infty_2 \ \diamondsuit$	$4_0 6_1 5_2 \ \clubsuit$	$1_0 2_0 4_0 \ \diamondsuit$	$2_0 3_0 5_0 \ \clubsuit$	$2_2 7_2 0_2 \ \clubsuit$	$2_1 3_1 5_1 \ \clubsuit$
$2_1 5_2 \infty_0 \ \clubsuit$	$5_0 7_1 6_2 \ \clubsuit$	$3_1 4_1 6_1 \ \clubsuit$	$0_1 1_1 3_1 \ \diamondsuit$	$3_0 4_0 6_0 \ \clubsuit$	$3_2 0_2 1_2 \ \clubsuit$
$2_0 3_1 4_2 \ \diamondsuit$	$6_0 0_1 7_2 \ \clubsuit$	$4_2 1_2 2_2 \ \clubsuit$	$4_1 5_1 7_1 \ \clubsuit$	$5_1 6_1 0_1 \ \diamondsuit$	$4_0 5_0 7_0 \ \clubsuit$
$7_0 7_1 7_2 \ \diamondsuit$	$0_0 2_1 1_2 \ \clubsuit$	$0_2 5_2 6_2 \ \diamondsuit$	$6_0 7_0 1_0 \ \clubsuit$	$6_2 3_2 4_2 \ \clubsuit$	$6_1 7_1 1_1 \ \clubsuit$
$5_0 1_1 3_2 \ \clubsuit$	$1_0 3_1 2_2 \ \clubsuit$	$7_1 0_1 2_1 \ \clubsuit$	$5_2 2_2 3_2 \ \diamondsuit$	$7_0 0_0 2_0 \ \clubsuit$	$7_2 4_2 5_2 \ \clubsuit$
$6_0 5_1 2_2 \ \clubsuit$	$2_0 4_1 3_2 \ \diamondsuit$	$3_0 7_0 \infty_0 \ \clubsuit$	$1_2 6_2 7_2 \ \diamondsuit$	$5_2 1_2 \infty_2 \ \clubsuit$	$4_1 0_1 \infty_1 \ \clubsuit$
$1_0 4_1 0_2 \ \clubsuit$	$3_0 5_1 4_2 \ \diamondsuit$	$1_1 5_1 \infty_1 \ \clubsuit$	$4_0 0_0 \infty_0 \ \clubsuit$	$1_1 2_1 4_1 \ \diamondsuit$	$6_2 2_2 \infty_2 \ \clubsuit$
$3_0 0_1 6_2 \ \clubsuit$	$\infty_0 \infty_1 \infty_2 \ \diamondsuit$	$3_2 7_2 \infty_2 \ \clubsuit$	$2_1 6_1 \infty_1 \ \clubsuit$	$5_0 1_0 \infty_0 \ \clubsuit$	$0_0 1_0 3_0 \ \diamondsuit$
$4_0 1_2 \infty_1 \ \diamondsuit$	$7_0 1_1 0_2 \ \diamondsuit$	$5_0 6_0 0_0 \ \diamondsuit$	$4_2 0_2 \infty_2 \ \clubsuit$	$3_1 7_1 \infty_1 \ \clubsuit$	$6_0 2_0 \infty_0 \ \clubsuit$

Fig. 4.2: A 2-$*$colorable special $\mathrm{GBTD}_1(3,9)$ $(X,\mathcal{A})$, where $X = (\mathbb{Z}_8 \times \mathbb{Z}_3) \cup \{\infty_0, \infty_1, \infty_2\}$ and colors $\{\clubsuit, \diamondsuit\}$. The cell $(1,5)$, occupied by the block $7_0 0_1 1_2$, is special. For succinctness, a set $\{x,y,z\}$ is written xyz.

Example 4.3.9. The $\mathrm{GBTD}_1(3,9)$ in Fig. 4.2 is a special $\mathrm{GBTD}_1(3,9)$ with special cell $(1,5)$.

A few more classes of auxiliary designs are also required.

4.3.3 Group Divisible Designs and Transversal Designs

Definition 4.3.10. Let $(X,\mathcal{A})$ be a set system and let $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ be a partition of X into subsets, called *groups*. The triple $(X,\mathcal{G},\mathcal{A})$ is a *group divisible design* (GDD) when every 2-subset of X not contained in a group appears in exactly one block, and $|A \cap G| \leq 1$

A	B

where A is the array

–	–	–	–	–	$2,13$	$3,14$	$4,15$	$5,16$	$6,17$	$7,18$	$8,19$	$9,0$
–	–	–	–	–	$12,16$	$13,17$	$14,18$	$15,19$	$16,0$	$17,1$	$18,2$	$19,3$
–	–	–	–	–	$15,18$	$16,19$	$17,0$	$18,1$	$19,2$	$0,3$	$1,4$	$2,5$
–	–	–	–	–	$1,3$	$2,4$	$3,5$	$4,6$	$5,7$	$6,8$	$7,9$	$8,10$
$0,10$	$2,7$	$12,17$	$4,16$	$14,6$	$4,5,11$	$i,18$	$h,1$	$g,12$	$f,18$	$e,13$	$d,16$	$c,13$
$1,11$	$3,8$	$13,18$	$5,17$	$15,7$	$a,0$	$5,6,12$	$i,19$	$h,2$	$g,13$	$f,19$	$e,14$	$d,17$
$2,12$	$4,9$	$14,19$	$6,18$	$16,8$	$b,7$	$a,1$	$6,7,13$	$i,0$	$h,3$	$g,14$	$f,0$	$e,15$
$3,13$	$5,10$	$15,0$	$7,19$	$17,9$	$c,6$	$b,8$	$a,2$	$7,8,14$	$i,1$	$h,4$	$g,15$	$f,1$
$4,14$	$6,11$	$16,1$	$8,0$	$18,10$	$d,10$	$c,7$	$b,9$	$a,3$	$8,9,15$	$i,2$	$h,5$	$g,16$
$5,15$	$7,12$	$17,2$	$9,1$	$19,11$	$e,8$	$d,11$	$c,8$	$b,10$	$a,4$	$9,10,16$	$i,3$	$h,6$
$6,16$	$8,13$	$18,3$	$10,2$	$0,12$	$f,14$	$e,9$	$d,12$	$c,9$	$b,11$	$a,5$	$10,11,17$	$i,4$
$7,17$	$9,14$	$19,4$	$11,3$	$1,13$	$g,9$	$f,15$	$e,10$	$d,13$	$c,10$	$b,12$	$a,6$	$11,12,18$
$8,18$	$10,15$	$0,5$	$12,4$	$2,14$	$h,19$	$g,10$	$f,16$	$e,11$	$d,14$	$c,11$	$b,13$	$a,7$
$9,19$	$11,16$	$1,6$	$13,5$	$3,15$	$i,17$	$h,0$	$g,11$	$f,17$	$e,12$	$d,15$	$c,12$	$b,14$

where B is the array

$10,1$	$11,2$	$12,3$	$13,4$	$14,5$	$15,6$	$16,7$	$17,8$	$18,9$	$19,10$	$0,11$	$1,12$
$0,4$	$1,5$	$2,6$	$3,7$	$4,8$	$5,9$	$6,10$	$7,11$	$8,12$	$9,13$	$10,14$	$11,15$
$3,6$	$4,7$	$5,8$	$6,9$	$7,10$	$8,11$	$9,12$	$10,13$	$11,14$	$12,15$	$13,16$	$14,17$
$9,11$	$10,12$	$11,13$	$12,14$	$13,15$	$14,16$	$15,17$	$16,18$	$17,19$	$18,0$	$19,1$	$0,2$
$b,15$	$a,9$	$14,15,1$	$i,8$	$h,11$	$g,2$	$f,8$	$e,3$	$d,6$	$c,3$	$b,5$	$a,19$
$c,14$	$b,16$	$a,10$	$15,16,2$	$i,9$	$h,12$	$g,3$	$f,9$	$e,4$	$d,7$	$c,4$	$b,6$
$d,18$	$c,15$	$b,17$	$a,11$	$16,17,3$	$i,10$	$h,13$	$g,4$	$f,10$	$e,5$	$d,8$	$c,5$
$e,16$	$d,19$	$c,16$	$b,18$	$a,12$	$17,18,4$	$i,11$	$h,14$	$g,5$	$f,11$	$e,6$	$d,9$
$f,2$	$e,17$	$d,0$	$c,17$	$b,19$	$a,13$	$18,19,5$	$i,12$	$h,15$	$g,6$	$f,12$	$e,7$
$g,17$	$f,3$	$e,18$	$d,1$	$c,18$	$b,0$	$a,14$	$19,0,6$	$i,13$	$h,16$	$g,7$	$f,13$
$h,7$	$g,18$	$f,4$	$e,19$	$d,2$	$c,19$	$b,1$	$a,15$	$0,1,7$	$i,14$	$h,17$	$g,8$
$i,5$	$h,8$	$g,19$	$f,5$	$e,0$	$d,3$	$c,0$	$b,2$	$a,16$	$1,2,8$	$i,15$	$h,18$
$12,13,19$	$i,6$	$h,9$	$g,0$	$f,6$	$e,1$	$d,4$	$c,1$	$b,3$	$a,17$	$2,3,9$	$i,16$
$a,8$	$13,14,0$	$i,7$	$h,10$	$g,1$	$f,7$	$e,2$	$d,5$	$c,2$	$b,4$	$a,18$	$3,4,10$

Fig. 4.3: An $\mathrm{IGBTP}_1(\{2,3\},29,14\times 25;9,4\times 5)$ $(X,\mathcal{A})$, where $X = \mathbb{Z}_{20} \cup \{a,b,c,d,e,f,g,h,i\}$ and $W = \{a,b,c,d,e,f,g,h,i\}$. For succinctness, a block $\{x,y,z\}$ is written x,y,z.

for $A \in \mathcal{A}$ and $G \in \mathcal{G}$.

We denote a GDD $(X,\mathcal{G},\mathcal{A})$ by K-GDD if $(X,\mathcal{A})$ is K-uniform. The *type* of a GDD $(X,\mathcal{G},\mathcal{A})$ is the multiset $\langle |G| : G \in \mathcal{G}\rangle$. For convenience, the exponential notation is used to

A	B

where A is the array

–	–	–	$4_01_07_0$	$4_11_17_1$	$4_21_27_2$	$6_03_09_0$	$6_13_19_1$	$6_23_29_2$
–	–	–	$6_07_28_0$	$6_17_08_1$	$6_27_18_2$	$8_09_20_0$	$8_19_00_1$	$8_29_10_2$
$2_08_11_0$	$2_18_21_1$	$2_28_01_2$	–	–	–	$4_18_2\infty_4$	$4_28_0\infty_3$	$4_08_1\infty_5$
$6_27_23_1$	$6_07_03_2$	$6_17_13_0$	–	–	–	$9_11_2\infty_5$	$9_21_0\infty_4$	$9_01_1\infty_3$
$4_00_13_0$	$4_10_23_1$	$4_20_03_2$	$1_13_09_1$	$1_23_19_2$	$1_03_29_0$	–	–	–
$8_29_25_1$	$8_09_05_2$	$8_19_15_0$	$4_26_18_2$	$4_06_28_0$	$4_16_08_1$	–	–	–
$6_02_15_0$	$6_12_25_1$	$6_22_05_2$	$8_11_2\infty_0$	$8_21_0\infty_1$	$8_01_1\infty_2$	$3_15_01_1$	$3_25_11_2$	$3_05_21_0$
$0_21_27_1$	$0_01_07_2$	$0_11_17_0$	$2_03_1\infty_1$	$2_13_2\infty_2$	$2_23_0\infty_0$	$6_28_10_2$	$6_08_20_0$	$6_18_00_1$
$8_04_17_0$	$8_14_27_1$	$8_24_07_2$	$2_29_0\infty_2$	$2_09_1\infty_0$	$2_19_2\infty_1$	$0_13_2\infty_0$	$0_23_0\infty_1$	$0_03_1\infty_2$
$2_23_29_1$	$2_03_09_2$	$2_13_19_0$	$3_24_1\infty_3$	$3_04_2\infty_5$	$3_14_0\infty_4$	$4_05_1\infty_1$	$4_15_2\infty_2$	$4_25_0\infty_0$
$0_06_19_0$	$0_16_29_1$	$0_26_09_2$	$2_16_2\infty_4$	$2_26_0\infty_3$	$2_06_1\infty_5$	$4_21_0\infty_2$	$4_01_1\infty_0$	$4_11_2\infty_1$
$4_25_21_1$	$4_05_01_2$	$4_15_11_0$	$7_19_2\infty_5$	$7_29_0\infty_4$	$7_09_1\infty_3$	$5_26_1\infty_3$	$5_06_2\infty_5$	$5_16_0\infty_4$

where B is the array

$8_05_01_0$	$8_15_11_1$	$8_25_21_2$	$0_07_03_0$	$0_17_13_1$	$0_27_23_2$	$2_09_05_0$	$2_19_15_1$	$2_29_25_2$
$0_01_22_0$	$0_11_02_1$	$0_21_12_2$	$2_03_24_0$	$2_13_04_1$	$2_23_14_2$	$4_05_26_0$	$4_15_06_1$	$4_25_16_2$
$6_23_0\infty_2$	$6_03_1\infty_0$	$6_13_2\infty_1$	$4_17_2\infty_0$	$4_27_0\infty_1$	$4_07_1\infty_2$	$9_11_07_1$	$9_21_17_2$	$9_01_27_0$
$7_28_1\infty_3$	$7_08_2\infty_5$	$7_18_0\infty_4$	$8_09_1\infty_1$	$8_19_2\infty_2$	$8_29_0\infty_0$	$2_24_16_2$	$2_04_26_0$	$2_14_06_1$
$6_10_2\infty_4$	$6_20_0\infty_3$	$6_00_1\infty_5$	$8_25_0\infty_2$	$8_05_1\infty_0$	$8_15_2\infty_1$	$6_19_2\infty_0$	$6_29_0\infty_1$	$6_09_1\infty_2$
$1_13_2\infty_5$	$1_23_0\infty_4$	$1_03_1\infty_3$	$9_20_1\infty_3$	$9_00_2\infty_5$	$9_10_0\infty_4$	$0_01_1\infty_1$	$0_11_2\infty_2$	$0_21_0\infty_0$
–	–	–	$8_12_2\infty_4$	$8_22_0\infty_3$	$8_02_1\infty_5$	$0_27_0\infty_2$	$0_07_1\infty_0$	$0_17_2\infty_1$
–	–	–	$3_15_2\infty_5$	$3_25_0\infty_4$	$3_05_1\infty_3$	$1_22_1\infty_3$	$1_02_2\infty_5$	$1_12_0\infty_4$
$5_17_03_1$	$5_27_13_2$	$5_07_23_0$	–	–	–	$0_14_2\infty_4$	$0_24_0\infty_3$	$0_04_1\infty_5$
$8_20_12_2$	$8_00_22_0$	$8_10_02_1$	–	–	–	$5_17_2\infty_5$	$5_27_0\infty_4$	$5_07_1\infty_3$
$2_15_2\infty_0$	$2_25_0\infty_1$	$2_05_1\infty_2$	$7_19_05_1$	$7_29_15_2$	$7_09_25_0$	–	–	–
$6_07_1\infty_1$	$6_17_2\infty_2$	$6_27_0\infty_0$	$0_22_14_2$	$0_02_24_0$	$0_12_04_1$	–	–	–

Fig. 4.4: An $\mathrm{FGBTD}_1(3,6^6)$ $(X,\mathcal{G},\mathcal{A})$, where $X=(\mathbb{Z}_{10}\times\mathbb{Z}_3)\cup\{\infty_i : i\in\mathbb{Z}_6\}$ and $\mathcal{G}=\{\{t_0,t_1,t_2,(5+t)_0,(5+t)_1,(5+t)_2\} : t\in\mathbb{Z}_5\}\cup\{\infty_i : i\in\mathbb{Z}_6\}$. For succinctness, a set $\{x,y,z\}$ is written xyz.

describe the type of a GDD: a GDD of type $g_1^{t_1}g_2^{t_2}\cdots g_s^{t_s}$ is a GDD with exactly t_i groups of size g_i, $i\in[s]$.

