Bounded Randomness*

Paul Brodhead[1], Rod Downey[2], and Keng Meng Ng[3]

[1] Indian River State College, Fort Pierce, Florida
pbrodhea@irsc.edu
[2] School of Math, Statistics, & Operations Research, Victoria University
Cotton Building, Room 358, Gate 7, Kelburn Parade, Wellington, New Zealand
Rod.Downey@vuw.ac.nz
[3] School of Physical & Mathematical Sciences
Nanyang Technological University
21 Nanyang Link, Singapore
kmng@ntu.edu.sg

Abstract. We introduce some new variations of the notions of being Martin-Löf random where the tests are all clopen sets. We explore how these randomness notions relate to classical randomness notions and to degrees of unsolvability.

1 Introduction

The underlying idea behind algorithmic randomness is that to understand randomness you should tie the notion to computational considerations. Randomness means that the object in question avoids simpler algorithmic descriptions, either through effective betting, effective regularities or effective compression. Exactly what we mean here by "effective" delineates notions of algorithmic randomness. A major theme in the area of algorithmic randomness seeks to calibrate notions of randomness by varying the notion of effectivity. For example, classical Martin-Löf randomness[1] uses tests, shrinking connections of c.e. open sets whose measure is bounded by effective bounds, whereas Schnorr randomness has the tests of some precise effective measure. We then see that Schnorr and Martin-Löf randomness are related but can have very different properties; for example outside the high degrees they coincide, but the lowness concepts are completely disjoint.

Another major theme in the study of algorithmic randomness is the intimate relationship of randomness concepts with calibrations of computational power as given by measures of relative computability, like the Turing degrees. If something is random, can it have high computational power, for instance? A classic result in this area is Stephan's theorem [14] that if a Martin-Löf real is random and has

* Supported by the Marsden Fund of New Zealand. We wish to dedicate this to Cris Calude on the occasion of his 60th Birthday.

[1] We assume that the reader is familiar with the basic notions of algorithmic randomness as found in the early chapters of either Downey-Hirschfeldt [6] or Nies [13].

M.J. Dinneen et al. (Eds.): WTCS 2012 (Calude Festschrift), LNCS 7160, pp. 59–70, 2012.

enough computational power to be able to compute a $\{0,1\}$-valued fixed point free function then it must be Turing complete.

The goal of the present paper is to introduce some new variations in these studies, and to explore both themes. In particular, we will introduce what we call bounded variations of the notion of Martin-Löf randomness where the tests are all finite. These notions generalize the notion of Kurtz (or weak) randomness but are incomparable with both Schnorr and computable randomness.

More precisely, if W is a finite set then $\#W$ denotes the cardinality of W. $|\sigma|$ denotes the length of a finite string σ. We work in the Cantor space 2^ω with the usual clopen topology. The basic open sets are of the form $[\sigma]$ where σ is a finite string, and $[\sigma] = \{X \in 2^\omega \mid X \supset \sigma\}$. We fix some effective coding of the set of finite strings, and we freely identify finite strings with their code numbers. We denote $[W] = \cup\{[\sigma] : \sigma \in W\}$ as the Σ_1 open set associated with the c.e. set W. $\mu([W])$ denotes Lebesgue measure, and we write $\mu(W)$ instead of $\mu([W])$.

Definition 1. *(a) A Martin-Löf (ML) test is a uniform c.e. sequence $\{U_n\}_{n\in\omega}$ of sets U_n such that $\mu(U_n) < 2^{-n}$.*

(b) A Martin-Löf test $\{U_n\}_{n\in\omega}$ is finitely bounded (FB) if $\#U_n < \infty$ for every n.

(c) A Martin-Löf test $\{U_n\}_{n\in\omega}$ is computably bounded (CB) if there is some total computable function f such that $\#U_n \leq f(n)$ for every n.

(d) A real $X \in 2^\omega$ passes a CB-test (FB-test) $\{U_n\}_{n\in\omega}$ if $X \notin \bigcap_n [U_n]$.

A real $X \in 2^\omega$ is computably bounded random *if X passes every CB-test. X is* finitely bounded random *if it passes every FB-test.*

These two notions of randomness are weaker than Martin-Löf randomness, although they imply Kurtz randomness. The obvious implications are:

$$\begin{tikzcd} ML\text{-random} \arrow[r] \arrow[dr] & FB\text{-random} \arrow[r] & CB\text{-random} \arrow[r] & \text{Kurtz random} \\ & \text{Schnorr random} \arrow[urr] & & \end{tikzcd}$$

No implications hold other than those stated in the diagram. This can be derived from the following facts: There is a Δ^0_3 1-generic real which is FB-random (see the remarks after Proposition 3), while no Schnorr random is weakly 1-generic. No incomplete c.e. degree can compute a FB-random (Proposition 1(i)) while some incomplete c.e. degree bounds a CB-random (Theorem 2). Lathrop and Lutz [12] showed that there is a computably random set X such that for every order function g, $K(X \restriction n) \leq K(n) + g(n)$ for almost every n. Hence X cannot be CB-random, by Proposition 3.

There is a non-zero Δ^0_2 degree containing no CB-random (Theorem 2) while every hyperimmune degree contains a Kurtz random.

What is interesting is that these notions of randomness turn out to have strong relationships with degrees classes hitherto unrelated to algorithmic randomness.

