COMPUTABLE TORSION ABELIAN GROUPS

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Abstract. We prove that c.c. torsion abelian groups can be described by a Π^0_4-predicate that describes the failure of a brute-force diagonalisation attempt on such groups. We show that there is no simpler description since their index set is Π^0_4-complete. The results can be viewed as a solution to a 60 year-old problem of Mal′cev in the case of torsion abelian groups. We prove that a computable torsion abelian group has one or infinitely many computable copies, up to computable isomorphism. The result confirms a conjecture of Goncharov from the early 1980s for the case of torsion abelian groups.

1. Introduction

This paper lies within the general area called *computable algebra* which seeks to understand the extent to which classical algebra can be made effective. The fundamental objects of computable algebra are groups, rings, Boolean algebras and other algebraic structures that admit an algorithmic presentation (to be clarified shortly). As a separate area of mathematical endeavour, computable algebra goes back to van der Waerden [vdW30], Dehn [Deh11], Hermann [Her26] and others. Such studies predate the formal definition of an algorithm. In the 1960s, Rabin [Rab60] and Mal′cev [Mal61] used the language of computable function theory ([Soa87, Rog87]) to clarify and extend these early ideas. In particular, Rabin and Mal′cev suggested the following formal definition of an algorithmically presented algebraic structure. A *computable presentation* (a computable copy, a constructivisation) of a countably infinite algebraic structure $\mathcal{A}$ is an isomorphic copy of $\mathcal{A}$ whose domain is a Turing computable set and whose functions, relations, and constants are all Turing computable. Any natural countable algebraic structure encountered by the working mathematician will have a computable copy.

Much of classical algebra is devoted to the classification of structures up to isomorphism. In computable algebra, it is natural to view structures up to computable isomorphism which is of course more fine-grained. For instance, Mal′cev constructed two computably presented torsion-free abelian groups of infinite rank which were not computably isomorphic [Mal61]. Mal′cev also realised that there were structures where the classical and the computable isomorphism types coincided. For example, any two computable copies of the order type of the rationals are computably isomorphic. Mal′cev called such structures *autostable*, but nowadays we call them *computably categorical*. Mal′cev asked a general question: *Which structures are computably categorical?* In fact Mal′cev was mainly interested in abelian groups and specifically asked:

Problem 1.1 (Mal′cev)**.** Describe computably categorical abelian groups[1].

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[1]Downey and Remmel [DR00] incorrectly attribute the question to Goncharov who perhaps was the first to advertise the question outside the USSR. According to Nazif G. Khisamiev (personal communication with the first author), this old question goes back to Mal′cev. See survey [Mel14] for a discussion.

Over the past 60 years, computable categoricity has become one of the central topics in computable structure theory, see e.g. [Rem81, LaR77, Gon97] and books [AK00, EG00]. Although the notion of computable categoricity arose from the example of an abelian group suggested by Mal′cev, *it is still unknown which abelian groups are computably categorical.*

How can we answer such a question? The most pleasing answer would be along the following lines. Remmel [Rem81] proved that a computably presented linear ordering is computably categorical iff it has a finite number of adjacencies. The beauty of Remmel's theorem is that it gives an algebraic invariant to classify computable linear orderings up to to computable isomorphism. Our first hope was that Mal′cev's question might be answered similarly. Down through the years, partial results have supported this hope. In 1974, Nurtazin [Nur74] proved that a torsion-free abelian group is computably categorical iff it has finite rank. Around 1980, Smith [Smi81] and Goncharov [Gon80], proved that an abelian p-group is computably categorical iff it is isomorphic to a direct sum of primary cyclic and quasi-cyclic p-groups, almost all of which have the same isomorphism type, and Goncharov [Gon80] established that an abelian group of infinite rank is never computably categorical. Logicians would call descriptions like those above semantic since they specify algebraic invariant properties of the structures.

Since around 1980, progress has more or less stopped, but these early investigations covered three large subclasses of abelian groups, leaving only two cases where the problem remains open:

(i) torsion abelian groups,
(ii) mixed abelian groups of finite rank.

