Improved Asymptotic Sphere-Packing Bounds for Subblock-Constrained Codes

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***Abstract*—Subblock-constrained codes are an important class of constrained codes, having applications in many diverse fields. In this paper, we provide closed-form expressions for the best known upper bounds on the asymptotic rates of subblock-constrained codes for a range of relative distance values via a generalized sphere-packing approach. In particular, we study binary *subblock energy-constrained codes* (SECCs), characterized by the property that the number of ones in each subblock exceeds a certain threshold, and binary *constant subblock-composition codes* (CSCCs), characterized by the property that the number of ones in each subblock is constant. Improved bounds on the optimal asymptotic rate for SECCs and CSCCs are obtained by applying a generalized sphere-packing approach and judiciously choosing appropriate constrained spaces for estimating asymptotic ball sizes. We also use numerical examples to highlight the improvement.**

I. Introduction

Subblock-constrained codes are a class of constrained codes where each codeword is divided into smaller non-overlapping subblocks, and each subblock satisfies a certain application dependent constraint. Subblock-constrained codes have recently gained attention as they are suitable candidates for applications such as simultaneous energy and information transfer [1], visible light communication [2], low-cost authentication [3], and powerline communications [4]. In this paper, we discuss two important subclasses of subblock-constrained codes.

The first subclass are the *subblock energy-constrained codes* (SECCs) which ensure that the energy content in every subblock of each codeword exceeds a certain threshold [1], [5]. SECCs have application in simultaneous energy and information transfer [5], and binary SECCs are characterized by the property that the number of ones in each subblock is at least w_s [6], [7]. Bounds on the capacity and error exponent for SECCs over noisy channels were presented in [1], while bounds on the SECC code size and asymptotic rate, with minimum distance constraint, were analyzed in [6], [7].

The second subclass of subblock-constrained codes that we study are the *constant subblock-composition codes* (CSCCs). Binary CSCCs have varied applications [2], [3], [8], and are characterized by the property that each subblock in every codeword has the same *weight*, i.e. each subblock has the same number of ones. Bounds on the capacity and error exponent for CSCCs over noisy channels were presented in [1], while bounds on the CSCC code size and asymptotic rate, with minimum distance constraint, were analyzed in [7], [9].

In this paper, we extend the results in [6], [7], [9] to present the *best known upper bounds* on the asymptotic rates for SECCs and CSCCs for a range of relative distance values. These results are obtained by applying a generalized version of the sphere-packing bound (Sec. II) to the above discussed subclasses of subblock-constrained codes (Sec. III and Sec. IV, respectively).

We remark that an alternate approach to generalized sphere-packing was presented in [10], [11], where bounds on the optimal size of fixed blocklength constrained codes are presented. However, the results in [10], [11] are obtained numerically via solving certain linear programs, and are not useful in providing upper bounds on the *asymptotic rate* of constrained codes, when the blocklength tends to infinity. In contrast, we provide improved bounds on the optimal asymptotic rates for SECCs and CSCCs by applying a generalized sphere-packing approach to judiciously choose appropriate constrained spaces for estimating asymptotic ball sizes, which lead to *closed-form expressions* for asymptotic bounds.

II. Improved Sphere-Packing Bounds

We give a version of the sphere-packing bound in full generality, and then specialize it to the class of codes that we are interested in.

Let τ be a distance metric defined over Σ^n and pick $\mathcal{S} \subseteq \Sigma^n$. A subset $\mathscr{C} \subseteq \mathcal{S}$ is an $(n, d; \mathcal{S})$-code if $d = \min\{\tau(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x}, \boldsymbol{y} \in \mathscr{C},\ \boldsymbol{x} \neq \boldsymbol{y}\}$ and we are interested in determining the value $A(n, d; \mathcal{S}) \triangleq \max\{|\mathscr{C}| : \mathscr{C} \text{ is an } (n, d; \mathcal{S})\text{-code}\}$.

Our theorem is motivated by Freiman's and Berger's methods [12], [13] that improve the usual sphere-packing bounds for constant weight codes. Choose $\tilde{\mathcal{S}} \subseteq \Sigma^n$, a subset possibly different from $\mathcal{S}$. For $\boldsymbol{x} \in \mathcal{S}$, define $\mathcal{B}_{\tilde{\mathcal{S}}}(\boldsymbol{x}, t) \triangleq \{\boldsymbol{y} \in \tilde{\mathcal{S}} : \tau(\boldsymbol{x}, \boldsymbol{y}) \leq t\}$ and when $\tilde{\mathcal{S}}$ is the whole space Σ^n, we drop the subscript and simply write $\mathcal{B}(\boldsymbol{x}, t)$. Set $V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t) \triangleq \min\{|\mathcal{B}_{\tilde{\mathcal{S}}}(\boldsymbol{x}, t)| : \boldsymbol{x} \in \mathcal{S}\}$.