We will show that FB-randomness and Martin-Löf -randomness coincide on the Δ_2^0 sets but are distinct on the Δ_3^0 sets (Theorem 1). There is some restriction on the degrees of these reals in that they cannot be c.e. traceable (Theorem 2). It is not clear exactly what the degrees of such reals can be.

In the case of CB-randomness there can be incomplete c.e. degrees containing such reals. We know that every c.e. degree contains a Kurtz random real, but the degrees containing a CB-random form a subclass of the c.e. degrees : those that are not totally ω-c.a.. This is a class of c.e. degress introduced by Downey, Greenberg and Weber [5] to explain certain "multiple permitting" phenomena in degree constructions such as "critical triples" in the c.e. degrees, and a number of other constructions as witnessed in the subsequent papers Barmpalias, Downey and Greenberg [1] and Downey and Greenberg [4]. This class extends the notion of array noncomputable reals, and correlates to the fact that all CB random reals have effective packing dimension 1 (Theorem 3). Downey and Greenberg [3] having previously showed that the c.e. degrees containing reals of packing dimension 1 are exactly the array noncomputable reals. We also show that if a c.e. degree $\mathbf{a}$ contains a CB random then every (not necessarily c.e.) degree above $\mathbf{a}$ contains a CB random as well. From all of this, we see that there remains a lot to understand for this class.

Some other results which space restrictions preclude us from including concern lowness for the classes we have introduced. We know that if A is K-trivial (i.e. low for Martin-Löf randomness) then A is low for FB-randomness. Also we know that if A is low for FB-randomness then A is Low(Ω). Finally in the case of CB-randomness, we know that if A is low for CB-randomness then A is of hyperimmune-free degree. However, we have a reasonably intricate construction which constructs a Δ_3^0 real which is low for CB-randomness.

2 Basic Results

We first show that the notions of FB-randomness and Martin-Löf -randomness coincide on the Δ_2^0 sets, and they differ on the Δ_3^0 sets.

Proposition 1. *(i) Suppose $Z \leq_T \emptyset'$. Then Z is ML-random iff Z is FB-random.*
(ii) There is some $Z \leq_T \emptyset''$ such that Z is FB-random but not ML-random.

Proof. (i): Given an approximation Z_s of Z, and suppose $\{U_x\}$ is the universal ML-test where $Z \in \cap_x [U_x]$. Enumerate an FB-test $\{V_x\}$ by the following: at stage s, enumerate into V_x, the string $Z_s \upharpoonright n$ for the least n such that $Z_s \upharpoonright n \in U_x[s]$. Then, $\{V_x\}$ is uniformly c.e., where $\mu(V_x) \leq \mu(U_x) < 2^{-x}$ for all x. Clearly $Z \in [V_x]$ for all x. We know $Z \upharpoonright n \in U_x$ for some least n, and let s be a stage such that $Z_s \upharpoonright n$ is correct and $Z \upharpoonright n$ has appeared in $U_x[s]$. Then, $Z \upharpoonright n$ will be in V_x by stage s, and we will never enumerate again into V_x after stage s.

(ii): We build $Z = \cup_s \sigma_s$ by finite extension. Let $\{U_x\}$ be the universal ML-test, and $\{V_x^e\}_x$ be the e^{th} ML-test. Assume we have defined σ_s, where for all $e < s$, we have

- all infinite extensions of σ_s are in U_e,
- if $\#V_x^e < \infty$ for all x, then there exists k such that no infinite extension of σ_s can be in U_k^e.

Now we define $\sigma_{s+1} \supset \sigma_s$. Firstly, find some $\tau \supseteq \sigma_s$ such that all infinite extensions of τ are in U_s; such τ exists because $\{U_e\}$ is universal. Let $k = |\tau|$. Next, ask if $\#V_k^s < \infty$. If not, let $\sigma_{s+1} = \tau^\frown 0$ and we are done. If yes, then figure out exactly the strings ρ_i such that $[V_k^s] = \cup\{[\rho_1], [\rho_2], \cdots, [\rho_n]\}$. We cannot have $[V_k^s] \supseteq [\tau]$ since $\mu(V_k^s) < 2^{-k}$, so there has to be some $\sigma_{s+1} \supset \tau$ such that $[\sigma_{s+1}] \cap [V_k^s] = \emptyset$, by the finiteness of V_k^s. We can figure σ_{s+1} out effectively from $\rho_1, \rho_2, \cdots, \rho_n$. Clearly the properties above continue to hold for σ_{s+1}. All questions asked can be answered by the oracle $\emptyset''$.

Note that there is no way of making $\{V_x\}$ computably bounded in (i), even if $Z \leq_{tt} \emptyset'$. It is easy to construct a low left c.e. real which is CB-random, while from Theorem 2 below, no superlow c.e. real can be CB-random. Hence CB-randomness and FB-randomness differ even on the c.e. reals.

CB-randomness is still sufficiently strong as a notion of randomness to exclude being traceable:

Proposition 2. *No CB-random is c.e. traceable.*

Proof. Suppose that A is c.e. traceable, and that A is coinfinite (otherwise we are done). We define the functional Φ by evaluating $\Phi^X(n)$ as σ where $\sigma \subset X$ is the shortest string such that $\#\{k : \sigma(k) = 1\} = 2n$, for any X and n. Since Φ^A is total, there is a c.e. trace $\{T_x\}_{x\in\mathbb{N}}$, such that $\#T_x \leq x$ and $\Phi^A(x) \in T_x$ for every x. We define the CB-test $\{U_x\}$ by the following: we enumerate σ into U_x if $|\sigma| \geq 2x$ and $\sigma \in T_x$. Then $\#U_x \leq x$ and $\mu(U_x) \leq x2^{-2x} < 2^{-x}$ for every x and $A \in \cap_x[U_x]$.