In the present paper we concentrate on (i). More specifically we attack the following question:

Which torsion abelian groups are computably categorical?

We will see that a semantic (algebraic) description, like the ones above, is highly unlikely. So we will look towards another method of classification.

How can we illustrate that an algebraic description is impossible, or at least is highly unlikely? For that, we need to isolate a general enough property that unites all known examples in the literature where an algebraic description *is* known. In each of these examples computable categoricity is equivalent to *relative computable categoricity*; that is, any isomorphic (not necessarily computable) copy B of the computable structure A is isomorphic to A via a B-computable isomorphism [AK00]. What is so special about relative computable categoricity? Using forcing, we can express relative computable categoricity as an *internal* syntactical property of the structure [AK00]. In contrast, "plain" computable categoricity is a computability-theoretic property of the whole class of computable presentations of the structure. The complexity difference between these two notions is rather significant. One way to see the difference is to compare their *index sets* [GK02], in the following sense. Fix an effective enumeration $(A_i)_{i\in\omega}$ of all partial computable structures in a computable signature; for example, fix the language of graphs. It has been shown that the index set $\{i : A_i \text{ is c.c.}\}$ is Π^1_1-complete [DKL+15], and thus computable categoricity is a second-order property in general. In contrast, the index set $\{i : A_i \text{ is relatively c.c.}\}$ is merely Σ^0_3-complete [DALD], showing that relative categoricity is a first-order arithmetical property of a structure.

We return to torsion abelian groups. If some algebraic property could describe computable categoricity of such groups, then plain computable categoricity and relative computable categoricity would most definitely coincide in the class. Indeed, this same algebraic property could be applied to non-computable copies of a computably categorical group, thus witnessing its relative computable categoricity. This intuition is supported by numerous examples in the literature [AK00, EG00]. Rather surprisingly, relative and plain computable categoricity differ in the class of torsion abelian groups:

Theorem 1.2. *There exists a computable torsion abelian group which is computably categorical but not relatively computably categorical.*

We note that the theorem above is new for general abelian groups.

Does Theorem 1.2 imply that there cannot be any structure theory of computably categorical torsion abelian groups (c.c. TAGs)? Some researchers suggest that plain computable categoricity is so badly behaved that it should not be studied at all. The Π^1_1-completeness result [DKL$^+$15] mentioned above supports this claim. However, when restricted to some nice subclass of structures, this index set may become arithmetical; it makes a perfect sense to study computable categoricity *within this subclass*. It is not too difficult to see that the set $\{A_i : A_i \text{ is a c.c. TAG}\}$ is arithmetical (to be explained in due course).

What kind of answer should we expect? We cannot hope for an algebraic description. Any criterion for computable categoricity of torsion abelian groups should appeal to the enumeration of a computable copy of the group. There is only one example in the literature where a plain categoricity notion (even more general than computable categoricity!) admits a nice explicit characterisation. More specifically, Downey and Melnikov [DM13] described plain Δ^0_2-categoricity of completely decomposable groups in terms of semi-lowness [Soa87]. We omit the definitions, but we note that semi-lowness is a rather specific *index set property* which arose from the study of the lattice of c.e. sets [Soa87].

1.1. **The main results.** No standard technique or notion seemed to help in obtaining any valuable structural information about computably categorical TAGs. For example, the index set seemed to be Π^0_5-complete, and it was not clear whether there could be any way of pushing this complexity down. Quite unexpectedly, there is a subtle Π^0_4-property that does describe c.c. TAGs.

Theorem 1.3. *A computable torsion abelian group G is computably categorical if and only if the computable index of G satisfies a certain Π^0_4 predicate Ψ which describes the failure of the brute-force diagonalisation attempt on G.*

The diagonalisation attempt from Theorem 1.3 is brute-force in the following sense. Its basic strategy is the most straightforward diagonalisation module that monitors two cyclic summands in $G[s]$ and tries to swap them in another copy of G which it attempts to build. We delay the formal description of Ψ until Section 3. The complexity of Ψ is optimal:

Theorem 1.4. *The index set of computably categorical torsion abelian groups is Π^0_4-complete.*

Theorem 1.3 and Theorem 1.4 show that plain and relative computable categoricity differ only very slightly in the class of TAGs. They are only one quantifier apart from each other (Σ^0_3 vs. Π^0_4). Also, both relative and plain categoricity notions are effectivisations of the same purely algebraic *weak homogeneity property* within the class (Def. 3.1, Prop. 3.10).

