Binary Subblock Energy-Constrained Codes: Bounds on Code Size and Asymptotic Rate

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***Abstract*—The *subblock energy-constrained codes* (SECCs) have recently been shown to be suitable candidates for simultaneous energy and information transfer, where bounds on SECC capacity were presented for communication over noisy channels. In this paper, we study binary SECCs with given error correction capability, by considering codes with a certain minimum distance. Binary SECCs are a class of constrained codes where each codeword is partitioned into equal sized subblocks, and every subblock has weight exceeding a given threshold. We present several upper and lower bounds on the optimal SECC code size, and also derive the asymptotic Gilbert-Varshamov (GV) and sphere-packing bounds for SECCs. A related class of codes are the *heavy weight codes* (HWCs) where the weight of each codeword exceeds a given threshold. We show that for a fixed subblock length, the asymptotic rate for SECCs is *strictly* lower than the corresponding rate for HWCs when the relative distance of the code is small. The rate gap between HWCs and SECCs denotes the penalty due to imposition of weight constraint per subblock, relative to the codeword based weight constraint.**

I. Introduction

For providing regular energy content in a codeword for the application of simultaneous energy and information transfer from a powered transmitter to an energy harvesting receiver, the use of *subblock energy-constrained codes* (SECCs) was proposed in [1], [2]. When *on-off keying* is employed, with bit-1 (bit-0) represented by the presence (absence) of a high energy signal, higher energy content in the transmitted signal is achieved by using relatively high weight codewords. Binary SECCs are a class of constrained codes where each codeword is partitioned into equal sized subblocks, and every subblock has weight exceeding a given threshold. The weight constraint *per subblock* in binary SECCs helps to avoid sequences which carry limited energy over long durations, thereby preventing energy outage at a receiver having finite energy storage capability. Bounds on SECC capacity and error exponent for communication over noisy channels were presented in [2].

In this paper, we study bounds on the optimal code size and asymptotic rate for binary SECCs with given error correction capability, by considering codes with a certain minimum distance. We present several upper and lower bounds on the SECC code size and asymptotic rate. Additionally, for fixed subblock length and small code distance, we show that the asymptotic rate for SECCs is strictly lower than the corresponding rate for heavy weight codes (HWCs) [3], [4].

The notation used is as follows. The input alphabet is denoted by $\mathcal{X}$ which comprises q symbols. An n-length, q-ary *code* $\mathscr{C}$ over $\mathcal{X}$ is a subset of $\mathcal{X}^n$. The elements of $\mathscr{C}$ are called *codewords* and $\mathscr{C}$ is said to have *minimum distance* d if the *Hamming distance* between any two distinct codewords is at least d. A q-ary code of length n and distance d is called an $(n,d)_q$-code, and the largest size of an $(n,d)_q$-code is denoted by $A_q(n,d)$. For binary alphabet, an $(n,d)_2$-code is just called an (n,d)-code, and its largest size is simply denoted $A(n,d)$.

A *constant weight code* (CWC) with parameter w is a binary code where each codeword has weight exactly w. We denote a CWC with weight parameter w, blocklength n, and distance d by (n,d,w)-CWC, and denote its maximum possible size by $A(n,d,w)$. A *heavy weight code* (HWC) with parameter w is a binary code where each codeword has weight *at least* w. We denote a HWC with weight parameter w, blocklength n, and distance d by (n,d,w)-HWC, and denote its maximum possible size by $H(n,d,w)$.

A binary SECC with codeword length $n = mL$, minimum distance d, subblock length L, and weight per subblock *at least* w_s, is called a (m,L,d,w_s)-SECC. We denote the maximum possible size of (m,L,d,w_s)-SECC by $S(m,L,d,w_s)$. Since an (m,L,d,w_s)-SECC is an (mL,d,mw_s)-HWC, we have that $S(m,L,d,w_s) \le H(mL,d,mw_s)$.

We also analyze bounds on the asymptotic SECC rate where the number of subblocks m tends to infinity, d scales linearly with m, but L and w_s are fixed. In the following, the base for $\log$ is assumed to be 2. Formally, for fixed $0 < \delta < 1$, the asymptotic rate for SECCs with subblock length L, subblock weight at least w_s, number of subblocks in a codeword $m \to \infty$, and distance d scaling as $d = \lfloor mL\delta \rfloor$ is defined as

$$\sigma(L,\delta,w_s/L) \triangleq \limsup_{m\to\infty} \frac{\log S(m,L,\lfloor mL\delta\rfloor,w_s)}{mL}. \quad (1)$$

