Product Construction of Affine Codes

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Motivation

Design Objective

We construct codes over matrices.
Fix 'good' codes $\mathcal{C}$ and $\mathcal{D}$ that satisfy certain constraints.
We construct a code such that for each matrix codeword,

(i) each row belongs to $\mathcal{C}$, **and**

(ii) each column belongs to $\mathcal{D}$.

$$\begin{matrix} \in\mathcal{D} & \in\mathcal{D} & \in\mathcal{D} & \in\mathcal{D} & \\ \end{matrix}$$
$$\begin{pmatrix} \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \end{pmatrix} \begin{matrix} \in\mathcal{C} \\ \in\mathcal{C} \\ \in\mathcal{C} \\ \in\mathcal{C} \end{matrix}$$

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Classical Solution: If $\mathcal{C}$ and $\mathcal{D}$ are linear codes, use **Product Construction**!

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$$\begin{array}{cccccc} \in\mathcal{D} & \in\mathcal{D} & \in\mathcal{D} & \in\mathcal{D} & & \\ \left(\begin{array}{cccc} \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \end{array}\right) & & & & \begin{array}{c} \in\mathcal{C} \\ \in\mathcal{C} \\ \in\mathcal{C} \\ \in\mathcal{C} \end{array} \end{array}$$

Classical Solution: If $\mathcal{C}$ and $\mathcal{D}$ are linear codes, use **Product Construction**!

Question: What if $\mathcal{C}$ and $\mathcal{D}$ are **not linear**?
This talk gives a partial solution...

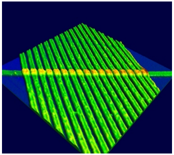
Motivation

Practical Applications for nonlinear constraints

Codes over matrices with weight constraints on **both** rows and columns and with “good” error-correcting capabilities:

(i) coded modulation schemes for power line channels [Chee *et al.* 2013] (considered only columns with weight constraints),

(ii) crossbar arrays of resistive devices [Ordentlich Roth 2000, 2011] (considered only efficient encoding without error correction).



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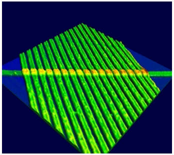

Potential for other applications with "**nonlinear**" constraints...

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Why Product Codes?

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Previous Work

- Amrani '07: guarantees that all the columns belong to the column code; however **only the first few rows** are guaranteed to belong to the row code.

$$\begin{array}{cccc} \in \mathcal{D} & \in \mathcal{D} & \in \mathcal{D} & \in \mathcal{D} \end{array}$$

$$\begin{pmatrix} \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \end{pmatrix} \begin{array}{l} \in \mathcal{C} \\ \in \mathcal{C} \\ ??? \\ ??? \end{array}$$

A Toy Problem

Consider the two codes,

$$\begin{aligned}\mathcal{C}^* &= \{\cancel{0000}, 0011, 0101, 0110, 1001, 1010, 1100, \cancel{1111}\} \\ &= \text{span}\{0011, 0101, 1001\} \setminus \{0000, 1111\}, \\ \mathcal{D}^* &= \{\cancel{0000}, 0011, 0101, 0110, 1001, 1010, 1100, \cancel{1111}\} \\ &= \text{span}\{0011, 0101, 1001\} \setminus \{0000, 1111\}.\end{aligned}$$

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&= \mathrm{span}\{0011, 0101, 1001\} \setminus \{0000, 1111\}.
\end{aligned}
$$

Let us construct a code such that for any matrix codeword

(i) each row belongs to $\mathcal{C}^*$, and

(ii) each column belongs to $\mathcal{D}^*$.

$$
\begin{array}{cccc}
\in \mathcal{D}^* & \in \mathcal{D}^* & \in \mathcal{D}^* & \in \mathcal{D}^*
\end{array}
$$

$$
\begin{pmatrix}
\triangle & \triangle & \triangle & \triangle \\
\triangle & \triangle & \triangle & \triangle \\
\triangle & \triangle & \triangle & \triangle \\
\triangle & \triangle & \triangle & \triangle
\end{pmatrix}
\begin{array}{l}
\in \mathcal{C}^* \\
\in \mathcal{C}^* \\
\in \mathcal{C}^* \\
\in \mathcal{C}^*
\end{array}
$$

In other words, each row and column has weight exactly two.

Attempt: Constrain the Information Array?

Recall that $\mathcal{C}^*$, $\mathcal{D}^*$ come from linear codes of dimension three, say $\mathcal{C}'$, $\mathcal{D}'$. Then a typical codeword from the product code of $\mathcal{C}'$ and $\mathcal{D}'$ is of the form:

$$\begin{pmatrix} x & x & x & \triangle \\ x & x & x & \triangle \\ x & x & x & \triangle \\ \triangle & \triangle & \triangle & \triangle \end{pmatrix}.$$

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(i) Observation: the *'bad'* codewords 0000 and 1111 have systematic parts 000 and 111.