We next investigate the connection between CB-randomness and effective dimension.

Proposition 3. *Every CB-random is of effective packing dimension* 1.

Proof. Suppose $K(\alpha \upharpoonright n) \leq cn$ for all $n \geq N$ for some $N \in \mathbb{N}$ and $c < 1$ is rational. Fix a computable increasing sequence of natural numbers $\{n_i\}$ all larger than N, such that $n_i > \frac{i}{1-c}$ for all i. Now define a CB-test $\{V_i\}$ by the following: $V_i := \{\sigma \in 2^{n_i} \mid K(\sigma) \leq cn_i\}$. Here we have $\#V_i \leq 2^{cn_i}$.

In contrast, every incomplete c.e. real which is CB-random cannot be of d.n.c. degree (and hence has effective Hausdorff dimension 0). The proof of Theorem 1(ii) constructs a FB-random real by finite extensions. It is straightforward to modify the construction to build a Δ_3^0 FB-random which is 1-generic, and hence not of d.n.c. degree.

Next we investigate the upward closure of CB-random degrees.

Theorem 1. *If A is a c.e. real and is CB-random, and $A \leq_T B$, then $deg_T(B)$ contains a CB-random.*

Proof. Fix a left-c.e. approximation A_s to A. Let $h : \mathbb{N} \mapsto \mathbb{N}$ be a strictly increasing function such that $h(n+1) \geq h(n) + 2$ for every n. For any real X we let $(A \oplus_h X)(z)$ be defined by the following: if $z = h(n)$ for some n then $(A \oplus_h X)(z) = X(n)$, otherwise let $(A \oplus_h X)(z) = A(z-n-1)$ where $h(n) < z < h(n+1)$. This is the "sparse" join of A and X, and is obtained by copying the first $h(0)$ many digits of A followed by $B(0)$, the next $h(1)-h(0)-1$ many digits of A followed by $B(1)$, and so on. For numbers n, s we denote α_s^n as the finite string $A_s \upharpoonright h_s(n) - n$. This represents the A portion of the current approximation to $A \oplus_h X$ below $h_s(n)$.

The construction builds a function $h \leq_T A$ such that $A \oplus_h X$ is CB-random for any path $X \in 2^\omega$. This is achieved by specifying an effective approximation $h_s(n)$ which is non-decreasing in each variable n, s. We let $h(n) = \lim_s h_s(n)$. We also ensure that for every n, s if $h_{s+1}(n) > h_s(n)$ then $A_{s+1} \not\supset \alpha_s^n$. Intuitively $h_s(n)$ is the stage s coding location for $X(n)$, and we are insuring that before moving the coding location $h_s(n)$ we need to first obtain a change in α_s^n. The theorem is then satisfied by taking $A \oplus_h B$, for given $A \oplus_h B$ as oracle, to figure out $B(n)$, one can run the construction until a stage s is found such that α_s^n agrees with the true α^n of the oracle string. Then each of the coding location $h_s(0), \cdots, h_s(n)$ must already be stable at s.

Construction of h: Let $\{U_x^e\}$ be the e^{th} Martin-Löf test, and φ_e be the e^{th} partial computable function. We set $h_0(n) = 2n$ for every n. At stage $s > 0$ find the least $n < s$ such that $A_s \not\supset \alpha_{s-1}^n$, and there is some $e, x \leq n$ and some $\sigma \in U_x^e[s-1]$ such that $\varphi_e(x) \downarrow$ and $\# U_x^e[s-1] \leq \varphi_e(x)$. We also require that $\alpha_{s-1}^n \supseteq \sigma$ but $A_s \not\supset \sigma$. If such n is found we set $h_s(n+i) = s + n + 2i$ for every i.

We now verify the construction works. Clearly h_s has the above-mentioned properties and $\lim_s h_s(n)$ exists. The only thing left is to check that $A \oplus_h X$ is CB-random. Suppose this fails for some $X \in 2^\omega$. Let $\{U_x\}$ be a CB-test such that $A \oplus_h X \in [U_x]$ for every x.

For a finite string σ and stage s, we let $\sigma^*(s)$ be the string obtained by removing the $(h_s(0)+1)^{th}, (h_s(1)+1)^{th}, \cdots$ digits from σ. We define a new CB-test $\{V_x\}$ by the following. At a stage s if we find some $\sigma \in U_{x^2}[s]$ and $\sigma^*(s) \subset A_s$ we enumerate $\sigma^*(s) \upharpoonright (h_s(x) - x)$ into V_x (unless some comparable string is already in V_x). That is, we enumerate the A-part of $A \oplus_h \emptyset$ below $h_s(x)$ into V_x, unless $\sigma^*(s)$ is shorter, in which case we enumerate $\sigma^*(s)$ instead.