This rate can be compared with related exponents:

$$\alpha(\delta) \triangleq \limsup_{n\to\infty} \frac{\log A\left(n,\lfloor n\delta\rfloor\right)}{n}, \quad (2)$$

$$\alpha(\delta,w_s/L) \triangleq \limsup_{n\to\infty} \frac{\log A\left(n,\lfloor n\delta\rfloor,\lfloor nw_s/L\rfloor\right)}{n}, \quad (3)$$

$$\eta(\delta,w_s/L) \triangleq \limsup_{n\to\infty} \frac{\log H\left(n,\lfloor n\delta\rfloor,\lfloor nw_s/L\rfloor\right)}{n}. \quad (4)$$

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In a related work, Cohen *et al.* [3] introduced the class of HWCs, motivated by certain asynchronous communication problems. Later Bachoc *et al.* [4] established the asymptotic rate for HWCs, showing that for $0 \le \delta, \omega \le 1$, we have

$$\eta(\delta,\omega) = \begin{cases} \alpha(\delta), & \text{when } 0 \le \omega \le 1/2, \\ \alpha(\delta,\omega), & \text{when } 1/2 \le \omega \le 1. \end{cases} \quad (5)$$

A. Our Contributions

The contributions of this paper are as follows:

- We provide both upper and lower bounds for $S(m, L, d, w_s)$ in Section II.
- We derive bounds on the asymptotic rate for SECCs in Section III. Additionally, for given L and w_s, in Section IV we demonstrate the existence of an $\hat{\delta}_L$ such that $\eta(\delta, w_s/L) > \sigma(L, \delta, w_s/L)$ for all $\delta < \hat{\delta}_L$.
- We provide numerical lower bounds on the asymptotic rate gap between HWCs and SECCs in Section V.

II. Bounds on optimal SECC code size

Among other bounds, we present the GV and sphere-packing bounds on the optimal SECC code size, $S(m, L, d, w_s)$, in this section, and their respective asymptotic versions in Sec. III. Let $\mathcal{S}(m, L, w_s)$ denote the space of all binary SECC words comprising of m subblocks, each subblock having length L, with weight at least w_s per subblock. For $\mathbf{x} \in \mathcal{S}(m, L, w_s)$, we define a ball of radius t, centered at $\mathbf{x}$, as $\mathcal{B}_{\mathcal{S}}(\mathbf{x}, t; m, L, w_s) \triangleq \{\mathbf{y} \in \mathcal{S}(m, L, w_s) : d(\mathbf{x}, \mathbf{y}) \le t\}$.

Unfortunately, the size of $\mathcal{B}_{\mathcal{S}}(\mathbf{x}, t; m, L, w_s)$ depends on $\mathbf{x}$. Take for example, $m = 1$, $L = 4$, $w_s = 2$ and $t = 1$. We have that $\mathcal{B}_{\mathcal{S}}(0111, t; m, L, w_s) = \{0111, 1111, 0011, 0101, 0110\}$, while $\mathcal{B}_{\mathcal{S}}(1001, t; m, L, w_s) = \{1001, 1101, 1011\}$.

We denote the smallest and the average ball size in the SECC space as follows:

$$|\mathcal{B}_{\mathcal{S}}^{\min}(t; m, L, w_s)| \triangleq \min_{\mathbf{x} \in \mathcal{S}(m,L,w_s)} |\mathcal{B}_{\mathcal{S}}(\mathbf{x}, t; m, L, w_s)|, \quad (6)$$

$$|\mathcal{B}_{\mathcal{S}}^{\text{avg}}(t; m, L, w_s)| \triangleq \sum_{\mathbf{x} \in \mathcal{S}(m,L,w_s)} \frac{|\mathcal{B}_{\mathcal{S}}(\mathbf{x}, t; m, L, w_s)|}{|\mathcal{S}(m, L, w_s)|}. \quad (7)$$

The total number of words in SECC space, $\mathcal{S}(m, L, w_s)$, are $\left(\sum_{i=w_s}^{L} \binom{L}{i}\right)^m$. The *generalized Gilbert-Varshamov bound* [5, Thm. 4] (for spaces where balls with fixed radius and different centers may have different sizes) when applied to the SECC space gives us the following lower bound on the optimal SECC code size, $S(m, L, d, w_s)$.

Proposition 1. *We have*

$$S(m, L, d, w_s) \ge \frac{\left(\sum_{i=w_s}^{L} \binom{L}{i}\right)^m}{|\mathcal{B}_{\mathcal{S}}^{\text{avg}}(d-1; m, L, w_s)|}. \quad (8)$$

The next proposition extends the concatenation approach [6] for SECCs.

Proposition 2. *If $q \le H(L, d_1, w_s)$, then*

$$S(m, L, d_1 d_2, w_s) \ge A_q(m, d_2). \quad (9)$$

Proof: Adapt the concatenated code construction scheme in [7, Prop. 4.1] by replacing the constant weight inner code by a heavy weight inner code. ∎

We extend the Elias-Bassalygo bound (see for example, [8, eq. 2.7]) for SECCs.