(ii) Set the information array such that it consists of no all-zero or all-one row or column.

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Bad Idea

Consider the following example:

$$\begin{pmatrix} 1 & 0 & 0 & \triangle \\ 0 & 1 & 0 & \triangle \\ 0 & 0 & 1 & \triangle \\ \triangle & \triangle & \triangle & \triangle \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Classical Product Construction of Linear Codes

Consider linear codes:

$\mathcal{C}$ - length n, dimension k, with generator matrix $(\mathbf{I}_k, \mathbf{A})$,

$\mathcal{D}$ - length m, dimension ℓ, with generator matrix $(\mathbf{I}_\ell, \mathbf{B})$.

Classical Product Construction of Linear Codes

Consider linear codes:

$\mathcal{C}$ - length n, dimension k, with generator matrix $(\mathbf{I}_k, \mathbf{A})$,

$\mathcal{D}$ - length m, dimension ℓ, with generator matrix $(\mathbf{I}_\ell, \mathbf{B})$.

Then codewords (of the product code $\mathcal{C} \otimes \mathcal{D}$) are of the form:

$$\left(\begin{array}{c|c} \mathbf{M} & \mathbf{MA} \\ \hline \mathbf{B}^T\mathbf{M} & \mathbf{B}^T\mathbf{MA} \end{array}\right),$$

where $\mathbf{M}$ is an $\ell \times k$ matrix.

Systematic Representation of Affine Codes

Affine code

- Of the form $\mathcal{C} + \mathbf{u}$, where $\mathcal{C}$ is linear and $\mathbf{u}$ any vector of length n.
- WLOG, assume $\mathbf{u} = (\mathbf{0}_k, \mathbf{a})$.
- Let $\mathcal{C}$ have generator matrix $(\mathbf{I}_k, \mathbf{A})$.
- Any codeword in $\mathcal{C} + \mathbf{u}$ may be written as

$$(\mathbf{x}, \mathbf{x}\mathbf{A} + \mathbf{a}).$$

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$$(\mathbf{x}, \mathbf{x}\mathbf{A} + \mathbf{a}).$$

Consider affine codes:

$\mathcal{C} + \mathbf{u}$ - length n, size 2^k, with codewords $(\mathbf{x}, \mathbf{x}\mathbf{A} + \mathbf{a})$,

$\mathcal{D} + \mathbf{v}$ - length m, size 2^ℓ, with codewords $(\mathbf{x}, \mathbf{x}\mathbf{B} + \mathbf{b})$.

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$\mathcal{C} + \mathbf{u}$ - length n, size 2^k, with codewords $(\mathbf{x}, \mathbf{x}\mathbf{A} + \mathbf{a})$,

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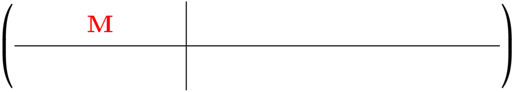
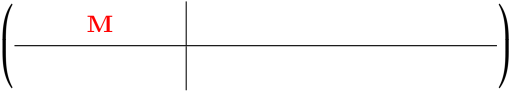
Def 1: 

Def 2: 

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Def 1: $$\left(\begin{array}{c|c} \mathbf{M} & \\ \hline \mathbf{B}^T\mathbf{M} + \mathbf{b}^T\mathbf{1}_k & \end{array}\right)$$

Def 2: $$\left(\begin{array}{c|c} \mathbf{M} & \\ \hline & \end{array}\right)$$

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$\mathcal{C} + \mathbf{u}$ - length n, size 2^k, with codewords $(\mathbf{x}, \mathbf{xA} + \mathbf{a})$,
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Def 1: $$\left(\begin{array}{c|c} \mathbf{M} & \mathbf{MA} + \mathbf{1}_\ell^T \mathbf{a} \\ \hline \mathbf{B}^T\mathbf{M} + \mathbf{b}^T\mathbf{1}_k & \left(\mathbf{B}^T\mathbf{M} + \mathbf{b}^T\mathbf{1}_k\right)\mathbf{A} + \mathbf{1}_{m-\ell}^T\mathbf{a} \end{array}\right)$$

Def 2: $$\left(\begin{array}{c|c} \mathbf{M} & \\ \hline & \end{array}\right)$$

Definition 1 guarantees all rows belong to $\mathcal{C} + \mathbf{u}$.