We consider a large x. Clearly $\# V_x \leq \# U_{x^2}$, since each $\sigma \in U_{x^2}$ causes at most one $\sigma^*(s)$ (or part of) to be enumerated in V_x. We need to compute a bound on the measure of $[V_x]$. Each string enumerated into V_x is either $\sigma^*(s)$ or part of $\sigma^*(s)$ for some s and $\sigma \in U_{x^2}[s]$. Each string of the first type satisfies $|\sigma| - |\sigma^*(s)| \leq x$, while it is easy to see that strings of the second type must all be of different length greater than x. Hence the measure of $[V_x]$ is bounded by $2^x \mu(U_{x^2}) + 2^{-x+1} < 2^{-x}$. Since $\{V_x\}$ is a CB-test A must escape this test, a contradiction.

We conclude this section with several questions.

Question 1. 1. If $A \leq_T B$ and A is CB-random, must $deg_T(B)$ contain a CB-random?
2. Are there characterizations of CB-randomness and FB-randomness in terms of prefix-free complexity and martingales?
3. Are there minimal Turing degrees which contain CB-randoms?

3 A Characterization of the Left c.e. Reals Containing a $\boldsymbol{CB}$-Random

The class of array computably c.e. sets was introduced by Downey, Jockusch and Stob [8,9] to explain a number of multiple permitting arguments in computability theory. Recall that a degree $\mathbf{a}$ is array non-computable[2] if for every function $f \leq_{wtt} \emptyset'$ there is a function $g \leq_T \mathbf{a}$ such that $f(x) < g(x)$ infinitely often. Downey, Greenberg and Weber [5] later introduced the totally ω-c.a.[3] sets to explain the construction needed for a weak critical triple, for which array non-computability seems too weak.

Definition 2 ([5]). *A c.e. degree* $\mathbf{a}$ *is totally* ω*-c.a. if every* $f \leq_T \mathbf{a}$ *is* ω*-c.e..*

Note that array computability can be viewed as a uniform version of this notion where the computable bound (for the mind changes) can be chosen independently of f; hence every c.e. array computable set is totally ω-c.e.. The class of totally ω-c.e. degrees capture a number of natural constructions. Downey, Greenberg and Weber [5] proved that a c.e. degree is not totally ω-c.e. iff it bounds a weak critical triple in the c.e. degrees.

In Theorem 2 we show that the non totally ω-c.e. degrees are exactly the class of c.e. degrees which permit the construction of a CB-random real:

Theorem 2. *Suppose A is a c.e. real. The following are equivalent.*

(i) $deg_T(A)$ *is not totally* ω*-c.a.,*
(ii) $deg_T(A)$ *contains a* CB*-random,*
(iii) *There is some c.e. real* $B \leq_T A$ *which is* CB*-random,*
(iv) *There is some* $B \leq_T A$ *which is* CB*-random.*

We fix a computable enumeration$\{\varphi_n\}_{n\in\omega}$ of all partial computable functions.We let $\{W_n^m\}_{n\in\omega}$ be the m^{th} Martin-Löf test. We use $<_L$ to denote the left-to-right lexicographical ordering on finite strings σ, τ, with 0 being to the left of 1 and $\sigma <_L \tau$ meaning that σ is to the left of τ. This ordering is extended naturally to $x <_L y$ for infinite strings x, y. We assume for any c.e. set U, that if $\sigma \in U_s$ then $|\sigma| < s$.

[2] This was not the original definition, but a later equivalent characterization, which is convenient for us to take as the definition.

[3] The original paper [5] called these *totally* ω*-c.e.*. However this terminology is somewhat at odds with Ershov's hierarchy of Δ_2^0 sets [10,11] and causes a problem when we work at various levels of the computable ordinals. Hence we will adopt the new name being used in Downey and Greenberg [4].

3.1 (i) ⇒ (iii)

Assume that $f = \Delta^A$ and that f is not ω-c.e. We will build $B \leq_T A$ and ensure that B is CB-random. We must ensure that $R_{m,i}$ holds for every m, i:

$$R_{m,i}: \quad B \notin \cap_n [W_n^m] \quad \text{if } \varphi_i \text{ is total and for all } n,\ \#W_n^m \leq \varphi_i(n).$$

To ensure that each requirement R is satisfied, suppose that R is the k^{th} requirement, where $k = \langle m, i \rangle$. Our construction will implement a sequence of modules $\{M_j^k\}_{j \in \omega}$ for R and each module is given infinitely many opportunities to act. At any particular stage, the construction attempts to satisfy at most one requirement through the implementation of at most one module. Associated with each module M_j^k is an integer $n = n_j^k$, and the module aims to ensure that if $\{W_e^m\}_{e \in \omega}$ is a CB-test then $B \notin [W_n^m]$ as follows. (Note that as long as some module succeeds, the requirement succeeds.)

Suppose at the current stage s of the construction that it is module M_j^k's turn to act and B is in $[W_n^m]$— that is, $B_{s-1} \in [W_{n,s}^m]$. The module's strategy is to redefine B to the right (outside of $[W_n^m]$), but on precondition that it receives an A-permission, due to certain conditions related to Δ^A.

To be more precise: throughout the construction, the modules $\{M_j^k\}_{j \in \omega}$ will collectively be defining an approximating function f_k for Δ^A towards ensuring that, for some j, module M_j^k's strategy succeeds (so that R_k is satisfied). We further discuss f_k and the A-permission below.

Module M_j^k is responsible for defining $f_k(j, s)$ for all s; it does so as follows. Whenever $B_{s-1} \in [W_n^m]$ as above, then— supposing this is the ${t_s}^{th}$ time it acts— M_j^k defines $f_k(j, t_s) := \Delta^A(j)[t_s]$. Module M_j^k waits to act at a later stage $q > s$ when either

- B remained in $[W_n^m]$ throughout all intermediate stages $\leq q$ and A changes below the use $\delta(j)$ for $\Delta^A(j)$, or
- B does not remain in $[W_n^m]$ until stage q due to an A-permission being granted to some other module, or perhaps some other requirement.