Proposition 3. *We have*

$$S(m, L, d, w_s) \ge \frac{\left(\sum_{i=w_s}^{L} \binom{L}{i}\right)^m}{2^{mL}} A(mL, d). \quad (10)$$

Proof: Let $\mathscr{C}$ be a (mL, d)-code with $A(mL, d)$ codewords. Let $\mathbb{F}_2^{mL}$ denote the space of binary vectors of length mL, and $\mathbf{x} \in \mathbb{F}_2^{mL}$ be chosen so that $|\mathcal{S}(m, L, w_s) \cap (\mathbf{x} + \mathscr{C})|$ is maximal. Then

$$\begin{aligned} S(m, L, d, w_s) &\ge |\mathcal{S}(m, L, w_s) \cap (\mathbf{x} + \mathscr{C})| \\ &\ge \frac{1}{2^{mL}} \sum_{\mathbf{y} \in \mathbb{F}_2^{mL}} |\mathcal{S}(m, L, w_s) \cap (\mathbf{y} + \mathscr{C})| \\ &= \frac{1}{2^{mL}} \sum_{\mathbf{y} \in \mathbb{F}_2^{mL}} \sum_{\mathbf{b} \in \mathcal{S}(m,L,w_s)} \sum_{\mathbf{c} \in \mathscr{C}} |\{\mathbf{b}\} \cap \{\mathbf{y} + \mathbf{c}\}| \\ &= \frac{1}{2^{mL}} \sum_{\mathbf{b} \in \mathcal{S}(m,L,w_s)} \sum_{\mathbf{c} \in \mathscr{C}} 1 \\ &= \frac{|\mathcal{S}(m, L, w_s)||\mathscr{C}|}{2^{mL}}. \end{aligned}$$

∎

Observing that balls of radius $t = \lfloor (d-1)/2 \rfloor$ around codewords are non-intersecting in an (m, L, d, w_s)-SECC, we have the following *sphere-packing bound* for SECCs.

Proposition 4. *Let $t \triangleq \lfloor (d-1)/2 \rfloor$. Then, we have*

$$S(m, L, d, w_s) \le \frac{\left(\sum_{i=w_s}^{L} \binom{L}{i}\right)^m}{|\mathcal{B}_{\mathcal{S}}^{\min}(t; m, L, w_s)|}. \quad (11)$$

As discussed earlier, for a given radius t, different SECC balls may have different sizes, depending on the center word. In view of this, note that the SECC sphere-packing upper bound (11) is obtained by considering the smallest ball size of radius t. The *generalized sphere-packing bound*, for spaces where different balls of same radius have different sizes, was investigated in [9], [10]. However, it is unclear if the techniques in [9], [10] are able to yield tighter *asymptotic* upper bound than that given in the next section via Theorem 1.

Furthermore, we point out that the average sphere-packing value is *not* an upper bound for the optimal SECC code size. Specifically, for a t-error-correcting code, the *average sphere-packing value* was defined in [9] to be the ratio of the size of the space, to the average ball size of radius t. It was observed that for many spaces, this average sphere-packing value is an *upper bound* for the optimal code size. However, we claim that this value is *not* an upper bound for the optimal SECC code size, and prove the claim by providing a counter-example.

Consider the space, $\mathcal{S}(m, L, w_s)$, corresponding to $m = 1$, $L = 3$, and $w_s = 1$. Here, the size $|\mathcal{S}(m, L, w_s)|$ is 7, while the average ball size, $|\mathcal{B}_{\mathcal{S}}^{\text{avg}}(t; m, L, w_s)|$, for $t = 1$, is equal to 25/7. In this case, the *average sphere-packing value*, for a single error correcting code, is 49/25. But this value is readily seen to be strictly less than the size of the SECC code $\mathscr{C} = \{100, 011\}$, thereby providing the required counter-example.

We next present a Johnson-type bound [11] for SECCs. Towards this, we consider a generalization of SECC where different subblocks in a codeword may have different length and weight constraints. Let $T(m, [L_1, \ldots, L_m], d, [w_1, \ldots, w_m])$ denote the largest size of a binary code where each codeword has m subblocks, the ith subblock has length L_i and weight at least w_i, and the minimum distance of the code is d. Here, the length of each codeword is $n = \sum_{i=1}^{m} L_i$.