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Def 2:
$$\left(\begin{array}{c|c} \mathbf{M} & \mathbf{M}\mathbf{A} + \mathbf{1}_\ell^T\mathbf{a} \\ \hline \mathbf{B}^T\mathbf{M} + \mathbf{b}^T\mathbf{1}_k & \mathbf{B}^T\left(\mathbf{M}\mathbf{A} + \mathbf{1}_\ell^T\mathbf{a}\right) + \mathbf{b}^T\mathbf{1}_{n-k} \end{array}\right)$$

Definition 1 guarantees all rows belong to $\mathcal{C} + \mathbf{u}$.
Definition 2 guarantees all columns belong to $\mathcal{D} + \mathbf{v}$.

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Def 2:
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Definition 1 guarantees all rows belong to $\mathcal{C} + \mathbf{u}$.
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Proposition (Sufficient Condition)

If $\mathbf{1}_n \in \mathcal{C}$ and $\mathbf{1}_m \in \mathcal{D}$, then both definitions coincide!

Product Construction of Affine Codes

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Pick $\mathbf{u} = (\mathbf{0}_k, \mathbf{a})$ and $\mathbf{v} = (\mathbf{0}_\ell, \mathbf{b})$.

Theorem (Construction I)

If $\mathbf{1}_n \in \mathcal{C}$ and $\mathbf{1}_m \in \mathcal{D}$, then the code

$$\left\{ \left(\begin{array}{c|c} \mathbf{M} & \mathbf{M}\mathbf{A} + \mathbf{1}_\ell^T \mathbf{a} \\ \hline \mathbf{B}^T\mathbf{M} + \mathbf{b}^T\mathbf{1}_k & \left(\mathbf{B}^T\mathbf{M} + \mathbf{b}^T\mathbf{1}_k\right)\mathbf{A} + \mathbf{1}_{m-\ell}^T \mathbf{a} \end{array} \right) : \mathbf{M} \text{ is a } \ell \times k \text{ matrix} \right\}$$

is a systematic code of size 2^{kl} and is a coset of $\mathcal{C} \otimes \mathcal{D}$ with coset leader

$$\mathbf{U} \triangleq \left(\begin{array}{c|c} \mathbf{0}_{\ell\times k} & \mathbf{1}_\ell^T \mathbf{a} \\ \hline \mathbf{b}^T\mathbf{1}_k & \mathbf{b}^T\mathbf{1}_{n-k} + \mathbf{1}_{m-\ell}^T \mathbf{a} \end{array} \right).$$

For any codeword $\mathbf{N}$,

(i) *each row of $\mathbf{N}$ belongs to $\mathcal{C} + \mathbf{u}$, and*

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Product Construction of Affine Codes

$\mathcal{C}$ - length n, dimension k, with generator matrix $(\mathbf{I}_k, \mathbf{A})$,

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is a systematic code of size $2^{k\ell}$ and is a coset of $\mathcal{C} \otimes \mathcal{D}$ with coset leader

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Product Construction of Affine Codes

Remarks

- Encoding and decoding complexities are very similar to usual product codes.

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Remarks

- Many well-known codes contain $\mathbf{1}$. Examples: primitive narrow-sense BCH, Reed-Muller, extended Golay.

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Toy Problem - Continued

Recall the two codes,

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Let $\mathcal{C} = \mathcal{D} = \mathrm{span}\{0101, 1010\}$ and $\mathbf{u} = \mathbf{v} = 0011$.
Then

$$\mathcal{C} + \mathbf{u} \subseteq \mathcal{C}^*, \quad \mathcal{D} + \mathbf{v} \subseteq \mathcal{D}^*.$$

Toy Problem - Continued

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Then

$$\mathcal{C} + \mathbf{u} \subseteq \mathcal{C}^*, \quad \mathcal{D} + \mathbf{v} \subseteq \mathcal{D}^*.$$

Applying Construction I, we have

$$\begin{pmatrix} 0&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1\\0&1&1&0\\1&1&0&0\\1&0&0&1 \end{pmatrix} \begin{pmatrix} 0&0&1&1\\1&0&0&1\\1&1&0&0\\0&1&1&0 \end{pmatrix} \begin{pmatrix} 0&0&1&1\\1&1&0&0\\1&1&0&0\\0&0&1&1 \end{pmatrix} \begin{pmatrix} 0&1&1&0\\0&0&1&1\\1&0&0&1\\1&1&0&0 \end{pmatrix} \begin{pmatrix} 0&1&1&0\\0&1&1&0\\1&0&0&1\\1&0&0&1 \end{pmatrix} \begin{pmatrix} 0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0 \end{pmatrix} \begin{pmatrix} 0&1&1&0\\1&1&0&0\\1&0&0&1\\0&0&1&1 \end{pmatrix}$$