In either of these two cases, an A-permission is granted and M_j^k moves B to the right.

Now suppose $\{W_n^m\}$ is a CB-test so that $\#W_n^m \leq \varphi_i(n)$. Since B is only ever redefined to the right, it follows that there can be at most $\varphi_i(n) = \varphi_i(n_j^k)$ A-permissions associated with module M_j^k so that

$$\#\{s : f_k(j, s) \neq f_k(j, s+1)\} \leq \varphi_i(n) = \varphi_i(n_j^k).$$

It follows that if $B \in [W_{n_j^k}^m]$ for all j, then eventually no A-permission occurs for module M_j^k to act, for all j. Consequently, $f_k(j, t) = \Delta^A(j)[t] = f(j)$ for sufficiently large t and f_k must be an approximating function for $\Delta^A = f$. This means that f is ω-c.e., a contradiction, and thus requirement $R = R_k$ must be satisfied.

We are ready to describe the stage-by-stage construction.

Construction. The construction will proceed in stages of the form $\langle a+1, \langle j,k\rangle\rangle$. The intention is that stage $\langle a+1, \langle j,k\rangle\rangle$ is the a^{th} time in which module M_j^k is allowed to act. Consequently, in what follows, we will use ℓ to denote $\ell = \langle j,k\rangle$. We also define the integer $n_j^k = \langle k,j\rangle + 1$ associated with module M_j^k of the k^{th} requirement. Since Δ^A is total, we assume that $\Delta^A(j)[s]\downarrow$ at every stage $s > j$.

At stage $s = 0$, define $B_0 = 0^\omega$ and goto stage $s+1$.

At stage $s = \langle 0,\ell\rangle > 0$ define $f_k(j,0) = \Delta^A(j)[0]$ and goto stage $s+1$.

At stage $s = \langle a+1,\ell\rangle$, implement the j^{th} module M_j^k of requirement R_k defined as follows.

Module M_j^k.

1. If $\varphi_{i,s}(n_j^k)\uparrow$, or $\#W^m_{n_j^k,s} \not\leq \varphi_{i,s}(n_j^k)$, or $B_{s-1} \notin [W^m_{n_j^k,s}]$, then no non-trivial action is needed for M_j^k. We simply define $f_k(j,a+1) := f_k(j,a)$, define $B_s := B_{s-1}$ and go to stage $s+1$.
2. Otherwise, define $f_k(j,a+1) = \Delta^A(j)[a+1]$, let $r = \langle a,\ell\rangle$, and implement the following. If $A_{a+1}\restriction\delta(j) \neq A_a\restriction\delta(j)$, then do the following. Let $\sigma \subset B_{s-1}$ be maximal such that $N_\sigma := ([\sigma] \cap \{x : B_{s-1} <_L x\}) \setminus [W^m_{n_j^k,s}]$ is nonempty. Define B_s to be the left-most path of N_σ, and go to stage $s+1$. Otherwise define $B_s := B_{s-1}$ and go to stage $s+1$.

This completes the construction.

Verification. First observe that for any module M_j^k, whenever it changes B, it only adds an amount $q \in \mathbb{Q}$ to B_s where q can be accounted against a distinct part of $W^m_{n_j^k}$. Therefore M_j^k contributes at most $2^{-n_j^k}$ to B. Consequently the total effect of all the modules can contribute at most $\sum_{k,j\in\omega} 2^{-n_j^k} \leq \frac{1}{2}$ to B, which means that σ in the construction, at every stage, can always be found so that N_σ is non-empty.

Lemma 1. *Every requirement is satisfied.*

Proof. Suppose to the contrary that for some pair m,i, $B \in \cap_n[W_n^m]$, φ_i is total, and $\#W_n^m \leq \varphi_i(n)$ for all n. We first observe that $\lim_a f_k(j,a) = \Delta^A(j)$ for each j. Let $W = W^m_{n_j^k}$. Since $B \in [W]$, hence at almost every stage of the construction when M_j^k acts, we have case 2 holds; hence we will set $f_k(j,a) = \Delta^A(j)[a]$ at almost every a. Next, we want to show that the f_k-changes is bounded by $O(\varphi_i(n_j^k))$. We fix a j, and argue that if $\langle a_0+1,\ell\rangle < \langle a_1+1,\ell\rangle$ are two stages in the construction such that M_j^k acts under case 2, and $f_k(j,a_0+1) \neq f_k(j,a_1+1)$, then $B_s \notin [W_{\langle a_0+1,\ell\rangle}]$ for some $\langle a_0+1,\ell\rangle < s \leq \langle a_1+1,\ell\rangle$. This is because there must be some $a_0 < a \leq a_1$ such that $A_{a+1}\restriction\delta(j) \neq A_a\restriction\delta(j)$. At stage $\langle a+1,\ell\rangle$ of the construction we may assume case 2 holds (otherwise we are done). Hence we will define $B_{\langle a+1,\ell\rangle}$ to avoid $W_{\langle a+1,\ell\rangle} \supseteq W_{\langle a_0+1,\ell\rangle}$. This proves the claim. Now to see that the number of changes in $f_k(-,a)$ is bounded by $O(\varphi_i(n_-^k))$, observe that if $f_k(j,a) \neq f_k(j,a+1)$, we must have case 2 applies at stage $\langle a+1,\ell\rangle$ of the construction.