Now, let $\mathscr{C}$ be such a generalized code of size $T(m, [L_1, \ldots, L_m], d, [w_1, \ldots, w_m])$. Consider a matrix with n columns, whose rows comprise of the $T(m, [L_1, \ldots, L_m], d, [w_1, \ldots, w_m])$ codewords of $\mathscr{C}$. By focusing on the ith subblock of each codeword, we observe that there exists a column, say column l, having at least $T(m, [L_1, \ldots, L_m], d, [w_1, \ldots, w_m]) \times (w_i/L_i)$ ones. Pick a subcode of $\mathscr{C}$ where each codeword has a 1 in the l-th position. Delete the l-th component in the subcode to obtain

$$T(m, [L_1, \ldots, L_m], d, [w_1, \ldots, w_m]) \leq \frac{L_i}{w_i} T(m, [L_1, \ldots, L_i - 1, \ldots L_m], d, [w_1, \ldots, w_i - 1, \ldots, w_m]). \tag{12}$$

By varying i from 1 to m and recursively applying (12),

$$T(m, [L_1, \ldots, L_m], d, [w_1, \ldots, w_m]) \leq \left(\prod_{i=1}^{m} \frac{L_i}{w_i}\right) T(m, [L_1 - 1, \ldots, L_m - 1], d, [w_1 - 1, \ldots, w_m - 1]). \tag{13}$$

Specializing (13) to the case when each $L_i = L$ and $w_i = w_s$, we obtain the following Johnson-type upper bound for SECCs.

Proposition 5. *We have*

$$S(m, L, d, w_s) \leq \frac{L^m}{w_s^m} S(m, L-1, d, w_s - 1). \tag{14}$$

III. Asymptotic Bound on SECC Rate

We fix the relative distance δ, and the subblock length L, and provide estimates of the asymptotic rate as number of subblocks $m \to \infty$. The motivation for fixing L to relatively small values comes from the application of SECCs to *simultaneous energy and information transfer* [1], where SECCs with appropriate weight were shown to avoid energy outage if the subblock length is less than a certain threshold.

Recall the definitions of $\sigma(L, \delta, w_s/L)$ and $\eta(\delta, w_s/L)$ given by (1) and (4). We have the following inequality.

$$\sigma(L, \delta, w_s/L) \leq \eta(\delta, w_s/L). \tag{15}$$

The gap $\eta(\delta, w_s/L) - \sigma(L, \delta, w_s/L)$ denotes the rate penalty on HWC due to the additional constraint on sufficient weight within every *subblock duration*. Now, if we define

$$\delta^* \triangleq 2\left(\frac{w_s}{L}\right)\left(1 - \frac{w_s}{L}\right), \tag{16}$$

then from MRRW bound [8, Eq. (2.16)] for CWCs, we have

$$\alpha(\delta, w_s/L) = 0, \text{ if } \delta \geq \delta^*. \tag{17}$$

Therefore, combining (5) and (15), for $w_s \geq L/2$, we have

$$\sigma(L, \delta, w_s/L) = 0, \text{ if } \delta \geq \delta^*. \tag{18}$$

The following proposition presents the asymptotic GV lower bound for $\sigma(L, \delta, w_s/L)$.

Proposition 6 (Asymptotic GV bound for SECCs)**.** *We have $\sigma(L, \delta, w_s/L) \geq \sigma_{GV}(L, \delta, w_s/L)$ where*

$$\sigma_{GV}(L, \delta, w_s/L) \triangleq \frac{1}{L} \log\left(\sum_{j=w_s}^{L} \binom{L}{j}\right) - h(\delta), \tag{19}$$

with $h(\delta) \triangleq -\delta \log_2 \delta - (1-\delta)\log_2(1-\delta)$.

Proof: A simple upper bound on $|\mathcal{B}_{\mathcal{S}}^{\text{avg}}(d-1; m, L, w_s)|$ is given by $\sum_{i=1}^{d-1}\binom{mL}{i}$. Hence, using Prop. 1, we get

$$S(m, L, d, w_s) \geq \frac{\left(\sum_{j=w_s}^{L}\binom{L}{j}\right)^m}{\sum_{i=1}^{d-1}\binom{mL}{i}}. \tag{20}$$

The proposition now follows by combining (1) and (20). ∎

Next, Theorem 1 presents the sphere-packing upper bound on $\sigma(L, \delta, w_s/L)$ for relatively small values of δ. We will use the following lemma towards proving this theorem.

