A CUPPABLE NON-BOUNDING DEGREE

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The classical examples of naturally occuring definable Turing ideals in $\mathcal{R}$ are the ideals generated by the cappable, noncuppable, and non-bounding degrees. We will provide a proof for the conjecture put forward by Nies in [18], that there is a cuppable, non-bounding r.e. degree. This implies that the ideals generated by the non-bounding and/or noncuppable degrees are new, and different from the known ones.

1. Introduction

In recent years, a major area of research in computability theory has been the study of definability in the Turing degree structure. One would be interested to ask which relations and properties of the Turing degrees are expressible in the first order language of degrees, with the partial ordering $\leq_T$. Nies, Shore and Slaman [21], Nerode, Jockusch, Simpson, Woodin and many others have contributed to this end. A particularly interesting result is the characterization of the definable relations in the r.e. degrees (similarly, in all Turing degrees) which are invariant under the double jump, by looking at their definability in first order (second order) arithmetic.

Much work has also been done on lattice-theoretic related problems of $\mathcal{R}$ (the upper semi-lattice of r.e. degrees), such as lattice embedding, the study of automorphisms, and attempts to give a full algebraic breakdown of $\mathcal{R}$. Various natural (definable) subsets in $\mathcal{R}$ arose, such as the cappable, cuppable, and promptly simple degrees. The cappable and noncappable degrees gave the first algebraic decomposition of $\mathcal{R}$, into a proper ideal and a strong filter. It soon became apparent that more attention had to be paid to how fast an element is enumerated into an r.e. set (relative to other r.e. sets), instead of whether or not an element is eventually enumerated. Classes such as the h-simple degrees, hh-simple degrees and promptly simple degrees were introduced, with the promptly simple degrees surprisingly coinciding with various other unrelated classes.

In this paper, we will construct a cuppable, non-bounding r.e. degree. This settles the conjecture by Nies in [18], and shows that the two ideals generated by the union and intersection of the non-cuppable and non-bounding ideals are new. We state this here as our main theorem :

Main Theorem. *There is an r.e. degree* $\boldsymbol{a} > \mathbf{0}$ *that is cuppable, and non-bounding.*

We begin with a few definitions. A coinfinite r.e. set A is said to be *promptly simple*, if there is a recursive function p, and an enumeration $\{A_s\}_{s\in\omega}$ of A, such that for every infinite r.e. set W_e, there is some s, x where x is enumerated into W_e at stage s, and $x \in A_{p(s)}$. An r.e. degree is said to be promptly simple, if it contains a promptly simple r.e. set.

(i) Let **M** be the set of all cappable r.e. degrees. Let **NCAP** be $\mathcal{R}\backslash\mathbf{M}$, the set of non-cappable r.e. degrees, and **ENCAP** be the set of effectively non-cappable r.e. degrees.
(ii) Let **LCUP** be the set of all low cuppable r.e. degrees, and **NCUP** be the set of non-cuppable r.e. degrees.
(iii) **PS** is the set of promptly simple r.e. degrees.
(iv) **NB** is the set of non-bounding r.e. degrees, i.e. those r.e. degrees which do not bound a minimal pair.
(v) $\mathbf{SP\bar{H}}$ is the set of r.e. sets definable in $\mathcal{R}$ as the non-hh-simple r.e. sets with the splitting property.

Theorem 1.1.

(i) (Ambos-Spies et al. [1]) $\boldsymbol{ENCAP}$=$\boldsymbol{NCAP}$=$\boldsymbol{LCUP}$=$\boldsymbol{PS}$=$\boldsymbol{SP\bar{H}}$.
(ii) $\mathcal{R} = \boldsymbol{M} \cup \boldsymbol{NCAP}$*, where* $\boldsymbol{M}$ *is a proper ideal in* $\mathcal{R}$*, and* $\boldsymbol{NCAP}$ *is a strong filter in* $\mathcal{R}$*.*

As pointed out by Slaman, **NCUP** trivially forms an ideal in $\mathcal{R}$. The question thus arose as to whether there are any other naturally occuring, definable ideals in $\mathcal{R}$ (Shore [22]). Nies answered the question by proving that the ideal generated by a definable subset of $\mathcal{R}$, is itself also definable.

Thus, any naturally defined class of r.e. degrees will generate a definable ideal, the only concern being whether or not it is new.

Theorem 1.2.

(i) (Nies [18]) If $D \subseteq \mathcal{R}$ is definable, then so is the ideal in $\mathcal{R}$ generated by D:

$$[D]_{id} := \{\boldsymbol{x} \in \mathcal{R} \mid \exists C \subseteq D \ \wedge \ C \text{ is finite } \wedge \ \boldsymbol{x} \leq \sup C\}.$$

(ii) There is a cappable degree not in $[\boldsymbol{NB}]_{id}$, as well as a r.e. degree which is both noncuppable, and non-bounding.

Theorem 1.3.

(i) (Yang, Yu [26]) There is a noncuppable r.e. degree which is not below the join of finitely many non-bounding degrees.

(ii) (Yang, Yu [26]) There is a cappable degree which is not below the join of a noncuppable degree and finitely many non-bounding degrees.

Therefore the ideals $\mathbf{M}$, $\mathbf{NCUP}$, and $[\mathbf{NB}]_{id}$ are related in the following way :

$$\mathbf{NCUP} \subsetneqq \mathbf{M}, \qquad [\mathbf{NB}]_{id} \subsetneqq \mathbf{M}, \qquad \mathbf{NCUP} \neq [\mathbf{NB}]_{id},$$

with all three being pairwise distinct. Furthermore, we also have

$$[\mathbf{NCUP} \cap \mathbf{NB}]_{id} \neq \emptyset, \qquad [\mathbf{NCUP} \cup \mathbf{NB}]_{id} \subsetneqq \mathbf{M}.$$

These are the classical examples of elementarily definable Turing ideals. Even with the result by Nies in Theorem 1.2, it is still very difficult to find new examples of such ideals, since one still has the task of obtaining an elementary (lattice-theoretic) characterization of the subsets of $\mathcal{R}$. We will give examples below, of some of the other ideals identified so far.