Definition 4.3.11. A *transversal design* $\mathrm{TD}(k,n)$ is a $\{k\}$-GDD of type n^k.

The following result on the existence of transversal designs (see [**?**, 1]) is sometimes used without explicit reference throughout this chapter.

Theorem 4.3.12. *Let TD(k) denote the set of positive integers n such that there exists a TD(k, n). Then, we have*

(i) $TD(4) \supseteq \mathbb{Z}_{>0} \setminus \{2, 6\}$,

(ii) $TD(5) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 6, 10\}$,

(iii) $TD(6) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 6, 10, 14, 22\}$,

(iv) $TD(7) \supseteq \mathbb{Z}_{>0} \setminus \{2, 3, 4, 5, 6, 10, 14, 15, 20, 22, 26, 30, 34, 38, 46\}$,

(v) $TD(k) \supseteq \{q : q \geq k - 1$ *is a prime power*$\}$.

Definition 4.3.13. A *doubly resolvable* TD(k, n), denoted by DRTD(k, n), is a TD(k, n) whose blocks can be arranged in an $n \times n$ array such that each point appears exactly once in each row and once in each column.

Colbourn *et al.* [22] established the following.

Proposition 4.3.14 (Colbourn *et al.* [22])**.** *There exists a TD$(k+2, n)$ if and only if there exists a DRTD(k, n).*

Corollary 4.3.15. *A DRTD$(3, n)$ exists for all $n \geq 4$ and $n \notin \{6, 10\}$.*

Proof. A TD$(5, n)$ exists if $n \geq 4$ and $n \notin \{6, 10\}$ by Theorem 4.3.12. ■

4.3.4 Frame Generalized Balanced Tournament Design

Let $(X, \mathcal{G}, \mathcal{A})$ be a $\{k\}$-GDD with $\mathcal{G} = \{G_1, G_2, \ldots, G_s\}$ and $|G_i| \equiv 0 \bmod k(k-1)$ for all $i \in [s]$. Let $R = \frac{1}{k}\sum_{i=1}^{s} |G_i|$ and $C = \frac{1}{k-1}\sum_{i=1}^{s} |G_i|$. Suppose there exists a partition $[R] = \bigcup_{i=1}^{s} R_i$ and a partition $[C] = \bigcup_{i=1}^{s} C_i$ such that for each $i \in [s]$, we have $|R_i| = |G_i|/k$ and $|C_i| = |G_i|/(k-1)$.

We say that $(X, \mathcal{G}, \mathcal{A})$ is a *frame generalized balanced tournament design* (FGBTD) if its blocks can be arranged in an $R \times C$ array such that the following conditions hold:

(i) the cell (r, c) is empty when $(r, c) \in R_i \times C_i$ for $i \in [s]$,

(ii) for any row $r \in R_i$, each point in $X \setminus G_i$ appears either once or twice and the points in G_i do not appear,

(iii) for any column $c \in C_i$, each point in $X \setminus G_i$ appears exactly once.

Denote this FGBTD by FGBTD(k, T), where $T = \langle |G_i| : i \in [s] \rangle$.

Example 4.3.16. An FGBTD$(3, 6^6)$ is given in Fig. 4.4.

4.4 Recursive Constructions

In this section, we develop the necessary recursive constructions.

4.4.1 Recursive Constructions for GBTPs

First, for block size three, we have the following tripling construction for GBTDs.

Proposition 4.4.1 (Tripling Construction)**.** *Suppose a* 3-$*$*colorable RBIBD*$(m, 3, 1)$ *and a DRTD*$(3, m)$ *exist. Then there exists a* 2-$*$*colorable GBTD*$_1(3, m)$*. Suppose further that the RBIBD*$(m, 3, 1)$ *is* 3-$*$*colorable with property* Π*. Then the GBTD*$_1(3, m)$ *is a special GBTD*$_1(3, m)$*.*

Proof. Consider a 3-$*$colorable RBIBD$(m, 3, 1)$ $(X, \mathcal{A})$ with colors from $\mathbb{Z}_3$ and let

$$X' = \{x_i : x \in X \text{ and } i \in \mathbb{Z}_3\}.$$

Make three copies of the 3-$*$colorable RBIBD$(m, 3, 1)$ as follows: for the jth copy, $j \in \{1, 2, 3\}$, each block $\{x, y, z\}$ of color i in the 3-$*$colorable RBIBD$(m, 3, 1)$ is replaced by block $\{x_{i+j}, y_{i+j}, z_{i+j}\}$, where arithmetic in the subscripts is performed modulo three. Stacking these three $\frac{m}{3} \times \frac{m-1}{2}$ arrays together gives an $m \times \frac{m-1}{2}$ array A with the property that

(i) each point in X' appears exactly once in each column,

(ii) each point in X' appears at most once in each row.

Now take a DRTD$(3, m)$ $(X', \mathcal{G}, \mathcal{A})$, where

$$\mathcal{G} = \{\{x_i : x \in X\} : i \in \mathbb{Z}_3\},$$

and adjoin it to A. This gives an $m \times \frac{3m-1}{2}$ array, which we claim is a $\mathrm{GBTD}_1(3,m)$. Indeed it is easy to see that in this array, each point in X' appears exactly once in each column and either once or twice in each row. It remains to show that this array is a $\mathrm{BIBD}(3m,3,1)$. To see this, observe that any pair of points contained in a group of the $\mathrm{DRTD}(3,m)$ is contained in a block of one of the copies of the 3-$*$colorable $\mathrm{RBIBD}(m,3,1)$. This $\mathrm{GBTD}_1(3,m)$ is 2-$*$colorable by giving the blocks from the $\mathrm{DRTD}(3,m)$ one color and the remaining blocks (from the three copies of the $\mathrm{RBIBD}(m,3,1)$) another color.

If, in addition, the $\mathrm{RBIBD}(m,3,1)$ is 3-$*$colorable with property Π, and that in row r of this $\mathrm{RBIBD}(m,3,1)$, the points x,y,z (not necessarily distinct) are witnesses for colors $0,1,2$, respectively, then we assume that the $\mathrm{DRTD}(3,m)$ used has the block $\{x_0,y_1,z_2\}$ and that this block can be made to appear in row r, by permuting rows if necessary. The cell that contains $\{x_0,y_1,z_2\}$ is a special cell of the $\mathrm{GBTD}_1(3,m)$. ■

Corollary 4.4.2. *Let $m > 3$ and suppose an RBIBD$(m,3,1)$ that is 3-$*$colorable with property Π exists. Then there exists a special GBTD$_1(3,3^km)$, for all $k \geq 0$.*

Proof. First note that $m \equiv 3 \bmod 6$ since this is a necessary condition for the existence of an $\mathrm{RBIBD}(m,3,1)$. Hence, there exists a $\mathrm{DRTD}(3,m)$ by Corollary 4.3.15. By Proposition 4.4.1, there exists a 2-$*$colorable special $\mathrm{GBTD}_1(3,m)$, which may be regarded as an $\mathrm{RBIBD}(3m,3,1)$ that is 3-$*$colorable with property Π. The corollary then follows by induction. ■

The following is a simple, but useful construction.

Proposition 4.4.3. *If an IGBTP$_\lambda(K,v,m\times n;w,m'\times n')$ and a GBTP$_\lambda(K,w,m'\times n')$ exists, then a GBTP$_\lambda(K,v,m\times n)$ exists.*

Proof. Let $(X,\mathcal{A})$ be an $\mathrm{IGBTP}_\lambda(K,v,m\times n;w,m'\times n')$. Fill in the empty subarray of this IGBTP with an a $\mathrm{GBTP}_\lambda(K,w,m'\times n')$, $(X',\mathcal{A}')$. The resulting array is a $\mathrm{GBTP}_\lambda(K,v,m\times n)$, $(X,\mathcal{A}\cup\mathcal{A}')$. ■

FGBTD is a useful tool to construct larger GBTPs from smaller ones.

Proposition 4.4.4 (FGBTD Construction for GBTP)**.** *Let $k \in K$. Suppose there exists an* FGBTD(k,T) $(X,\mathcal{G},\mathcal{A})$*, where $\mathcal{G}=\{G_1,G_2,\ldots,G_s\}$, and let $r_i=|G_i|/k$ and $c_i=|G_i|/(k-1)$, for $i\in[s]$. If there exists an IGBTP$_1(K,|G_i|+w,(r_i+m)\times(c_i+n);w,m\times n)$ for all $i\in[s]$, then there exists an IGBTP$_1(K,\sum_{i=1}^{s}|G_i|+w,(\sum_{i=1}^{s}r_i+m)\times(\sum_{i=1}^{s}c_i+n);w,m\times n)$. Furthermore, if a GBTP$_1(K,w,m\times n)$ exists, then an GBTP$_1(K,\sum_{i=1}^{s}|G_i|+w,(\sum_{i=1}^{s}r_i+m)\times(\sum_{i=1}^{s}c_i+n))$ exists.*

Proof. We use the notations as in the definition of FGBTD in Section 4.3.4, and assume that the blocks of the FGBTD(k,T) are arranged in an $R\times C$ array, with rows and columns indexed by $[R]$ and $[C]$, respectively.

Let P and Q be two sets satisfying $|P|=m, |Q|=n, P\cap[R]=\emptyset, Q\cap[C]=\emptyset$.

For each $i\in[s]$, consider an IGBTP$_1(K,|G_i|+w,(r_i+m)\times(c_i+n);w,m\times n)$ $(X_i,\mathcal{A}_i)$, where $X_i=G_i\cup\{\infty_1,\infty_2,\cdots,\infty_w\}$, and whose rows and columns are indexed by $P\cup R_i$ and $Q\cup C_i$, respectively. It can be verified that $(X',\mathcal{A}')$, where

$$X'=X\cup\{\infty_1,\infty_2,\cdots,\infty_w\},$$
$$\mathcal{A}'=\mathcal{A}\cup\left(\bigcup_{i=1}^{s}\mathcal{A}_i\right),$$

is an IRP$(\sum_{i=1}^{s}|G_i|+w,K,1)$.

Arrange the blocks of $(X',\mathcal{A}')$ into an $(R+m')\times(C+n')$ array A, whose rows and columns are indexed by $P\cup[R]$ and $Q\cup[C]$, respectively, such that each block in $\mathcal{A}$ that appears in cell (i,j) of either the FGBTD or the IGBTP, is placed in cell (i,j) of A.

The definition of an FGBTD ensures that no cells are occupied by two blocks. It is also easily checked that every point in X' appears exactly once in each column and either once or twice in each row. In addition, the $m\times n$ subarray indexed by $P\times Q$ is empty. This gives an IGBTP$_1(K,\sum_{i=1}^{s}|G_i|+w,(\sum_{i=1}^{s}r_i+m)\times(\sum_{i=1}^{s}c_i+n);w,m\times n)$.

The last statement follows from Proposition 4.4.3. ■

Since a GBTD is an instance of GBTP, we have the following recursive construction for GBTDs.

Corollary 4.4.5 (FGBTD Construction for GBTD)**.** *Suppose an* FGBTD(k, T) *exists with groups* $\{G_1, G_2, \ldots, G_s\}$*. Let* $g_i = |G_i|/k$*, for* $i \in [s]$*. If there exists a special GBTD*$_1(k, g_i + 1)$ *for all* $i \in [s]$*, then there exists a special GBTD*$_1(k, \sum_{i=1}^{s} g_i + 1)$*.*

When the groups are of the same size, we have the following corollary.

Corollary 4.4.6. *If there exists an* FGBTD$(3, (3g)^t)$ *and a special* GBTD$_1(3, g + 1)$*, then there exists a special* GBTD$_1(3, gt + 1)$*.*

For Proposition 4.4.3 and Corollary 4.4.5 to be useful, we require large classes of FGBTDs. We give three recursive constructions for FGBTDs next.

4.4.2 Recursive Constructions for FGBTDs

Proposition 4.4.7 (Inflation)**.** *Suppose an* FGBTD(k, T) *and a* DRTD(k, n) *exists. Then there exists an* FGBTD(k, nT)*.*

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be an FGBTD(k, T) arranged in an $R \times C$ array A, with rows and columns indexed by $[R]$ and $[C]$, respectively. Define

$$X' = X \times [n],$$
$$\mathcal{G}' = \{G \times [n] : G \in \mathcal{G}\},$$

and for each block $A \in \mathcal{A}$, let

$$X_A = A \times [n],$$
$$\mathcal{G}_A = \{\{x\} \times [n] : x \in A\}.$$

and let $(X_A, \mathcal{G}_A, \mathcal{B}_A)$ be a DRTD(k, n) whose blocks are arranged in an $n \times n$ array with rows and columns both indexed by $[n]$. Let $\mathcal{A}' = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A$ and the blocks in $\mathcal{A}'$ can be arranged, as follows, in an $Rn \times Cn$ array, whose rows and columns are indexed by $[R] \times n$ and $[C] \times n$, respectively: a block $B \in \mathcal{B}_A$ is placed in cell $((i, a), (j, b))$ if A appears in cell (i, j) of the FGBTD(k, T) and B appears in cell (a, b) of the DRTD(k, n). Hence, $(X', \mathcal{G}', \mathcal{A}')$ gives an FGBTD(k, nT). ■

Input:	(master) GDD $\mathcal{D} = (X, \mathcal{G}, \mathcal{A})$;
	weight function $w \to \mathbb{Z}_{\geq 0}$;
	(ingredient) FGBTD(k, T_A) $\mathcal{D}_A = (X_A, \mathcal{G}_A, \mathcal{B}_A)$ for each $A \in \mathcal{A}$, where
	$T_A = \langle w(x) : x \in A \rangle$,
	$X_A = \bigcup_{x \in A}(\{x\} \times [w(a)])$,
	$\mathcal{G}_A = \{\{x\} \times [w(x)] : x \in A\}$,
	and the blocks in $\mathcal{B}_A$ are arranged in a $\frac{1}{k}\sum_{x\in A} w(x) \times \frac{1}{k-1}\sum_{x\in A} w(x)$ array, whose rows and columns are indexed by $\bigcup_{x\in A}(\{x\} \times [w(x)/k])$ and $\bigcup_{x\in A}(\{x\} \times [w(x)/(k-1)])$, respectively.
Output:	FGBTD$(k, \langle \sum_{x\in G} w(x) : G \in \mathcal{G}\rangle)$ $\mathcal{D}^* = (X^*, \mathcal{G}^*, \mathcal{A}^*)$, where
	$X^* = \bigcup_{x\in X}(\{x\} \times [w(x)])$,
	$\mathcal{G}^* = \{\bigcup_{x\in G}(\{x\} \times [w(x)]) : G \in \mathcal{G}\}$,
	$\mathcal{A}^* = \bigcup_{A\in\mathcal{A}} \mathcal{B}_A$, and
	the blocks in $\mathcal{A}^*$ are arranged in a $\frac{1}{k}\sum_{x\in X} w(x) \times \frac{1}{k-1}\sum_{x\in X} w(x)$ array, whose rows and columns are indexed by $\bigcup_{x\in X}(\{x\} \times [w(x)/k])$ and $\bigcup_{x\in X}(\{x\} \times [w(x)/(k-1)])$, respectively,
	by placing a block $B \in \mathcal{B}_A$ in cell (i, j) of $\mathcal{D}^*$ if it appears in cell (i, j) of $\mathcal{D}_A$.
Note:	By convention, for $x \in X$, $\{x\} \times [w(x)] = \varnothing$ if $w(x) = 0$.