Theorem 1. *Set $t = \lfloor (d-1)/2 \rfloor$. For any $\tilde{\mathcal{S}} \subseteq \Sigma^n$, if $V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t) \geq 1$, then*

$$A(n, d; \mathcal{S}) \leq \frac{|\tilde{\mathcal{S}}|}{V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t)}. \tag{1}$$

To prove this theorem, we show that the righthand side of (1) corresponds to the objective value of a certain optimization program. Before we provide the detailed proof, we make some remarks on the computation aspects of Theorem 1.

- *Choice of the space $\tilde{\mathcal{S}}$.* Following (1), we have that $A(n, d; \mathcal{S}) \leq \min_{\tilde{\mathcal{S}} \subseteq \Sigma^n} |\tilde{\mathcal{S}}|/V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t)$. However, the minimization problem of the right-hand side is computationally infeasible as the number of choices for $\tilde{\mathcal{S}}$ is exponential in $|\mathcal{S}|$. Hence, for specific spaces $\mathcal{S}$, we choose a family of subspaces $\tilde{\mathcal{S}}$ and minimize the value of (1).

- *Lower Bounding* $V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t)$. Suppose that $\tilde{\mathcal{S}}$ and $\mathcal{S}$ are given. To compute $V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t)$, we have to look at $|\mathcal{S}|$ spheres centered at the points in $\mathcal{S}$. Since $|\mathcal{S}|$ is usually exponential in n, determining the exact value of $V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t)$ remains difficult. Hence, in specific cases, we provide a lower bound V' for $V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t)$ and so, (1) implies that $A(n,d;\mathcal{S}) \le |\tilde{\mathcal{S}}|/V'$.
- In summary, to apply Theorem 1, we choose a family of subspaces $\tilde{\mathcal{S}}$ such that both $|\tilde{\mathcal{S}}|$ and $V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t)$ can be computed or estimated efficiently.

To prove Theorem 1, we apply a modified version of generalized sphere-packing bound *a la* Fazelli *et al.* [11]. A specialized version of the generalized bound was introduced by Kulkarni and Kiyavash [10] in the context of deletion-correcting code and since then, variants of their method were applied to a myriad of coding problems (see [11] for a survey). Fazeli *et al.* then studied their method in a general setup and provided what is called the *generalized sphere-packing bound*. We provide a short exposition and derivation of our modified bound.

Fix d and set $t = \lfloor (d-1)/2 \rfloor$. Define $T \triangleq \bigcup_{\boldsymbol{x}\in\mathcal{S}} \mathcal{B}(\boldsymbol{x},t)$. In other words, T is the set of all words whose distance is at most t from some word in $\mathcal{S}$.

We consider a binary matrix $\boldsymbol{M}$ whose rows are indexed by $\mathcal{S}$ and columns are indexed by T. Set

$$\boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}} = \begin{cases} 1 & \text{if } \tau(\boldsymbol{x},\boldsymbol{y}) \le t, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A(n,d;\mathcal{S})$ can be upper bounded as follows:

$$A(n,d;\mathcal{S}) \le \max\left\{ \sum_{\boldsymbol{x}\in\mathcal{S}} X_{\boldsymbol{x}} : \boldsymbol{X}\boldsymbol{M} \le \boldsymbol{1}, X_{\boldsymbol{x}} \in \{0,1\} \text{ for } \boldsymbol{x}\in\mathcal{S} \right\}.$$

When we relax $\boldsymbol{X}$ to be a real-valued vector, we get

$$A(n,d;\mathcal{S}) \le \max\left\{ \sum_{\boldsymbol{x}\in\mathcal{S}} X_{\boldsymbol{x}} : \boldsymbol{X}\boldsymbol{M} \le \boldsymbol{1}, X_{\boldsymbol{x}} \ge 0 \text{ for } \boldsymbol{x}\in\mathcal{S} \right\}.$$

We have the following inequality via strong duality [14].

$$A(n,d;\mathcal{S}) \le \min\left\{ \sum_{\boldsymbol{y}\in T} Y_{\boldsymbol{y}} : \boldsymbol{M}\boldsymbol{Y} \ge \boldsymbol{1}, Y_{\boldsymbol{y}} \ge 0 \text{ for } \boldsymbol{y}\in T \right\}. \quad (2)$$