- Chaitin [3] and Solovay [25] studied the class of K-trivial sets. Recall that a set A is K-trivial if the prefix complexity of each initial segment of A is as low as it can be. That is,

  $$\forall n \ \Big(K(A\restriction_n) \leq K(n) + b\Big)$$

  for some constant b. This class is, in some sense, far away from the notion of being random, since a set A is Martin-Löf random iff $\forall n \ \ K(A\restriction_n) \geq n - b$ for some constant b. Thus, the K-trivial sets behave in the same way as the recursive sets when we examine their complexity, yet Solovay [25], and Downey, Hirschfeldt, Nies and Stephan [9] gave constructions of a non-recursive K-trivial set. This interesting class has been studied extensively, and Chaitin [3] has shown that every K-trivial set is Δ^0_2. Nies [19] showed that the class of K-trivial sets are the same as the sets which are low for 1-randomness, and Downey, Hirschfeldt, Nies and Stephan [9] also showed that the K-trivial sets form a natural solution to Post's Problem (i.e. every K-trivial set is Turing incomplete), and also constructed a promptly simple K-trivial set.

It is also interesting to note that the r.e. K-trivial reals are closed under join [9], and downwards closed [19]. It is also easy to see that $\{e \mid \Phi_e \text{ total} \wedge \Phi_e^{\emptyset'} \text{ is } K\text{-trivial } \} \in \Sigma^0_3$, hence the ideal of the r.e. K-trivial degrees is a Σ^0_3 ideal. Furthermore, Nies [20] and Downey, Hirschfeldt [8] also showed that every non-trivial Σ^0_3 ideal in $\mathcal{R}$ is bounded above by a low_2 r.e. set.

- Bickford, Mills [2] defined an r.e. degree $\boldsymbol{a}$ to be deep, if for all other r.e. degrees $\boldsymbol{b}$, we have $(\boldsymbol{a} \cup \boldsymbol{b})' = \boldsymbol{b}'$. The deep degrees forms a definable ideal in $\mathcal{R}$, but Lempp, Slaman [15] showed that this ideal is trivial, i.e. there is no deep degree other than $\mathbf{0}$. In [4], Cholak, Groszek and Slaman weakened the requirement for a degree to be deep, by introducing the notion of an almost deep degree. That is, instead of requiring that for every $\boldsymbol{b}$, when $\boldsymbol{b}$ is joined with $\boldsymbol{a}$, the jump is preserved, we will instead look at what happens if only lowness was preserved. Say that an r.e. degree $\boldsymbol{a}$ is almost deep, if for every low r.e. degree $\boldsymbol{b}$, the join $\boldsymbol{a} \cup \boldsymbol{b}$ is also low. [4] also contains the construction of a non recursive, almost deep r.e. degree. This shows that the ideal of the almost deep degrees is non-trivial, and is contained in $\mathbf{M}$.
- An r.e. degree is said to be contiguous, if it contains a single r.e. wtt degree. The concept of a contiguous degree was first introduced by Ladner [13], and Ladner, Sasso [14] and used in the study of transfer techniques (i.e. results which transfer from wtt degrees to the r.e. degrees). This was taken up by Downey [5] who introduced the notion of a strongly contiguous degree (i.e. an r.e. degree with only a single wtt degree), and showed by transfer techniques, that there is an r.e. (Turing) degree with the strong anti-cupping property.

  The most significant difference between $\mathcal{R}_{wtt}$ (r.e. wtt degrees) and $\mathcal{R}$ is probably the fact that $\mathcal{R}_{wtt}$, but not $\mathcal{R}$ forms a distributive upper semi-lattice. Even so, the distributivity of $\mathcal{R}_{wtt}$ transfer locally to the contiguous degrees in $\mathcal{R}$. Downey, Lempp in [11] proved that an r.e. degree $\boldsymbol{a}$ is contiguous if and only if it is locally distributive, i.e.

$$\forall \boldsymbol{a}_0 \forall \boldsymbol{a}_1 \forall \boldsymbol{b} \ \Big((\boldsymbol{a}_0 \cup \boldsymbol{a}_1 = \boldsymbol{a}) \ \wedge \ (\boldsymbol{b} \leq \boldsymbol{a}) \ \Rightarrow \\ \exists \boldsymbol{b}_0 \exists \boldsymbol{b}_1 \ \ (\boldsymbol{b}_0 \cup \boldsymbol{b}_1 = \boldsymbol{b} \ \wedge \ \boldsymbol{b}_0 \leq \boldsymbol{a}_0 \ \wedge \ \boldsymbol{b}_1 \leq \boldsymbol{a}_1) \Big).$$

Hence, the r.e. contiguous degrees are elementarily definable in $\mathcal{R}$, and generates a definable ideal. In unpublished work, Downey and the author showed that $\mathbf{0'}$ is the join of two contiguous degrees, and hence the generated ideal is non-proper.
Other related work includes the study of totally ω-r.e. and totally ω^ω-r.e. degrees, the array recursive degrees, and some lattice embedding results by Downey, Greenberg, Walk, and Weber. In [7], Downey, Greenberg, Walk and Weber showed that the totally ω-r.e. degrees forms a definable subset of $\mathcal{R}$ (with a non-proper generated ideal, since every contiguous degree was also totally ω-r.e.), while it is still not known if the array recursive degrees are definable. For more details, the reader is referred to [6] and [10].

2. Overview of the construction

We shall build r.e. sets A, B, such that A is non-bounding, and $K \leq_T A \oplus B$. Our notation is standard, and follows Soare [24]. We fix an effective listing $\{(\Phi_i, \Psi_i, X_i, Y_i)\}_{i\in\omega}$ of all 4−tuples where Φ_i, Ψ_i are p.r. functionals, and X_i, Y_i are r.e. sets. We will build r.e. sets A, B, C, and Turing functional Γ to satisfy the requirements

$\mathcal{P}_e : A \neq \overline{W_e}$ *(A is non-recursive)*

$\mathcal{R}_e : (\Phi_e^A = X_e) \wedge (\Psi_e^A = Y_e) \wedge (X_e, Y_e$ are both non-recursive$)$
$\quad \Rightarrow (\exists$ an r.e. $D_e)(D_e \leq_T X_e \ \wedge \ D_e \leq_T Y_e \ \wedge \ D_e$ is non-recursive$)$
$\quad$ *(non-bounding strategy)*

$\mathcal{N}_e : \Phi_e^B \neq C$ *(incompleteness of B),*

and such that $\Gamma(A \oplus B) = K$.