Lemma 1. *Let $\mathbf{z}$ be a length L vector whose weight $\tilde{w}_s$ satisfies $\tilde{w}_s \geq w_s$. Then the number of binary vectors with length L, weight at least w_s, which are at a distance of either 1 or 2 from $\mathbf{z}$ is lower bounded by $(L - w_s)(w_s + 1)$.*

Proof: Let N_1 (resp. N_2) be the number of L length binary vectors of weight at least w_s, which are at a distance of 1 (resp. 2) from $\mathbf{z}$. We consider three different cases:

1) $\tilde{w}_s = w_s$: In this case $N_1 = L - w_s$. If $(L - w_s) \geq 2$, then $N_2 = (L-w_s)w_s + \binom{L-w_s}{2}$, else $N_2 = (L-w_s)w_s$.
2) $\tilde{w}_s = w_s+1$: In this case $N_1 = L$. If $(L-w_s) \geq 2$, then $N_2 = (L-w_s)w_s + \binom{L-w_s}{2}$, else $N_2 = (L-w_s)w_s$.
3) $\tilde{w}_s \geq w_s + 2$: In this scenario, $N_1 = L$ and $N_2 = \binom{L}{2}$.

For all the above three cases, it can easily be verified that $N_1 + N_2 \geq (L - w_s)(w_s + 1)$. ∎

Theorem 1 (Asymptotic sphere-packing bound for SECCs)**.** *For $0 < \delta < \min\{\delta^*, 4/L\}$, we have $\sigma(L, \delta, w_s/L) \leq \sigma_{SP}(L, \delta, w_s/L)$ where*

$$\sigma_{SP}(L, \delta, w_s/L) \triangleq \frac{1}{L}\log\left(\sum_{j=w_s}^{L}\binom{L}{j}\right) - \frac{1}{L}h\left(\frac{L\delta}{4}\right) - \frac{\delta}{4}\log\left((L-w_s)(w_s+1)\right). \tag{21}$$

Proof: Let $t = \lfloor (d-1)/2 \rfloor$, where distance $d = \lfloor \delta m L \rfloor$. Define $\tilde{m} \triangleq \lfloor t/2 \rfloor$ and note that the constraint $\delta < 4/L$ implies that $\tilde{m} < m$. For a given $\mathbf{x} \in \mathcal{S}(m, L, w_s)$, let $\mathbf{x}_{[j]}$ denote the j-th subblock of $\mathbf{x}$, i.e. $\mathbf{x} = (\mathbf{x}_{[1]}\,\mathbf{x}_{[2]} \cdots \mathbf{x}_{[m]})$. Let $\Lambda_{\mathbf{x}} \subset \mathcal{S}(m, L, w_s)$ be the set of vectors which satisfy:

(i) For every $\mathbf{y} \in \Lambda_{\mathbf{x}}$, exactly $\tilde{m}$ subblocks of $\mathbf{y}$ differ from corresponding subblocks of $\mathbf{x}$, and (ii) If $\mathbf{y}_{[j]} \neq \mathbf{x}_{[j]}$, then $d(\mathbf{x}_{[j]}, \mathbf{y}_{[j]}) \in \{1, 2\}$. Thus, if $\mathbf{y} \in \Lambda_{\mathbf{x}}$, then $d(\mathbf{x}, \mathbf{y}) \leq 2\tilde{m} \leq t$, and hence $\Lambda_{\mathbf{x}} \subseteq \mathcal{B}_{\mathcal{S}}(\mathbf{x}, t; m, L, w_s)$ with

$$|\mathcal{B}_{\mathcal{S}}(\mathbf{x}, t; m, L, w_s)| \geq |\Lambda_{\mathbf{x}}| \overset{\text{(i)}}{\geq} \binom{m}{\tilde{m}} \left[(L - w_s)(w_s + 1)\right]^{\tilde{m}}, \tag{22}$$

where (i) follows from Lemma 1. Because the above inequality holds for all $\mathbf{x} \in \mathcal{S}(m, L, w_s)$, we have

$$|\mathcal{B}_{\mathcal{S}}^{\min}(t; m, L, w_s)| \geq \binom{m}{\tilde{m}} \left[(L - w_s)(w_s + 1)\right]^{\tilde{m}}. \tag{23}$$

Now $\lim_{m \to \infty} \frac{\tilde{m}}{m} = \frac{L\delta}{4}$ and hence the claim is proved by combining (1), Prop. 4, and (23). ■

IV. Rate Penalty Due to Subblock Constraints

In SECCs, the fraction of ones in every subblock is at least w_s/L, and hence the fraction of ones in the entire codeword is also at least w_s/L. Relative to the constraint requiring at least w_s/L fraction of bits to be 1 for all *codewords*, the rate penalty due to the constraint requiring minimum weight w_s per *subblock* is quantified by $G_{\eta-\sigma}(L, \delta, w_s/L)$, defined as

$$G_{\eta-\sigma}(L, \delta, w_s/L) \triangleq \eta(\delta, w_s/L) - \sigma(L, \delta, w_s/L). \tag{24}$$