Proof of Theorem 1: Abbreviate $V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t)$ with V and we consider the vector

$$\boldsymbol{Y} = \begin{cases} 1/V & \text{if } \boldsymbol{y}\in\tilde{\mathcal{S}}, \\ 0 & \text{otherwise.} \end{cases}$$

We first show that vector $\boldsymbol{Y}$ above is a feasible point in the optimization program (2). In other words, we claim that $\boldsymbol{M}\boldsymbol{Y} \ge \boldsymbol{1}$. Indeed, for $\boldsymbol{x}\in\mathcal{S}$, let $\boldsymbol{M}_{\boldsymbol{x}}$ denote the row of $\boldsymbol{M}$ that corresponds to $\boldsymbol{x}$. We have that

$$\boldsymbol{M}_{\boldsymbol{x}}\boldsymbol{Y} = |\mathcal{B}_{\tilde{\mathcal{S}}\cap T}(\boldsymbol{x},t)|/V = |\mathcal{B}_{\tilde{\mathcal{S}}}(\boldsymbol{x},t)|/V \ge 1,$$

since V corresponds to the smallest ball volume.

To complete the proof, it remains to compute the objective value that is $\sum_{\boldsymbol{y}\in T} Y_{\boldsymbol{y}} = \sum_{\boldsymbol{y}\in\tilde{\mathcal{S}}\cap T} Y_{\boldsymbol{y}} \le |\tilde{\mathcal{S}}|/V$. ■

Remark: We point out the differences between (2) and the upper bound derived by Fazelli *et al.* [11]. Specifically, in [11], Fazelli *et al.* considered the binary $\boldsymbol{M}_F$ whose rows are indexed by $\mathcal{S}$ and columns are indexed by $\mathcal{S}$ and set $\boldsymbol{M}_{\boldsymbol{x},\boldsymbol{y}}$ to be one if and only if $\tau(\boldsymbol{x},\boldsymbol{y}) \le t$. Then their upper bound is given by the optimal value of the following program.

$$\min\left\{ \sum_{\boldsymbol{y}\in\mathcal{S}} Y_{\boldsymbol{y}} : \boldsymbol{M}_F\boldsymbol{Y} \ge \boldsymbol{1}, Y_{\boldsymbol{y}} \ge 0 \text{ for } \boldsymbol{y}\in\mathcal{S} \right\}. \quad (3)$$

Since $\boldsymbol{M}_F$ is a submatrix of $\boldsymbol{M}$ obtained by deleting certain columns of $\boldsymbol{M}$, any feasible solution of (3) is also a feasible solution of (2). Furthermore, in certain cases, Theorem 1 provides a strictly better upper bound than (3). Consider the space $\mathcal{C}$ of all binary words of length n and weight w and set $d=4$, or equivalently, $t=1$. Then $\boldsymbol{M}_F$ is the identity matrix whose rows and columns are indexed by $\mathcal{C}$. Hence, the optimal value of the program defined by (3) is $\binom{n}{w}$.

In contrast, we apply Theorem 1 by setting $\tilde{\mathcal{S}}$ to be the space of all binary words of length n and weight $w+1$. Then $V^{\min}_{\mathcal{C},\tilde{\mathcal{S}}}(1) = n-w$. Therefore, the upper bound in Theorem 1 is $\binom{n}{w+1}/(n-w) = \binom{n}{w}/(w+1)$, which is a strict improvement. In the following sections, we judiciously choose the space $\tilde{\mathcal{S}}$ and apply Theorem 1 to improve the upper bound on the code sizes for certain classes of codes.

III. Subblock Energy-Constrained Codes

Let $\mathcal{S}(m,L,w_s)$ denote the space of all binary words comprising of m subblocks, each subblock having length L, with weight per subblock *at least* w_s. A binary SECC with codeword length $n = mL$, subblock length L, minimum distance d, and weight at least w_s per subblock is called an (m,L,d,w_s)-SECC. We denote the maximum possible size of an (m,L,d,w_s)-SECC by $S(m,L,d,w_s) \triangleq A(mL,d;\mathcal{S}(m,L,w_s))$. Further, we introduce the notation $\binom{L}{\ge w_s}$ which we define as

$$\binom{L}{\ge w_s} \triangleq \sum_{j=w_s}^{L} \binom{L}{j}.$$

A. Generalized Sphere Packing Bound for SECCs

We now present an upper bound on the optimal code size for SECCs via the following proposition.