Lemma 2. $B \leq_T A$.

Proof. Next we describe how to compute $B \leq_T A$. To compute $B(x)$, we would like to say that only modules M_j^k for $n_j^k \leq x$ can change $B(x)$. This is unfortunately not true, because of the "carry-over" in the addition. Instead we have to compute B from A in a slightly more elaborate fashion. Define the total function $g \leq_A$ by the following. Let $g(0) = x$, and given $g(z)$ we define $g(z+1)$ by first searching recursively in A for some number a such that $A_a \upharpoonright \delta(g(z))$ is stable and correct. Let $g(z+1) = \max\{\langle a+1, \langle j,k\rangle\rangle \mid n_j^k \leq g(z)\}$. Hence the function g is defined so that after stage $g(z+1)$ of the construction, no module M_j^k for $n_j^k \leq g(z)$ can change B.

Assume we have computed $\sigma = B \upharpoonright x$. Now search for the least z such that either $B_{g(z+2)}(x) = 1$, or else $B_{g(z+2)}(y) = 0$ for some $x < y < g(z+1)$. This search will terminate because otherwise $B = \sigma 011111\cdots$ which means B is computable. Let z be the first found. If $B_{g(z+2)}(x) = 1$ then $B(x) = 1$. Otherwise we claim that $B(x) = 0$. After stage $g(z+2)$, only modules M_j^k for $n_j^k > g(z+1)$ can contribute to B, and the sum of their total contribution to B is $< 2^{-g(z+1)}$. On the other hand if $B_t(x) = 1$ at some $t > g(z+2)$, then the amount added to B after $g(z+2)$ is at least $2^{-x-1} - (2^{-x-2} + \cdots + 2^{-y-1}) = 2^{-y-1} \geq 2^{-g(z+1)}$.

3.2 (iv) ⇒ (i)

Suppose $B = \Delta^A$ and B is CB-random. Let φ_e be the e^{th} partial computable function. Fix a left c.e. approximation $\{A_s\}$ to A. Define $f(\langle e,k\rangle)$ by the following. Search for the first stage s such that $A_s \upharpoonright \delta(\langle e,k\rangle) = A \upharpoonright \delta(\langle e,k\rangle)$. If $\varphi_e(\langle e,k\rangle)[s] \uparrow$ then output $A \upharpoonright \delta(\langle e,k\rangle)$; otherwise output $A \upharpoonright \delta(\varphi_e(\langle e,k\rangle) + \langle e,k\rangle)$. Clearly f is total and $f \leq_T A$. Note that the use of the computation is not (and cannot be) computable. We claim f is not ω-c.e.; suppose the contrary we have $f(x) = \lim_s g(x,s)$ where $g(x,-)$ has at most $\varphi_e(x)$ mind changes for some total computable functions g and φ_e. We build a CB-test $\{V_k\}$ capturing B, contrary to assumption. For each k we find a stage s_0 such that $\varphi_e(\langle e,k\rangle)[s_0] \downarrow$, and $\Delta^A \upharpoonright \langle e,k\rangle[s_0] \downarrow$. We then enumerate $\Delta^A \upharpoonright \langle e,k\rangle[s_0]$ into V_k, and for every $s > s_0$ such that $\Delta^A \upharpoonright \langle e,k\rangle + \varphi_e(\langle e,k\rangle)[s] \downarrow$ with $g(\langle e,k\rangle, s) \supseteq A \upharpoonright \delta(\langle e,k\rangle + \varphi_e(\langle e,k\rangle))[s]$, we enumerate $\Delta^A \upharpoonright \langle e,k\rangle + \varphi_e(\langle e,k\rangle)[s]$ into V_k.

Clearly for each k we have $\#V_k \leq 1 + \varphi_e(\langle e,k\rangle)$, and that $\mu(V_k)$ is at most $2^{-\langle e,k\rangle} + \varphi_e(\langle e,k\rangle) 2^{-\langle e,k\rangle - \varphi_e(\langle e,k\rangle)} < 2^{-\langle e,k\rangle+1} \leq 2^{-k}$. We claim that $B \in [V_k]$. At stage s_0 we threw in $\Delta^A \upharpoonright \langle e,k\rangle[s_0]$, and if $A \upharpoonright \delta(\langle e,k\rangle)$ is stable at s_0 then clearly $B \in [V_k]$. Since $\{A_s\}$ is a monotonic approximation to A, we therefore may assume that A was not stable at s_0, hence $f(\langle e,k\rangle) = A \upharpoonright \delta(\varphi_e(\langle e,k\rangle) + \langle e,k\rangle)$. Since g approximates f correctly, at some large enough stage we will enumerate $B \upharpoonright \langle e,k\rangle + \varphi_e(\langle e,k\rangle)$ into V_k.

Finally the proof of Theorem 2 is complete upon observing that (iii) implies (ii) follows from Theorem 1.

