Rates of Constant-Composition Codes that Mitigate Intercell Interference

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***Abstract*—For certain families of substrings $\mathcal{F}$, we provide a closed formula for the maximum size of a q-ary $\mathcal{F}$-avoiding code with a given composition. In addition, we provide numerical procedures to determine the asymptotic information rate for $\mathcal{F}$-avoiding codes with certain composition ratios. Using our procedures, we recover known results and compute the information rates for certain classes of $\mathcal{F}$-avoiding constant-composition codes for $2 \leqslant q \leqslant 8$. For these values of q, we find composition ratios such that the rates of $\mathcal{F}$-avoiding codes with constant composition achieve the capacity of the $\mathcal{F}$-avoiding channel.**

I. Introduction

Flash memories have become a popular nonvolatile storage of information owing to its advantage of high speed, low noise, low power consumption, compact form factor, and good physical reliability. The basic information storage element of a flash memory is called a *cell*, which consists of a floating-gate (FG) transistor. The amount of charge on an FG transistor is discretized into *charge levels* as a way to store information. The operation of injecting charge into an FG transistor to a desired level is called *programming*.

Multilevel cell (MLC) flash memories have cells with $q > 2$ charge levels, with the ability to store $\log_2 q$ bits per cell. More specifically, we use qLC to refer to cells with q charge levels. The cells of a flash memory are further organized into blocks, each containing a constant number of cells. Hence, a block in a qLC flash memory stores a q-ary word (where symbol i is used to represent charge level i of a cell), and such a flash memory stores a collection of q-ary words. MLC technology increases the storage density of flash memories. However, precise programming is needed. There are two main challenges to reliable programming and storage: namely, *intercell interference* (ICI) and *charge leakage*.

Different techniques have been explored to mitigate ICI. Physical methods [1] and programming methods [2] have been investigated but the approach that is most effective has been the constrained coding method of Berman and Birk [3], [4], [5]. In their approach, certain words are forbidden to be stored, since the programming required to store such a word is highly unreliable, owing to ICI.

To mitigate the effect of charge leakage, a straightforward way is to adopt asymmetric error-correcting codes [6], [7]. Dynamic threshold techniques were later introduced by Zhou *et al.* [8] and extended by Sala *et al.* [9]; and the method is shown to be not only highly effective against asymmetric errors caused by charge leakage but also offer some protection against over-programming. In error-correcting schemes with dynamic threshold, the codes have constant composition, and in particular, the case when the codes are balanced (where the number of times a symbol appears in a codeword is as close as possible) was studied in detail by Zhou *et al.* and Sala *et al.* [8], [9].

Recent approaches have combined constrained coding and dynamic threshold techniques [10], [11]. Before we give an account of these results, we introduce some necessary notations and terminology.

A. Notations

Let $\Sigma \triangleq \{0, 1, \ldots, q-1\}$ be an alphabet of $q \geqslant 2$ symbols. A q-ary word of length n over Σ is an element $\mathsf{u} \in \Sigma^n$. The ith coordinate of u is denoted u_i, so that $\mathsf{u} = (\mathsf{u}_1, \mathsf{u}_2, \ldots, \mathsf{u}_n)$. There is a natural correspondence between the data represented by the charge levels of a block of n cells in a qLC flash memory and a q-ary word $\mathsf{u} \in \Sigma^n$: u_i is the charge level of the ith cell in the block.

For a positive integer n, a *composition of n into q parts* is a q-tuple $\overline{w} = [w_0, w_1, \ldots, w_{q-1}]$ of nonnegative integers such that $\sum_{i=0}^{q-1} w_i = n$. A q-ary word is said to have *composition* $\overline{w}$ if the frequency of symbol $i \in \Sigma$ in u is w_i. The *weight* of a word $\mathsf{u} \in \Sigma^n$ with composition $\overline{w}$ is $w = \sum_{i=1}^{q-1} w_i$. A word $\mathsf{u} \in \Sigma^n$ is said to be *balanced* if it has composition $\overline{w}$ such that $w_i \in \{\lfloor n/q \rfloor, \lceil n/q \rceil\}$ for all $i \in \Sigma$.

A q-ary code of length n is a nonempty subset $\mathcal{C} \subseteq \Sigma^n$. Elements of $\mathcal{C}$ are called *codewords*. The size of $\mathcal{C}$ is the number of codewords in $\mathcal{C}$. A code $\mathcal{C}$ is said to have *constant composition* $\overline{w}$, if each codeword in $\mathcal{C}$ has composition $\overline{w}$. A code is *balanced* if each of its codewords is balanced.