For any r.e. set Z, we write $x \searrow Z_s$ to mean that $x \in Z_s - Z_{s-1}$. We will assume that for all x, e and s,

$$x \searrow X_{e,s} \Rightarrow \Phi_{e,s}^{A_s}(x) = 1 \qquad \text{and} \qquad x \searrow Y_{e,s} \Rightarrow \Psi_{e,s}^{A_s}(x) = 1,$$

since we are only interested in the $\mathcal{R}_e$'s where the premise holds. Fix an enumeration $\{K_s\}$ of K, in which $\forall s \left|\{x \mid x \searrow K_s\}\right| \leq 1$, and $\exists^\infty s(K_s = K_{s-1})$. We also define the length agreement

$$\begin{aligned} l^{\Phi_e}(s) &= \max\{x \mid (\forall y < x)(X_{e,s}(y) = \Phi_{e,s}^{A_s}(y))\}, \\ l^{\Psi_e}(s) &= \max\{x \mid (\forall y < x)(Y_{e,s}(y) = \Psi_{e,s}^{A_s}(y))\}, \\ l^e(s) &= \min\{l^{\Phi_e}(s), l^{\Psi_e}(s)\}. \end{aligned}$$

3. Strategy of a single requirement

For the requirements $\mathcal{P}_e$ and $\mathcal{N}_e$, the individual strategy is the standard one. In the case of $\mathcal{P}_e$ (*positive requirement*), we wait for an element $x \searrow W_{e,s}$, then we will put $x \searrow A_s$, otherwise we keep x out of A. In the case of $\mathcal{N}_e$ (*negative requirement*), we wait for $\Phi_{e,s}^{B_s}(x) \downarrow= 0$, then we will put x into C, and preserve the computation $\Phi_{e,s}^{B_s}(x)$ by keeping elements out of B. At the end of every stage s in the construction, we will move one step closer to building $\Gamma(A \oplus B) = K$, by either extending the definition of Γ_s, or by correcting any wrong approximation made by Γ_s, due to changes in K (i.e. some $x \searrow K_s$). To this end, we will maintain a set of markers $\{\gamma(n, s)\}$, and $\gamma(n)$ will be enumerated into B for the sake of the correctness of $\Gamma(A \oplus B; n)$. Finally, for the requirement $\mathcal{R}_i$, we will subdivide it into infinitely many sub-requirements

$$\mathcal{R}_{i,j} : D_i \neq \overline{W_j}$$

The requirement $\mathcal{R}_i$ will construct an r.e. set D_i below X_i and Y_i, and attempt to satisfy all sub-requirements $\mathcal{R}_{i,j}$ for every j. Either it will succeed in doing so, or else some $\mathcal{R}_{i,j}$ fails, and we can make use of this to compute either X_i or Y_i. Fix a j, and we will now describe the strategy of a single sub-requirement $\mathcal{R}_{i,j}$. Let x denote the current witness that $\mathcal{R}_{i,j}$ is working on, and $r_1(s), r_2(s)$ be the restraint imposed at stage s by X_i and Y_i respectively. Let $r(s) := \max\{r_1(s), r_2(s)\}$. The action of $\mathcal{R}_{i,j}$ consists of the following steps:

(Step 1): Wait for a stage s such that $x \searrow W_{j,s}$. If no such stage exists, then $\mathcal{R}_{i,j}$ is satisfied. Otherwise, set $r_1(s+1) = 0$, and go to step 2. This action is also called *opening of an X-gap*.

(Step 2) : Wait for the next stage $t > s$, where $l^{\Phi_i}(t) > x$. This will happen if the premise in $\mathcal{R}_i$ holds. If $X_{i,s}\restriction_x \neq X_{i,t}\restriction_x$, then we *close the X-gap successfully* by performing the following : Set $r_2(t+1) = 0$ (*open a Y-gap*), and go to step 3. Otherwise if $X_{i,s}\restriction_x = X_{i,t}\restriction_x$,

then we *close the X-gap unsuccessfully* by defining $r_1(t+1) = t$, reset x by choosing another witness $> t+1$, and return back to step 1.

(Step 3) : Wait for the next stage $u > t$, where $l^{\Psi_i}(u) > x$. If $Y_{i,t}\upharpoonright_x \neq Y_{i,u}\upharpoonright_x$, then we *close the Y-gap successfully* by enumerating x into D_i, and then halt. This satisfies $\mathcal{R}_{i,j}$. Otherwise if $Y_{i,t}\upharpoonright_x = Y_{i,u}\upharpoonright_x$, then we *close the Y-gap unsuccessfully* by defining $r_2(u+1) = u$, reset x by choosing another witness $> u+1$, and return back to step 1.

Note that if there are only finitely many opening of X-gaps and Y-gaps, then $\mathcal{R}_{i,j}$ is satisfied, and $\lim_s r(s) < \infty$. Furthermore, any element x that enters D_i can only do so if it obtains permission from both X_i and Y_i, and hence $D_i \leq_T X_i$ and $D_i \leq_T Y_i$. Hence $\mathcal{R}_i$ will be satisfied, if $\mathcal{R}_{i,j}$ is satisfied for all j. Suppose that there are infinitely many Y-gaps, then $\liminf_s r(s) = 0$ and for any y, $Y_{i,s}\upharpoonright_y = Y_i\upharpoonright_y$ for the least stage s such that the witness $x(s) > y$ and a Y-gap is open. Hence, Y_i is recursive. If there are finitely many Y-gaps but infinitely many X-gaps, then $\liminf_s r(s) = \lim_s r_2(s) < \infty$, and X_i is recursive. Therefore, $\mathcal{R}_i$ is satisfied even when $\mathcal{R}_{i,j}$ fails for some j.

4. Interaction of strategies

Now we look at the possible conflicts amongst the different requirements. The non-bounding strategy may impose an unbounded restraint on the nodes below it if X or Y is recursive, thus there is a need for the strategy to enter the gap stages, in which the nodes (on the true path) below it has only a finite restraint (on A) to work with. These gap stages provide an opportunity for the strategies below the node working on the non-bounding strategy to enumerate numbers into A.