4 Lowness

Theorem 3. *There is a non-computable Δ^0_3 set A which is low for CB-randomness.*

Proof (Sketch of proof). The construction involves building a Δ^0_3 approximation to A. We will specify a computable approximation α_s and at the end we will take $A = \liminf_s \alpha_s$. We need to meet the requirements

$$P_e : A \neq \varphi_e$$

$$R_{e,i} : \text{If } \{U^A_{e,i}\}_{i\in\omega} \text{ is an } A\text{-relative } CB\text{-test with bound } \Psi^A_e, \text{ there is a } CB\text{-test } \{V_{e,i}\}_{i\in\omega} \text{ such that } \cap_{i\in\omega}[U^A_{e,i}] \subseteq \cap_{i\in\omega}[V_{e,i}]$$

Here we let $\{U^X_{e,i}\}$ be the e^{th} oracle CB-test, and Ψ_e be the e^{th} Turing functional. φ_e is the e^{th} partial computable function. The construction builds A of hyperimmune-free degree. For more details on how to construct a non-computable real of hyperimmune-free degree by a full Δ^0_3 approximation we refer the reader to Downey [2]. We sketch the main ideas here.

To make A of hyperimmune-free degree, for each Turing functional Ψ, we need to find a computable function δ that dominates Ψ^A. We begin by letting α_s be a string of zeroes. The aim is to build a perfect computable tree $T : 2^{<\omega} \mapsto 2^{<\omega}$ such that for every σ, $\Psi^{T(\sigma)}(|\sigma|)\downarrow$. We need to also ensure that A is in the range of T. If this fails then we will force Ψ^A to be non-total. In the former case we can read δ off T, and in the latter case we satisfy the requirement automatically. At every stage we let α_s extend $T(\sigma)$ for some σ of maximal length such that $T(\sigma)$ has been defined. If we never encounter a convergent Ψ^α we keep $\alpha \supset T(\sigma)$. If we find a convergent computation $\Psi^{\alpha_t \upharpoonright u}(|\sigma|+1)$ at some stage t, we set $T(\sigma^\frown 0)\downarrow = \alpha_t \upharpoonright u$ and move α to an incomparable string extending $T(\sigma)$ and search for a way to define $T(\sigma^\frown 1)$. In this way we define $T(\sigma)$ level by level, starting with $|\sigma| = 0$, and then $|\sigma| = 1$, and so on. It is clear that if we get stuck searching above some $T(\sigma)$ then $A = \liminf \alpha_s$ will extend $T(\sigma)$ and hence Ψ^A is not total. On the other hand if the procedure builds a total computable perfect tree T then $A = T(0^\omega)$. A lower priority requirement working for another Ψ' and believing in the totality of T will take the tree T as parameter and work to build a perfect subtree T' of T. A lower priority requirement working for P will be assigned a string σ_P in the domain of T, which is consistent with P's belief about the outcomes of higher priority requirements, and P will then later delete either $T(\sigma_P{}^\frown 0)$ or $T(\sigma_P{}^\frown 1)$ (or neither) depending on the value of φ_e.

When more requirements are considered it will become necessary to define A not as the direct $\liminf$ of α_s, but as the $\liminf$ with respect to the "true stages" of the construction, namely, the stages where the true path is visited.

How do we implement the R-requirements in this framework? Let us consider a top requirement working for R_0. It seeks to define a single CB-test $\{V_{0,i}\}_{i\in\omega}$ covering $\cap_{i\in\omega}[U^A_{0,i}]$. R_0 would pursue the abovementioned strategy to obtain a computable function dominating Ψ^A_0. Additionally it has to build the test $\{V_{0,i}\}_{i\in\omega}$. For $i = 0$ we wait for $T_0(\emptyset)$ to converge. If we ever discover some σ

entering $U^{\tau}_{0,0}$ with oracle string τ on T_0, we will delete every path on T_0 not extending τ, and enumerate σ into $V_{0,0}$. Since the cardinality of $U_{0,0}$ cannot exceed $\Psi_0^{T_0(\emptyset)}$, we will only act for $U_{0,0}$ finitely often, and succeed in making $V_{0,0} = U^A_{0,0}$.

Of course we cannot allow $U_{0,i}$ to delete paths in this way for every i, because we will end up with a computable path A. Suppose P is a requirement believing that R_0 has outcome ∞, i.e. R_0 succeeds in making T_0 total. The requirement P (and all other positive requirements) of lower priority will need to be assigned a diagonalization location σ_P. Suppose that P has been assigned σ_P for diagonalization. Each time $U_{0,0}$ acts as described above it will move σ_P. It is crucial to ensure that σ_P is moved only finitely often. We arrange for $U_{0,1}$ to respect σ_P, so $U_{0,1}$ will be prohibited from deleting prefixes of $T(\sigma_P{}^\frown 0)$ and $T(\sigma_P{}^\frown 1)$. If we consider infinitely many positive requirements $P_0 < P_1 < \cdots$ below R_0, we can arrange for a local priority ordering $U_{0,0} < P_0 < U_{0,1} < P_1 < U_{0,2} < \cdots$, where each $U_{0,i}$ has to respect $\sigma_{P_0}, \cdots \sigma_{P_{i-1}}$. This resolves the (potentially) infinitary conflicts between R_0 and lower priority P requirements. A computable bound for $V_{0,i}$ can then be easily computed from upperbounds for $\Psi_0^A(0), \cdots, \Psi_0^A(i)$.