A *substring* of a word u is a word $(\mathsf{u}_{i+1}, \mathsf{u}_{i+2}, \ldots, \mathsf{u}_{i+\ell}) \in \Sigma^\ell$, where $i \geqslant 0$ and $i + \ell \leqslant n$. Let $\mathcal{F}$ be a set of words over Σ. A word u is said to *avoid* $\mathcal{F}$ or *$\mathcal{F}$-avoiding* if no word in $\mathcal{F}$ is a substring of u. A code $\mathcal{C}$ is said to *avoid* $\mathcal{F}$ if every codeword in $\mathcal{C}$ avoids $\mathcal{F}$. We denote the set of all q-ary words of length n that avoid $\mathcal{F}$ by $\mathcal{A}(n; \mathcal{F})$.

The *rate* of a code $\mathcal{C}$ is $R \triangleq \log_2 |\mathcal{C}|/n$, and intuitively, the rate measures the number of information bits stored in each multilevel cell. Henceforth, we adopt the notation log to mean logarithm base two.

Let $\mathcal{F}$ be a set of words over Σ. An *$\mathcal{F}$-avoiding channel* is a channel whose input codewords avoids $\mathcal{F}$. The *capacity* of an $\mathcal{F}$-avoiding channel or the *capacity of the $\mathcal{F}$-constraint* is given by the value

$$C(\mathcal{F}) \triangleq \limsup_{n\to\infty} \frac{\log |\mathcal{A}(n; \mathcal{F})|}{n}.$$

Recent approaches combine constrained coding and dynamic threshold techniques, leading to the consideration of codes that both avoid $\mathcal{F}$ and have constant composition. We denote an

$\mathcal{F}$-avoiding code of length n and constant composition $\overline{w}$ by $\mathcal{C}(n;\overline{w},\mathcal{F})$. The maximum size of a $\mathcal{C}(n;\overline{w},\mathcal{F})$, that is, the size of the set of all $\mathcal{F}$-avoiding words of composition $\overline{w}$, is denoted by $A(n;\overline{w},\mathcal{F})$ and the set is denoted by $\mathcal{A}(n;\overline{w},\mathcal{F})$.

Let $\overline{\rho}=[\rho_0,\rho_1,\ldots,\rho_{q-1}]$ be a real-valued vector such that $\sum_{i=0}^{q-1}\rho_i=1$. Let $(\overline{w}(n))_{n=1}^{\infty}$ be a sequence of compositions of n such that $\lim_{n\to\infty} w_i(n)/n=\rho_i$ for all $i\in\Sigma$. We define the *asymptotic information rate of* $(\overline{\rho},\mathcal{F})$ to be

$$R(\overline{\rho},\mathcal{F})\triangleq\limsup_{n\to\infty}\frac{\log A(n;\overline{w}(n),\mathcal{F})}{n},$$

and refer to $\overline{\rho}$ as the *composition ratio*.

Notice for the family of balanced codes, the sequence $\overline{w}(n)$ converges to the ratio $\overline{\rho}=[1/q,1/q,\ldots,1/q]$. In this case, we write $R([1/q,1/q,\ldots,1/q],\mathcal{F})$ simply as $R_{\rm bal}(\mathcal{F})$.

B. Previous Work

As mentioned earlier, a number of proposals for the avoidance set $\mathcal{F}$ have been put forth to mitigate the effects of ICI. In view of these proposals, we consider the following set of words over Σ. Fix $0\leqslant a<b\leqslant q-1$ and let $\mathcal{I}(a,b)\triangleq\{(c_1,c_2,c_3):0\leqslant c_2\leqslant a \text{ and } b\leqslant c_1,c_3\leqslant q-1\}$.

Taranalli *et al.* [12] proposed the avoidance set $\mathcal{I}_1(q)\triangleq\mathcal{I}(q-2,q-1)$, while Qin *et al.* [10] proposed the avoidance set $\mathcal{I}_2(q)\triangleq\mathcal{I}(0,q-1)$.

Example 1. $\mathcal{I}_1(2)=\mathcal{I}_2(2)=\{(1,0,1)\}$. $\mathcal{I}_1(4)=\{(3,0,3),(3,1,3),(3,2,3)\}$, while $\mathcal{I}_2(4)=\{(3,0,3)\}$.

In general, the capacity of the $\mathcal{F}$-constraint may be computed using the standard techniques detailed in [13]. For the purposes of mitigating ICI, the following results are known[1].

Proposition 1 ([11], [14])**.**

(i) $C(\mathcal{I}_1(2))=C(\mathcal{I}_2(2))=\log\lambda\approx 0.81137$*, where* λ *is the unique real root of the polynomial* X^3-2X^2+X-1.

(ii) $C(\mathcal{I}_1(4))\approx 1.9374$.

For completeness, we state the following proposition without proof. Selected capacity values are provided in Table I, where we benchmark the rates of certain $\mathcal{I}(a,b)$-avoiding codes with constant composition.