Fig. 4.5: Fundamental Construction for FGBTDs

Wilson's Fundamental Construction for GDDs [82] can also be modified to construct FGBTDs. Fig. 4.5 describes this construction.

Proposition 4.4.8 (Fundamental Construction)**.** *Suppose there exists a (master) GDD* $(X, \mathcal{G}, \mathcal{A})$ *of type* T *and let* $w : X \to \mathbb{Z}_{\geq 0}$ *be a weight function. If for each* $A \in \mathcal{A}$*, an (ingredient)* FGBTD$(k, \langle w(a) : a \in A\rangle)$ *exists, then an* FGBTD$(k, \langle \sum_{x\in G} w(x) : G \in \mathcal{G}\rangle)$ *exists.*

Proof. The Fundamental Construction in Fig. 4.5 constructs the desired FGBTD from the master GDD and ingredient FGBTDs. ∎

Proposition 4.4.8 admits the following specialization.

Proposition 4.4.9 (FGBTD from Truncated TD)**.** *Suppose there exists a TD*$(u + s, m)$*, and* $g_1, g_2, \ldots, g_s$ *are nonnegative integers at most* m*. If there exists an* FGBTD(k, g^t) *for each* $t \in \{u, u + 1, \ldots, u + s\}$*, then there exists an* FGBTD(k, T)*, where* $T = (g \cdot m)^u(g \cdot g_1)(g \cdot g_2) \cdots (g \cdot g_s)$*.*

Proof. For each $i \in [s]$, delete $m - g_i$ points from the ith group of the TD$(u+s, m)$. This results in a $\{u, u+1, \ldots, u+s\}$-GDD of type $m^u g_1 g_2 \cdots g_s$. Use this as the master GDD and apply the fundamental construction with weight function w that assigns weight g to all points. ■

4.5 Direct Constructions

This section constructs some small GBTDs and FGBTDs that are required to seed the recursive constructions given in Section 4.4. The main tool in our constructions is the *method of differences*.

Let Γ be an additive abelian group and let n be a positive integer. For a set system $(\Gamma, \mathcal{S})$, the *difference list* of $\mathcal{S}$ is the multiset

$$\Delta\mathcal{S} = \langle x - y : x, y \in A,\ x \neq y, \text{ and } A \in \mathcal{S} \rangle.$$

For a set-system $(\Gamma \times [n], \mathcal{S})$ and $i, j \in [n]$, the multiset

$$\Delta_{ij}\mathcal{S} = \langle x - y : x_i, y_j \in A,\ x_i \neq y_j, \text{ and } A \in \mathcal{S} \rangle$$

is called a list of *pure differences* when $i = j$, and called a list of *mixed differences* when $i \neq j$.

4.5.1 Direct Constructions for GBTDs

We use the notion of starters to construct GBTDs of block size three.

Definition 4.5.1 (Starter for GBTD)**.** Let m be an odd positive integer, Γ be an additive abelian group of size m. Let T be an index set of size $(m-1)/2$. Let $(\Gamma \times [3], \mathcal{S})$ be a $\{3\}$-uniform set system of size $(3m-1)/2$, where

$$\mathcal{S} = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}.$$

$\mathcal{S}$ is called a $(\Gamma \times [3])$-*GBTD-starter* if the following conditions hold:

(i) $\Delta_{ii}\mathcal{S} = \Gamma \setminus \{0\}$, for $i \in [3]$,

(ii) $\Delta_{ij}\mathcal{S} = \Gamma$, for $i, j \in [3]$, $i \neq j$,

(iii) $\bigcup_{\alpha\in\Gamma} A_\alpha = \Gamma \times [3]$,

(iv) $\{j : \alpha_j \in B_t \text{ for some } \alpha \in \Gamma\} = [3]$, for $t \in T$,

(v) each element in $\Gamma \times [3]$ appears either once or twice in the multiset

$$R = \left(\bigcup_{\alpha\in\Gamma} A_\alpha - \alpha\right) \cup \left(\bigcup_{t\in T} B_t\right).$$

Furthermore, $\mathcal{S}$ is said to be *special* if

(vi) each element in A_0 appears exactly once in R.

Also, $\mathcal{S}$ is said to be 3-$*$*colorable with property* Π if each of the blocks in

$$\{A_\alpha - \alpha : \alpha \in \Gamma\} \text{ and } \{B_t : t \in T\},$$

can be colored with one of three colors so that

(vii) blocks of the same color are pairwise disjoint,

(viii) for each color c, there exists a point (a *witness* for c) that is not contained in any block assigned color c.

Proposition 4.5.2. *If a* $(\Gamma \times [k])$*-GBTD-starter exists, then a* $\mathrm{GBTD}_1(k, m)$ *exists. Similarly, if there exists a special* $(\Gamma\times[3])$*-GBTD-starter, then there exists a special GBTD*$_1(3, m)$*; and if there exists a* 3-$*$*colorable* $(\Gamma \times [3])$*-GBTD-starter with property* Π*, then there exists a* 3-$*$*colorable GBTD*$_1(3, m)$ *with property* Π.

Proof. Let $X = \Gamma \times [k]$, and suppose $\mathcal{S} = \{A_\alpha : \alpha \in \Gamma\} \cup \{B_t : t \in T\}$ is an $(\Gamma \times [k])$-GBTD-starter. Let

$$\mathcal{A} = \bigcup_{A\in\mathcal{S}} \{A + \alpha : \alpha \in \Gamma\}.$$

$$\begin{array}{|c|c|}\hline \mathsf{A} & \mathsf{B} \\ \hline \end{array}$$

where A is the array

$$\begin{array}{|ccccc|}\hline A_0 & A_{-\alpha_1}+\alpha_1 & A_{-\alpha_2}+\alpha_2 & \cdots & A_{-\alpha_{m-1}}+\alpha_{m-1} \\ A_{\alpha_1} & A_0+\alpha_1 & A_{\alpha_1-\alpha_2}+\alpha_2 & \cdots & A_{\alpha_1-\alpha_{m-1}}+\alpha_{m-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{\alpha_{m-1}} & A_{\alpha_{m-1}-\alpha_1}+\alpha_1 & A_{\alpha_{m-1}-\alpha_2}+\alpha_2 & \cdots & A_0+\alpha_{m-1} \\ \hline \end{array}$$

and B is the array

$$\begin{array}{|cccc|}\hline B_1 & B_2 & \cdots & B_{(m-1)/(k-1)} \\ B_1+\alpha_1 & B_2+\alpha_1 & \cdots & B_{(m-1)/(k-1)}+\alpha_1 \\ \vdots & \vdots & \ddots & \vdots \\ B_1+\alpha_{m-1} & B_2+\alpha_{m-1} & \cdots & B_{(m-1)/(k-1)}+\alpha_{m-1} \\ \hline \end{array}.$$

Fig. 4.6: A $\mathrm{GBTD}_1(k,m)$ from $(\Gamma\times[k])$-GBTD-starter $\mathcal{S}=\{A_\alpha:\alpha\in\Gamma\}\cup\{B_t:t\in T\}$, where $\Gamma=\{0,\alpha_1,\dots,\alpha_{m-1}\}$ and $T=[(m-1)/(k-1)]$.

Then $(X,\mathcal{A})$ is a $\mathrm{BIBD}(km,k,1)$, whose blocks can be arranged in an $m\times\frac{(km-1)}{k-1}$ array, whose rows and columns are indexed by Γ and $\Gamma\cup T$, respectively, as follows:

- for $\alpha,\beta\in\Gamma$, the block $A_\alpha+\beta$ is placed in cell $(\alpha+\beta,\beta)$, and
- for $t\in T$ and $\alpha\in\Gamma$, the block $B_t+\alpha$ is placed in cell (α,t).

Fig. 4.6 depicts the placement of blocks in the array.

For $\beta\in\Gamma$, the set of blocks occupying column β is $\{A_\alpha+\beta:\alpha\in\Gamma\}$, which forms a resolution class by condition (iii) of Definition 4.5.1. Similarly, for $t\in T$, the set of blocks occupying column t is $\{B_t+\alpha:\alpha\in\Gamma\}$, which forms a resolution class by condition (iv) in Definition 4.5.1.

The set of blocks occupying row 0 is given by R, and by condition (v) of Definition 4.5.1, each point in X appears either once or twice in row 0. Since the blocks occupying row α ($\alpha\in\Gamma$) are exactly the translates of the blocks in R by α, every point in X also appears either once or twice in row α.

Suppose $\mathcal{S}=\{A_\alpha:\alpha\in\Gamma\}\cup\{B_t:t\in T\}$ is a special $(\Gamma\times[3])$-GBTD-starter. Then condition (vi) of 4.5.1 ensures that the cell $(0,0)$ is special.

On the other hand, if $\mathcal{S}$ be a 3-$*$colorable $(\Gamma \times [3])$-GBTD-starter and let

$$c_i \text{ be the color assigned to } \begin{cases} A_i - i, & \text{if } i \in \Gamma, \\ B_i, & \text{otherwise.} \end{cases}$$

For $\alpha, \beta \in \Gamma$ and $t \in T$, assign the block $A_\alpha + \beta$ color c_α and the block $B_t + \beta$ color c_t. Then conditions (vii) and (viii) of Definition 4.5.1 ensure that the $\mathrm{GBTD}_1(3, m)$ is 3-$*$colorable with property Π. ■

Proposition 4.5.3. *Let* $q \equiv 1 \bmod 6$. *Then there exists a special* $(\mathbb{F}_q \times [3])$*-GBTD starter that is* 3*-*$*$*colorable with property* Π.

Proof. Let $s = (q-1)/6$ and ω be a primitive element of $\mathbb{F}_q$. Consider $\gamma \in \mathbb{F}_q$ that satisfies the following conditions (note that ω^{2s} has order three):

(A) $\gamma \notin \{0, -1, -\omega^{2s}, -\omega^{4s}\}$;

(B) $\gamma \notin \left\{ \dfrac{\omega^{2is} - \omega^{t+2js}}{\omega^t - 1} : i \neq j \in [3], t \in [s-1] \right\}$.

The existence of γ is guaranteed since the cardinality of the union of sets in (A) and (B) is at most $4 + 6(s-1) < 6s + 1 = q$.

Define Λ to be $\{-\gamma\omega^{t-1+2(j-1)s} : t \in [s], j \in [3]\}$ and construct the following $q + 3s = (3q-1)/2$ blocks. For $\alpha \in \mathbb{F}_q$, let

$$A_\alpha = \begin{cases} \left\{ \left(\omega^{t-1+2(j-1)s}\right)_i : j \in [3] \right\}, & \text{if } \alpha = -\gamma\omega^{t-1+2(i-1)s} \text{ where } t \in [s],\ i \in [3], \\ \left\{ \left(-\frac{\alpha}{\gamma}\omega^{2(i-1)s}\right)_i : i \in [3]) \right\}, & \text{otherwise.} \end{cases}$$

For $(t, j) \in [s] \times [3]$, let

$$B_{(t,j)} = \left\{ \left(\omega^{t-1+2(j-1)s}\left(\omega^{2(i-1)s} + \gamma\right)\right)_i : i \in [3] \right\}$$

Let $\mathcal{S} = \{A_\alpha : \alpha \in \mathbb{F}_q\} \cup \{B_{(t,j)} : (t, j) \in [s] \times [3]\}$ and we claim that $\mathcal{S}$ is the desired starter.

Define

$$\mathcal{D} = \{\{\omega^{t-1+2(j-1)s} : j \in [3]\} : t \in [s]\},$$

and Wilson [81] showed that the blocks in $\mathcal{D}$ are mutually disjoint and $\Delta\mathcal{D} = \mathbb{F}_q \setminus \{0\}$.

Hence, for condition (i) of Definition 4.5.1, we check for $i \in [3]$,

$$\begin{aligned}\Delta_{ii}\mathcal{S} &= \Delta_{ii}\{A_\alpha : \alpha = -\gamma\omega^{t-1+2(i-1)s}, t \in [s], i \in [3]\} \\ &= \Delta\mathcal{D} = \mathbb{F}_q \setminus \{0\}.\end{aligned}$$

For condition (ii), we verify for $i \neq i' \in [3]$,

$$\begin{aligned}\Delta_{ii'}\mathcal{S} &= \bigcup_{\alpha\notin\Lambda}\left(-\frac{\alpha}{\gamma}\left(\omega^{2(i-1)s} - \omega^{2(i'-1)s}\right)\right) \cup \bigcup_{(t,j)\in[s]\times[3]} \omega^{t-1+2(j-1)s}\left(\omega^{2(i-1)s} - \omega^{2(i'-1)s}\right) \\ &= \left(\omega^{2(i-1)s} - \omega^{2(i'-1)s}\right)\left(\bigcup_{\alpha\notin\Lambda} -\frac{\alpha}{\gamma} \cup \bigcup_{(t,j)\in[s]\times[3]} \omega^{t-1+2(j-1)s}\right) \\ &= \left(\omega^{2(i-1)s} - \omega^{2(i'-1)s}\right)\mathbb{F}_q = \mathbb{F}_q.\end{aligned}$$

For condition (iii) of Definition 4.5.1, since the number of points in $\bigcup_{\alpha\in\mathbb{F}_q} A_\alpha$ is kq, it suffices to check that each point $\beta_i \in \mathbb{F}_q \times [k]$ belongs to some block A_α. Indeed, if $\beta/\omega^{2(i-1)s} = \omega^{(t-1)+2(j-1)s}$ for some $(t,j) \in [s] \times [3]$, then let $\alpha = -\gamma\omega^{t-1+2(i-1)s}$ and so, $\beta_i = \left(\omega^{t-1+2(i+j-2)s}\right)_i$ belongs to A_α. Otherwise, $-\gamma\beta/\omega^{2(i-1)s} \notin \Lambda$. Let $\alpha = -\gamma\beta/\omega^{2(i-1)s}$ and $\beta_i \in A_\alpha$ as desired.

Condition (iv) of Definition 4.5.1 is clearly true from the definition of $B_{(t,j)}$. We establish condition (v) of Definition 4.5.1 through the following claims:

Claim 4.5.4. The blocks in $\bigcup_{\alpha\notin\Lambda}(A_\alpha - \alpha) \cup \bigcup_{(t,j)\in[s]\times[3]} B_{(t,j)}$ form a resolution class.

As above, it suffices to check that each point $\beta_i \in \mathbb{F}_q \times [3]$ belongs to some block in $\bigcup_{\alpha\notin\Lambda}(A_\alpha - \alpha) \cup \bigcup_{(t,j)\in[s]\times[k]} B_{(t,j)}$ as the total number of points is kq.

Indeed, if $\beta/(\omega^{2(i-1)s} + \gamma) = \omega^{t-1+2(j-1)s}$ for some $(t,j) \in [s] \times [k]$, then $\beta_i \in B_{(t,j)}$. Otherwise, $-\gamma\beta/(\omega^{2(i-1)s} + \gamma) \notin \Lambda$. Let $\alpha = -\gamma\beta/(\omega^{2(i-1)s} + \gamma)$ (note that α is well-defined by Condition (A)) and $\beta_i \in A_\alpha - \alpha$.

Claim 4.5.5. Each point in $\mathbb{F}_q \times [k]$ appears at most once in $\bigcup_{\alpha\in\Lambda}(A_\alpha - \alpha)$.