Now consider requirements R_0, R_1 and P, where R_1 believes that R_0 has outcome ∞, and P believes that R_1 has outcome ∞. Say we arrange for the local priority ordering $U_{0,0} < U_{1,0} < P < U_{0,1} < \cdots$. Since R_0 cannot assume knowledge about the outcomes of the nodes of lower priority, $U_{0,1}$ cannot possibly wait for the tree T_1 to converge before fixing an upperbound for $V_{0,1}$. Furthermore $U_{0,1}$ has to respect σ_P, so we might enumerate a large number of elements into $V_{0,1}$ while α was extending $T(\sigma_P{}^\frown 0)$. Suppose α is next moved to extend $T(\sigma_P{}^\frown 1)$, and $U_{1,0}$ obtains an upperbound for $V_{1,0}$ after seeing T_1 grow. Now if we later discover some σ in $U^{\tau}_{1,0}$ with oracle $\tau \supseteq T(\sigma_P{}^\frown 1)$ we will have to move σ_P to make $T(\sigma_p) \supset \tau$, since $U_{1,0}$ is of higher local priority than P. This means that all the elements enumerated into $V_{0,1}$ so far are no longer possible elements of $U^A_{0,1}$, and the cardinality of $V_{0,1}$ has gone up unnecessarily. This wastage can be compounded each time $U_{1,0}$ moves σ_P, and since the bound for $V_{0,1}$ was computed with no knowledge of $\Psi_1^A(0)$, we might run out of space and exceed our declared cardinality bound for $V_{0,1}$.

Observe that we need not have fixed the local priority ordering beforehand. The solution is to assign the local priority of P *only when P is visited*. Let us consider the situation above again. Suppose P has not yet been visited by the construction (hence the local priority of P has not yet been decided). Suppose T_0 has been growing and we are currently waiting for T_1 to be defined at the root. At this point the local priority list reads $U_{0,0} < U_{1,0} < U_{0,1} < U_{0,2}, \cdots, < U_{0,i}$. If now $T_1(\emptyset)$ finds a definition, we will play outcome ∞ for R_0 and outcome ∞ for R_1, and visit P, who will now be queued after $U_{0,i}$.

The key point is that $U_{0,i+1}, U_{0,i+2}, \cdots$ will only be considered after this stage, so they can compute upperbounds for $V_{0,i+1}, V_{0,i+2}, \cdots$ using information about $\Psi_1^A(0)$. They are therefore safe from the actions of $U_{1,0}$ (and of course, from $U_{0,0}, \cdots, U_{0,i}$). On the other hand even though the upperbounds for $V_{0,0}, \cdots, V_{0,i}$ have been declared without any knowledge of $\Psi_1^A(0)$, they too, are safe from the actions of $U_{1,0}$ because $U_{0,0}, \cdots, U_{0,i}$ are all allowed to move σ_P whenever we enumerate new elements into $V_{0,0}, \cdots, V_{0,i}$. The only downside is that σ_P gets injured a lot more times. Since the local priority of P once fixed, is never again changed, this means that σ_P will be eventually stable.

The interactions between other requirements present no new difficulty, and a formal construction proceeds in a more or less routine fashion. A complete proof will appear in the journal version of this paper.

References

1. Barmpalias, G., Downey, R., Greenberg, N.: Working with strong reducibilities above totally ω-c.e. degrees. Transactions of the American Mathematical Society 362, 777–813 (2010)
2. Downey, R.: On Π_1^0 classes and their ranked points. Notre Dame Journal of Formal Logic 32(4), 499–512 (1991)
3. Downey, R., Greenberg, N.: Turing degrees of reals of positive effective packing dimension. Information Processing Letters 108, 298–303 (2008)
4. Downey, R., Greenberg, N.: A Hierarchy of Computably Enumerable Degrees, Unifying Classes and Natural Definability (in preparation)
5. Downey, R., Greenberg, N., Weber, R.: Totally $< \omega$ computably enumerable degrees and bounding critical triples. Journal of Mathematical Logic 7, 145–171 (2007)
6. Downey, R., Hirschfeldt, D.: Algorithmic Randomness and Complexity. Springer, Berlin (2010)
7. Downey, R., Hirschfeldt, D., Nies, A., Terwijn, S.: Calibrating randomness. Bulletin of Symbolic Logic 3, 411–491 (2006)
8. Downey, R., Jockusch, C., Stob, M.: Array nonrecursive sets and multiple permitting arguments. In: Ambos-Spies, K., Muller, G.H., Sacks, G.E. (eds.) Recursion Theory Week. Lecture Notes in Mathematics, vol. 1432, pp. 141–174. Springer, Heidelberg (1990)
9. Downey, R., Jockusch, C., Stob, M.: Array nonrecursive degrees and genericity. In: Cooper, S.B., Slaman, T.A., Wainer, S.S. (eds.) Computability, Enumerability, Unsolvability. London Mathematical Society Lecture Notes Series, vol. 224, pp. 93–105. Cambridge University Press (1996)
10. Ershov, Y.: A hierarchy of sets, Part 1. Algebra i Logika 7, 47–73 (1968)
11. Ershov, Y.: A hierarchy of sets, Part 2. Algebra i Logika 7, 15–47 (1968)
12. Lathrop, J., Lutz, J.: Recursive computational depth. Information and Computation 153, 139–172 (1999)
13. Nies, A.: Computability and Randomness. Oxford University Press (in preparation)
14. Stephan, F.: Martin-Löf random sets and PA complete sets. In: Chatzidakis, Z., Koepke, P., Pohlers, W. (eds.) Logic Colloquium 2002, pp. 342–348. ASL and A. K. Peters, La Jolla (2006)