Proposition 2. *Fix* q *and* $0\leqslant a<b\leqslant q-1$. *We have* $C(\mathcal{I}(a,b))=\log\lambda_{a,b}$*, where* $\lambda_{a,b}$ *is the maximum real root of the polynomial* $X^3-qX^2+(q-b)(a+1)X-(q-b)(a+1)b$.

The asymptotic rate of balanced $\mathcal{I}_1(2)$-avoiding codes were investigated by Qin *et al.* and in the same paper, they documented the asymptotic rate of balanced $\mathcal{I}_2(3)$-avoiding codes.

Proposition 3 (Qin *et al.* [10])**.** $R_{\rm bal}(\mathcal{I}_1(2))=(\log 3)/2\approx 0.79428$ *and* $R_{\rm bal}(\mathcal{I}_2(3))\approx 1.52576$.

Observe that the balanced $\mathcal{I}_1(2)$-avoiding codes have rates that fall short of over 2% of the capacity of the $\mathcal{I}_1(2)$-constraint. We state our question of interest: is there a ratio $\overline{\rho}$ where the asymptotic rate of $\mathcal{I}_1(2)$-avoiding codes with composition ratio $\overline{\rho}$ achieves capacity?

[1]Berman and Birk computed $C(\mathcal{F})$ for a variety of avoidance sets $\mathcal{F}$ in the cases where $q\in\{4,8,16\}$ [5].

C. Our Contributions

Our first contribution is a closed formula for the number of $\mathcal{I}(a,b)$-avoiding words with composition $\overline{w}$.

Theorem 4. *Fix* q, n, $\mathcal{I}(a,b)$ *with* $a<b$ *and* $\overline{w}$. *Then*

$$\begin{aligned}&A(n;\overline{w},\mathcal{I}(a,b))\\&=\binom{s_1}{w_0,\cdots,w_a}\binom{s_2}{w_{a+1},\cdots,w_{b-1}}\binom{s_3}{w_b,\cdots,w_{q-1}}\\&\times\sum_{m=0}^{\min(s_2,s_3-1)}\binom{n-s_3-m}{s_1}B_n^{(m,s_3)},\end{aligned}$$

where $s_1=\sum_{i=0}^{a}w_i$, $s_2=\sum_{i=a+1}^{b-1}w_i$, $s_3=\sum_{i=b}^{q-1}w_i$, *and*

$$B_n^{(m,s_3)}=\binom{s_3-1}{m}\sum_{i=0}^{s_3-m-1}\binom{s_3-m-1}{i}\binom{n-s_3-m-i+1}{n-s_3-m-2i}.\quad(1)$$

In the instance where $b=a+1$*, we have* $s_2=0$ *and so we have only one summand in the outer summation. Therefore,*

$$A(n;\overline{w},\mathcal{I}(a,b))=\binom{s_1}{w_0,\cdots,w_a}\binom{s_3}{w_b,\cdots,w_{q-1}}B_n^{(0,s_3)}.$$

We defer the proof of Theorem 4 to Section II and explain the significance of the term $B_n^{(m,s_3)}$ therein.

While it is difficult to derive a closed expression for $R(\overline{\rho},\mathcal{I}(a,b))$ from Theorem 4 for general $\overline{\rho}$ and $\mathcal{I}(a,b)$, it is possible to compute *numerically* $R(\overline{\rho},\mathcal{I}(a,b))$ for specific values. Our next contributions are numerical procedures that:

- determine the rates $R(\overline{\rho},\mathcal{I}_1(q))$ and $R(\overline{\rho},\mathcal{I}_2(q))$ for specific values of $\overline{\rho}$;
- find composition ratios $\overline{\rho}$ that yield high rates $R(\overline{\rho},\mathcal{I}_1(q))$ and $R(\overline{\rho},\mathcal{I}_2(q))$. Interestingly, these rates coincide with their respective channel capacity in certain cases.

Section III provides a detailed description of the procedure and the numerical computations of certain rates.

II. Proof of Theorem 4

We enumerate the set of all q-ary $\mathcal{I}(a,b)$-avoiding words of composition $\overline{w}$, and hence, prove Theorem 4. To do so, we first enumerate *binary* words that obey certain properties in Section II-A, and then provide a mapping from these binary words to q-ary $\mathcal{I}(a,b)$-avoiding words in Section II-B .

A. A Family of Binary Words

Let $0\leqslant m\leqslant s_3$. Define $\mathcal{B}_n^{(m,s_3)}$ to be the set of words over the alphabet $\{\circ,\bullet\}$ of length n with the following properties:

(i) each word has exactly s_3 $\bullet$'s;
(ii) each word has exactly m substrings of the form $(\bullet,\circ,\bullet)$.