The end of stage action will maintain the correctness of the functional $\Gamma(A \oplus B)$, and whenever a correction needs to be taken, we will put $\gamma(n, s)$ into B to make $\Gamma_s(A \oplus B; n) \uparrow$, in preparation for it to receive a new definition later. The problem is that if we *always* put numbers into B whenever a K-change is observed, we might end up coding K into B, and hence there is no chance for the $\mathcal{N}$-strategies (incompleteness strategies) to work. To prevent this situation, we will require both A and B to be involved in carrying the information contained in K. The important point here is that the end of stage actions taken to construct Γ has the highest priority over any strategy on the tree. So, whenever an incompleteness strategy observes a B-computation and wants to preserve it, the strategy can only try to limit the damage done to itself by the end of stage actions. This is done as follow:

Each incompleteness node $\alpha = \mathcal{N}_e$ will choose a number $n(\alpha, s)$, in which it believes that $K_s\restriction_{n(\alpha,s)} = K\restriction_{n(\alpha,s)}$. Whenever α witnesses $\Phi^{B_s}_{e,s}(z)\downarrow$, it will attempt a diagonalization by enumerating z into C, move all markers $\gamma(p, s)$ for $p \geq n(\alpha, s)$ to values larger than $u :=$ the use of $\Phi^{B_s}_{e,s}(z)$. In order to do this, α will enumerate $\gamma(n(\alpha, s), s)$ into A, and trust that $K_s\restriction_{n(\alpha,s)} = K\restriction_{n(\alpha,s)}$. In future, this diagonalization will be preserved, unless the end of stage action puts in some number $< u$, which can only happen if $K_s\restriction_{n(\alpha,s)} \neq K\restriction_{n(\alpha,s)}$. At the next visit to α, we will give up the diagonalization attempt, and start on a new one by choosing a new witness z', and wait for $\Phi^{B_t}_{e,t}(z')\downarrow$ again. This new attempt can once again be ruined, but only if $K_t\restriction_{n(\alpha,s)} \neq K\restriction_{n(\alpha,s)}$. Hence, there will be at most $n(\alpha, s)$ many failed attempts.

Note that the incompleteness strategies will make enumerations into A in an attempt to guarantee their success, so that in future, the enumerations made by the end of stage action can only ruin them finitely often. Hence, for each x, the final coding location $\lim_{s\to\infty} \gamma(x, s)$ depends on *both* A and B.

5. Priority tree layout

Our requirements will be laid out on the priority tree, where the nodes are ordered by the standard ordering $<_L$:

- For a node α on the tree, $|\alpha| = 4e$, we assign the requirement $\mathcal{P}_e$, with possible outcomes *succ* (success) $<_L$ w (waiting).
- If $|\alpha| = 4e + 1$, we have the requirement $\mathcal{N}_e$ with possible outcomes $0 <_L 1 <_L \cdots$ indicating the restraint imposed by the negative requirement $\mathcal{N}_e$ on B.
- For $|\alpha| = 4e + 2$, we will assign it to some $\mathcal{R}_{i(\alpha)}$ which attempts to guess if $(\Phi^A_{i(\alpha)} = X_{i(\alpha)}) \wedge (\Psi^A_{i(\alpha)} = Y_{i(\alpha)})$ holds. The outcomes are 0 (stands for infinitely many expansionary stages) $<_L$ 1 (finitely many expansionary stages).
- For $|\alpha| = 4e + 3$, we assign it some sub-requirement $\mathcal{R}_{i(\alpha),j(\alpha)}$. The possible outcomes are : *succ* (success in putting some x in $D_{i(\alpha)}$) $<_L$ g_2 (infinitely many Y-gaps) $<_L$ g_1 (infinitely many X-gaps) $<_L$ w (waiting for $x \searrow W_{j(\alpha)}$).

To complete the description of the priority tree, we need to define the functions i, j. We use the functions L_0, L_1 (where for all $\alpha \in dom(L_0)$, $L_0(\alpha), L_1(\alpha) \subseteq \omega$) to help us. For $|\alpha| = 2$, define $i(\alpha) = 0$, and let $L_0(\alpha) =$

$L_1(\alpha) = \omega$. If $|\alpha| = 4n$ or $4n+1 (n > 0)$, we let $i(\alpha)\uparrow, j(\alpha)\uparrow$, $L_0(\alpha)\uparrow$, and $L_1(\alpha)\uparrow$. If $|\alpha| = 4n+2$ or $4n+3$, we let β be the maximal node such that $\beta \prec \alpha$ and $|\beta| = 4n'+2$ or $4n'+3$ for some n'. Let $\beta^\wedge a \preceq \alpha$, and we define $L_0(\alpha), L_1(\alpha)$ as follow:

- $a \in \{1, g_1, g_2\}$:

$$L_0(\alpha) = \Big(L_0(\beta) \setminus \{i(\beta)\}\Big) \cup \Big\{k \mid k > i(\beta)\Big\},$$
$$L_1(\alpha) = \Big(L_1(\beta) \setminus \{\langle i(\beta), m\rangle \mid m \in \omega\}\Big) \cup \Big\{\langle k, m\rangle \mid k > i(\beta), m \in \omega\Big\}.$$

- $a = 0$:

$$L_0(\alpha) = L_0(\beta) \setminus \{i(\beta)\},$$
$$L_1(\alpha) = L_1(\beta).$$

- $a \in \{succ, w\}$:

$$L_0(\alpha) = L_0(\beta),$$
$$L_1(\alpha) = L_1(\beta) \setminus \{\langle i(\beta), j(\beta)\rangle\}.$$

If $|\alpha| = 4n+2$, then define $i(\alpha) = \min L_0(\alpha)$ and $j(\alpha)\uparrow$. If $|\alpha| = 4n+3$, then define $i(\alpha) = i', j(\alpha) = j'$, where $\langle i', j'\rangle = \min L_1(\alpha)$.

For convenience of notation, we will say that a node $\alpha = \mathcal{P}_e$ if $|\alpha| = 4e$, and $\alpha = \mathcal{N}_e$ if $|\alpha| = 4e+1$. Similarly, we say that $\alpha = \mathcal{R}_i$ if $|\alpha| = 4e+2$ for some e, and $i(\alpha) = i$. Also, say that $\alpha = \mathcal{R}_{i,j}$ if $|\alpha| = 4e+3$ for some e, $i(\alpha) = i$, and $j(\alpha) = j$. We say that $\alpha \neq \mathcal{P}_e$ (similarly for $\mathcal{N}_e, \mathcal{R}_i, \mathcal{R}_{i,j}$), if $\forall k(|\alpha| \neq 4k)$. We let

$$\tau(\alpha) = (\mu\beta \preceq \alpha)\Big(\beta = \mathcal{R}_{i(\alpha)} \wedge$$
$$(\neg\exists\gamma)(\beta \preceq \gamma \prec \alpha)\Big[i(\gamma) < i(\alpha) \quad \wedge \quad \alpha(|\gamma|) \in \{1, g_1, g_2\}\Big]\Big),$$

and if no such β exists, let $\tau(\alpha)\uparrow$. For any infinite path h, and $i \in \omega$, let $\tau(h, i) =$ maximal node α such that $\alpha \prec h$ and $\alpha = \mathcal{R}_i$, and let $E(h, i) = \{\beta \mid \beta \succeq \tau(h, i) \quad \wedge \quad \tau(\beta)\downarrow = \tau(h, i)\}$.