But is Theorem 1.3 really a description of c.c. TAGs? We conjecture that one cannot obtain a criterion significantly better than the one in Theorem 1.3. First of all, Theorem 1.4 shows that the syntactical complexity of Ψ is optimal. Also, any such criterion must appeal to the computable enumeration of the group, otherwise the criterion would be relativisable, contradicting Theorem 1.2. On the other hand, the predicate Ψ can be used to derive non-obvious information about c.c. TAGs. For instance, Ψ allowed us to push the seemingly optimal index set complexity (Π^0_5) down to Π^0_4, and to show that almost all primary summands of a c.c. TAG are weakly homogeneous. It is not clear how to extract this information avoiding the use of Ψ. Thus, Theorem 1.3 is not just a reformulation of the definition of categoricity. We conclude that Theorem 1.3 and Theorem 1.4 settle the 60 year-old problem of Mal′cev for torsion abelian groups in the sense that *our analysis is optimal*.

We briefly discuss the proofs of Theorems 1.4 and 1.3. There have been enough arguments in recursion theory exploiting the failure of a diagonalisation attempt, the most closely related examples can be found in [Gon75, HKMS15, Mos84]. In our proof both the diagonalisation attempt itself and its role in the proof are more subtle than in any other proof in computable structure theory that we are aware of. Although the diagonalisation attempt is merely a finite injury construction, we will face significant combinatorial difficulties even in meeting one basic module in isolation. To deal with this combinatorial nightmare we introduce the technique of *tangles* which extends the clique technique from [DMN15]. In fact, the situation is so complicated that we would not know how to run the diagonalisation attempt on the group itself. Instead, we use a careful uniform reduction from torsion abelian groups to cardinal sums of equivalence structures. The reduction relies on specific group-theoretic techniques similar to p-basic subgroup analysis, see [Fuc70]. In fact, the main algebraic Proposition 3.4 extends the main result of [ADH$^+$], but via a totally different proof.

Assuming Theorem 1.3, to obtain Theorem 1.4 it is sufficient to prove that the index set is Π^0_4-hard. The proof of Π^0_4-hardness is not too sophisticated, but it uses a new idea. It is not hard to see that the index set of relatively c.c. TAGs is Σ^0_3-complete (Proposition 6.4). Thus, Theorem 1.2 is follows from Theorem 1.4.

1.2. **Computable dimension.** Our last result contributes to the theory of computable dimension, see book [EG00]. In [Gon81], Goncharov conjectured that every abelian group that is not c.c. must have infinitely many computable copies, up to computable isomorphism. The conjecture has been verified for broad subclasses of abelian groups (see [Gon80, Mel14]). There are only two classes of abelian groups where the conjecture has not been verified. These are again the torsion abelian groups and the mixed abelian groups of finite rank. We apply our techniques to confirm the 30 year-old conjecture of Goncharov in the case of torsion abelian groups:

Theorem 1.5. *If a computable torsion abelian group is not computably categorical, then it has infinitely many computable copies up to computable isomorphism.*

The proof of Theorem 1.5 is not that hard, put it does require a new idea. The result does not follow from the well-known sufficient condition involving two Δ^0_2-isomorphic but not computably isomorphic copies [EG00]. In our case the isomorphisms are not necessarily Δ^0_2.

The paper is organised as follows. Section 2 will be a short preliminary section. In Section 3 we prove that there exists a Π^0_4-predicate describing categoricity (Theorem 1.3), and Section 4 contains the proof of the Π^0_4-hardness of the index set (Theorem 1.4). We prove Theorem 1.5 in Section 5. In Section 6 we discuss relative computable categoricity of TAGs.