Note that the blocks are of the form

$$\left\{\left(\omega^{t-1+2(j-1)s} + \gamma\omega^{t-1+2(i-1)s}\right)_i : j \in [3]\right\}$$

for $(t, i) \in [s] \times [3]$. Suppose otherwise that a point appears twice. That is, there exist $j, j' \in [3]$, $(t, i), (t', i) \in [s] \times [3]$ with $t > t'$ such that

$$\omega^{t-1+2(j-1)s} + \gamma\omega^{t-1+2(i-1)s} = \omega^{t'-1+2(j'-1)s} + \gamma\omega^{t'-1+2(i-1)s}.$$

Hence,

$$\gamma = \frac{\omega^{2(j'-i)s} - \omega^{2(j-i)s+(t-t')}}{\omega^{t-t'} - 1}.$$

Since $t \neq t'$, we have $t - t' \in [s-1]$. If $j \neq j'$, this contradicts Condition (B). Otherwise $j = j'$ implies $\gamma = -\omega^{2(j-i)s}$ contradicting (A).

Next, observe that $A_0 = \{(0, i) : i \in [3]\}$. By Claim 4.5.4, to establish condition (vi) of Definition 4.5.1, it suffices to show that $0_i \notin A_\alpha - \alpha$ for $\alpha \in \Lambda$ and $i \in [3]$. Suppose otherwise. Then there exists $(t, j) \in [s] \times [3]$ and $i \in [3]$ such that

$$(\omega^{(j-1)s} + \gamma)\omega^{t+(i-1)s} = 0,$$

contradicting (A).

Finally, we exhibit that $\mathcal{S}$ is 3-$*$colorable with property Π by assigning the block A_0 color $\clubsuit$, the blocks $A_\alpha - \alpha$ for $\alpha \notin \Lambda$ and B_t for $t \in T$ color $\heartsuit$ and the blocks $A_\alpha - \alpha$ for $\alpha \in \Lambda$ color $\diamondsuit$. Then this assignment satisfies condition (vii) of Definition 4.5.1. In addition, 0_1 is a witness for both $\heartsuit$ and $\diamondsuit$ and α_1 is a witness for $\clubsuit$ for some $\alpha \neq 0$, satisfying condition (viii) of Definition 4.5.1. ■

Corollary 4.5.6. *Let $q \equiv 1 \bmod 6$. Then a 3-$*$colorable $\mathrm{GBTD}_1(3, m)$ with property Π exists.*

Proof. This follows from Proposition 4.5.2 and Proposition 4.5.3. ■

Corollary 4.5.7. *A special* $\mathrm{GBTD}_1(3, m)$ *exists for* $m \in \{1, 17, 29, 35, 47, 53, 55\}$*, a 3-$*$colorable special* $\mathrm{GBTD}_1(3, m)$ *with property* Π *for* $m \in \{9, 11, 23\}$ *and a 3-$*$colorable* $\mathrm{RBIBD}(15, 3, 1)$ *with property* Π.

Proof. A special $\mathrm{GBTD}_1(3, 1)$ exists trivially. Also, a 3-$*$colorable special $\mathrm{GBTD}_1(3, 9)$ with property Π is given by Example 4.3.9, and a 3-$*$colorable $\mathrm{RBIBD}(15, 3, 1)$ with property Π is given by Example 4.3.4.

For $m \in \{11, 17, 23, 29, 35, 47, 53, 55\}$, apply Proposition 4.5.2 with special $(\mathbb{Z}_m \times [3])$-GBTD-starters and 3-$*$colorable special $(\mathbb{Z}_m \times [3])$-GBTD-starters with property Π given in [11]. ■

4.5.2 Direct Constructions for IGBTPs

As with GBTDs, we use a set of starters to construct IGBTPs. To construct this starters, we need the notion of infinity elements.

Given an abelian group Γ, we augment the point set with *infinite* elements, denoted by ∞_i where i belongs to some index set I. The infinite elements are fixed under addition by elements in Γ. That is, $\infty_i + \gamma = \infty_i$ for $\gamma \in \Gamma$. Let w be a positive integer and $W_w := \{\infty_i : i \in [w]\}$. So, given a block $A \subset \Gamma \cup W_w$ and $\gamma \in \Gamma$, $A + \gamma = \{a + \gamma : a \in A \setminus W_w\} \cup (A \cap W_w)$.

We also extend the definition of difference lists. For a set system $(\Gamma \cup W_w, \mathcal{S})$, then the difference list of $\mathcal{S}$ is given by the multiset

$$\Delta\mathcal{S} = \langle x - y : x, y \in A \setminus W_w, x \neq y, A \in \mathcal{S}\rangle.$$

Definition 4.5.8. Let m be an odd integer with $m \geq 11$ Let $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w, \mathcal{S})$ be a $\{2, 3\}$-uniform set system of size $w - 3 + m$, where

$$\mathcal{S} = \{A_i : i \in [(w-5)/2]\} \cup \{B_i : i \in [(w-1)/2]\} \cup \{C_i : i \in \mathbb{Z}_m\}.$$

satisfying $|A_i| = 2$ for $i \in [(w-5)/2]$, $|B_i| = 2$ for $i \in [(w-1)/2]$, $|C_0| = 3$, and $|C_i| = 2$ for

$$\begin{array}{|c|c|c|}\hline \mathsf{W} & \mathsf{B} & \mathsf{B}+0_1 \\ \hline \mathsf{A} & \mathsf{C} & \mathsf{C}+0_1 \\ \hline \end{array}$$

where W is a $(w-1)/2 \times (w-4)$ empty array, A is an $m \times (w-4)$ array,

$$\begin{array}{|c|ccccccc|}\hline \{0_0,0_1\} & A_1 & A_1+0_1 & A_2 & A_2+0_1 & \cdots & A_{(w-5)/2} & A_{(w-5)/2}+0_1 \\ \{1_0,1_1\} & A_1+1_0 & A_1+1_1 & A_2+1_0 & A_2+1_1 & \cdots & A_{(w-5)/2}+1_0 & A_{(w-5)/2}+1_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \{(m-1)_0,(m-1)_1\} & A_1-1_0 & A_1-1_1 & A_2-1_0 & A_2-1_1 & \cdots & A_{(w-5)/2}-1_0 & A_{(w-5)/2}-1_1 \\ \hline \end{array},$$

B and C are the following $(w-1)/2 \times m$ and $m \times m$ arrays,

$$\begin{array}{|cccc|}\hline B_1 & B_1+1_0 & \cdots & B_1-1_0 \\ B_2 & B_1+1_0 & \cdots & B_1-1_0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{(w-1)/2} & B_{(w-1)/2}+1_0 & \cdots & B_{(w-1)/2}-1_0 \\ \hline \end{array}, \quad \begin{array}{|cccc|}\hline C_0 & C_{m-1}+1_0 & \cdots & C_1-1_0 \\ C_1 & C_0+1_0 & \cdots & C_2-1_0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{m-1} & C_{m-2}+1_0 & \cdots & C_0-1_0 \\ \hline \end{array}.$$

Fig. 4.7: An $\mathrm{IGBTP}_1(\{2,3^*\}, 2m+w, (m+(w-1)/2)\times(2m+w-4); w, (w-1)/2\times(w-4))$ from a $((\mathbb{Z}_m\times\mathbb{Z}_2)\cup W_w)$-GBTP-starter.

$i \in \mathbb{Z}_m \setminus \{0\}$.

$\mathcal{S}$ is called a $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$-*IGBTP-starter* if the following conditions hold:

(i) $\Delta\mathcal{S} = \mathbb{Z}_m \times \mathbb{Z}_2 \setminus \{0_0, 0_1\}$,

(ii) $\{j : a_j \in A_i\} = \mathbb{Z}_2$ for $i \in [(w-5)/2]$,

(iii) $\{B_i : i \in [(w-1)/2]\} \cup \{C_j : j \in \mathbb{Z}_m\} = (\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$,

(iv) $|C_i \cap W_w| \le 1$ for $i \in \mathbb{Z}_m$,

(v) each element in $(\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$ appears either once or twice in the multiset

$$R = \{0_0, 0_1\} \cup \left(\bigcup_{\substack{i\in[(w-5)/2]\\ j\in\mathbb{Z}_2}} A_i + 0_j \right) \cup \left(\bigcup_{i_j \in \mathbb{Z}_m\times\mathbb{Z}_2} C_i - i_j \right).$$

Proposition 4.5.9. *Suppose there exists a $((\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w)$-IGBTP-starter. Then there exists an $IGBTP_1(\{2,3^*\}, 2m+w, (m+(w-1)/2)\times(2m+w-4); w, (w-1)/2\times(w-4))$.*

<table>
<tr><td rowspan="1">W</td><td>B</td><td>B + 0_1</td><td>B + 0_2</td><td>B + 0_3</td></tr>
<tr><td rowspan="2">A</td><td>C</td><td>D + 0_1</td><td>C + 0_2</td><td>D + 0_3</td></tr>
<tr><td>D</td><td>C + 0_1</td><td>D + 0_2</td><td>C + 0_3</td></tr>
</table>

where W is a 4×5 empty array, A is a $2m \times 5$ array,

$$\begin{array}{|ccc|cc|}
\hline
\{0_0, 0_1\} & \{x_0, x_2\} & \{y_0, y_3\} & A & A + 0_2 \\
\{1_0, 1_1\} & \{(x+1)_0, x_2\} & \{(y+1)_0, (y+1)_3\} & A + 1_0 & A + 1_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\{(m-1)_0, (m-1)_1\} & \{(x-1)_0, x_2\} & \{(y-1)_0, (y-1)_3\} & A + (m-1)_0 & A + (m-1)_2 \\
\hline
\{0_2, 0_3\} & \{x_1, x_3\} & \{y_1, y_2\} & A + 0_1 & A + 0_3 \\
\{1_2, 1_3\} & \{(x+1)_1, x_3\} & \{(y+1)_1, (y+1)_2\} & A + 1_1 & A + 1_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\{(m-1)_2, (m-1)_3\} & \{(x-1)_1, x_3\} & \{(y-1)_1, (y-1)_2\} & A + (m-1)_1 & A + (m-1)_3 \\
\hline
\end{array},$$

B, C and D are the following $4 \times m$, $m \times m$ and $m \times m$ arrays respectively,

$$\begin{array}{|cccc|}
\hline
B_1 & B_1 + 1_0 & \cdots & B_1 - 1_0 \\
B_2 & B_2 + 1_0 & \cdots & B_2 - 1_0 \\
B_3 & B_3 + 1_0 & \cdots & B_3 - 1_0 \\
B_4 & B_4 + 1_0 & \cdots & B_4 - 1_0 \\
\hline
\end{array}, \quad
\begin{array}{|cccc|}
\hline
C_0 & C_{m-1} + 1_0 & \cdots & C_1 - 1_0 \\
C_1 & C_0 + 1_0 & \cdots & C_2 - 1_0 \\
\vdots & \vdots & \ddots & \vdots \\
C_{m-1} & C_{m-2} + 1_0 & \cdots & C_0 - 1_0 \\
\hline
\end{array}, \quad
\begin{array}{|cccc|}
\hline
D_0 & D_{m-1} + 1_0 & \cdots & D_1 - 1_0 \\
D_1 & D_0 + 1_0 & \cdots & D_2 - 1_0 \\
\vdots & \vdots & \ddots & \vdots \\
D_{m-1} & D_{m-2} + 1_0 & \cdots & D_0 - 1_0 \\
\hline
\end{array}.$$

Fig. 4.8: An $\mathrm{IGBTP}_1(\{2, 3^*\}, 4m+9, (2m+4) \times (4m+5); 9, 4 \times 5)$ from a $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)$-GBTP-starter.

Proof. Let

$$X = \mathbb{Z}_m \times \mathbb{Z}_2 \cup W_w,$$

$$\mathcal{A} = \{S + j : S \in \mathcal{S} \text{ and } j \in \mathbb{Z}_m \times \mathbb{Z}_2\} \cup \{\{i_0, i_1\} : i \in \mathbb{Z}_m\}.$$

Then $(X, W_w, \mathcal{A})$ is an IRP$(2m+w, K, 1; w)$, whose blocks can be arranged in an $(m+(w-1)/2) \times (2m+w-4)$ array as in Figure 4.8. We index the rows by $[(w-1)/2] \cup \mathbb{Z}_m$ and the columns by $[w-4] \cup (\mathbb{Z}_m \times \mathbb{Z}_2)$.

First, check that the cell (r, c) is empty for $(r, c) \in [(w-1)/2] \times [w-4]$.

For $j \in [w-4]$, the set of blocks occupying column j is $\mathbb{Z}_m \times \mathbb{Z}_2$ by condition (ii) of Definition 4.5.8. For $j \in \mathbb{Z}_m \times \mathbb{Z}_2$, first observe that the set of the blocks occupying the column 0_0 by condition (iii) of Definition 4.5.8 is $(\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$. Since the blocks of column j are translates (by j) of the blocks in column 0_0, the union of the blocks in column j is also $(\mathbb{Z}_m \times \mathbb{Z}_2) \cup W_w$.

For $i \in [(w-1)/2]$, each element in $\mathbb{Z}_m \times \mathbb{Z}_2$ appears exactly twice in row i by construction. For $i \in \mathbb{Z}_m$, let R_i denote the multiset containing all the points appearing in the blocks of row i. Then $R_0 = R$ and $R_i = R_0 + i_0$, for all $i \in \mathbb{Z}_m$. Hence, it suffices each element in X appears either once or twice in R, which follows immediately from conditions (v) in Definition 4.5.8. ■

Definition 4.5.10. Let m be an odd integer with $m \geq 11$. Let $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9, \mathcal{S})$ be a $\{1, 2, 3\}$-uniform set system of size $7 + 2m$, where

$$\mathcal{S} = \{x_0\} \cup \{y_0\} \cup A \cup \{B_i : i \in [4]\} \cup \{C_i : i \in \mathbb{Z}_m\} \cup \{D_i : i \in \mathbb{Z}_m\}.$$

satisfying $|A| = 2$, $|B_i| = 2$ for $i \in [4]$, $|C_0| = 3$, $|C_i| = 2$ for $i \in \mathbb{Z}_m \setminus \{0\}$ and $|D_i| = 2$ for $i \in \mathbb{Z}_m$.

$\mathcal{S}$ is called a $((\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9)$-*IGBTP-starter* if the following conditions hold:

(i) $\Delta\mathcal{S} = (\mathbb{Z}_m \times \mathbb{Z}_4) \setminus \{0_0, 0_1, 0_2, 0_3\}$,

(ii) $\{j : a_j \in A\} = \{0, 2\}$,

(iii) $\{B_i : i \in [(w-1)/2]\} \cup \{C_i : i \in \mathbb{Z}_m\} \cup \{D_i : i \in \mathbb{Z}_m\} = (\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$,

(iv) $|C_i \cap W_9| \leq 1$ and $|D_i \cap W_9| \leq 1$ for $i \in \mathbb{Z}_m$,

(v) each element in $(\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$ appears either once or twice in the multisets

$$R_\circ = \{0_0, 0_1, x_0, x_2, y_0, y_3\} \cup A \cup A + 0_2 \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{0,2\}} C_i - i_j \right) \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{1,3\}} D_i - i_j \right),$$

$$R_\bullet = \{0_2, 0_3, x_1, x_3, y_1, y_2\} \cup A + 0_1 \cup A + 0_3 \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{1,3\}} C_i - i_j \right) \cup \left(\bigcup_{i \in \mathbb{Z}_m, j \in \{0,2\}} D_i - i_j \right).$$

Proposition 4.5.11. *Suppose there exists a $(\mathbb{Z}_m \times \mathbb{Z}_4 \cup W_9)$-IGBTP-starter. Then there exists an $IGBTP_1(\{2, 3^*\}, 4m+9, (2m+4) \times (4m+5); 9, 4 \times 5)$.*

Proof. Let

$$X = (\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9,$$

$$\mathcal{A} = \{S + j : S \in \mathcal{S},\ |S| \neq 1,\ j \in \mathbb{Z}_m \times \mathbb{Z}_2\} \cup \{\{i_0, i_1\} : i \in \mathbb{Z}_m\} \cup \{\{i_2, i_3\} : i \in \mathbb{Z}_m\}$$
$$\cup \{\{(x+i)_0, (x+i)_2\} : i \in \mathbb{Z}_m\} \cup \{\{(x+i)_1, (x+i)_3\} : i \in \mathbb{Z}_m\}$$
$$\cup \{\{(y+i)_0, (y+i)_3\} : i \in \mathbb{Z}_m\} \cup \{\{(y+i)_1, (y+i)_2\} : i \in \mathbb{Z}_m\}.$$

Then $(X, W_9, \mathcal{A})$ is an IRP$(4m+9, K, 1; 9)$, whose blocks can be arranged in a $(2m+4) \times (4m+5)$ array as in Figure 4.7. We index the rows by $[4] \cup (\mathbb{Z}_m \times \{\circ, \bullet\})$ and the columns by $[5] \cup (\mathbb{Z}_m \times \mathbb{Z}_4)$.