Proposition 1. *For $d \le 2m+1$, $0 \le m_0 \le m$, and $t = \lfloor (d-1)/2 \rfloor$, we have*

$$S(m,L,d,w_s) \le \frac{\binom{L}{\ge w_s - 1}^{m_0} \binom{L}{\ge w_s}^{m-m_0}}{\sum_{\substack{t_1,t_2 \\ t_1+t_2\le t}} \binom{m_0}{t_1}\binom{m-m_0}{t_2} L^{t_1}(L-w_s)^{t_2}}. \quad (4)$$

Proof: We will apply Theorem 1, and choose $\tilde{\mathcal{S}} \subset \{0,1\}^{mL}$ to be the space where the first m_0 subblocks have weight at least $w_s - 1$, and the remaining $m - m_0$ subblocks have weight at least w_s, with fixed subblock length L. Thus $|\tilde{\mathcal{S}}| = \binom{L}{\geq w_s - 1}^{m_0}\binom{L}{\geq w_s}^{m-m_0}$, and using Theorem 1, it suffices to show that

$$V_{\mathcal{S},\tilde{\mathcal{S}}}^{\min}(t) \geq \sum_{\substack{t_1,t_2\\ t_1+t_2\leq t}} \binom{m_0}{t_1}\binom{m-m_0}{t_2} L^{t_1}(L-w_s)^{t_2}. \quad (5)$$

For $\boldsymbol{x} \in \mathcal{S}(m,L,w_s)$, let $\boldsymbol{x}_{[i]}$ denote the ith subblock of $\boldsymbol{x}$, and hence $\boldsymbol{x} = (\boldsymbol{x}_{[1]}\boldsymbol{x}_{[2]}\ldots\boldsymbol{x}_{[m]})$. Let $\Lambda_{\boldsymbol{x}}$ be defined as

$$\Lambda_{\boldsymbol{x}} \triangleq \{\boldsymbol{y} \in \tilde{\mathcal{S}} : \tau(\boldsymbol{x},\boldsymbol{y}) \leq t, \tau(\boldsymbol{x}_{[i]},\boldsymbol{y}_{[i]}) \leq 1, i \in \{1,\ldots,m\}\}.$$

Let $\boldsymbol{y} \in \tilde{S}$ be such that t_1 (resp. t_2) subblocks out of the first m_0 (resp. last $m - m_0$) subblocks of $\boldsymbol{y}$ differ in exactly one bit from corresponding subblocks of $\boldsymbol{x}$, with $t_1 + t_2 \leq t$. Then $\boldsymbol{y} \in \Lambda_{\boldsymbol{x}}$, and

$$|\Lambda_{\boldsymbol{x}}| \geq \sum_{\substack{t_1,t_2\\ t_1+t_2\leq t}} \binom{m_0}{t_1}\binom{m-m_0}{t_2} L^{t_1}(L-w_s)^{t_2}.$$

Note that the inequality above holds for every $\boldsymbol{x} \in \mathcal{S}(m,L,w_s)$. Finally, the inequality in (5) follows because $\Lambda_{\boldsymbol{x}} \subseteq \mathcal{B}_{\tilde{\mathcal{S}}}(\boldsymbol{x},t)$ for every $\boldsymbol{x} \in \mathcal{S}(m,L,w_s)$. ■

We will apply Prop. 1 to provide an upper bound on the asymptotic rate for SECCs. We are interested in the asymptotic setting where the number of subblocks m tends to infinity, minimum distance d scales linearly with m, and parameters L, w_s are fixed. In the following, the base for $\log$ is assumed to be 2. Formally, for fixed $0 < \delta < 1$, the asymptotic rate for SECCs is defined as

$$\sigma(L,\delta,w_s/L) \triangleq \limsup_{m\to\infty} \frac{\log S\left(m,L,\lfloor mL\delta\rfloor, w_s\right)}{mL}. \quad (6)$$

The following theorem gives an upper bound on the SECC rate $\sigma(L,\delta,w_s/L)$.

Theorem 2. *For $0 < \delta < 2/L$, we have*

$$\sigma(L,\delta,w_s/L) \leq R_1 - \hat{\alpha}\nu, \quad (7)$$

where

$$R_1 \triangleq \frac{\log\binom{L}{\geq w_s}}{L} - \frac{h(\delta L/2)}{L} - \frac{\delta}{2}\log(L-w_s) \quad (8)$$

$$h(x) \triangleq -x\log(x) - (1-x)\log(1-x)$$

$$\nu \triangleq \frac{\delta}{2}\log\left(\frac{L}{L-w_s}\right) - \frac{1}{L}\log\left[\frac{\binom{L}{\geq w_s-1}}{\binom{L}{\geq w_s}}\right] \quad (9)$$

$$\hat{\alpha} \triangleq \begin{cases} 0, & \text{if } \nu \leq 0\\ 1, & \text{if } \nu > 0\end{cases} \quad (10)$$