For $w_s \geq L/2$, using (5), we note that a lower bound for $G_{\eta-\sigma}(L, \delta, w_s/L)$ is given by

$$G_{\eta-\sigma}^{LB}(L, \delta, w_s/L) \triangleq \left[\alpha_{GV}(\delta, w_s/L) - \sigma_{SP}(L, \delta, w_s/L)\right]^{+}, \tag{25}$$

where the notation $[z]^{+}$ implies $\max\{0, z\}$, and $\alpha_{GV}(\delta, w_s/L)$ denotes the asymptotic GV lower bound for CWCs [8], [12]

$$\alpha_{GV}(\delta, \omega) \triangleq h(\omega) - \omega h\left(\frac{\delta}{2\omega}\right) - (1-\omega) h\left(\frac{\delta}{2(1-\omega)}\right). \tag{26}$$

When $w_s \leq L/2$, we have $\eta(\delta, w_s/L) = \alpha(\delta) = \alpha(\delta, 0.5)$, and the corresponding rate gap lower bound is defined as

$$G_{\eta-\sigma}^{LB}(L, \delta, w_s/L) \triangleq \left[\alpha_{GV}(\delta, 0.5) - \sigma_{SP}(L, \delta, w_s/L)\right]^{+}. \tag{27}$$

The following theorem shows that rate gap between HWCs and SECCs is strictly positive when δ is sufficiently small.

Theorem 2. *For even L with $L \geq 4$, we have the strict inequality $G_{\eta-\sigma}^{LB}(L, \delta, 0.5) > 0$ for $0 < \delta < \hat{\delta}_L$, where $\hat{\delta}_L$ is the smallest positive root of $\hat{f}_L(\delta)$ defined as*

$$\hat{f}_L(\delta) \triangleq 1 - h(\delta) - \frac{1}{L} \log\left(\sum_{j=L/2}^{L} \binom{L}{j}\right) + \frac{1}{L} h\left(\frac{L\delta}{4}\right) + \frac{\delta}{4} \log\left(\frac{L(L+2)}{4}\right). \tag{28}$$

Proof: Using (25), (26), and (21), we have $G_{\eta-\sigma}^{LB}(L, \delta, 0.5) = \hat{f}_L(\delta)$ for $\delta < 2/L$. We show in the full paper [13] that $\hat{f}_L(0) > 0$ and $\hat{f}_L(2/L) < 0$, and hence the equation $\hat{f}_L(\delta) = 0$ has a solution in the interval $(0, 2/L)$, as $\hat{f}_L$ is a continuous function of δ. The theorem now follows by denoting the smallest positive root of $\hat{f}_L(\delta)$ by $\hat{\delta}_L$. ■

Remark: For $L = 2$ and $w_s = 1$, it can be numerically verified using (21) that $G_{\eta-\sigma}^{LB}(2, \delta, 0.5) > 0$ for $0 \leq \delta < 0.056$.

Although Thm. 2 only considers the case $w_s = L/2$, a similar argument can be used to show that, in general for $0 < w_s < L$, we have $G_{\eta-\sigma}^{LB}(L, \delta, w_s/L) > 0$ for small δ. The following proposition addresses the converse question on identifying δ where the rate gap is provably zero.

Proposition 7. *For $w_s \leq L/2$, the rate gap between HWCs and SECCs, $G_{\eta-\sigma}(L, \delta, w_s/L)$ is identically zero when $1/2 \leq \delta \leq 1$, while for $w_s \geq L/2$, this gap is zero when $\delta^* \leq \delta \leq 1$.*

Proof: The claim for $w_s \leq L/2$ follows from (5) and the asymptotic Plotkin bound, while the claim for $w_s \geq L/2$ follows from (5) and (17). ■

Proposition 8. *The lower bound on the rate gap between HWCs and SECCs, $G_{\eta-\sigma}^{LB}(L, \delta, w_s/L)$, is tight when $\delta \to 0$.*

Proof: For $w_s \geq L/2$, from (25) we have that

$$G_{\eta-\sigma}^{LB}(L, 0, w_s/L) = h(w_s/L) - \frac{1}{L} \log\left(\sum_{j=w_s}^{L} \binom{L}{j}\right). \tag{29}$$

The asymptotic sphere-packing bound for CWCs [14] is

$$\alpha_{SP}(\delta, \omega) \triangleq h(\omega) - \omega h\left(\frac{\delta}{4\omega}\right) - (1-\omega) h\left(\frac{\delta}{4(1-\omega)}\right). \tag{30}$$

For $w_s \geq L/2$, an upper bound on $G_{\eta-\sigma}(L, \delta, w_s/L)$ is given by $\alpha_{SP}(\delta, w_s/L) - \sigma_{GV}(L, \delta, w_s/L)$ (using (5)). This upper bound tends to the right hand side of (29) as $\delta \to 0$, thereby proving the tightness of $G_{\eta-\sigma}^{LB}(L, \delta, w_s/L)$ for $w_s \geq L/2$.