First, check that the cell (r, c) is empty for $(r, c) \in [4] \times [5]$.

For $j \in [5]$, the set of blocks occupying column j is $\mathbb{Z}_m \times \mathbb{Z}_4$ by condition (ii) of Definition 4.5.10. For $j \in \mathbb{Z}_m \times \mathbb{Z}_4$, first observe that the set of the blocks occupying the column 0_0 by condition (iii) of Definition 4.5.10 is $(\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$. Since the blocks of column j are translates (by j) of the blocks in column 0_0, the union of the blocks in column j is also $(\mathbb{Z}_m \times \mathbb{Z}_4) \cup W_9$.

For $i \in [4]$, each element in $\mathbb{Z}_m \times \mathbb{Z}_4$ appears exactly twice in row i by construction. For $(i, *) \in \mathbb{Z}_m \times \{\circ, \bullet\}$, let $R_{(i,*)}$ denote the multiset containing all the points appearing in the blocks of row $(i, *)$. Then $R_{(0,*)} = R_*$ and $R_{(i,*)} = R_{(0,*)} + i_0$, for all $i \in \mathbb{Z}_m$. Hence, it suffices each element in X appears either once or twice in R_*, which follows immediately from conditions (v) in Definition 4.5.10. ■

Corollary 4.5.12. *There exists an* $IGBTP_1(\{2,3^*\},2m+9,(m+4)\times(2m+5);9,4\times 5)$ *exists for* $m \in \{s : 10 \le s \le 45\} \cup \{47,49,53,57,77\} \setminus \{16,20,24,28,36,40,44\}$, *and an* $IGBTP_1(\{2,3^*\},2m+11,(m+5)\times(2m+7);11,5\times 7)$ *for* $m \in \{15,19,23,27,31, 35,45,49\}$.

Proof. The required $((\mathbb{Z}_m\times\mathbb{Z}_2)\cup W_9)$-IGBTP-starter for $m \in \{s : 11 \le s \le 49, s \text{ odd}\} \cup \{53,57,77\}$ and $((\mathbb{Z}_m\times\mathbb{Z}_4)\cup W_9)$-IGBTP starter for $m \in \{s : 5 \le s \le 21, s \text{ odd}\}$ is given in [11] and we apply Proposition 4.5.9 and Proposition 4.5.11 to obtain the corresponding IGBTP.

Similarly, to construct an $\mathrm{IGBTP}_1(\{2,3^*\},2m+11,(m+5)\times(2m+7);11,5\times 7)$ for $m \in \{15,19,23,27,31,35,45,49\}$, we apply Proposition 4.5.9 to $(\mathbb{Z}_m\times\mathbb{Z}_2\cup W_{11})$-IGBTP starters listed in [11].

It remains to construct an $\mathrm{IGBTP}_1(\{2,3^*\},33,16\times 29;9,4\times 5)$. Consider $((\mathbb{Z}_3\times\mathbb{Z}_8)\cup W_9,\mathcal{S})$, a $\{2,3\}$-uniform set system of size 36, where $\mathcal{S}$ comprise the blocks below:

$$
\begin{array}{llll}
A_1=\{1_0,1_2\} & A_2=\{1_1,1_5\} & A_3=\{0_0,0_4\} & A_4=\{1_3,1_6\}\\
A_5=\{0_3,0_5\} & A_6=\{1_1,1_3\} & A_7=\{1_4,1_7\} & A_8=\{0_1,0_6\}\\
A_9=\{0_0,0_5\} & A_{10}=\{0_2,0_4\} & A_{11}=\{1_4,1_6\} & A_{12}=\{1_0,1_3\}\\
A_{13}=\{0_2,0_5\} & A_{14}=\{1_2,1_7\} & A_{15}=\{0_1,0_7\} & A_{16}=\{1_5,1_7\}\\
A_{17}=\{0_2,0_6\} & A_{18}=\{0_3,0_7\} & A_{19}=\{1_1,1_4\} & A_{20}=\{1_0,1_6\}\\
B_1=\{0_0,0_1\} & B_2=\{0_5,1_5\} & B_3=\{1_1,2_4\} & B_4=\{0_7,1_3\}\\
C_0^1=\{1_0,2_1,2_6\} & C_1^1=\{1_0,2_1\} & C_2^1=\{1_0,2_1\} & \\
C_0^2=\{0_2,\infty_1\} & C_1^2=\{0_4,\infty_2\} & C_2^2=\{1_2,\infty_3\} & \\
C_0^3=\{2_0,\infty_4\} & C_1^3=\{2_3,\infty_5\} & C_2^3=\{1_6,\infty_6\} & \\
C_0^4=\{2_7,\infty_7\} & C_1^4=\{2_2,\infty_8\} & C_2^4=\{2_5,\infty_9\}. &
\end{array}
$$

Let

$$X = (\mathbb{Z}_3 \times \mathbb{Z}_8) \cup W$$
$$\mathcal{A} = \{S + j : S \in \mathcal{S}, j \in \mathbb{Z}_3 \times \mathbb{Z}_8\}.$$

Then $(X, W, \mathcal{A})$ is an IRP$(33, \{2, 3^*\}, 1; 9)$, whose blocks can be arranged in a 16×29 array as in Figure 4.9. It can be readily verified that this arrangement results in an IGBTP$_1(\{2, 3^*\}, 33, 16 \times 29; 9, 4 \times 5)$. ■

4.5.3 Direct Constructions for FGBTDs

Lemma 4.5.13. *There exists an FGBTD$(2, 2^t)$ for $t \in \{4, 5\}$.*

Proof. The desired FGBTDs are given in Fig. 4.10 and Fig. 4.11. ■

Definition 4.5.14. Let t be a positive integer, and let $I = [t-1] \times [2]$. Let $(\mathbb{Z}_{3t} \times [2], \mathcal{S})$ be a 3-uniform set system of size $2(t-1)$, where $\mathcal{S} = \{A_i : i \in I\}$. $\mathcal{S}$ is called a $(\mathbb{Z}_{3t} \times [2])$-*FGBTD-starter* if the following conditions hold:

(i) $\Delta_{ij}\mathcal{S} = \mathbb{Z}_{3t} \setminus \{0, t, 2t\}$ for $i, j \in [2]$,

(ii) $\cup_{i \in I} A_i = (\mathbb{Z}_{3t} \setminus \{0, t, 2t\}) \times [2]$,

(iii) for $j \in [2]$, each element in $(\mathbb{Z}_t \setminus \{0\}) \times [2]$ appears either once or twice in the multiset

$$R_j = \bigcup_{i=1}^{t-1} A_{(i,j)} - i \bmod t,$$

(iv) $r \in (\mathbb{Z}_t \setminus \{0\}) \times [2]$ for each $r \in R_1 \cup R_2$.

Proposition 4.5.15. *If a $(\mathbb{Z}_{3t} \times [2], 6^t)$-FGBTD-starter exists, then an* FGBTD$(3, 6^t)$ *exists.*

W	B	$\mathsf{B}+0_1$	$\mathsf{B}+0_2$	$\mathsf{B}+0_3$	$\mathsf{B}+0_4$	$\mathsf{B}+0_5$	$\mathsf{B}+0_6$	$\mathsf{B}+0_7$
A	C_1	C_4+0_1	C_3+0_2	C_2+0_3	C_1+0_4	C_4+0_5	C_3+0_6	C_2+0_7
	C_2	C_1+0_1	C_4+0_2	C_3+0_3	C_2+0_4	C_1+0_5	C_4+0_6	C_3+0_7
	C_3	C_2+0_1	C_1+0_2	C_4+0_3	C_3+0_4	C_2+0_5	C_1+0_6	C_4+0_7
	C_4	C_3+0_1	C_2+0_2	C_1+0_3	C_4+0_4	C_3+0_5	C_2+0_6	C_1+0_7

where W is a 4×5 empty array, A is a 12×5 array,

$$\begin{array}{|ccccc|}\hline A_1 & A_2 & A_3 & A_4 & A_5 \\ A_1+1_0 & A_2+1_0 & A_3+1_0 & A_4+1_0 & A_5+1_0 \\ A_1+2_0 & A_2+2_0 & A_3+2_0 & A_4+2_0 & A_5+2_0 \\ A_6 & A_7 & A_8 & A_9 & A_{10} \\ A_6+1_0 & A_7+1_0 & A_8+1_0 & A_9+1_0 & A_{10}+1_0 \\ A_6+2_0 & A_7+2_0 & A_8+2_0 & A_9+2_0 & A_{10}+2_0 \\ A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{11}+1_0 & A_{12}+1_0 & A_{13}+1_0 & A_{14}+1_0 & A_{15}+1_0 \\ A_{11}+2_0 & A_{12}+2_0 & A_{13}+2_0 & A_{14}+2_0 & A_{15}+2_0 \\ A_{16} & A_{17} & A_{18} & A_{19} & A_{20} \\ A_{16}+1_0 & A_{17}+1_0 & A_{18}+1_0 & A_{19}+1_0 & A_{20}+1_0 \\ A_{16}+2_0 & A_{17}+2_0 & A_{18}+2_0 & A_{19}+2_0 & A_{20}+2_0 \\ \hline \end{array},$$

B is a 4×3 array,

$$\begin{array}{|ccc|}\hline B_1 & B_1+1_0 & B_1+2_0 \\ B_2 & B_2+1_0 & B_2+2_0 \\ B_3 & B_3+1_0 & B_3+2_0 \\ B_4 & B_4+1_0 & B_4+2_0 \\ \hline \end{array},$$

C_i for $i \in [4]$ is a 3×3 array,

$$\begin{array}{|ccc|}\hline C_0^i & C_2^i+1_0 & C_1^i+2_0 \\ C_1^i & C_0^i+1_0 & C_2^i+2_0 \\ C_2^i & C_1^i+1_0 & C_0^i+2_0 \\ \hline \end{array}.$$

Fig. 4.9: An $\mathrm{IGBTP}_1(\{2,3^*\},33,16\times 29;9,4\times 5)$.

—	—	{2,7}	{6,3}	{7,1}	{3,5}	{5,6}	{1,2}
{2,3}	{6,7}	—	—	{3,0}	{7,4}	{0,2}	{4,6}
{5,7}	{1,3}	{3,4}	{7,0}	—	—	{4,1}	{0,5}
{1,6}	{5,2}	{6,0}	{2,4}	{4,5}	{0,1}	—	—

Fig. 4.10: An $\mathrm{FGBTD}_1(2,2^4)$ $(X,\mathcal{G},\mathcal{A})$, where $X=\mathbb{Z}_8$ and $\mathcal{G}=\{\{i,4+i\}:i\in\mathbb{Z}_4\}$.

—	—	{7,9}	{2,4}	{3,4}	{8,9}	{6,2}	{1,7}	{1,8}	{6,3}
{7,4}	{2,9}	—	—	{8,0}	{3,5}	{4,5}	{9,0}	{7,3}	{2,8}
{3,9}	{8,4}	{8,5}	{3,0}	—	—	{9,1}	{4,6}	{5,6}	{0,1}
{1,2}	{6,7}	{4,0}	{9,5}	{9,6}	{4,1}	—	—	{0,2}	{5,7}
{6,8}	{1,3}	{2,3}	{7,8}	{5,1}	{0,6}	{0,7}	{5,2}	—	—

Fig. 4.11: An $\mathrm{FGBTD}_1(2,2^5)$ $(X,\mathcal{G},\mathcal{A})$, where $X=\mathbb{Z}_{10}$ and $\mathcal{G}=\{\{i,5+i\}:i\in\mathbb{Z}_5\}$.

Proof. Let

$$X=\mathbb{Z}_{3t}\times[2],$$

$$\mathcal{G}=\{G_i=\{i_1,(t+i)_1,(2t+i)_1,i_2,(t+i)_2,(2t+i)_2\}:i\in\mathbb{Z}_t\},$$

$$\mathcal{A}=\{A_i+j:i\in I\text{ and }j\in\mathbb{Z}_{3t}\}.$$

Then $(X,\mathcal{G},\mathcal{A})$ is a $\{3\}$-GDD of type 6^t, whose blocks can be arranged in a $2t\times 3t$ array, with rows and columns indexed by $\mathbb{Z}_t\times[2]$ and $\mathbb{Z}_{3t}$, respectively, as follows: the block $A_{(i,j)}+k$ is placed in cell $((i+k,j),k)$.

The set of blocks occupying column zero are $\{A_i : i\in I\}$ and by condition (ii) of Definition 4.5.14, $\bigcup_{i\in I}A_i=X\setminus G_0$. For other $j\in\mathbb{Z}_{3t}$, observe that the blocks occupying column j are translates (by j) of the blocks in column zero, and hence the union of the blocks in column j is $X\setminus G_{j'}$, where $j'\equiv j \bmod t$.

For $(i,j)\in\mathbb{Z}_t\times[2]$, let $R_{(i,j)}$ denote the multiset containing all the points appearing in the blocks of row (i,j). Then $R_{(i,j)}=R_{(0,j)}+i$, for all $i\in\mathbb{Z}_t$. Hence, it suffices to check that each element of $X\setminus G_0$ appears either once or twice in $R_{(0,j)}$ and the elements of $R_{(0,j)}$ belong to $X\setminus G_0$ for $j\in[2]$. This, however, follows immediately from conditions (iii) and (iv) in Definition 4.5.14, since $R_{(0,j)}=R_j\cup(R_j+t)\cup(R_j+2t)$ for $j\in[2]$. ■

Corollary 4.5.16. *There exist an* $\mathrm{FGBTD}(3,6^t)$ *for all* $t\in\{5,6,7,8\}$*, an* $\mathrm{FGBTD}(3,24^t)$ *for all* $t\in\{5,8\}$ *and an* $\mathrm{FGBTD}(3,30^t)$ *for all* $t\in\{5,7\}$*.*

Proof. An $\mathrm{FGBTD}_1(3, 6^6)$ is given by Example 4.3.16. An $\mathrm{FGBTD}(3, 6^t)$ for $t \in \{5, 7\}$ exists by applying Proposition 4.5.15 with FGBTD-starters given in [11].

The existence of an $\mathrm{FGBTD}(3, 24^t)$, $t \in \{5, 8\}$ follows by applying Proposition 4.4.7 with an $\mathrm{FGBTD}(3, 6^t)$ (constructed in this proof) and a $\mathrm{DRTD}(3, 4)$, whose existence is provided by Corollary 4.3.15. The existence of an $\mathrm{FGBTD}(3, 30^t)$, $t \in \{5, 7\}$ follows by applying Proposition 4.4.7 similarly.