Proof: For $0 \leq m_0 \leq m$, let $\alpha = m_0/m$ with $t_1 = \lfloor t\alpha\rfloor$ and $t_2 = \lfloor t(1-\alpha)\rfloor$. Then $t_1 + t_2 \leq t$, and it follows from (4) that

$$S(m,L,d,w_s) \leq \frac{\binom{L}{\geq w_s-1}^{m_0}\binom{L}{\geq w_s}^{m-m_0}}{\binom{m_0}{\lfloor t\alpha\rfloor}\binom{m-m_0}{\lfloor t(1-\alpha)\rfloor} L^{\lfloor t\alpha\rfloor}(L-w_s)^{\lfloor t(1-\alpha)\rfloor}}, \quad (11)$$

Combining (6) and (11), we get

$$\begin{aligned}\sigma(L,\delta,w_s/L) \leq\ & \frac{\alpha}{L}\log\binom{L}{\geq w_s-1} + \frac{1-\alpha}{L}\log\binom{L}{\geq w_s}\\ & - \frac{\alpha}{L}h\left(\frac{\delta L}{2}\right) - \frac{(1-\alpha)}{L}h\left(\frac{\delta L}{2}\right)\\ & - \frac{\alpha\delta}{2}\log(L) - \frac{(1-\alpha)\delta}{2}\log(L-w_s).\end{aligned}$$

By combining the coefficients of α, the above inequality can be expressed as

$$\sigma(L,\delta,w_s/L) \leq R_1 - \alpha\nu, \quad (12)$$

where R_1 and ν are given by (8) and (9), respectively. The above bound on SECC rate holds for all $\alpha \in [0,1]$, and hence the right side in (12) is minimized by choosing $\alpha = \hat{\alpha}$, with $\hat{\alpha}$ given by (10). ■

We observe that the upper bound on $\sigma(L,\delta,w_s/L)$, given by Theorem 2, can equivalently be expressed as

$$\sigma(L,\delta,w_s/L) \leq \min\{R_1, R_1 - \nu\},$$

where R_1 corresponds to the sphere packing bound on the asymptotic rate $\sigma(L,\delta,w_s/L)$ when space $\tilde{\mathcal{S}}$ is chosen to be $\tilde{\mathcal{S}} = \mathcal{S}(m,L,w_s)$, while $R_1-\nu$ corresponds to the sphere packing bound when space $\tilde{\mathcal{S}}$ is chosen to be $\tilde{\mathcal{S}} = \mathcal{S}(m,L,w_s-1)$.

Corollary 1. *$R_1 - \nu$, the upper bound on SECC rate obtained by choosing $\tilde{\mathcal{S}} = \mathcal{S}(m,L,w_s-1)$, is less than R_1, the upper bound on SECC rate obtained by choosing $\tilde{\mathcal{S}} = \mathcal{S}(m,L,w_s)$, for the following range of δ values*

$$\frac{2}{L\log[L/(L-w_s)]}\log\left[\frac{\binom{L}{\geq w_s-1}}{\binom{L}{\geq w_s}}\right] < \delta < \frac{2}{L}. \quad (13)$$

Proof: Follows from (9). ■

An alternate sphere-packing bound was presented in [6], where it was shown that $\sigma(L,\delta,w_s/L)$, the asymptotic rate for SECCs, is upper bounded by

$$\sigma_{SP} \triangleq \frac{\log\binom{L}{\geq w_s}}{L} - \frac{h(\delta L/4)}{L} - \frac{\delta}{4}\log\left((L-w_s)(w_s+1)\right). \quad (14)$$

Fig. 1 compares different sphere-packing bounds for the SECC asymptotic rate $\sigma(L,\delta,w_s/L)$ as a function of δ with fixed $L = 10$, and $w_s = 5$. As shows in Cor. 1, it is observed in Fig. 1 that the upper bound given by $R_1 - \nu$ is less than R_1 for $\delta > \frac{2}{L\log[L/(L-w_s)]}\log\left[\frac{\binom{L}{\geq w_s-1}}{\binom{L}{\geq w_s}}\right] = 0.0821$.

IV. Constant Subblock-Composition Codes

A binary CSCC with codeword length $n = mL$, subblock length L, minimum distance d, and weight exactly w_s per subblock is called an (m,L,d,w_s)-CSCC. We denote the space of all binary words comprising of m subblocks, each subblock having length L, with weight exactly w_s per subblock, by $\mathcal{C}(m,L,w_s)$. We denote the maximum possible size of (m,L,d,w_s)-CSCC by $C(m,L,d,w_s) \triangleq A\left(mL,d;\mathcal{C}(m,L,w_s)\right)$.