We demonstrate the following lemma.

Lemma 5. *Let* $0\leqslant m\leqslant s_3-1$. *Then*

$$\sum_{n\geqslant 0}\frac{\left|\mathcal{B}_n^{(m,s_3)}\right|}{\binom{s_3-1}{m}}X^n=\frac{X^{s_3+m}(1-X+X^2)^{s_3-m-1}}{(1-X)^{s_3-m+1}}.$$

To prove this lemma, we map $\mathsf{u}\in\mathcal{B}_n^{(m,s_3)}$ to an integer-valued (s_3+1)-tuple $\boldsymbol{d}_{\mathsf{u}}=(d_1,d_2,\ldots,d_{s_3+1})$ such that $\{t_j=\sum_{i=1}^{j}d_i:1\leqslant j\leqslant s_3\}$ is the set of coordinates where $\mathsf{u}_{t_j}=\bullet$, and $d_{s_3+1}=n-\sum_{i=1}^{s_3}d_i$.

Example 2. The word $\mathsf{u} = (\bullet, \circ, \bullet, \bullet, \circ, \bullet, \bullet, \circ)$ belongs to $\mathcal{B}_8^{(2,5)}$, where $m = 2$, $s_3 = 5$, $n = 8$. Hence, $\boldsymbol{d}_\mathsf{u} = (1,2,1,2,1,1)$ and $\{1,3,4,6,7\}$ is the set of coordinates where u has the symbol $\bullet$.

It is not difficult to see that $\boldsymbol{d}_\mathsf{u} = \boldsymbol{d}_{\mathsf{u}'}$ implies $\mathsf{u} = \mathsf{u}'$. We observe further that for $\mathsf{u} \in \mathcal{B}_n^{(m,s_3)}$, the (s_3+1)-tuple $\boldsymbol{d}_\mathsf{u}$ has the following properties:

(C1) the sum of entries in $\boldsymbol{d}_\mathsf{u}$ is n;
(C2) exactly m entries of $d_2, d_3, \ldots, d_{s_3}$ are two;
(C3) all entries except d_{s_3+1} of $\boldsymbol{d}_\mathsf{u}$ are positive, and d_{s_3+1} is nonnegative.

Conversely, for each (s_3+1)-tuple $\boldsymbol{c}$ that obeys the properties (C1), (C2) and (C3), there exists a $\mathsf{u} \in \mathcal{B}_n^{(m,s_3)}$ such that $\boldsymbol{d}_\mathsf{u} = \boldsymbol{c}$. Therefore, the cardinality of $\mathcal{B}_n^{(m,s_3)}$ is equal to the number of (s_3+1)-tuples satisfying these properties.

From (C1) and (C3), such (s_3+1)-tuples are compositions of n with s_3+1 parts and in general, the combinatorics of compositions have been well studied (see Heubach and Mansour [15] for a survey). If we impose restrictions for each part of the composition, we have what is known as *compositions with restricted parts* and the following theorem.

Theorem 6 (Folklore, see [15, Ch. 3])**.** *Let* $\boldsymbol{P} = (P_1, P_2, \ldots, P_k)$ *be an ordered collection of subsets of integers. Define* $\mathrm{Comp}(n;\boldsymbol{P}) \triangleq \{\boldsymbol{c} = (c_1, c_2, \ldots, c_k) : \sum_{j=1}^k c_j = n$ *and* $c_j \in P_j$ *for* $1 \leqslant j \leqslant k\}$*. Then*

$$\sum_{n\geqslant 0} |\mathrm{Comp}(n;\boldsymbol{P})| X^n = \prod_{j=1}^{k} \sum_{i\in P_j} X^i.$$

For each (s_3+1)-tuple $\boldsymbol{c}$ satisfying properties (C1), (C2) and (C3), we have $\binom{s_3-1}{m}$ ways to choose exactly m entries of $c_2, c_3, \ldots, c_{s_3}$ to be two. Without loss of generality, we assume $c_2 = c_3 = \cdots = c_{m+1} = 2$. Set $k = s_3+1$ and consider the ordered collection $\boldsymbol{P}$ be such that

$$P_j = \begin{cases} \mathbb{Z}_{\geqslant 1}, & \text{if } j = 1,\\ \{2\}, & \text{if } 2 \leqslant j \leqslant m+1,\\ \mathbb{Z}_{\geqslant 1}\setminus\{2\}, & \text{if } m+2 \leqslant j \leqslant s_3,\\ \mathbb{Z}_{\geqslant 0}, & j = s_3+1. \end{cases}$$

where $\mathbb{Z}_{\geqslant t}$ denote the set of integers at least t. Then, we have

$$\left|\mathcal{B}_n^{(m,s_3)}\right| = |\mathrm{Comp}(n;\boldsymbol{P})| \binom{s_3-1}{m}.$$