It follows by a simple induction that for every infinite path h and every $i \in \omega$, $(\exists^{<\infty}\alpha \prec h)(\alpha = \mathcal{R}_i)$. Thus it follows that either $\forall n(h(n) \neq \mathcal{R}_i)$, or else $\tau(h, i)\downarrow$. If $\alpha = \mathcal{R}_j$, then $\tau(\alpha) = \alpha$, and it follows that $\alpha = \mathcal{R}_j \wedge \alpha \in E(h, i) \Rightarrow \alpha = \tau(h, i)$.

6. The construction

We shall construct r.e. sets A, B, C and Turing functional Γ, with A_s, B_s, C_s, Γ_s to denote the finite sets of elements enumerated so far at stage s. For a node α on the tree, let $\langle\alpha\rangle$ be the number assigned to α under some effective coding of the tree. $\gamma(x, s)$ will denote the approximation at stage s, of the final use $u(A \oplus B; x)$, and the current state of the module α is denoted by $F(\alpha, s)$. We will now state down the rest of the parameters used in the construction.

- For a node $\alpha = \mathcal{P}_e$, let $r(\alpha, s) = 0$ (restraint on A contributed by α).
- For each node $\alpha = \mathcal{N}_e$, $n(\alpha, s)$ will mark the location of x in which α believes that $K_s\restriction_x = K\restriction_x$, and $z(\alpha, s)$ denotes the witness z at stage s that attempts to make $\Phi_e^B(z) \neq C(z)$. We let $r(\alpha, s) = 0$ (since $\mathcal{N}_e$ only keeps elements out of B.
- At each node $\alpha = \mathcal{R}_i$, the r.e. set D_α will be formed by elements contributed by the nodes $\{\beta \mid \beta \succeq \alpha \;\wedge\; \tau(\beta)\downarrow = \alpha\}$. We always define $r(\alpha, s) = 0$.
- For a node $\alpha = \mathcal{R}_{i,j}$, we let $r_1(\alpha, s)$ and $r_2(\alpha, s)$ denote the restraint put up at stage s by X_i and Y_i respectively, in the attempt to make X_i and Y_i computable. We let $r(\alpha, s) = \max\{r_1(\alpha, s), r_2(\alpha, s)\}$, and let $x(\alpha, s)$ be the current witness of the basic module $\mathcal{R}_{i,j}$.

All parameters will remain in force until re-assigned (or initialized). Hence, we may drop s from the notation without ambiguity, and refer to the parameters as $\gamma(x), p(\alpha), n(\alpha), A, B, C$, etc. Define the restraint function $\bar{r}(\alpha, s) = \max\{r(\beta, s) \mid \beta \leq_L \alpha\}$.

To *reset the witness* $x(\alpha)$ (and $z(\alpha)$) at stage s, is to re-define $x(\alpha, s)$ (similarly $z(\alpha, s)$) to be the least $x \in \omega^{[\alpha]}, x > s$, and $x > x(\alpha, s-1)$.

To *reset* $\gamma(x)$ *above* y at stage s is to do the following. Cancel the existing value of $\gamma(x)$, and redefine $\gamma(x)$ to be the least value in $\omega^{[x]}$ larger than $y, \max A_s, \max B_s$ and all the previous values of $\gamma(x)$. Next, for each $z > x$, cancel the existing value of $\gamma(z)$ and redefine $\gamma(z)$ to be the least value in $\omega^{[z]}$ larger than $\max\{y, \max A_s, \max B_s, \gamma(x), \cdots, \gamma(z-1)\}$ and all previous values of $\gamma(z)$.

To *initialize a node* α at stage s means the following. If $\alpha = \mathcal{P}_e$ and $s = 0$, we set $F(\alpha) = w$. If $\alpha = \mathcal{N}_e$, we set $n(\alpha)$ to be the least $x > n(\alpha, s-1)$, $x \in \omega^{[\alpha]}$, such that $\gamma(x) > \bar{r}(\alpha)$, reset $z(\alpha)$, and set $F(\alpha) = 0$. For $\alpha = \mathcal{R}_i$, we will remove any link with top α. Lastly, if $\alpha = \mathcal{R}_{i,j}$, we will reset $x(\alpha)$, set $r_1(\alpha) = r_2(\alpha) = 0$, remove any links with bottom α, and if $F(\alpha) \neq succ$ we set $F(\alpha) = w$.

The construction will proceed by induction on stage s. If $s = 0$, we initialize all nodes on the tree, reset $\gamma(0)$ above 0, and do nothing else. For each stage $s > 0$, we will define $\delta_{s,t}$ at each substage $t < s$, and state the action of the node $\delta_{s,t}$. We will have $\delta_{s,0} \prec \cdots \prec \delta_{s,s-1} = \delta_s$. We sometimes refer to a substage t of a stage s by (s,t), and order the substages (s,t) lexicographically. We will say that a node α is *visited* at substage (s,t), if $\delta_{s,t} = \alpha$, and that α is visited at stage s, if it is visited at some substage of s. For a node $\alpha = \mathcal{R}_i$, we say that a stage $s > 0$ is α-*expansionary*, if $l^i(s) > \max\{l^i(t) \mid t < s \ \wedge \ \alpha \text{ is visited at stage } t\}$. Let $s > 0$, and proceed with the construction as follow.

(Substage $t = 0$) : Define $\delta_{s,0} = \mathcal{P}_0$, and check if $F(\mathcal{P}_0) = w$, and there is some $y > \gamma(\langle \mathcal{P}_0 \rangle)$ such that $y \in W_0 \cap \omega^{[\mathcal{P}_0]}$. If there is such y, enumerate the least such into A, initialize all $\beta >_L \mathcal{P}_0$, set $F(\mathcal{P}_0) = succ$, reset $\gamma(z)$ (where z is the least such that $y \leq \gamma(z)$) above 0, and go to the next substage. If there is no such y, or $F(\alpha) = succ$, proceed to the next substage with no action needed.