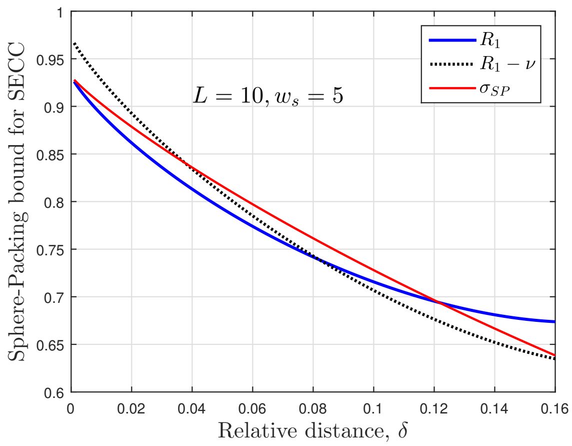

Fig. 1: Comparison of sphere-packing upper bounds for the SECC asymptotic rate $\sigma(L, \delta, w_s/L)$.

A. Generalized Sphere Packing Bound for CSCCs

We will provide improved sphere-packing bounds for the CSCC rate in the asymptotic setting where the number of subblocks m tends to infinity, minimum distance d scales linearly with m, but L and w_s are fixed. Formally, for fixed $0 < \delta < 1$, the asymptotic rate for CSCCs with fixed subblock length L, subblock weight parameter w_s, number of subblocks in a codeword $m \to \infty$, and minimum distance d scaling as $d = \lfloor mL\delta \rfloor$ is defined as

$$\gamma(L, \delta, w_s/L) \triangleq \limsup_{m \to \infty} \frac{\log C(m, L, \lfloor mL\delta \rfloor, w_s)}{mL}. \quad (15)$$

The asymptotic CSCC rate, $\gamma(L, \delta, w_s/L)$, was studied in [6], [7] and it was shown that $\gamma(L, \delta, w_s/L) = 0$ when $\delta \geq \delta^*(w_s/L)$, where $\delta^*(w_s/L)$ is defined as

$$\delta^*(w_s/L) \triangleq 2\left(\frac{w_s}{L}\right)\left(1 - \frac{w_s}{L}\right).$$

Further, in [7] the following sphere-packing upper bound on $\gamma(L, \delta, w_s/L)$ was presented.

Theorem 3 ([7]). *For $0 < \delta < \delta^*(w_s/L)$, we have*

$$\gamma(L, \delta, w_s/L) \leq \gamma_{SP}(L, \delta, w_s/L), \quad (16)$$

where $\gamma_{SP}(L, \delta, w_s/L)$ is defined as

$$\frac{1}{L}\log\binom{L}{w_s} - \left(\frac{1 + \tilde{u} - \lceil \tilde{u} \rceil}{L}\right)\log\binom{w_s}{\lceil \tilde{u} \rceil} - \left(\frac{\lceil \tilde{u} \rceil - \tilde{u}}{L}\right)\log\binom{w_s}{\lfloor \tilde{u} \rfloor} - \left(\frac{\lceil \tilde{u} \rceil - \tilde{u}}{L}\right)\log\binom{L - w_s}{\lfloor \tilde{u} \rfloor} - \left(\frac{1 + \tilde{u} - \lceil \tilde{u} \rceil}{L}\right)\log\binom{L - w_s}{\lceil \tilde{u} \rceil} - \frac{1}{L}h(\lceil \tilde{u} \rceil - \tilde{u}), \quad (17)$$

where $\tilde{u} \triangleq \delta L/4$.

We will show that for certain parameters, the above result can be improved by applying the generalized sphere packing formulation in Theorem 1. The bound on the asymptotic CSCC rate in Theorem 3 was obtained by estimating the ball size in the space $\mathcal{C}(m, L, w_s)$, and therefore corresponds to the case where $\tilde{\mathcal{S}} = \mathcal{S} = \mathcal{C}(m, L, w_s)$. In Prop. 2, we present an upper bound on the optimal CSCC code-size, $C(m, L, d, w_s)$, by choosing the space $\tilde{\mathcal{S}} = \mathcal{C}(m, L, w_s + 1)$.