2. Preliminaries

All structures in this section, and throughout the paper, are at most countable. All groups in the rest of the paper are abelian and torsion, unless otherwise stated.

2.1. **Abelian groups.** The standard reference for this is [Fuc70]. Recall that every abelian group can be viewed as a $\mathbb{Z}$-module. An abelian group is *divisible* if for any integer n and any element g of the group, the equation $nx = g$ has a solution in the group. For example, the Prüfer p-group (also known as the quasi-cyclic p-group)

$$\mathbb{Z}_{p^\infty} = \langle a_i : i \in \omega \,|\, pa_0 = 0, pa_{i+1} = a_i : i \in \omega \rangle$$

is divisible for any prime p. A group is *reduced* if it contains no non-zero divisible subgroup. It is well-known that any divisible subgroup H of an abelian A detaches as a direct summand of A. In particular, A can be split into its divisible and reduced parts, but the maximal reduced subgroup is not uniquely defined *as a subset of A*. It uniquely determined up to isomorphism.

Every torsion abelian group splits into the direct sum of its maximal p-subgroups, and this splitting is uniformly effective [Khi98]. We discuss abelian p-groups. For a non-zero $g \in A$, its p-height is usually defined to be the maximum n such that $p^n x = g$ has a solution in A. If no such maximal n exists, we set the p-height equal to ∞. For reasons that will become clear later, we will slightly adjust this standard definition by always adding 1 to the height.

Convention 2.1. *We agree that the p-height of an element $g \in A$ is equal to $n + 1$ if n is largest such that $\exists x \in A$ $(p^n x = g)$, and if no such largest n exists then the p-height will be ∞. We denote this by $h_p^A(g)$, or simply $h_p(g)$ when the context is clear.*

For example, the p-height of any non-zero element of $A = \mathbb{Z}_p$ is 1, and thus it matches the p-order (i.e., $log_p(|A|)$) of this cyclic group.

A subgroup H of A is *p-pure* if for any $h \in H$ and $n \in \mathbb{Z}$, $\exists x \in A$ $(p^n x = h)$ implies $\exists x \in H$ $(p^n x = h)$. In other words, the p-height of $h \in H$ within G is always witnessed within H. A subgroup is pure if it is p-pure for every p. A p-pure subgroup of an abelian p-group is in fact pure. It is well-known that a pure cyclic subgroup always detaches as a direct summand [Fuc70].

Given an abelian p-group A, we define A' to be its subgroup consisting of elements having infinite p-height. Iterating this process, we define $A^{(i)}$ for $i \in \omega$, and taking intersections we define $A^{(\alpha)}$ for any ordinal α. Since A is countable, the sequence must stabilize, and the stable $A^{(\alpha)}$ must clearly be divisible. The least α such that $A^{(\alpha)} = A^{(\alpha+1)}$ is called the Ulm type of A. It is well-known that any abelian p-group of Ulm type 1 splits into a direct sum of its cyclic and quasi-cyclic subgroups. We will call any such decomposition *full* or *complete*. Any two full decompositions of an Ulm type 1 p-group must be isomorphic as decompositions, i.e. must have the same number cyclic or quesi-cyclic summands of a given isomorphism type [Fuc70].

Recall that the *socle* $[G]_p$ of an abelian p-group G is the collection of all its elements of order at most p. This is a $\mathbb{Z}_p$-vector space and thus it makes sense to speak of $\mathbb{Z}_p$-independence in the space. Since $\mathbb{Z}_p$ is a finite field, linear independence is decidable in the diagram of the vector space.

2.2. **Infinitary formulae.** See [AK00] for a rigorous definition and some basic properties of $\mathcal{L}^c_{\omega_1\omega}$ such as the remarkable Barwaise-Kreisel compactness. We note that infinitary computable formulae over a computable A can be re-written into a first-order form but over $\mathbb{HF}(A)$, the hereditarily finite extension of the structure A [Ers00]. The role of $\mathcal{L}^c_{\omega_1\omega}$ is quite significant in computable structure theory, see [AK00]. We will be using only some standard operations on infinitary computable formulae (such as calculating their complexity), these can be found in the first half of [AK00].