(Substage $0 < t < s$) : Assume that $\delta_{s,t-1}$ has been defined, and its action taken. Define $\alpha = \delta_{s,t}$ by the following : if $\delta_{s,t-1} = \mathcal{R}_i, F(\delta_{s,t-1}) = 0$, and there exists a link $(\delta_{s,t-1}, \beta)$ for some β, then let $\alpha = \beta$, otherwise let $\alpha = \delta_{s,t-1}{}^\wedge F(\delta_{s,t-1})$. The corresponding action to be taken by α is listed below.

$(\alpha = \mathcal{P}_e)$: If $F(\alpha) = w$, and there is some y such that $y \in W_e \cap \omega^{[\alpha]}$ with $y > \max\{\bar{r}(\alpha), \gamma(\langle \alpha \rangle)\}$, enumerate the least such y into A, initialize all $\beta >_L \alpha$, set $F(\alpha) = succ$, reset $\gamma(z)$ (where z is the least such that $y \leq \gamma(z)$) above 0, and go to the next substage. Otherwise, go to the next substage with no action needed.

$(\alpha = \mathcal{N}_e)$: There are four possibilities, and their corresponding actions are listed below.

$(\mathcal{N}.1)$: If $F(\alpha) = 0$ and $\Phi_{e,s}^{B_s}(z(\alpha)) \downarrow = 0$, then we will perform the following actions: enumerate $z(\alpha)$ into C, make $\Gamma_s(A_s \oplus B_s; y) \uparrow$ (for all $y \geq n(\alpha)$) by enumerating $\gamma(n(\alpha))$ into A, reset $\gamma(n(\alpha))$ above u, where u is the use of $\Phi_{e,s}^{B_s}(z(\alpha))$, set $F(\alpha) = u$, and initialize all $\beta >_L \alpha$.

$(\mathcal{N}.2)$: If we have $F(\alpha) = 0$ and $\Phi_{e,s}^{B_s}(z(\alpha)) \neq 0$, we proceed to the next substage with no action.

$(\mathcal{N}.3)$: Suppose that $F(\alpha) > 0$, and there is some $y < n(\alpha)$ such that $y \searrow K_{u'}$ for some $u \leq u' < s$, and u is the previous stage in which α is visited. Then, reset $z(\alpha)$ and set $F(\alpha) = 0$.

$(\mathcal{N}.4)$: Suppose that $F(\alpha) > 0$, and $K_{u-1}\restriction_{n(\alpha)} = K_{s-1}\restriction_{n(\alpha)}$. Then, proceed to the next substage with no action.

$(\alpha = \mathcal{R}_i)$: If stage s is α-expansionary, set $F(\alpha) = 0$, otherwise set $F(\alpha) = 1$.

$(\alpha = \mathcal{R}_{i,j})$: Choose the first clause from the following list $(\mathcal{R}.1)$ – $(\mathcal{R}.3)$ that applies, and perform the action stated.

$(\mathcal{R}.1)$: α is ready to open an X-gap : that is, $\tau(\alpha) \downarrow$, $F(\alpha) = w$, $x(\alpha) \in W_{j,s}$, and $x(\alpha) < l^i(s)$. The action to be taken is to open an X-gap by setting $r_1(\alpha) = 0$, $F(\alpha) = g_1$, initialize all $\beta \geq_L \alpha^\wedge w$, and create a link $(\tau(\alpha), \alpha)$.

$(\mathcal{R}.2)$: α is ready to close an X-gap : that is, $F(\alpha) = g_1$ and s is $\tau(\alpha)$-expansionary. Let $u < s$ be the stage where the current X-gap was opened. If $X_{i,u}\restriction_{x(\alpha)} \neq X_{i,s}\restriction_{x(\alpha)}$ we close the X-gap successfully and open a Y-gap by setting $r_2(\alpha) = 0$, $F(\alpha) = g_2$, and initializing all nodes $\beta \geq_L \alpha^\wedge g_1$. Otherwise if $X_{i,u}\restriction_{x(\alpha)} = X_{i,s}\restriction_{x(\alpha)}$, we close the X-gap unsuccessfully by setting $r_1(\alpha) = s$, $F(\alpha) = w$, resetting $x(\alpha)$, initializing all nodes $\beta \geq_L \alpha^\wedge w$, and removing the link $(\tau(\alpha), \alpha)$. If the X-gap was closed unsuccessfully, go directly to substage s, and set $\delta_s = \delta_{s,t}$.

$(\mathcal{R}.3)$: α is ready to close a Y-gap : $F(\alpha) = g_2$ and s is $\tau(\alpha)$-expansionary. As above, let $u < s$ be the stage where the current Y-gap was opened. If $Y_{i,u}\restriction_{x(\alpha)} \neq Y_{i,s}\restriction_{x(\alpha)}$ we close the Y-gap successfully by enumerating $x(\alpha)$ into $D_{\tau(\alpha)}$, setting $F(\alpha) = succ$, $r_1(\alpha) = r_2(\alpha) = 0$, and initializing all $\beta \geq_L \alpha^\wedge g_2$. Otherwise if $Y_{i,u}\restriction_{x(\alpha)} = Y_{i,s}\restriction_{x(\alpha)}$, we close the Y-gap unsuccessfully by setting $F(\alpha) = w$, $r_2(\alpha) = s$, resetting $x(\alpha)$, and initializing all nodes $\beta \geq_L \alpha^\wedge g_1$. In either case, we remove the link $(\tau(\alpha), \alpha)$, set $\delta_s = \delta_{s,t}$, and go directly to substage s.

If none of the clauses $(\mathcal{R}.1)$ – $(\mathcal{R}.3)$ holds, do nothing and proceed to the next substage.

(Substage $t = s$) : If there is some $y \searrow K_s$, we enumerate $\gamma(y)$ into B and reset $\gamma(y)$ above 0. Otherwise if $K_s = K_{s-1}$, we pick the least y such that $\Gamma_s(A_s \oplus B_s; y) \uparrow$, and set $\Gamma_s(A_s \oplus B_s; y) \downarrow = K_s(y)$ with use $2\gamma(y) + 1$.

This ends the construction.

7. Verification

We will now state some straightforward properties exhibited by the links, which can be shown by induction.