Since $\sum_{i\in\mathbb{Z}_{\geqslant t}} X^i = X^t/(1-X)$, we have

$$\sum_{n\geqslant 0} \frac{\left|\mathcal{B}_n^{(m,s_3)}\right|}{\binom{s_3-1}{m}} X^n = \sum_{n\geqslant 0} |\mathrm{Comp}(n;\boldsymbol{P})|\, X^n$$
$$= \left(\frac{X}{1-X}\right) (X^2)^m \left(X + \frac{X^3}{1-X}\right)^{s_3-m-1} \left(\frac{1}{1-X}\right)$$
$$= \frac{X^{s_3+m}(1-X+X^2)^{s_3-m-1}}{(1-X)^{s_3-m+1}}.$$

This completes the proof of Lemma 5. To compute $\left|\mathcal{B}_n^{(m,s_3)}\right|$, we extract the coefficient of X^n and multiply it by $\binom{s_3-1}{m}$. For convenience, we let $[X^j]\Big\{g(X)\Big\}$ denote the coefficient of X^j in $g(X)$. Hence,

$$[X^n]\left\{X^{s_3+m}(1-X+X^2)^{s_3-m-1}(1-X)^{-s_3+m-1}\right\}$$
$$= \left[X^{n-s_3-m}\right]\left\{(1-X+X^2)^{s_3-m-1}(1-X)^{-s_3+m-1}\right\}$$
$$= \sum_{i=0}^{s_3-m-1} \binom{s_3-m-1}{i} \left[X^{n-s_3-m-2i}\right]\left\{(1-X)^{-2-i}\right\}$$
$$= \sum_{i=0}^{s_3-m-1} \binom{s_3-m-1}{i} \binom{n-s_3-m-i+1}{n-s_3-m-2i}.$$

Setting $B_n^{(m,s_3)} = \left|\mathcal{B}_n^{(m,s_3)}\right|$ yields (1).

B. *Mapping to q-ary Words*

Finally, to complete the proof of Theorem 4, we take a word in $\mathcal{B}_n^{(m,s_3)}$ and replace the symbols in $\{\bullet, \circ\}$ with symbols in Σ. For convenience, we partition Σ into three parts:

$\Sigma_1 = \{0, \ldots, a\}$, $\Sigma_2 = \{a+1, \ldots, b-1\}$, $\Sigma_3 = \{b, \ldots, q-1\}$.

In addition, for $i = 1,2,3$, we consider $\mathcal{E}_i$ to be a set of words over Σ_i of length s_i such that $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ are the sets of all words with compositions $[w_0, \ldots, w_a]$, $[w_{a+1}, \ldots, w_{b-1}]$, and $[w_b, \ldots, w_{q-1}]$, respectively.

Example 3. Let $q = 5$, $a = 1$, $b = 4$. So, $\Sigma_1 = \{0,1\}$, $\Sigma_1 = \{2,3\}$, and $\Sigma_3 = \{4\}$. Furthermore, let $n = 8$ with $\overline{w} = (1,1,1,2,3)$. Hence, $(s_1, s_2, s_3) = (2,3,3)$ and

$$\mathcal{E}_1 = \{(0,1),(1,0)\},$$
$$\mathcal{E}_2 = \{(2,3,3),(3,2,3),(3,3,2)\},$$
$$\mathcal{E}_3 = \{(4,4,4)\}.$$

For $\mathsf{u} \in \mathcal{B}_n^{(m,s_3)}$, we further define $T(\mathsf{u})$ to be the set of $n-s_3-m$ coordinates such that $t \in T(\mathsf{u})$ implies that $\mathsf{u}_t = \circ$, but $(\mathsf{u}_{t-1}, \mathsf{u}_t, \mathsf{u}_{t+1}) \neq (\bullet, \circ, \bullet)$. In other words, $T(\mathsf{u})$ is the set of $n-s_3-m$ $\circ$'s in u that do not belong to the substrings $(\bullet, \circ, \bullet)$. Let $\mathcal{D}(\mathsf{u})$ be the collection of all subsets of $T(\mathsf{u})$ of size s_1.

Example 4. Let $\mathsf{u} = (\bullet, \circ, \bullet, \circ, \bullet, \circ, \circ, \circ)$ with $n = 8$, $s_3 = 3$, $m = 2$. Then $T(\mathsf{u}) = \{6,7,8\}$ and for $s_1 = 2$, we have $\mathcal{D}(\mathsf{u}) = \{\{6,7\},\{6,8\},\{7,8\}\}$.