In this paper, $\mathcal{L}$ is the language of additive groups. For example, the following $\mathcal{L}_{\omega_1\omega}$-sentence describes divisibility:

$$\bigwedge_{n \in \omega} \forall g \exists x \ (nx = g),$$

Informally, when determining the complexity of a formula, the infinite disjunctions should be counted as existential quantifiers, and the infinite conjunctions should be thought of as universal quantifiers. For example, the syntactical complexity of the above sentence is Π^c_2.

Clearly, if Φ is a Π^c_α $\mathcal{L}_{\omega_1\omega}$-sentence, the the index set $\{i : A_i \models \Phi\}$ is Π^0_α (and similarly for Σ^c_α), see [AK00]. Furthermore, we can uniformly replace any such formula by a predicate upon ω of the respective complexity that isolates the computable structures satisfying the formula. We will use this property without explicit reference.

3. A Π^0_4-description of computable categoricity

3.1. **A brute-force attempt to describe categoricity.** Throughout this section, we identify a computable structure with its index, e.g. we write $\forall G(\dots G \dots)$ but really mean $(\forall e)(\dots M_e \dots)$. The most naive attempt to characterise c.c. TAG would be to say

$$(\forall G)(G \cong A \implies G \cong_c A),$$

but this is way too complicated as it is Π^1_1. It is not hard to see that we can do better. Each p-component A_p of $A = \bigoplus_{p\in\omega} A_p$ must be c.c., since otherwise A is clearly not c.c. It is also known that c.c. implies relative c.c. for abelian p-groups (this easily follows from [Smi81]; simply run the diagonilasation procedure on this p-component and copy the rest). Each A_p must be of the form

$$F \oplus (\mathbb{Z}_{p^\lambda})^\alpha,$$

where F itself splits into finitely many cyclic and quasi-cyclic summands, $\lambda \in \omega \cup \{\infty\}$, and $\alpha \in \omega \cup \{\omega\}$ (the power indicates the number of direct summands). For example,

$$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^5} \oplus (\mathbb{Z}_{p^\infty})^\omega$$

is (relatively) c.c., where $\mathbb{Z}_{p^\infty}$ is the Prüfer group and $(\mathbb{Z}_{p^\infty})^\omega = \bigoplus_{i\in\omega} \mathbb{Z}_{p^\infty}$. As noted in [CCHM06], such groups naturally correspond to equivalence structures, and this correspondence will be heavily used throughout this paper. The reader might now think that the rest of the paper will be an elementary analysis of equivalence structures, unfortunately the situation is a lot more complicated.

The existence of an isomorphism for each distinct r.c.c. A_p is merely Σ^c_3, and thus the complexity of the statement

$$(\forall G)(G \cong A \implies G \cong_c A),$$

can be reduced to Π^0_5 by saying that for any torsion abelian p-group whose p-components are r.c.c., if the respective components are isomorphic (this is Σ^0_3), then the groups are computably isomorphic. Clearly, "A is a TAG" can be expressed by a Π^c_2-formula. We also say:

$$(\forall p)(A_p \text{ is r.c.c.}) \ \& \ (\forall G)\Big([G \text{ in } TAG \ \& \ (\forall p) G_p \text{ is r.c.c.} \ \& \ (\forall p) G_p \cong A_p] \implies G \cong_c A\Big).$$

As we already mentioned in the introduction, the property of being relatively c.c. is Σ^0_3 [DALD]. (Towards the end of the paper we will also produce a Σ^0_3 definition that does not appeal to a Scott family.) Note that "$\forall G$" ranges over all computable structures G. So the second conjunct is of complexity $(\forall)([\forall\Sigma^0_3] \implies \Sigma^0_3)$ which is Π^0_5.

However, a very simple construction shows that the property of having r.c.c. p-components fails to characterize c.c. TAGs (we skip it). The idea is that if a TAG is c.c. then, as we will illustrate later, almost all p-components of it must satisfy the weak homogeneity property. The weak homogeneity property (WHP) can be used to produce a d-Σ^0_4-definition of c.c. TAGs (we will skip it as well). Although the WHP fails to capture categoricity, it will help in our analysis. Quite a bit of work will be needed to push the complexity down to Π^0_4.