Remark: For SECCs, we applied the generalized sphere-packing approach to provide improved bound on the asymptotic rate (see Corollary 1). This result, in turn, was obtained via Prop. 1 by choosing $\tilde{\mathcal{S}}$ to be the space where the first m_0 subblocks have weight at least $w_s - 1$, while the remaining $m - m_0$ subblocks have weight at least w_s. On the other hand, for CSCCs, an analogous approach of choosing $\tilde{\mathcal{S}}$ to be the space where the first m_0 subblocks have weight exactly $w_s - 1$, while the remaining $m - m_0$ subblocks have weight exactly w_s, does not lead to improved bounds on the asymptotic rate, in general, for $\delta < 2/L$. However, we will show in Prop. 3 that for δ in the vicinity of $\delta = 4/L$, improved bounds for CSCCs can be obtained by choosing $\tilde{\mathcal{S}} = \mathcal{C}(m, L, w_s + 1)$ (rather than the default space $\tilde{\mathcal{S}} = \mathcal{C}(m, L, w_s)$).

Proposition 2. *For $2m < d \leq 6m$ and $L \geq w_s + 2$, with $t = \lfloor (d-1)/2 \rfloor$ and $\tilde{t} = \lfloor (t - m)/2 \rfloor$, we have*

$$C(m, L, d, w_s) \leq \frac{\binom{L}{w_s+1}^m}{\binom{m}{\tilde{t}}\left[\binom{L-w_s}{2}\binom{w_s}{1}\right]^{\tilde{t}}(L - w_s)^{m-\tilde{t}}}. \quad (18)$$

Proof: We will apply Theorem 1, where we choose $\tilde{\mathcal{S}} = \mathcal{C}(m, L, w_s + 1)$. Thus $|\tilde{\mathcal{S}}| = \binom{L}{w_s+1}^m$, and using Theorem 1, it suffices to show that

$$V^{\min}_{\mathcal{S},\tilde{\mathcal{S}}}(t) \geq \binom{m}{\tilde{t}}\left[\binom{L - w_s}{2}\binom{w_s}{1}\right]^{\tilde{t}}(L - w_s)^{m-\tilde{t}}, \quad (19)$$

where the constrained CSCC space is $\mathcal{S} = \mathcal{C}(m, L, w_s)$. For $\boldsymbol{x} \in \mathcal{S}$, let $\Lambda_{\boldsymbol{x}}$ consist of all words $\boldsymbol{y} \in \tilde{\mathcal{S}}$ which satisfy the following two properties:

(i) $\tilde{t}$ subblocks of $\boldsymbol{y}$ differ from corresponding subblocks of $\boldsymbol{x}$ in exactly three bit positions.

(ii) Remaining $m - \tilde{t}$ subblocks of $\boldsymbol{y}$ differ from corresponding subblocks of $\boldsymbol{x}$ in exactly one bit position.

The size of $\Lambda_{\boldsymbol{x}}$ is given by

$$|\Lambda_{\boldsymbol{x}}| = \binom{m}{\tilde{t}}\left[\binom{L - w_s}{2}\binom{w_s}{1}\right]^{\tilde{t}}(L - w_s)^{m-\tilde{t}}.$$

For any $\boldsymbol{y} \in \Lambda_{\boldsymbol{x}}$, we observe that $\tau(\boldsymbol{x}, \boldsymbol{y}) = 3\tilde{t} + (m - \tilde{t}) \leq t$, and thus $\Lambda_{\boldsymbol{x}} \subseteq \mathcal{B}_{\tilde{\mathcal{S}}}(\boldsymbol{x}, t)$. Finally, the inequality in (19) follows because $\mathcal{B}_{\tilde{\mathcal{S}}}(\boldsymbol{x}, t) \geq |\Lambda_{\boldsymbol{x}}|$ for all $\boldsymbol{x} \in \mathcal{S}$. ■

The following theorem applies Prop. 2 to provide an upper bound on the asymptotic rate for CSCCs.

Theorem 4. *For $2/L < \delta < 6/L \leq \delta^*(w_s/L)$, we have*

$$\gamma(L, \delta, w_s/L) \leq \acute{\gamma}_{SP}(L, \delta, w_s/L), \quad (20)$$

where $\acute{\gamma}_{SP}(L, \delta, w_s/L)$ is defined as

$$\frac{1}{L}\log\binom{L}{w_s + 1} - \left(\frac{\delta}{4} - \frac{1}{2L}\right)\log\left[\binom{L - w_s}{2}\binom{w_s}{1}\right] - \frac{1}{L}h\left(\frac{L\delta}{4} - \frac{1}{2}\right) - \left(\frac{3}{2L} - \frac{\delta}{4}\right)\log(L - w_s). \quad (21)$$