To prove the existence of an $\mathrm{FGBTD}(3, 6^8)$, consider $(\mathbb{Z}_{48}, \mathcal{S})$, a $\{3\}$-uniform set system of size seven, where $\mathcal{S}$ comprises the blocks below:

$$A_1 = \{2, 3, 5\} \qquad A_2 = \{4, 14, 31\} \qquad A_3 = \{9, 22, 45\} \qquad A_4 = \{15, 34, 43\}$$
$$A_5 = \{20, 35, 42\} \qquad A_6 = \{13, 17, 47\} \qquad A_7 = \{1, 6, 12\}.$$

Observe that $\mathcal{S}$ satisfies the following conditions:

(i) $\Delta\mathcal{S} = \mathbb{Z}_{48} \setminus \{0, 8, 16, 24, 32, 40\}$,

(ii) $\bigcup_{i\in[7]} A_i \bmod 24 = \mathbb{Z}_{24} \setminus \{0, 8, 16\}$,

(iii) each element in $\mathbb{Z}_{16} \setminus \{0, 8\}$ appears either once or twice in the multiset

$$R = \bigcup_{i\in[7]} A_i - i \bmod 16,$$

(iv) $r \in \mathbb{Z}_{16} \setminus \{0, 8\}$ for each $r \in R$.

Further, let

$$\begin{aligned} X &= \mathbb{Z}_{48}, \\ \mathcal{G} &= \{\{i + 8k : k \in \mathbb{Z}_6\} : i \in \mathbb{Z}_8\}, \\ \mathcal{A} &= \{A_i + j : i \in [7] \text{ and } j \in \mathbb{Z}_{48}\}. \end{aligned}$$

Then $(X, \mathcal{G}, \mathcal{A})$ is a $\{3\}$-GDD of type 6^8, whose blocks can be arranged in a 16×24 array, with rows and columns are indexed by $\mathbb{Z}_{16}$ and $\mathbb{Z}_{24}$, respectively, as follows: the block $A_i + j$ is placed in cell $(i + j, j)$. This array can be verified to be an $\mathrm{FGBTD}(3, 6^8)$. ∎

4.6 Existence of Two Classes of GBTPs

We apply recursive constructions in Section 4.4 with small designs directly constructed in Section 4.5 to completely settle the existence of $\mathrm{GBTD}_1(3,m)$ and $\mathrm{GBTP}_1(\{2,3^*\};2m+1,m\times(2m-3))$.

4.6.1 Existence of GBTD$_1(3,m)$

Lemma 4.6.1. *There exists a special* $\mathrm{GBTD}_1(3,3^rq)$ *for all* $r\geq 0$ *and* $q\in Q$*, where* $Q=\{q:q\equiv 1 \bmod 6$ *is a prime power*$\}\cup\{5,9,11,23\}$*, except when* $(r,q)=(0,5)$.

Proof. Existence of a special $\mathrm{GBTD}_1(3,q)$ for all $q\in Q\setminus\{5\}$ is provided by Corollary 4.5.6 and 4.5.7. These GBTDs are all 3-$*$colorable with property Π. The lemma then follows by considering these GBTDs as RBIBDs and applying Corollary 4.4.2. ■

Lemma 4.6.2. *Let* $s\in[2]$ *and suppose there exists a* $\mathrm{TD}(5+s,n)$*. If* $0\leq g_i\leq n$*,* $i\in[s]$ *and that there exists a special* $\mathrm{GBTD}_1(3,m)$ *for all* $m\in\{2n+1\}\cup\{2g_i+1:i\in[s]\}$*, then there exists a special* $\mathrm{GBTD}_1(3,10n+1+2\sum_{i=1}^{s}g_i)$.

Proof. By Corollary 4.5.16, there exists an $\mathrm{FGBTD}(3,6^t)$ for all $t\in\{5,6,7\}$. By Proposition 4.4.9, there exists an $\mathrm{FGBTD}(3,(6n)^5(6g_1)\cdots(6g_s))$. Now apply Corollary 4.4.5 to obtain a special $\mathrm{GBTD}_1(3,10n+1+2\sum_{i=1}^{s}g_i)$. ■

Lemma 4.6.3. *A special* $\mathrm{GBTD}_1(3,m)$ *exists for odd* $m\geq 7$.

Proof. First, a special $\mathrm{GBTD}_1(3,m)$ can be constructed for odd m, $7\leq m\leq 95$. Details are provided in Table 4.1.

We then prove the lemma by induction on $m\geq 97$.

Let $E=\{t:t\geq 9\}\setminus\{10,14,15,20,22,26,30,34,38,46\}$. By Theorem 4.3.12, a $\mathrm{TD}(7,n)$ exists for any $n\in E$. Hence, if there exists a special $\mathrm{GBTD}_1(3,m')$ for odd m', $7\leq m'\leq 2n+1$, then applying Lemma 4.6.2 with $3\leq g_1,g_2\leq n$ yields a special $\mathrm{GBTD}_1(3,m)$ for odd m, $10n+7\leq m\leq 14n+1$.

Suppose there exists a $\mathrm{GBTD}_1(3,m')$ for all odd $m'<m$. We claim there exists $n\in E$ with $10n+7\leq m\leq 14n+1$. Suppose otherwise. In other words, there exists $n_1\in E$ such

Tab. 4.1: Existence of special $\mathrm{GBTD}_1(3,m)$

Authority	m
Corollary 4.5.7	9, 11, 17, 23, 29, 35, 47, 53, 55
Lemma 4.6.1	7, 13, 15, 19, 21, 25, 27, 31, 33, 37, 39, 43, 45, 49, 57, 61, 63, 67, 69, 73, 75
Corollary 4.4.6 with (g,t) in $\{(8,5),(5,10),(8,8),(7,10)\}$	41, 51, 65, 71
Lemma 4.6.2 with $n=5$, $g_1=4$	59
Lemma 4.6.2 with $n=7$, $g_1,g_2 \in \{0\}\cup\{t:3\le t\le 7\}$	$\{s:77\le s\le 95, s \text{ odd}\}$

Tab. 4.2: Existence of $\mathrm{IGBTP}_1(\{2,3^*\},2m+9,(m+4)\times(2m+5);4\times 5)$

Authority	m
Corollary 4.5.12	$\{s:10\le s\le 57\}\setminus\{16,20,24,28,32,36,40,44,48,50,52,54,55,56\}$
Lemma 4.6.4 with $(n,g)\in\{(10,0),(11,0),(12,0),(13,0),(11,10),(11,11),(14,0)\}$	40, 44, 48, 52, 54, 55, 56

that $14n_1+1<10n_2+7$ for all $n_2>n_1$ and $n_2\in E$. This, together with the fact that $n_1\ge 9$, implies that $n_2-n_1>3$ for all $n_2\in E$ and $n_2>n_1$. However, a quick check on E gives a contradiction.

Since $n\in E$ and there exists a special $\mathrm{GBTD}_1(3,m')$ for all $m'\le 2n+1<10n+7\le m$ (induction hypothesis), there exists a special $\mathrm{GBTD}_1(3,m)$, completing the induction. ∎

Lemma 4.6.3 shows that a $\mathrm{GBTD}_1(3,m)$ exists for all odd $m\ne 3,5$. Theorem 3.3.1(vi) now follows.

4.6.2 Existence of GBTP$_1(\{2,3^*\};2m+1,m\times(2m-3))$

Lemma 4.6.4. *Suppose there exists a TD$(5,n)$. Suppose $0\le g\le n$ and that there exists an IGBTP$_1(\{2,3^*\},2m+9,(m+4)\times(2m+5);9,4\times 5)$ for $m\in\{n,g\}$. Then there exists an IGBTP$_1(\{2,3^*\},2M+9,(M+4)\times(2M+5);9,4\times 5)$, where $M=4n+g$.*

Proof. By Lemma 4.5.13, there exists an $\mathrm{FGBTD}(2,2^t)$ for all $t\in\{4,5\}$. By Proposition 4.4.9, there exists an $\mathrm{FGBTD}(2,(2n)^4(2g))$. Now apply Proposition 4.4.4 to obtain an $\mathrm{IGBTP}_1(\{2,3^*\},2M+9,(M+4)\times(2M+5);9,4\times 5)$. ∎

Lemma 4.6.5. *There exists an IGBTP$_1(\{2,3^*\},2m+9,(m+4)\times(2m+5);9,4\times 5)$ for any $m\ge 10$, except possibly for $m\in\{16,20,24,28,32,36,46,50\}$.*

Proof. Let $E = \{16, 20, 24, 28, 32, 36, 46, 50\}$. An IGBTP$_1(\{2, 3^*\}, 2m + 9, (m + 4) \times (2m + 5); 9, 4 \times 5)$ can be constructed for $10 \leq m \leq 57$ and $m \notin E$, except possibly for $m = 51$. Details are provided in Table 4.2. When $m = 51$, consider a TD$(5, 11)$ and delete four points from a block to form a $\{4, 5\}$-GDD of type $10^4 11$. Proposition 4.4.8 yields an FGBTD$(2, 20^4 22)$ and hence, Proposition 4.4.4 yields an IGBTP$_1(\{2, 3^*\}, 2m+9, (m+4) \times (2m + 5); 9, 4 \times 5)$ with $m = 51$.

We then prove the lemma by induction on $m \geq 57$. Let $E' = \{4n + g : n \in E, 10 \leq g \leq 13\}$ and assume the lemma is true for $n < m$.

When $m \notin E'$, then write $m = 4n + g$ with $13 \leq n < m$, $n \notin E$ and $g \in \{10, 11, 12, 13\}$. Since a TD$(5, n)$ which exists by Theorem 4.3.12, applying Lemma 4.6.4 with the corresponding n and g, we obtain the desired IGBTP.

When $m \in E'$, we have two cases.

- If $m = 77$, the required IGBTP is given by Corollary 4.5.12.

- Otherwise, apply Lemma 4.6.4 with (n, g) taking values in $\{(15, 14), (15, 15),$ $(19, 0), (18, 18), (19, 15), (23, 0), (19, 17), (22, 18), (22, 19), (27, 0), (22, 21),$ $(25, 22), (25, 23), (31, 0), (25, 25), (29, 22), (29, 23), (35, 0), (29, 25), (31, 30),$ $(31, 31), (39, 0), (33, 25), (39, 38), (39, 39), (49, 0), (40, 37), (42, 42), (43, 39),$ $(43, 40), (43, 41)\}$.

This completes the induction. ■

Lemma 4.6.6. *A GBTP$_1(\{2, 3^*\}, 2m + 1, m \times (2m - 3))$ exists for $m \geq 4$, except possibly for $m \in \{12, 13\}$.*

Proof. A GBTP$_1(\{2, 3^*\}; 2m + 1, m \times (2m - 3))$ can be found via computer search for $4 \leq m \leq 11$. The GBTPs are listed in [11].

For $m \in \{20, 24, 28, 32, 36, 40, 50, 54\}$, set $M = m-5$ and we apply Proposition 4.4.3 with the GBTP$_1(\{2, 3^*\}, 11, 5\times 7)$ and the IGBTP$_1(\{2, 3^*\}, 2M+11, (M+5)\times(2M+7); 11, 5\times 7)$ constructed in Corollary 4.5.12.

Finally, for $m \geq 14$ and $m \notin \{20, 24, 28, 32, 36, 40, 50, 54\}$, set $M = m - 4$ and apply Proposition 4.4.3 with GBTP$_1(\{2, 3^*\}, 9, 4 \times 5)$ and the IGBTP$_1(\{2, 3^*\}, 2M + 9, (M + 4) \times$

$(2M+5); 9, 4 \times 5)$ constructed in Lemma 4.6.5. ■

Lemma 4.6.6 shows that a $\mathrm{GBTP}_1(\{2, 3^*\}, 2m+1, m \times (2m-3))$ exists for all $m \geq 4$, except possibly for $m \in \{12, 13\}$. Theorem 3.3.1(vii) now follows.

4.7 Concluding Remarks

Using methods from combinatorial design theory, we establish infinite families of optimal equitable symbol weight codes and also reveal interesting interplays with generalized balanced tournament packings. In particular, we exploit the construction due to Semakov and Zinoviev [70] to construct equitable symbol weight codes from designs. Hence, it is interesting to ask if equitable symbol weight codes offer new problems to other areas of combinatorial design theory.

We determine completely the existence of $\mathrm{GBTD}_\lambda(k, m)$ for the case where $(k, \lambda) = (3, 1)$. In general, it would be difficult to completely determine the existence of $\mathrm{GBTD}_\lambda(k, m)$ for all k and λ. Instead, for fixed k and λ, we would like to determine if a $\mathrm{GBTD}_\lambda(k, m)$ exists provided that m is sufficiently large and m satisfies certain necessary conditions. This problem is addressed in our future work.

5. MATRIX CODES FOR MULTITONE FREQUENCY SHIFT KEYING

In Chapters 2 to 4 we consider a coded modulation scheme that uses an FSK system to transmit one information symbol at each time instance. We demonstrate the importance of symbol equity in combating narrowband noise under this scheme and this motivates our study of equitable symbol weight codes. Unfortunately, the codes constructed usually involve the use of codebooks and do not have efficient decoding algorithms.

In this chapter, we propose a modification to this FSK scheme, so as to achieve efficient decodability. In particular, we propose the use of a multitone FSK system to transmit a combination of information symbols are at each time instance.

For multitone FSK, various authors have studied its applications. Luo *et al.* [55] analyzed and compared the performance of multitone FSK and single-tone FSK schemes in which the signal energy is peaky both in time and frequency. Their results show that both single-tone FSK and multitone FSK, with simple hard-decision decoding, have comparable error performance, and furthermore, both approach the wideband capacity limit at large but finite bandwidths. Verdú [77] also showed that in order to achieve the capacity of a wideband noncoherent fading channel, the signaling must be peaky. Oshinomi *et al.* [59] studied a specific implementation of multitone FSK to demonstrate the spectral efficiency of the model. These results are encouraging with respect to the use of multitone FSK for narrowband PLC.

Hence, we adopt the use of multitone FSK modulation scheme with the understanding that the energy is concentrated on only a small fraction of the available frequencies. We consider the special case of using a combination of *exactly* w frequencies at any time instance.

Thus, to determine the set of frequencies to transmit, we use binary matrices as codewords, instead of q-ary vectors. Since these binary matrices are required to have the same number of ones in each column, we are unable to employ general burst-error correcting codes such as array codes [68] or Gabidulin codes [34].

Instead, we construct matrix codes meeting our requirements through concatenation and a simple modification of Gabidulin codes. As a result, we establish infinite families of *efficiently decodable codes* whose rate and relative distance are bounded away from zero. In addition, whenever possible, we use a logarithmic number of frequencies in the length of the code. Simulation results show our multitone modulation schemes outperform single-tone modulation schemes. This chapter has been presented in part at the IEEE International Symposium on Information Theory, 2013 [14].

5.1 Preliminaries

Let Σ be a set of q symbols. Recall that an $(n,q)_q$-code denotes a q-ary code of length n with distance d. For a codeword $\mathsf{u} \in \mathbb{F}_2^n$, the *weight* of a vector u is the number of nonzero components in u. An $(n,d)_2$-code whose codewords are all of weight w is called an $(n,d,w)_2$-*constant weight code*, and is denoted by $\mathrm{CW}(n,d,w)_2$.

5.1.1 Binary Matrix Codes

Let m,n be positive integers and let $\mathbb{F}_2^{m\times n}$ denote the set of $m\times n$ matrices over $\mathbb{F}_2$. Let $\mathsf{M}\in\mathbb{F}_2^{m\times n}$. We index the rows of M by $[m]$, the columns by $[n]$, and let $\mathsf{M}_{i,j}$ be the (i,j)-th entry of M. We denote the ith row by $\mathsf{M}_{i,*}$ and the jth column by $\mathsf{M}_{*,j}$. A *binary* $(m\times n)$-*matrix code* $\mathcal{C}$ is hence a subset of $\mathbb{F}_2^{m\times n}$. The code $\mathcal{C}$ is said to have *constant column weight* w if each column of a matrix in $\mathcal{C}$ has weight w.