3.2. **The weak homogeneity property.**

Definition 3.1. We say that an abelian p-group G satisfies *the weak homogeneity property (WHP)* if it is either divisible, or for each non-zero $a \in G$ of order p and $h_p(a) < \infty$ there exist at most finitely many elements of order p and height $> h_p(a)$.

Example. The group
$$\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^5} \oplus (\mathbb{Z}_{p^\infty})^\omega$$
does not satisfy the WHP because $2 < \infty$, but both
$$\mathbb{Z}_{p^5} \oplus \mathbb{Z}_{p^7} \oplus (\mathbb{Z}_{p^\infty})^4$$
and
$$\mathbb{Z}_{p^5} \oplus (\mathbb{Z}_{p^2})^\omega$$
satisfy the WHP. (Here A^ω stands for the infinite *direct sum* of A with itself, which we also call the infinite direct power of A.)

The algebraic characterisation of countable abelian p-groups satisfying the WHP is given by the lemma below:

Lemma 3.2. *Suppose A is a countable p-group satisfying the WHP. Then A is of the form*
$$U \oplus H,$$
where U is a finite direct sum of cyclic and quasi-cyclic p-groups, H is the direct power of some (fixed) cyclic or quasi-cyclic group Z_{p^λ}, and the least α such that $\mathbb{Z}_{p^\alpha}$ occurs in the decomposition of U (if there are any) is not less than λ.

Proof of Lemma. Observe that all groups of the isomorphism types claimed by the lemma satisfy the WHP. Now suppose A satisfies the WHP. If A is divisible, then there is nothing to prove. Otherwise, let $a \in A$ be an element of finite height and of order p. Clearly, in this case the divisible part must have finite rank, for otherwise the WHP fails (as witnessed by a and any basis of the divisible component). We prove that, furthermore, the reduced part of the group has Ulm type 1. Suppose, for the contrary, that x is not a divisible element but has infinite p-height. Then there exists an infinite collection of elements b_i, $i = 1, 2, \ldots$, such that $pb_i = x$ and $h_p(b_i) < h_p(b_{i+1})$. For almost all i, the height of $(b_i - b_{i+1})$ is greater than the height of a, contradicting the WHP.

Since the Ulm type of the reduced part is 1, it splits into a direct sum of cyclic p-groups. If there are infinitely many cyclic summands in this decomposition, then the WHP guarantees that almost all of these summands are of some fixed finite order. These summands will form H in the notation of the lemma. Then U will consist of the finitely many cyclic and quasi-cyclic summands that are left after forming H. Finally, if some cyclic summand in U is of a smaller order than the order of each summand in H, then the WHP fails. □

We conclude that *the WHP implies relative computable categoricity.* Also, the lemma justifies the term "weak" in the WHP, since if the finite U is the zero-group then we are left with a "homogeneous" H.

Lemma 3.3. *The syntactical complexity of the WHP is (at most) Π^c_3.*

Proof of Lemma. Let us first do a careful syntactical analysis of several properties that will be combined to define the WHP.

Saying that A is reduced involves asking for a non-zero element whose p-height is finite, this is clearly Σ^c_2. Also, divisibility of a group is Π^c_2-definable.

The property $h_p(x) = k'$ is equivalent to saying that $h_p(x) \geq k'$ (which is Σ^c_1) and not $h_p(x) \geq k'+1$ (Π^c_1). Thus, saying that no element has p-height exactly k' has complexity $\Sigma^c_1 \wedge \Pi^c_1$, and for our purposes the upper bound Π^c_2 is enough.

The property of having infinitely many b in the socle whose heights are greater than some fixed finite number k can be (informally) expressed as

$$\psi_k = (\exists^\infty b)\,(pb = 0 \,\&\, h_p(b) \geq k+1),$$

and since $h_p(b) \geq k+1$ is Σ^c_1, the property is Π^c_2.