For $w_s \leq L/2$, from (27) we have that

$$G_{\eta-\sigma}^{LB}(L, 0, w_s/L) = 1 - \frac{1}{L} \log\left(\sum_{j=w_s}^{L} \binom{L}{j}\right). \tag{31}$$

Now, from (5) and the relation $\alpha(\delta, 0.5) = \alpha(\delta)$ [8], an upper bound on $G_{\eta-\sigma}(L, \delta, w_s/L)$, for $w_s \leq L/2$, is given by $\alpha_{SP}(\delta, 0.5) - \sigma_{GV}(L, \delta, w_s/L)$. Using (19) and (30), we note that this upper bound tends to the right hand side of (31) as $\delta \to 0$. This proves the claim for $w_s \leq L/2$. ■

V. Numerical Results

Fig. 1 plots $G_{\eta-\sigma}^{LB}(L, \delta, w_s/L)$ as a function of L, when $w_s = L/2$. For a given δ, it is seen that $G_{\eta-\sigma}^{LB}(L, \delta, 0.5)$ decreases with L. Further, in the full paper [13], we show that for $w_s \geq L/2$, we have $G_{\eta-\sigma}^{LB}(L, \delta, w_s/L) \to 0$, as $L \to \infty$. Fig. 2 plots $G_{\eta-\sigma}^{LB}(L, \delta, w_s/L)$ versus w_s, for fixed $L = 16$.

The shaded area in Fig. 3 depicts the region where the rate gap between HWC and SECC is provably strictly positive. Here, $\hat{\delta}_L$ is the smallest value of δ for which the lower bound on the rate gap $G_{\eta-\sigma}^{LB}(L, \delta, 0.5)$ is zero (see Theorem 2).

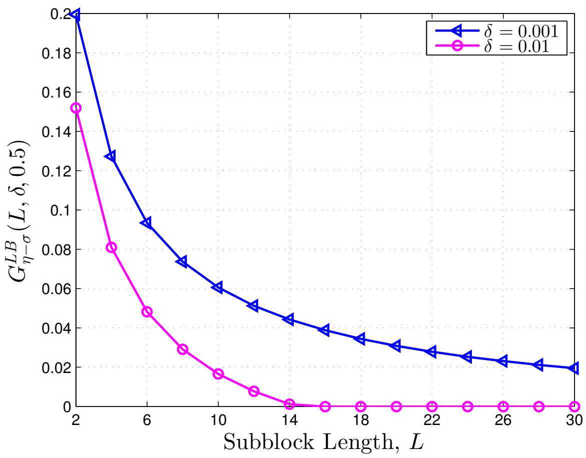

Fig. 1. $G^{LB}_{\eta-\sigma}(L,\delta,0.5)$ as a function of subblock length, L.

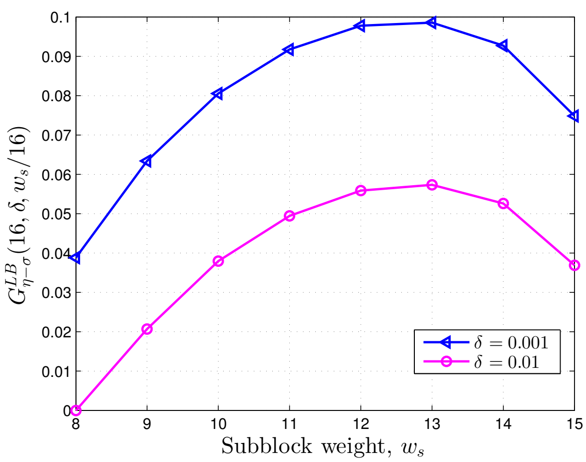

Fig. 2. $G^{LB}_{\eta-\sigma}(16,\delta,w_s/16)$ as a function of w_s.

Fig. 3 shows that $\hat{\delta}_L$ decreases with L, and in the full paper [13], we show that $\hat{\delta}_L \to 0$ as $L \to \infty$.

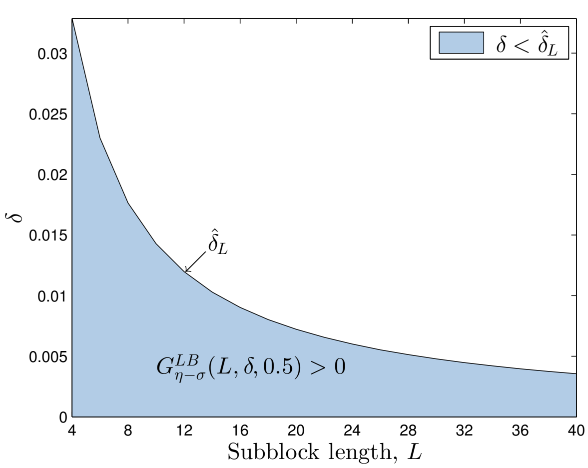

Fig. 3. Area where $G^{LB}_{\eta-\sigma}(L,\delta,0.5)$ is strictly positive.