5.1.2 Concatenated Codes

Let $\mathcal{B}$ be an $(n,d_{\mathcal{B}})_q$-code over Σ and $\mathcal{A}$ be an $(m,d_{\mathcal{A}})_2$-code with $|\mathcal{A}|\geq q$. Let $\psi:\Sigma\to\mathcal{A}$ be any injective mapping and we write $\psi(\sigma)$ as a binary column vector of length m. Then the *concatenated code* $\mathcal{A}\circ\mathcal{B}$ defined by *inner code* $\mathcal{A}$, *outer code* $\mathcal{B}$ and *mapping* ψ is the following set of $m \times n$ matrices over $\mathbb{F}_2$:

$$\mathcal{A} \circ \mathcal{B} = \{\mathsf{M} : \mathsf{M}_{*,j} = \psi(\mathsf{u}_j), j \in [n], \mathsf{u} \in \mathcal{B}\}.$$

The *inner distance* of $\mathcal{A} \circ \mathcal{B}$ is $d_{\mathcal{A}}$ and its *outer distance* is $d_{\mathcal{B}}$. The size of $\mathcal{A} \circ \mathcal{B}$ is $|\mathcal{B}|$. Note that elements of $\mathcal{A} \circ \mathcal{B}$ are binary $m \times n$ matrices, so $\mathcal{A} \circ \mathcal{B}$ is a binary matrix code. If in addition, $\mathcal{A}$ is a constant weight code of weight w, then $\mathcal{A} \circ \mathcal{B}$ has constant column weight w and $\mathcal{A} \circ \mathcal{B}$ is called an $(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$*-concatenated constant column weight code*, and is denoted by $\mathrm{CCW}(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$.

5.2 Coded Modulation with Multitone FSK

We modify the coded modulation scheme in Chapter 2 to use a binary matrix code in conjunction with multitone FSK, where each symbol is signaled by a combination of w different tones from an set of m tones. We call such a multitone FSK an $\binom{m}{w}$-FSK. An $\binom{m}{1}$-FSK corresponds to the single-tone FSK described in Chapter 2.

Consider a binary $(m \times n)$-matrix code $\mathcal{C}$ with constant column weight w. Each codeword in $\mathcal{C}$ corresponds to a message. We use an $\binom{m}{w}$-FSK with the frequency set $\{f_1, f_2, \ldots, f_m\}$. To transmit a message corresponding to $\mathsf{M} \in \mathcal{C}$, we transmit n symbols, each of which is signaled by a combination of w tones, $\{f_i : i \in [m], \mathsf{M}_{i,j} = 1\}$, $j \in [n]$, over n discrete time instances. We can therefore think of each codeword in $\mathcal{C}$ as having rows indexed by tones and columns indexed by time instances.

We note that in case where $w = 1$, a binary $(m \times n)$-matrix code $\mathcal{C}$ with constant column weight one is equivalent to a m-ary code over alphabet $\Sigma = \{f_1, f_2, \ldots, f_m\}$. The rate of $\mathcal{C}$ is then given by $\log_m |C|/n = \log |C|/(n \log m)$. In particular, in the case for permutation codes, this definition corresponds to the rate defined by Blake, Cohen and Deza [6]. Hence, for a general w, a binary $(m \times n)$-matrix code $\mathcal{C}$ with constant column weight w is equivalent to a $\binom{m}{w}$-ary code over alphabet $\binom{\Sigma}{w}$ and its rate is given by

$$R(\mathcal{C}) = \frac{\log |\mathcal{C}|}{n \log \binom{m}{w}}.$$

This definition of the rate hence captures the size of the "space" when we use w frequencies in n time instances. In addition, this definition "penalizes" the use of additional frequencies and makes it possible to compare between $\binom{m}{w}$-FSK schemes with different w. We note that this differs from the definition in [19, 31] where the rate is defined as the number of bits transmitted per channel use.

Example 5.2.1. The message corresponding to the codeword

$$\mathsf{M} = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

is transmitted via the sets of tones $\{f_3, f_4\}$, $\{f_2, f_4\}$, $\{f_2, f_3\}$, $\{f_1, f_4\}$, and $\{f_1, f_3\}$ over five discrete time steps.

Assuming a hard-decision threshold detector, the received signal (which may contain errors caused by noise) is demodulated to an output $\mathsf{N} \in \mathbb{F}_2^{m \times n}$. As with the case of single-tone FSK, we consider the effects on the detector output that arises from the different types of noise described in Section 2.2.

1. Narrowband noise introduces a tone over a prolonged period. For simplicity, we assume that this (unwanted) tone is present at all time instances. If $e \in [m]$ and e *narrowband noise errors* occur, then there is a set $\Gamma \in \binom{[m]}{e}$ of e rows, such that $\mathsf{N}_{i,j} = 1$ for $i \in \Gamma$, $j \in [n]$.

2. A channel fade event erases a particular tone. If $e \in [m]$, and e *signal fading errors* occur then there is a set $\Gamma \in \binom{[m]}{e}$ of e rows such that $N_{i,j} = 0$ for all $j \in [n]$.

3. Impulse noise results in the entire set of tones being received at a certain time instance. If $e \in [n]$ and e *impulse noise errors* occur, then there is a set $\Pi \in \binom{[n]}{e}$ of e columns such that $\mathsf{N}_{i,j} = 1$ for $i \in [m]$, $j \in \Pi$.

4. Background noise flips the bit value at a particular tone and time instance. If e

background noise occurs then there exists a set $\Omega \in \binom{[n]\times[m]}{e}$ such that $\mathsf{N}_{i,j} = \mathsf{M}_{i,j} + 1$, for all $(i,j) \in \Omega$.

In other words, a narrowband noise error turns an entire row of N to ones, an impulse noise error turns an entire column of N to ones, a channel fade event turns an entire row of N to zeros, and a background noise flips an entry of N.

Example 5.2.2. Continuing Example 5.2.1, if one narrowband noise error occur at frequency 1 and one impulse noise occur at time instance 2, the resulting demodulated matrix is

$$\mathsf{N} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

The following sections adapt some well-known matrix codes to this coded modulation scheme to achieve efficient decoding with asymptotically good parameters.

5.3 Concatenated Codes - Construction and Decoding

First, we consider the classical concatenation methods [29, 32] using a constant weight code as the inner code and a q-ary code as the outer code. We follow the usual method of decoding concatenated codes by decoding the inner code, followed by decoding the outer code. Below, we present the sufficient conditions under which correct decoding can be performed.

Let $\mathcal{A} \circ \mathcal{B}$ be an $(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$-concatenated constant column weight code. Let Σ be the alphabet for $\mathcal{B}$ and $\psi : \Sigma \to \mathcal{A}$ be the injective map defining $\mathcal{A} \circ \mathcal{B}$. For the code $\mathcal{B}$ we use a bounded distance decoder that corrects both errors and erasures, and for the code $\mathcal{A}$ we use a minimum distance decoder which corrects errors. Suppose the detector output is $\mathsf{N} \in \mathbb{F}_2^{m\times n}$. We decode N to $\mathsf{N}' \in \mathcal{A} \circ \mathcal{B}$ in two steps. First, we decode N to a codeword $\mathsf{v} \in (\Sigma \cup \{?\})^n$, where ? is the erasure symbol. For $j \in [n]$, if the column $\mathsf{N}_{*,j}$ is an all-one vector, we set the v_j to be ?. Otherwise, we decode the column $\mathsf{N}_{*,j}$ to a codeword in $\mathcal{A}$, and using ψ, convert this codeword to $\mathsf{v}_j \in \Sigma$. Next, we decode v to a codeword $\mathsf{u} \in \mathcal{B}$.

Input: detector output $\mathsf{N} \in \mathbb{F}_2^{m\times n}$
Output: $\mathsf{N}' \in \mathcal{A} \circ \mathcal{B}$

for $j \in [n]$ **do**
  if $\mathsf{N}_{i,j} = 1$ *for all* $i \in [m]$ **then**
    $\mathsf{v}_j \leftarrow ?$
  else
    decode $\mathsf{N}_{*,j}$ to $\mathsf{c}_j \in \mathcal{A}$
    $\mathsf{v}_j \leftarrow \psi^{-1}(\mathsf{c}_j)$
  end
end
decode v to $\mathsf{u} \in \mathcal{B}$
for $j \in [n]$ **do**
  $\mathsf{N}'_{*,j} \leftarrow \psi(\mathsf{u}_j)$
end
return N'

Fig. 5.1: decoder for concatenated codes

Using ψ again, we represent the codeword u as a matrix $\mathsf{N}' \in \mathcal{A} \circ \mathcal{B}$. See Algorithm 5.1 for details.

The conditions for correct decoding are given in the following proposition. For simplicity, consider the case where only narrowband noise and impulse noise are present. The sufficient conditions can be readily extended to the case when background noise and fading are also present.

Proposition 5.3.1. *Let $\mathcal{A}\circ\mathcal{B}$ be an $(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$-concatenated constant column weight code. Then $\mathcal{A}\circ\mathcal{B}$ is able to correct e_{N} narrowband noise errors and e_{IMP} impulse noise errors if $2e_{\mathsf{N}} < d_{\mathcal{A}}$, $e_{\mathsf{N}} + w < m$, and $e_{\mathsf{IMP}} < d_{\mathcal{B}}$.*

The inequality $e_{\mathsf{N}} + w < m$ captures the situation where a column of all ones is *not* introduced by the presence of narrowband noise errors.

We emphasize that we do not introduce erasure symbols in Algorithm 5.1 prior to the decoding of the inner code $\mathcal{A}$. There is no algorithm to “detect” narrowband noise as the codewords of $\mathcal{A}\circ\mathcal{B}$ may contain an all-one row. Indeed, a more careful construction of $\mathcal{A}\circ\mathcal{B}$ could restrict the weights of both rows and columns and hence, introduce erasure symbols at the decoding stage of the inner code.

5.3.1 Code Construction

We are after concatenated codes with relative outer and/or inner distances bounded away from zero (to guarantee good error-correcting capabilities implied by Proposition 5.3.1), have efficient decoding algorithms for decoding of outer and inner codes, and have rates bounded away from zero. To achieve this, we use Reed-Solomon codes as outer codes. Denote a q-ary Reed-Solomon code of length n, dimension k and minimum distance d by $\mathrm{RS}[n,k,d]_q$. The following theorem gives a general construction of an efficiently decodable concatenated constant column weight code.

Theorem 5.3.2. *Let $n+1$ be a prime power with $n+1 \leq |\mathcal{A}|$. Let the outer code $\mathcal{B}$ be an RS$[n, n-d_{\mathcal{B}}+1, d_{\mathcal{B}}]_{n+1}$ and inner code $\mathcal{A}$ be a CW$(m, d_{\mathcal{A}}, w)_2$. Then $\mathcal{A} \circ \mathcal{B}$ is a CCW$(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$ of rate $(1-(d_{\mathcal{B}}-1)/n)\log n/\log\binom{m}{w}$ and decoding complexity $O(n^2)$ + $O(n|\mathcal{A}|)$.*

We specialize Theorem 5.3.2 in the following ways to give families of asymptotically good codes.

Codes from Block Designs. Our first specialization of Theorem 5.3.2 comes from application of combinatorial designs. Recall that a $(v,k,1)$-BIBD (*balanced incomplete block design*) is a $\{k\}$-uniform set system $(X, \mathcal{S})$ with the property that every pair of distinct points in X is contained in exactly one block. Wilson [83] showed that for every fixed k, there exists a $(v,k,1)$-BIBD for all sufficiently large v satisfying the congruences $v(v-1) \equiv 0 \bmod k(k-1)$ and $v-1 \equiv 0 \bmod k-1$. The blocks of an $(n,w,1)$-BIBD form the supports of a $\mathrm{CW}(n, 2(w-1), w)_2$ of size $n(n-1)/(w(w-1))$.

Corollary 5.3.3 (Block Design Construction)**.** *Fix $w \geq 2$ and let $0 < \delta_{\mathcal{B}} < 1$. Then there exists a CCW$(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$ $\mathcal{C}_m$, for all sufficiently large m satisfying $m(m-1) \equiv 0 \bmod w(w-1)$ and $m-1 \equiv 0 \bmod w-1$, where*

(i) $n = \Theta(m^2)$,

(ii) $d_{\mathcal{A}} = 2(w-1)$,

(iii) $d_{\mathcal{B}} = \lceil \delta_{\mathcal{B}} n \rceil$.

Furthermore, this code family has the property that

$$\lim_{m\to\infty} R(\mathcal{C}_m) \geq \frac{2}{w}(1-\delta_{\mathcal{B}}).$$

Proof. For sufficiently large m satisfying $m(m-1) \equiv 0 \bmod w(w-1)$ and $m-1 \equiv 0 \bmod w-1$, take a $\mathrm{CW}(m, 2(w-1), w)_2$ of size $m(m-1)/(w(w-1))$ as inner code $\mathcal{A}$. Let $n+1$ be a prime power such that $m(m-1)/(2w(w-1)) \leq n+1 \leq m(m-1)/(w(w-1))$, and let $d_n = \lceil \delta_{\mathcal{B}} n \rceil$. Take an $\mathrm{RS}[n, n-d_{\mathcal{B}}+1, d_{\mathcal{B}}]_{n+1}$ as outer code $\mathcal{B}$. Then $\mathcal{A} \circ \mathcal{B}$ is the desired $\mathrm{CCW}(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$.

Conditions (i)–(iii) are immediate. The asymptotic rate of the code family can be verified as follows:

$$\begin{aligned}\lim_{m\to\infty}\left(1-\frac{d_{\mathcal{B}}-1}{n}\right)\frac{\log n}{\log\binom{m}{w}} &\geq (1-\delta_{\mathcal{B}})\lim_{m\to\infty}\frac{\log(m^2/(2w(w-1)))}{\log m^w}\\ &\geq 2(1-\delta_{\mathcal{B}})/w.\end{aligned}$$

■

Codes via Rödl's Nibble. The next specialization is based on Rödl's construction of constant weight codes. In particular, Rödl [67] showed that for fixed w and s, there exists a $\mathrm{CW}(n, w, 2(w-s+1))_2$ of size $(1-o(1))\binom{n}{s}/\binom{w}{s}$.

Corollary 5.3.4 (Rödl's Nibble Construction)**.** *Fix $1 \leq s < w$ and let $0 < \delta_{\mathcal{B}} < 1$. Then there exists a CCW$(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$ $\mathcal{C}_m$, for all sufficiently large m, where*

(i) $n = \Theta(m^s)$,

(ii) $d_{\mathcal{A}} = 2(w-s+1)$,

(iii) $d_{\mathcal{B}} = \lceil \delta_{\mathcal{B}} n \rceil$.

Furthermore, this code family has the property that

$$\lim_{m\to\infty} R(\mathcal{C}_m) \geq \frac{s}{w}(1-\delta_{\mathcal{B}}).$$

Proof. Take a $\mathrm{CW}(m, 2(w-s), w)_2$ of size $M = (1-o(1))\binom{m}{s}/\binom{w}{s}$ as inner code A. Let $n+1$ be a prime power such that $M/2 \leq n+1 \leq M$, and let $d_n = \lceil \delta_{\mathcal{B}} n \rceil$. Take an $\mathrm{RS}[n, n-d_{\mathcal{B}}+1, d_{\mathcal{B}}]_{n+1}$ as outer code $\mathcal{B}$. Then $\mathcal{A} \circ \mathcal{B}$ is the desired $\mathrm{CCW}(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$.