- If (ρ, γ) is a link that had been formed in the construction, then necessarily we must have $\gamma \succeq \rho^\frown 0$, and also

$$\nexists \sigma \Big(\rho \preceq \sigma \preceq \gamma \ \wedge \ \sigma = \mathcal{R}_j \text{ for some } j < i(\rho) \Big), \tag{7.1}$$

$$\forall \sigma \Big((\sigma \succeq \gamma^\frown g_1 \vee \sigma \succeq \gamma^\frown g_2) \wedge i(\sigma) \geq i(\rho) \wedge \tau(\sigma) \downarrow \ \Rightarrow \ \tau(\sigma) \succ \gamma \Big). \tag{7.2}$$

- It is also clear that for any node $\sigma \preceq \delta_s$, a link (ρ, γ) exists, and is travelled during stage s for some $\rho \prec \sigma \prec \gamma \Leftrightarrow \sigma$ is not visited during stage s.
- Combining the above fact with (7.1) and (7.2), we see that any links that exist simultaneously may be nested, but never crossing.
- Therefore for any node $\sigma \preceq \delta_s$, σ is visited at stage $s \Leftrightarrow \nexists$ a link (ρ, γ) at substage $(s, 0)$ with $\rho^\frown 0 \preceq \sigma$ and $\gamma \npreceq \sigma$. That is, any link (ρ, γ) existing at the beginning of stage s with $\rho^\frown 0 \preceq \sigma$, must have bottom $\gamma \preceq \sigma$.
- Any link that is created, can be travelled at most twice before it is removed. Also, any link that is removed must be travelled in the same stage.
- Nested links are removed from outermost inwards.
- No link may be formed and travelled in the same stage.
- If a link (ρ, γ) is formed during the construction, then every node α such that $\rho \prec \alpha^\frown g_1 \preceq \gamma$ or $\rho \prec \alpha^\frown g_2 \preceq \gamma$ must have a gap open at the instance of formation, and the gap will remain open until after the link (ρ, γ) is removed. Therefore at any stage s, every node α such that $\alpha^\frown g_1 \preceq \delta_s$ or $\alpha^\frown g_2 \preceq \delta_s$ must have a (X- or Y-gap, respectively) open at the end of stage s.

The true path (TP) of the construction is defined to be $TP(n) = \liminf_s F(TP\upharpoonright_n)$, for all n. The $\liminf_s F(\alpha)$ always exists for any node α, because α only has finitely many possible outcomes, with the exception of $\alpha = \mathcal{N}_e$, where we require that $F(\alpha, t) < F(\alpha, s) \Rightarrow \exists u(t \leq u < s \ \wedge \ F(\alpha, u) = 0)$.

Lemma 7.1. *The true path is the leftmost path visited infinitely often. That is for any n, there are infinitely many stages s, where $TP\upharpoonright_n$ is visited, and $\exists^{<\infty} s(\delta_s <_L TP\upharpoonright_n)$.*

Proof. The lemma obviously holds when $n = 0$. Let $n > 0$, and assume the lemma holds for $\alpha = TP\upharpoonright_{n-1}$. Let $a = TP(n-1)$, and choose a stage s_0 such that $\forall s > s_0(\delta_s \not<_L \alpha \wedge F(\alpha, s) \not<_L a)$. For any $s_1 > s_0$, if $\delta_{s_1} <_L \alpha^\wedge a$, then there must be a link (ρ, γ) with $\rho \prec \alpha$ and $\alpha^\wedge b \preceq \gamma$ for some $b <_L a$, which is travelled during stage s. Since α is visited infinitely often, there will be a $s_2 \geq s_1$ where α is visited. Hence by induction, we have for all $s \geq s_2$, we have $\delta_s \not<_L \alpha^\wedge a$, and there will be no link (ρ', γ') with $\rho' \prec \alpha$ and $\alpha^\wedge b \preceq \gamma'$ for any $b <_L a$ existing at the end of stage s.

To show that $\alpha^\wedge a$ is visited infinitely often, we first show that α can only be initialized finitely often. To see this, consider a stage $s_3 > s_0$, in which no $\beta \prec \alpha$ is initialized after stage s_3. After stage s_3, α can only be initialized by α^- where $\alpha = (\alpha^-)^\wedge b$ for some b. The case $\alpha^- = \mathcal{P}_e$ or $\mathcal{R}_e$ is trivial, and if $\alpha^- = \mathcal{N}_e$, we have $n(\alpha^-, t) = n(\alpha^-, s_3)$ for all $t \geq s_3$, and subsequently α can only be initialized by α^- at most $n(\alpha^-, s_3) + 1$ many times. If $\alpha^- = \mathcal{R}_{i,j}$ initializes α at infinitely many stages $t \geq s_3$, then there must be infinitely many stages $t' \geq s_3$ in which $F(\alpha^-, t') <_L b$. Therefore, we can let $s_4 > s_0$ be a stage after which α is never initialized, α is visited at stage s_4, where $F(\alpha, s_4)$ is set to a due to the action of α. We may assume that there is some link (α, γ) existing at stage s_4, and that $a = 0$, because otherwise $\alpha^\wedge a$ will be visited at stage s_4. This link will be travelled at most twice before it is removed at some stage $s_5 \geq s_4$. Let $s_6 > s_5$ be the least stage such that α is visited, and $F(\alpha, s_6)$ is set to 0. Then for all $s_5 < u < s_6$, we have $\delta_u \not\succeq \alpha^\wedge 0$, and there is no link (ρ', γ') with $\rho' \prec \alpha^\wedge 0 \preceq \gamma'$ existing at the end of stage u. Thus, $\alpha^\wedge 0$ will be visited at stage s_6. □

Lemma 7.2.

(i) *For each node $\alpha \neq \mathcal{R}_{i,j}$ on the true path, we have*

$$\lim_{s\to\infty} \{\bar{r}(\alpha, s) \mid \alpha \textit{ is visited at stage } s\} < \infty.$$

(ii) For all x, $\lim_{s\to\infty} \gamma(x,s) < \infty$.

Proof.
(i) Firstly we note that for any $\beta = \mathcal{R}_{i,j}$ on the true path, we have

$$TP(|\beta|) \in \{succ, w\} \Rightarrow \exists r_0 \forall^\infty s\big(r(\alpha,s) = r_0\big), \tag{7.3}$$
$$TP(|\beta|) = g_1 \Rightarrow \exists r_1 \forall^\infty s\big(r(\alpha,s) = r_1 \text{ when an } X\text{-gap is open}\big), \tag{7.4}$$
$$TP(|\beta|) = g_2 \Rightarrow \forall^\infty s\big(r(\alpha,s) = 0 \text{ whenever a } Y\text{-gap is open}\big). \tag{7.5}$$

Now fix $\alpha \neq \mathcal{R}_{i,j}$ on the true path. Let s_0 be a stage such that $\forall s > s_0$, $\delta_s \not<_L \alpha$ and the relevant clauses within the brackets on the right side of the implications in (7.3), (7.4) and (7.5) hold for every $\beta \prec \alpha$. For any $s > s_0$ in which α is visited, we have $r(\alpha,s) = 0$, and every $\beta \prec \alpha$ such that $TP(|\beta|) = g_1$ or g_2 will have a X- or respectively Y-gap open. Hence $\lim_{s\to\infty} \{\bar{r}(\alpha,s) \mid \alpha \text{ is visited at stage } s\}$ exists.