Proof: We will combine (15) and (18) to prove the theorem. Towards this, note that when d scales as $d = \lfloor mL\delta \rfloor$, and $\tilde{t} = \lfloor (t-m)/2 \rfloor$ with $t = \lfloor (d-1)/2 \rfloor$, then we have

$$\limsup_{m\to\infty} \frac{1}{mL} \log\left[\binom{m}{\tilde{t}}\right] = \frac{1}{L} h\left(\frac{L\delta}{4} - \frac{1}{2}\right), \tag{22}$$

$$\limsup_{m\to\infty} \frac{\tilde{t}}{mL} = \left(\frac{\delta}{4} - \frac{1}{2L}\right), \tag{23}$$

$$\limsup_{m\to\infty} \frac{m-\tilde{t}}{mL} = \left(\frac{3}{2L} - \frac{\delta}{4}\right). \tag{24}$$

The proof is now complete by combining (22), (23), (24), with (15) and (18). ■

Proposition 3. *For $L/2 \le w_s < L-1$ and $\delta = 4/L$, we have*

$$\acute{\gamma}_{SP}(L, \delta, w_s/L) < \gamma_{SP}(L, \delta, w_s/L)$$

Proof: When $\delta = 4/L$, using (17) we get

$$\gamma_{SP}(L, 4/L, w_s/L) = \frac{1}{L}\log\binom{L}{w_s} - \frac{1}{L}\log\left((L-w_s)w_s\right). \tag{25}$$

On the other hand, using (21) we observe that $\acute{\gamma}_{SP}(L, 4/L, w_s/L)$ is equal to

$$\frac{1}{L}\log\binom{L}{w_s+1} - \frac{1}{2L}\log\left(2(L-w_s)^2(L-w_s-1)w_s\right)$$
$$= \frac{1}{L}\log\binom{L}{w_s} - \frac{1}{2L}\log\left(2(w_s+1)^2(L-w_s-1)w_s\right). \tag{26}$$

The proposition is now proved by comparing (25) and (26), and observing that $2w_s(L-w_s-1) \ge (L-w_s)^2$ when $L/2 \le w_s < L-1$. ■

Remark: As $\acute{\gamma}_{SP}(L, \delta, w_s)$ and $\gamma_{SP}(L, \delta, w_s)$ are both continuous functions of δ, we observe that Prop. 3 implies that for a certain interval around $\delta = 4/L$, the upper bound on the CSCC asymptotic rate given by $\acute{\gamma}_{SP}(L, \delta, w_s)$ is an improved upper bound on the CSCC rate compared to $\gamma_{SP}(L, \delta, w_s)$.

The above observation is depicted in Fig. 2 for the case where $L = 20$ and $w_s \in \{10, 14\}$. Fig. 2 shows that $\acute{\gamma}_{SP}(L, \delta, w_s) < \gamma_{SP}(L, \delta, w_s)$ for a range of δ values around $\delta = 4/L = 0.2$.

V. Reflections

We provided closed-form expressions for best known upper bounds on the asymptotic rates of SECCs and CSCCs for a range of relative distance values via a generalized sphere-packing approach. These bounds were obtained by judiciously choosing appropriate constrained spaces for estimating asymptotic ball sizes.

Alternate approaches to generalized sphere-packing, presented in [10], [11], provide upper bounds on the size of fixed blocklength constrained codes via numerically solving certain linear programs. However, these results are not amenable to providing closed-form expressions for the asymptotic rate. An interesting area of future work would be to provide improved bounds on the asymptotic rate for other classes of constrained codes, apart from subblock-constrained codes, using the approach presented in our paper.

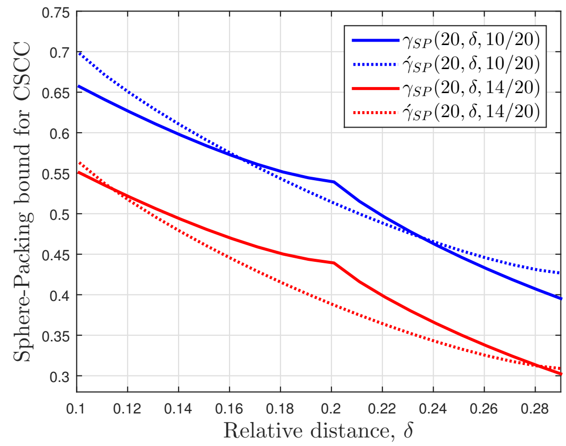

Fig. 2: Comparison of sphere-packing upper bounds for the CSCC asymptotic rate $\gamma(L, \delta, w_s/L)$ for $L = 20$.

Acknowledgments

H. M. Kiah was supported in part by the Singapore Ministry of Education under Research Grants MOE2016-T1-001-156 and MOE2015- T2-2-086.

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