Next, we define the following collection of pairs:

$$\mathcal{D}_n^{(m,s_3)} \triangleq \left\{(\mathsf{u}, D) : \mathsf{u} \in \mathcal{B}_n^{(m,s_3)}, D \in \mathcal{D}(\mathsf{u})\right\}.$$

Observe that $\left|\mathcal{D}_n^{(m,s_3)}\right| = B_n^{(m,s_3)}\binom{n-s_3-m}{s_1}$ and consider the following maps,

$$\Phi_1 : \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \times \bigcup_{m=0}^{\min(s_2,s_3-1)} \mathcal{D}_n^{(m,s_3)} \to \mathcal{A}(n;\overline{w},\mathcal{I}(a,b)),$$

$$\Phi_2 : \mathcal{A}(n;\overline{w},\mathcal{I}(a,b)) \to \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \times \bigcup_{m=0}^{\min(s_2,s_3-1)} \mathcal{D}_n^{(m,s_3)}.$$

To define Φ_1, consider $\mathsf{e}_i \in \mathcal{E}_i$ for $i = 1,2,3$, $\mathsf{u} \in \mathcal{B}_n^{(m,s_3)}$ and $D_1 \in \mathcal{D}(\mathsf{u})$. Let D_2 be the set of coordinates of $\circ$ in u that do not belong to D_1. Then $\Phi_1(\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3, (\mathsf{u}, D_1))$ is the q-ary word obtained by substituting

- the s_1 $\circ$'s of u at index set D_1 with e_1,
- the s_2 $\circ$'s of u at index set D_2 with e_2, and
- the s_3 $\bullet$'s of u with e_3.

Conversely, consider $\mathsf{v} \in \mathcal{A}(n;\overline{w},\mathcal{I}(a,b))$ and we set $\Phi_2(\mathsf{v}) = (\mathsf{e}_1,\mathsf{e}_2,\mathsf{e}_3,(\mathsf{u},D))$, where

- e_i is the subsequence of v whose symbols belong to Σ_i for $i = 1,2,3$,
- u is the word obtained by substituting symbols in $\Sigma_1 \cup \Sigma_2$ with $\circ$ and symbols in Σ_3 with $\bullet$, and
- D is the set of indices with symbols in Σ_1.

Example 5. Let $q,a,b,n,\overline{w}$, and u be as defined in Examples 3 and 4. Consider $\mathsf{e}_1 = (0,1)$, $\mathsf{e}_2 = (3,2,3)$, $\mathsf{e}_3 = (4,4,4)$ and $D = \{6,8\}$. Then $\Phi_1(\mathsf{e}_1,\mathsf{e}_2,\mathsf{e}_3,(\mathsf{u},D)) = (4,3,4,2,4,0,3,1)$. Conversely, if we set $\mathsf{v} = (4,3,4,2,4,0,3,1)$, then $\Phi_2(\mathsf{v})$ recovers $\mathsf{e}_1,\mathsf{e}_2,\mathsf{e}_3,\mathsf{u}$ and D.

Due to space constraints, we omit the detailed proof of the following lemma.

Lemma 7. *Let Φ_1 and Φ_2 be defined as above. Then the composite maps $\Phi_1 \circ \Phi_2$ and $\Phi_2 \circ \Phi_1$ are identity maps on their respective domains. Therefore, Φ_1 and Φ_2 are bijections.*

Combining Lemmas 5 and 7 yields Theorem 4.

III. Rates of Constant-Composition $\mathcal{F}$-avoiding Codes

In this section, we provide an efficient numerical procedure to determine the asymptotic information rates of certain $(\overline{\rho},\mathcal{F})$-pairs. Before we evaluate these rates, the following proposition is an analogue of a result by Kayser and Siegel [11].

Proposition 8. *Fix an avoidance set $\mathcal{F}$ over Σ. Then*

$$\lim_{n\to\infty} \max_{\sum w_i = n} \frac{\log\ A(n;\overline{w},\mathcal{F})}{n} = C(\mathcal{F}).$$

Proof. Let $D_{\max}(n) = \max\{A(n;\overline{w},\mathcal{F}) : \sum w_i = n\}$ for all n. Since $|\mathcal{A}(n;\mathcal{F})| = \sum_{\sum w_i = n} A(n;\overline{w},\mathcal{F})$ and we have at most n^q compositions of n into q parts, we have

$$D_{\max}(n) \leqslant |\mathcal{A}(n;\mathcal{F})| \leqslant n^q D_{\max}(n).$$

Taking logarithms, dividing by n and taking limits in n yields the proposition. ∎

Unfortunately, Proposition 8 does not guarantee the existence of a composition ratio $\overline{\rho}$ where $R(\overline{\rho},\mathcal{F}) = C(\mathcal{F})$. Indeed, if we set $\overline{w}(n) \in \arg\max_{\sum w_i = n} A(n;\overline{w}(n),\mathcal{F})$, the sequences $w_i(n)$ need not converge for all $i \in \Sigma$.