VI. Reflections

We presented several bounds for the optimal SECC code size, and also derived the asymptotic GV and sphere-packing bounds for SECCs. For a fixed subblock length L and weight parameter w_s, we demonstrated the existence of some $\hat{\delta}_L$, such that the rate gap between HWCs and SECCs, $G_{\eta-\sigma}(L,\delta,w_s/L)$, is strictly positive for $\delta < \hat{\delta}_L$. The rate gap reflects the penalty due to the imposition of subblock-based weight constraint, relative to the codeword-based constraint. Furthermore, we provided an estimate on $\hat{\delta}_L$ via Theorem 2.

The converse problem, on identifying an interval for δ where the rate gap is zero was addressed via Proposition 7. An interesting but unsolved problem in this regard is to establish the *smallest* δ beyond which the rate gap is zero.

In the full paper [13], we study another class of constrained codes called the *constant subblock-composition codes* (CSCCs) where the constraint on each subblock having weight *at least* w_s, is replaced by the constraint that each subblock has weight *exactly* w_s. Relative to SECCs, the CSCCs are more constrained in the choice of bits within each subblock. It can be shown [13] that for small δ, the SECCs result in asymptotic rate which is *strictly* greater than CSCC rate. Further, when $w_s \geq L/2$, the asymptotic SECC rate is sandwiched strictly between the asymptotic rates for CWCs and CSCCs.

References

[1] A. Tandon, M. Motani, and L. R. Varshney, “Subblock energy-constrained codes for simultaneous energy and information transfer,” in *Proc. 2016 IEEE Int. Symp. Inf. Theory*, Jul. 2016, pp. 1969–1973.

[2] ——, “Subblock-constrained codes for real-time simultaneous energy and information transfer,” *IEEE Trans. Inf. Theory*, vol. 62, no. 7, pp. 4212–4227, Jul. 2016.

[3] G. Cohen, P. Sole, and A. Tchamkerten, “Heavy weight codes,” in *Proc. 2010 IEEE Int. Symp. Inf. Theory*, Jun. 2010, pp. 1120–1124.

[4] C. Bachoc, V. Chandar, G. Cohen, P. Sole, and A. Tchamkerten, “On bounded weight codes,” *IEEE Trans. Inf. Theory*, vol. 57, no. 10, pp. 6780–6787, Oct. 2011.

[5] J. Gu and T. Fuja, “A generalized Gilbert-Varshamov bound derived via analysis of a code-search algorithm,” *IEEE Trans. Inf. Theory*, vol. 39, no. 3, pp. 1089–1093, May 1993.

[6] G. D. Forney, Jr., *Concatenated Codes*. Cambridge, MA, USA: MIT Press, 1966.

[7] Y. M. Chee *et al.*, “Multiply constant-weight codes and the reliability of loop physically unclonable functions,” *IEEE Trans. Inf. Theory*, vol. 60, no. 11, pp. 7026–7034, Nov. 2014.

[8] R. McEliece, E. Rodemich, H. Rumsey, and L. Welch, “New upper bounds on the rate of a code via the Delsarte-MacWilliams inequalities,” *IEEE Trans. Inf. Theory*, vol. 23, no. 2, pp. 157–166, Mar. 1977.

[9] A. Fazeli, A. Vardy, and E. Yaakobi, “Generalized sphere packing bound,” *IEEE Trans. Inf. Theory*, vol. 61, pp. 2313–2334, May 2015.

[10] D. Cullina and N. Kiyavash, “Generalized sphere-packing bounds on the size of codes for combinatorial channels,” *IEEE Trans. Inf. Theory*, vol. 62, no. 8, pp. 4454–4465, Aug. 2016.

[11] S. Johnson, “A new upper bound for error-correcting codes,” *IRE Trans. Inf. Theory*, vol. 8, no. 3, pp. 203–207, April 1962.

[12] R. Graham and N. Sloane, “Lower bounds for constant weight codes,” *IEEE Trans. Inf. Theory*, vol. 26, no. 1, pp. 37–43, Jan. 1980.

[13] A. Tandon, H. M. Kiah, and M. Motani, “Bounds on the size and asymptotic rate of subblock-constrained codes,” Jan. 2017, arXiv:1701.04954 [cs.IT].

[14] T. Etzion and M. Schwartz, “Perfect constant-weight codes,” *IEEE Trans. Inf. Theory*, vol. 50, no. 9, pp. 2156–2165, Sep. 2004.