Conditions (i)–(iii) are immediate. The asymptotic rate of the code family can be verified as follows:

$$\begin{aligned}
\lim_{m\to\infty}\left(1-\frac{d_{\mathcal{B}}-1}{n}\right)\frac{\log n}{\log\binom{m}{w}} &\geq (1-\delta_{\mathcal{B}})\lim_{m\to\infty}\frac{\log(1-o(1))\binom{m}{s}/(2\binom{w}{s})}{\log m^w} \\
&\geq s(1-\delta_{\mathcal{B}})/w.
\end{aligned}$$

■

Codes via GV Construction. Our final specialization is based on the GV construction. Levenshteĭn [51] showed that when applied to the space of constant weight vectors, the GV construction gives, for fixed positive $\delta, \kappa < 1$, a $\mathrm{CW}(m, \delta m, \kappa m)_2$ of size at least $2^{m(H(\kappa)-s(\delta,\kappa))}$, where

$$s(\delta,\kappa) = \max_{0\leq\sigma\leq\delta/2} \kappa H(\sigma/\kappa) + (1-\kappa)H(\sigma/(1-\kappa)).$$

Corollary 5.3.5 (GV Construction)**.** *Fix $0 < \delta_{\mathcal{A}} < \kappa < 1/2$, $0 < \delta_{\mathcal{B}} < 1$. Then for m sufficiently large, there exists a $CCW(m \times n, d_{\mathcal{A}}, d_{\mathcal{B}}, w)$ $\mathcal{C}_m$ such that*

(i) $n = \Theta\left(2^{m(H(\kappa)-s(\delta_{\mathcal{A}},\kappa))}\right)$,

(ii) $d_{\mathcal{A}} = \lceil \delta_{\mathcal{A}} m \rceil$,

(iii) $d_{\mathcal{B}} = \lceil \delta_{\mathcal{B}} n \rceil$,

(iv) $w = \lceil \kappa m \rceil$.

Furthermore, this code family has the property that

$$\lim_{m\to\infty} R(\mathcal{C}_m) \geq \left(1 - \frac{s(\delta_{\mathcal{A}},\kappa)}{H(\kappa)}\right)(1-\delta_{\mathcal{B}}).$$

Proof. Take a $\mathrm{CW}(m, \delta_{\mathcal{A}} m, \kappa m)_2$ of size M, with $M = 2^{m(H(\kappa)-s(\delta_{\mathcal{A}},\kappa))}$, as inner code A, and choose a prime power $n+1$ such that $M/2 \leq n+1 \leq M$. Let $d_B = \lceil \delta_B n \rceil$ and take an $\mathrm{RS}[n, n-d_B+1, d_B]_{n+1}$ as outer code B. Then $A \circ B$ is the desired $\mathrm{CCW}(m \times n, d_A, d_B, w)$.

Conditions (i)–(iv) are immediate. The asymptotic rate of the code family can be verified as follows:

$$\begin{aligned}
\lim_{m\to\infty} \left(1 - \frac{d_{\mathcal{B}}-1}{n}\right) \frac{\log n}{\log \binom{m}{w}} &\geq (1-\delta_{\mathcal{B}}) \lim_{m\to\infty} \frac{\log M - o(1)}{\log 2^{mH(w/m)}} \\
&\geq (1-\delta_{\mathcal{B}}) \lim_{m\to\infty} \frac{m(H(\kappa) - s(\delta_{\mathcal{A}}, \kappa) - o(1))}{m(H(\kappa) - o(1))} \\
&= (1-\delta_{\mathcal{B}}) \left(1 - \frac{s(\delta_{\mathcal{A}}, \kappa)}{H(\kappa)}\right).
\end{aligned}$$

■

5.3.2 Simulations

We simulate the performance of concatenated constant column weight codes in the presence of narrowband noise. The setup is similar to that in Section 2.4. Let m be the number of instances used, n be the number of discrete time steps taken to transmit a symbol, $L = \{bn : b \in [10]\}$ and $0 < p < 1$. We simulate a PLC channel with the following independent error characteristics:

1. for each $i \in [m]$, a narrowband noise error of duration $l \in L$ occurs at tone i with probability p,
2. for each $j \in [n]$, an impulse noise error occurs at time instance j with probability 0.05,
3. for each $i \in [m]$, a signal fading error occurs at frequency i with probability 0.05, and
4. for each $(i, j) \in [m] \times [n]$, a background noise occurs at frequency i and time instance j with probability 0.05.

We choose random codewords M from each code under comparison to transmit through the simulated PLC channel. At the receiver, we decode the detector output N to the codeword N' using Algorithm 5.1. The number of symbols in error when transmitting a codeword is then $|\{j \in [n] : \mathsf{M}_{*,j} \neq \mathsf{N}'_{*,j}\}|$, and in this cases, the *error rate* is the fraction of time instances in error.

We compare with the low symbol weight cosets of Reed-Solomon codes studied by Versfeld *et al.* [78,79]. In particular, we observe the error rates of concatenated constant column weight codes with symbol weight cosets of similar rates.

Recall that a $(n, d, r)_q$-symbol weight code is a q-ary code of length n with distance d and bounded symbol weight r. Versfeld *et al.* [78, 79] showed that there exists a coset of the Reed-Solomon code $\mathrm{RS}[n, k, n-k+1]_q$ with bounded symbol weight k. Denote such a code by $\mathrm{RSC}(n, d, k)_q$.

Consider an $\mathrm{RSC}(n, d, k)_q$. Identify the elements in $\mathbb{F}_q$ with elements in $[q]$ and for codeword u we transmit the matrix $\mathsf{M} \in \mathbb{F}_2^{q\times n}$, where

$$\mathsf{M}_{i,j} = \begin{cases} 1 & \text{if } \mathsf{u}_j = i \\ 0 & \text{otherwise.} \end{cases}$$

At the receiver, we decode the detector output N to a codeword u' using the algorithm described in Figure 2.2. The number of symbols in error is then $d(\mathsf{u}, \mathsf{u}')$ and in this case the *error rate* is the ratio of the total number of symbols in error to the total number of symbols transmitted.

We compare concatenated constant column weight codes and low weight cosets of Reed-Solomon codes of similar rates. The parameters of the codes under comparison are given in Table 5.1 and the results of the simulations are given in Fig. 5.2. Observe that concatenated constant column weight codes achieve significantly lower error rates as compared to the low weight cosets of Reed-Solomon codes.

5.4 Gabidulin Codes and Decoding

In this section, we modify Gabidulin codes so that the matrix codes can be used in conjunction with a multitone FSK. Gabidulin introduced these matrix codes [34] with the purpose of correcting "rank errors" and many efficient decoding algorithms have been studied since

Tab. 5.1: Comparing low symbol weight cosets of Reed-Solomon codes and concatenated constant column weight codes

Code	Number of tones	Rate	Remarks
$\mathrm{RSC}(15,8,8)_{16}$	16	≈ 0.533	coset of $\mathrm{RS}[15,8,8]_{16}$
$\mathrm{CW}(9,4,4)_2 \circ \mathrm{RS}[15,14,2]_{16}$	9	≈ 0.535	concatenated constant column weight code
$\mathrm{RSC}(15,11,5)_{16}$	16	≈ 0.333	coset of $\mathrm{RS}[15,5,11]_{16}$
$\mathrm{CW}(13,6,5)_2 \circ \mathrm{RS}[15,14,2]_{16}$	13	≈ 0.361	concatenated constant column weight code

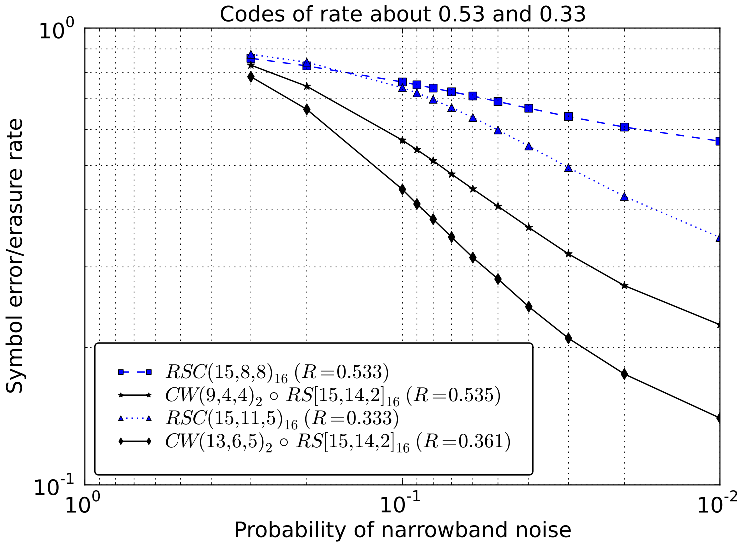


Fig. 5.2: Comparing concatenated and Reed Solomon codes of similar rates

(see for example [34,35,54,68,72]). For our purposes we look at the works of Gabidulin *et al.* [36] and Gabidulin and Pilipchuk [37] where row and column erasure-correction is addressed. In addition, Gabidulin and Pilipchuk [37] pointed to applicability of Gabidulin codes in addressing "narrowband" and "wideband impulse" noise in a multi-channel communication system.

Consider a binary $(m \times n)$-matrix code. Suppose N is a received code matrix. If $e \in [m]$ and e *row erasures* occur, then there is a set $\Gamma \in \binom{[m]}{e}$ of e rows, such that $\mathsf{N}_{i,j} = ?$ for $i \in \Gamma, j \in [n]$. Similarly, if $e \in [n]$ and e *column erasures* occur, then there is a set $\Pi \in \binom{[n]}{e}$ of e columns, such that $\mathsf{N}_{i,j} = ?$ for $i \in [m], j \in \Pi$.

Gabidulin and Pilipchuk then established the following.

Proposition 5.4.1. *Let $d \leq n \leq m$ and $k = n - d + 1$. Then there exists a binary $(m \times n)$-matrix code of size 2^{mk} that corrects r row erasures and c column erasures provided* *that*

$$r + c < d. \tag{5.1}$$

Furthermore, there exist efficient decoding algorithms that do so.

Unfortunately, the binary matrices given by Proposition 5.4.1 do not meet the requirement of having the same number of ones in each column. As such, we propose a simple modification to these matrices to obtain a binary $(m \times n)$-matrix code $\mathcal{C}$ with constant column weight.

Proposition 5.4.2. *Let $d \le n \le m$ and $k = n-d+1$. Then there exists a binary $(2m \times 2n)$-matrix code of size 2^{mk} with constant column weight m that corrects e_{N} narrowband noise and e_{IMP} impulse noise errors provided that $e_{\mathsf{N}} < m$, $e_{\mathsf{IMP}} < n$ and*

$$\left\lfloor \frac{e_{\mathsf{N}}}{2} \right\rfloor + \left\lfloor \frac{e_{\mathsf{IMP}}}{2} \right\rfloor < d. \tag{5.2}$$

Proof. Let $\mathcal{C}$ be the binary $(m \times n)$-matrix code of size 2^{mk} guaranteed by Proposition 5.4.1. Consider the following binary $(2m \times 2n)$-matrix code of size 2^{mk},

$$\mathcal{C}^* = \left\{ \begin{pmatrix} \mathsf{M} & \mathsf{M}+\mathsf{J} \\ \mathsf{M}+\mathsf{J} & \mathsf{M} \end{pmatrix} : \mathsf{M} \in \mathcal{C} \right\}.$$

Then clearly, $\mathcal{C}^*$ is a binary $(2m \times 2n)$-matrix code with constant column weight m. Also, each row of a matrix in $\mathcal{C}^*$ has weight exactly n.

Suppose

$$\begin{pmatrix} \mathsf{M} & \mathsf{M}+\mathsf{J} \\ \mathsf{M}+\mathsf{J} & \mathsf{M} \end{pmatrix} \in \mathcal{C}^*$$

is transmitted and for brevity, let $\mathsf{M}_{1,1} = \mathsf{M}$, $\mathsf{M}_{1,2} = \mathsf{M}+\mathsf{J}$, $\mathsf{M}_{2,1} = \mathsf{M}+\mathsf{J}$ and $\mathsf{M}_{2,2} = \mathsf{M}$.

Suppose N is received with e_{N} narrowband noise and e_{IMP} impulse noise errors. Then for $j \in [n]$, if the column $\mathsf{N}_{*,j}$ is an all-one vector, we set $\mathsf{N}_{i,j} = ?$ for all $i \in [m]$. Similarly, for $i \in [m]$, if the column $\mathsf{N}_{i,*}$ is an all-one vector, we set $\mathsf{N}_{i,j} = ?$ for all $j \in [n]$.

Since $e_{\mathsf{N}} < m$ and $e_{\mathsf{IMP}} < n$, exactly e_{N} rows and e_{IMP} columns are set to ?. Without loss of generality, we can assume $\mathsf{M}_{1,1}$ and $\mathsf{M}_{1,2}$ have at most $\lfloor e_{\mathsf{IMP}}/2 \rfloor$ column erasures. Similarly, we can then assume that $\mathsf{M}_{1,1}$ has at most $\lfloor e_{\mathsf{N}}/2 \rfloor$ row erasures. Therefore, since $\lfloor e_{\mathsf{N}}/2 \rfloor + \lfloor e_{\mathsf{IMP}}/2 \rfloor < d$, we are able to decode to M correctly. ■

Proposition 5.4.2 can be specialized to give a family of asymptotically good codes that are efficiently decodable.

Corollary 5.4.3 (Gabidulin Construction). *Fix $0 < \delta < 1$. Then for all m, there exists a binary $(2m \times 2m)$-matrix code $\mathcal{C}_m$ with constant column weight m, which is able to correct e_{N} narrowband noise and e_{IMP} impulse noise errors provided that $e_{\mathsf{N}} < m$, $e_{\mathsf{IMP}} < m$ and*

$$\left\lfloor \frac{e_{\mathsf{N}}}{2} \right\rfloor + \left\lfloor \frac{e_{\mathsf{IMP}}}{2} \right\rfloor < \lceil \delta m \rceil . \tag{5.3}$$

Furthermore, this code family has the property that

$$\lim_{m\to\infty} R(\mathcal{C}_m) \geq \frac{1}{4}(1-\delta).$$

Proof. Set $m = n$ and $d = \lceil \delta m \rceil$ for Proposition 5.4.2 to obtain $\mathcal{C}_m$. The asymptotic rate of the code family is given by:

$$\begin{aligned} \lim_{m\to\infty} \frac{\log 2^{mk}}{2m \log \binom{2m}{m}} &\geq \lim_{m\to\infty} \frac{mk}{4m^2} \\ &= \frac{1}{4} \lim_{m\to\infty} \frac{m-d+1}{m} \\ &= \frac{1}{4}(1-\delta). \end{aligned}$$

■

5.5 Concluding Remarks

We propose a coded modulation scheme for PLC based on multitone FSK and binary matrix codes. Using concatenation and a simple modification of Gabidulin codes, we construct families of efficiently decodable constant column weight codes for this scheme with rates and relative distances bounded away from zero. Simulation results show that concatenated constant column weight codes achieve lower error rates compared to low weight cosets of Reed-Solomon codes of similar rates. In this work the lengths are chosen with respect to the previous literature, and in our future work we examine simulations at longer lengths.

We observe that our proposal bears similarities with a current proposal for communication of narrowband power line channels that uses Orthogonal Frequency Division Multiplexing (OFDM) [58]. The transmission of the data can also be adaptively modulated depending on the channel characteristics. In this chapter, we do not assume that the channel state information is known to the transmitter or the receiver. Comparisons of our work with OFDM, and other extensions to include coherent demodulation and adaptive modulation techniques is an interesting avenue for future research.

Finally, we observe that Algorithm 5.1 does not attempt to "detect" narrowband noise as we do not exclude the possibility of an all-one row in the matrix codewords. Hence, a more careful construction could restrict the weights of both rows and columns and enable the introduction of the erasure symbol in the decoding of *both* inner and outer codes. This construction is addressed in our future work.

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