However, we conjecture the existence of such a composition ratio $\overline{\rho}$. Furthermore, in the following subsections, we look at the avoidance sets $\mathcal{I}_1(q)$ and $\mathcal{I}_2(q)$ and verify numerically the existence of such $\overline{\rho}$.

In what follows, we consider the usual binary entropy function $H_2(p) = -p\log p - (1-p)\log(1-p)$ for $0 \leqslant p \leqslant 1$.

A. Avoiding $\mathcal{I}_1(q)$

Our first theorem computes the asymptotic rate of a family of constant-composition codes.

Theorem 9. *Fix $0 \leqslant x \leqslant 1$. Define the function F_1 so that*

$$F_1(x,y) \triangleq (1-x)\log(q-1) + xH_2(y) + (1-x-xy)H_2\left(\frac{1-x-2xy}{1-x-xy}\right).$$

Let $\overline{\rho} \triangleq ((1-x)/(q-1),(1-x)/(q-1),\ldots,(1-x)/(q-1),x)$. Then the asymptotic rate $R(\overline{\rho},\mathcal{I}_1(q))$ is given by $\max_{0\leqslant y\leqslant 1} F_1(x,y)$.

Proof. For each n, let $\overline{w}(n)$ be such that $w_0 = \cdots = w_{q-2} = \lfloor (1-x)n/(q-1)\rfloor$ and $w_{q-1} = n-(q-1)w_0$. We verify that the sequence $\overline{w}(n)$ converges to $\overline{\rho}$ componentwise.

Applying Theorem 4 with $a = q-2$, $b = q-1$, $s_1 = (q-1)w_0$, $s_2 = 0$ and $s_3 = w_{q-1}$, we have the value of $A(n;\overline{w}(n),\mathcal{I}_1(q))$ given by

$$\sum_{i=0}^{w_{q-1}-1} \binom{(q-1)w_0}{w_0,\cdots,w_0}\binom{w_{q-1}-1}{i}\binom{n-w_{q-1}-i+1}{n-w_{q-1}-2i}.$$

Let D_i be the ith summand for $0 \leqslant i \leqslant w_{q-1}-1$ and $y^* \in \arg\max_{0\leqslant y\leqslant 1} F_1(x,y)$. Then by Stirling's approximation,

$$2^{nF_1(x,i/xn)-o(n)} \leqslant D_i \leqslant 2^{nF_1(x,i/xn)+o(n)} \text{ for all } i.$$

Let $i^* = \lfloor xy^*n\rfloor$. Then we have $A(n;\overline{w}(n),\mathcal{I}_1(q)) \geqslant D_{i^*} \geqslant 2^{nF_1(x,i^*/xn)-o(n)}$. Taking logarithms, dividing by n and taking limits in n yields the inequality $R(\overline{\rho},\mathcal{I}_1(q)) \geqslant F_1(x,y^*)$.

On the other hand, we have $A(n;\overline{w}(n),\mathcal{I}_1(q)) \leqslant \sum_i 2^{nF_1(x,i/xn)+o(n)} \leqslant n2^{nF_1(x,y^*)+o(n)}$. Taking logarithms, dividing by n and taking limits in n, we obtain $R(\overline{\rho},\mathcal{I}_1(q)) \leqslant F_1(x,y^*)$. This completes the proof. ∎

Example 6. Let $q = 2$ and $x = 1/2$. Then $\overline{\rho} = (1/2,1/2)$ and

$$F_1\left(\frac{1}{2},y\right) = \frac{1}{2}\left(H_2(y) + (1-y)H_2\left(\frac{1-2y}{1-y}\right)\right).$$

Now, $F_1(1/2,y)$ is maximized when $y = 1/3$ and achieves the value $(\log 3)/2$. This yields $R_{\rm bal}(\mathcal{I}_1(2))$ and recovers the result in Qin *et al.* [10]. Continuing this example, we compute the rates $R_{\rm bal}(\mathcal{I}_1(q))$ for $2 \leqslant q \leqslant 8$ and tabulate these values in Table I.

B. Avoiding $\mathcal{I}_2(q)$

The following is analogous to Theorem 9.

Theorem 10. *Let $q \geqslant 3$ and fix $0 \leqslant x \leqslant (q-2)/(2q-3)$. Define the function F_2 so that*

$$\begin{aligned}
&F_2(x,y,z)\\
&\triangleq \frac{(1-x)(q-2)}{q-1}\log(q-2)\\
&+ (1-x-xy)H_2\left(\frac{1-x}{(q-1)(1-x-xy)}\right)\\
&+ xH_2(y) + (x-xy)H_2(z)\\
&+ (1-x-xy-z(x-xy))H_2\left(\frac{1-x-xy-2z(x-xy)}{1-x-xy-z(x-xy)}\right).
\end{aligned}$$

Let $\overline{\rho} \triangleq ((1-x)/(q-1),(1-x)/(q-1),\ldots,(1-x)/(q-1),x)$ Then the asymptotic rate $R(\overline{\rho},\mathcal{I}_2(q))$ is given by $\max_{0\leqslant y,z\leqslant 1} F_2(x,y,z)$.