$$\begin{pmatrix} 1&0&0&1\\0&0&1&1\\0&1&1&0\\1&1&0&0 \end{pmatrix} \begin{pmatrix} 1&0&0&1\\0&1&1&0\\0&1&1&0\\1&0&0&1 \end{pmatrix} \begin{pmatrix} 1&0&0&1\\1&0&0&1\\0&1&1&0\\0&1&1&0 \end{pmatrix} \begin{pmatrix} 1&0&0&1\\1&1&0&0\\0&1&1&0\\0&0&1&1 \end{pmatrix} \begin{pmatrix} 1&1&0&0\\0&0&1&1\\0&0&1&1\\1&1&0&0 \end{pmatrix} \begin{pmatrix} 1&1&0&0\\0&1&1&0\\0&0&1&1\\1&0&0&1 \end{pmatrix} \begin{pmatrix} 1&1&0&0\\1&0&0&1\\0&0&1&1\\0&1&1&0 \end{pmatrix} \begin{pmatrix} 1&1&0&0\\1&1&0&0\\0&0&1&1\\0&0&1&1 \end{pmatrix}.$$

Matrices with Bounded Row and Column Weights

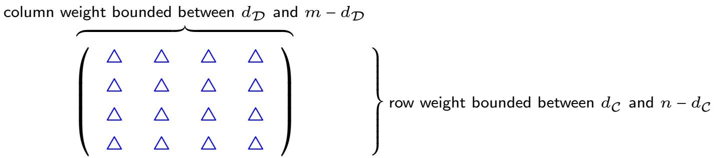
We generalize the previous construction.

Proposition

Let $\mathcal{C}$, $\mathcal{D}$ be binary linear $[n, k, d_\mathcal{C}]$, and $[m, \ell, d_\mathcal{D}]$ codes respectively. Suppose $\mathbf{1}_n \in \mathcal{C}$, and $\mathbf{1}_m \in \mathcal{D}$. Then we have a systematic code over binary $m \times n$ matrices of size $2^{(k-1)(\ell-1)}$ whose codeword matrices have

(i) *row weight bounded between $d_\mathcal{C}$ and $n - d_\mathcal{C}$,*

(ii) *column weight bounded between $d_\mathcal{D}$ and $m - d_\mathcal{D}$.*



Variants of Construction I

Construction I can be modified so that

(a) the component codes are unions of affine codes; i.e.

$$\mathcal{C}^* = \bigcup \mathcal{C} + \mathbf{u}, \text{ and } \mathcal{D}^* = \bigcup \mathcal{D} + \mathbf{v},$$

where $\mathcal{C}$ and $\mathcal{D}$ are linear codes.

(b) the component code is an expurgated code; i.e.

$$\mathcal{C}^* = \mathcal{C}_1 \setminus \mathcal{C}_2, \text{ and } \mathcal{D}^* = \mathcal{D}_1 \setminus \mathcal{D}_2,$$

where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_1, \mathcal{D}_2$ are linear codes such that $\mathcal{C}_2 \subseteq \mathcal{C}_1$ and $\mathcal{D}_2 \subseteq \mathcal{D}_1$.

$$\begin{matrix} \in \mathcal{D}^* & \in \mathcal{D}^* & \in \mathcal{D}^* & \in \mathcal{D}^* & \\ \end{matrix}$$

$$\begin{pmatrix} \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \end{pmatrix} \begin{matrix} \in \mathcal{C}^* \\ \in \mathcal{C}^* \\ \in \mathcal{C}^* \\ \in \mathcal{C}^* \end{matrix}$$

Variants of Construction I

Construction I can be modified so that

(c) each row (column) belongs to a different component affine code (*a la* Alipour *et al.* '12: Irregular product codes).

$$\begin{array}{cccc} \in \mathcal{D}_1^* & \in \mathcal{D}_2^* & \in \mathcal{D}_3^* & \in \mathcal{D}_4^* \end{array}$$

$$\begin{pmatrix} \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \\ \triangle & \triangle & \triangle & \triangle \end{pmatrix} \begin{array}{l} \in \mathcal{C}_1^* \\ \in \mathcal{C}_2^* \\ \in \mathcal{C}_3^* \\ \in \mathcal{C}_4^* \end{array}$$

Conclusion

(i) Construction of systematic affine matrix codes that are obtained by taking product of affine codes.
- Property: every row and every column belongs to the row code and column code, respectively.
- Construct matrix codes with restricted column and row weights and with error-correcting capabities.

(ii) Potential applications in array codes with “affine-like” constraints.

Questions?