(ii) During the construction, only the action taken by some node $\alpha = \mathcal{P}_e$ or $\mathcal{N}_e$, or the action at the end of a stage can move $\gamma(x)$. For a particular x, we choose s_0 so that $K_{s_0}\restriction_{x+1} = K\restriction_{x+1}$, and for all $s > s_0$, if $\alpha = \mathcal{P}_e$ and $\langle\alpha\rangle < x$, then α does not enumerate any element into A_s. Therefore after stage s_0, only the action of finitely many nodes α (where $\alpha = \mathcal{N}_e$) can move $\gamma(x)$. Let α be such a node, and let $n(\alpha,s) \le x$ for some $s > s_0$. If α is never initialized after stage s, then α will reset $\gamma(n(\alpha))$ at most $n(\alpha)+1$ many times. □

Lemma 7.3. *Along the true path, the requirements succeed.*

Proof. First consider the requirement $\mathcal{P}_e : A \neq \overline{W_e}$: let $\alpha = TP\restriction_{4e}$, and if $TP(4e) = succ$ then $A \cap W_e \neq \emptyset$. Thus, we suppose that $TP(4e) = w$. At all times in the construction, we have

$$\gamma(x) \text{ value is reassigned } \Leftrightarrow \exists y \le \gamma(x) \ \ (y \text{ is enumerated into } A \text{ or } B). \tag{7.6}$$

We claim that $A^{[\alpha]}$ is finite, since the only nodes that can enumerate elements into $A^{[\alpha]}$ is the node α itself, as well as the node β such that $\beta = \mathcal{N}_i$ and $\langle\alpha\rangle \in \omega^{[\beta]}$. Now, α contributes no elements to $\omega^{[\alpha]}$, while β can only put finitely many elements into $\omega^{[\alpha]}$ (by Lemma 7.2(ii), and (7.6)). On the other hand, $W_e^{[\alpha]}$ is also finite because of Lemma 7.2. Hence, $A \neq \overline{W_e}$.

$\mathcal{R}_e$ - *Non-bounding Strategy* : assume that $\Phi_e^A = X_e$, $\Psi_e^A = Y_e$, and X_e, Y_e are both non-recursive. We first show that there is no node α such that $\exists j(\alpha = \mathcal{R}_{e,j})$ and $\alpha^\wedge g_1 \prec TP$. Suppose such α exists, then $\tau(\alpha)\downarrow$ and we let s_0 be a stage after which α is never initialized, and $\forall s > s_0(\delta_s \not<_L \alpha)$. To compute $X_e(y)$, we wait for a stage $s > s_0$ such that $x(\alpha, s) > y$, α is visited at stage s, where a X-gap is opened. Then, $X_{e,s}\restriction_{y+1} = X_e\restriction_{y+1}$ because during any stage $t > s$ where α has no open X-gap, we must have $\delta_t \not\succ \alpha^\wedge g_1$. The same argument also shows that there is no node α such that $\exists j(\alpha = \mathcal{R}_{e,j})$ and $\alpha^\wedge g_2 \prec TP$. Therefore $\tau := \tau(TP, e)\downarrow$, and there is no β such that $\tau \prec \beta \prec TP$, $i(\beta) < e$ and $TP(|\beta|) \in \{1, g_1, g_2\}$. We also have $\tau^\wedge 0 \prec TP$, and for almost all j, there is some α such that $\alpha = \mathcal{R}_{e,j}$, $\alpha \prec TP$, and $\alpha \in E(TP, e)$. Therefore, we can conclude that the r.e. set D_τ is non-recursive. It only remains to show that $D_\tau \leq_T X_e$ and $D_\tau \leq_T Y_e$. A X_e-recursive test of whether $y \in D_\tau$ is the following. Wait for a stage s_0 such that $x(\alpha, s_0) = y$ for some $\alpha \in E(TP, e)$, and $X_{e,s_0}\restriction_y = X_e\restriction_y$. If no X-gap is open at stage s_0 then $x \notin D_\tau$, otherwise we wait for stage $s_1 > s_0$ where all gaps (X or Y) are closed, and we have $D_\tau(y) = D_{\tau,s_1}(y)$. A similar argument is used for $D_\tau \leq_T Y_e$.

$\mathcal{N}_e$ - $\Phi_e^B \neq C$: let $\alpha = TP\restriction_{4e+1}$, and let s_0 be a stage after which α is never initialized, $\forall s > s_0(\delta_s \not<_L \alpha)$, $K_{s_0}\restriction_{n(\alpha,s_0)} = K\restriction_{n(\alpha,s_0)}$, and α is visited at stage $s_0 + 1$. Let $s_1 > s_0 + 1$ be the next stage where α is visited, and let $z := z(\alpha, s_1) = \lim_{t\to\infty} z(\alpha, t)$. Thus we have

$$TP(|\alpha|) = 0 \Rightarrow z \notin C \ \wedge \ \Phi_e^B(z) \neq 0,$$
$$TP(|\alpha|) > 0 \Rightarrow z \in C \ \wedge \ \Phi_e^B(z) = 0,$$

which satisfies the requirement $\mathcal{N}_e$.

$\Gamma(A \oplus B) = K$: fix an x, and by Lemma 7.2 and (7.6) there is a stage s_0 where we set $\Gamma_{s_0}(A_{s_0} \oplus B_{s_0}; x)\downarrow = K_{s_0}(x)$ with use $2\gamma(x, s_0) + 1$, and $(\forall y \leq x)(\forall s > s_0)\Big(\gamma(y, s) = \lim_{t\to\infty} \gamma(y, t)\Big)$. We have $(A_{s_0} \oplus B_{s_0})\restriction_{2\gamma(x,s_0)+2} = (A \oplus B)\restriction_{2\gamma(x,s_0)+2}$, and thus $\Gamma(A \oplus B; x) = K_{s_0}(x) = K(x)$. □

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