Proof. The proof is similar to the proof of Theorem 9 and is omitted due to space constraints. ∎

q	$R_{\rm bal}(\mathcal{I}_1(q))$	ρ_{q-1}	$R(\overline{\rho},\mathcal{I}_1(q))$	$C(\mathcal{I}_1(q))$	$R_{\rm bal}(\mathcal{I}_2(q))$	ρ_{q-1}	$R(\overline{\rho},\mathcal{I}_2(q))$	$C(\mathcal{I}_2(q))$
2	0.79248	0.41150	0.81137	0.81137				
3	1.46127	0.25653	1.48353	1.48353	1.52576	0.29308	1.53145	1.53145
4	1.92207	0.19425	1.93743	1.93743	1.97589	0.22989	1.97758	1.97758
5	2.26928	0.15865	2.27945	2.27945	2.30984	0.18867	2.31046	2.31046
6	2.54732	0.13496	2.55420	2.55420	2.57805	0.15967	2.57832	2.57832
7	2.77921	0.11782	2.78403	2.78403	2.80304	0.13827	2.80317	2.80317
8	2.97821	0.10475	2.98169	2.98169	2.99713	0.12181	2.99719	2.99719

TABLE I: Rates of $\mathcal{I}_1(q)$ and $\mathcal{I}_2(q)$-avoiding codes with constant composition. Here, the composition ratio is $\overline{\rho} = [\rho,\rho,\ldots,\rho,\rho_{q-1}]$, where $\rho = (1-\rho_{q-1})/(q-1)$.

As before, for $3 \leqslant q \leqslant 8$, we compute $R_{\rm bal}(\mathcal{I}_2(q))$ and tabulate these results in Table I. Again, we recover the result $R_{\rm bal}(\mathcal{I}_2(3)) \approx 1.52576$ in Qin *et al.* [10].

C. Capacity-Achieving Codes with Constant Composition

Consider the functions F_1 and F_2 defined in Theorem 9 and Theorem 10, respectively. Since we are interested in constant-composition codes with high rates, a natural approach is to maximize $F_1(x,y)$ in both variables x and y, and $F_2(x,y,z)$ in all variables x, y and z.

We do so for $2 \leqslant q \leqslant 8$ and present the results in Table I. Interestingly, for the corresponding values of $\overline{\rho}$, the rates $R(\overline{\rho},\mathcal{I}_1(q))$ and $R(\overline{\rho},\mathcal{I}_2(q))$ achieve capacity and we conjecture this to be true for all q. We give a precise formulation of our conjecture.

Conjecture 11. *Consider the functions F_1 and F_2 defined in Theorem 9 and Theorem 10, respectively.*

(i) $C(\mathcal{I}_1(q)) = \max\{F_1(x,y) : 0 \leqslant x,y \leqslant 1\}$ *for* $q \geqslant 2$.
(ii) $C(\mathcal{I}_2(q)) = \max\{F_2(x,y,z) : 0 \leqslant x,y,z \leqslant 1\}$ *for* $q \geqslant 3$.

Furthermore, for a set $\mathcal{F}$ of words over Σ, there exists a composition ratio $\overline{\rho}$ such that $R(\overline{\rho},\mathcal{F}) = C(\mathcal{F})$. When $\mathcal{F} = \mathcal{I}_1(q)$ and $\mathcal{F} = \mathcal{I}_2(q)$, we can even conjecture the precise form of the composition ratio.

IV. Conclusion

We enumerated the set of all $\mathcal{F}$-avoiding words with a fixed composition for certain avoidance sets $\mathcal{F}$. Using this formula, we presented numerical procedures to determine the rates of $\mathcal{F}$-avoiding codes with certain composition ratios. We also determined the composition ratios that maximize the rates of $\mathcal{F}$-avoiding constant-composition codes for $\mathcal{F} = \mathcal{I}_1(q)$ or $\mathcal{F} = \mathcal{I}_2(q)$, and $2 \leqslant q \leqslant 8$. Interestingly, we observe that the $\mathcal{F}$-avoiding codes with the optimal composition ratio achieve the capacity of the $\mathcal{F}$-avoiding channel in all our numerical computations, and we conjecture this to be true in general.

The encoding and decoding algorithms for certain special classes of constant-composition $\mathcal{F}$-avoiding codes are discussed in our companion paper [16].

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