Now we go back to the syntactical analysis of the WHP. Recall that G satisfies the WHP if either G is divisible (which is Π^c_2, see above), or [for each non-zero $a \in G$ of order p and $h_p(a) < \infty$ there exists at most finitely many elements of order p and height $> h_p(a)$]. To express the second disjunct of the definition, we write

$$\bigwedge_{k\in\omega} ([(\exists a \neq 0) h_p(a) = k \text{ and } pa = 0] \implies \neg\psi_k),$$

which says that if k if the height of some element in A then there are at most finitely many elements in the socle having their height greater than k. According to the above analysis, the formula is of the form

$$\bigwedge_{k\in\omega} ([\exists\Pi^c_2] \implies \neg\Pi^c_2),$$

or

$$\bigwedge_{k\in\omega} \Pi^c_3,$$

which is Π^c_3. □

3.3. **An effective correspondence with equivalence structures.** It is clear that p-groups of Ulm type 1 naturally correspond to equivalence structures. The correspondence is defined as follows. Suppose

$$G = \bigoplus_{i\in I} G_i,$$

where for each i the summand G_i is either a cyclic p-group or $\mathbb{Z}_{p^\infty}$. The λ such that $\mathbb{Z}_\lambda \cong G_i$ is either a natural umber n or the symbol ∞, and it will be denoted by $\#G_i$.

The definition of $G \to E_G$. In the notation as above, define E_G to be the equivalence structure in which the $i'th$ equivalence class E_i has size exactly $\#G_i$. (We write $\#E_i$ to denote the size of E_i.)

It is well-known that any two full decompositions of any fixed Ulm type 1 abelian p-group are isomorphic (as decompositions), see e.g. [Fuc70]. Thus, the isomorphism type of E_G does not depend on the given full decomposition of G. We can pass from an equivalence structure to a group using the following dual rule.

The definition of $E \to G_E$. Given an equivalence structure $E = \sum_{i\in I} E_i$, define

$$G_E = \bigoplus_{i\in I} G_i,$$

where G_i is either cyclic or quasi-cyclic and $\#G_i = \#E_i$ for each $i \in I$.

It follows that $G_{E_A} \cong A$ and $E_{G_U} \cong U$ for any equivalence structure U and any abelian p-group A of Ulm type 1. There is nothing specifically deep in this observation. Nonetheless, it turns out that the effective properties of these functors can be quite intricate. The algorithmic content of the functors was first investigated in [CCHM06] and then more recently in [DMN14]. For example, it follows that the functors do not preserve Δ^0_2-categoricity [DMN15, DMN14] which is a rather counter-intuitive feature.

Clearly, the functor $E \to G_E$ is uniformly computable. Although it is not hard to show that $G \to E_G$ maps isomorphism types of computable groups to isomorphism types of computable equivalence structures (follows from [Mel14]), is was not clear whether $G \to E_G$ is *uniformly* computable. It follows from [ADH$^+$] that it is uniformly effective when restricted to G with finite socle. The proof in [ADH$^+$] exploits combinatorial techniques specific of abelian group theory and is not completely straightforward. We will need an extension of this result. We note that the proof below is quite different from the one contained in [ADH$^+$].

Proposition 3.4. *The functor $G \to E_G$ defined above is uniformly effective. Furthermore, regardless of the Ulm type of the input abelian p-group G, the output of the uniform procedure is always an equivalence structure.*

Proof. The definition of the Turing functional representing $G \to E_G$ is fairly straightforward. We work computably relative to the open diagram of G. Recall that $\mathbb{Z}_p$-independence is decidable in $[G]_p$, relative to the diagram. First, initiate a uniformly effective enumeration of any basis $x_0, x_1, x_2, \dots$ of the socle of G. For each i such that x_i has been found, define

$$s_i = \sup_{m_0,\dots,m_{i-1} \in \mathbb{Z}_p} h_p(x_i - \sum_{j=0}^{i-1} m_j x_j),$$

which is clearly non-computable but can be effectively approximated from below. We allow $s_i = \infty$. Recall that a function $\omega \to \omega \cup \{\infty\}$ is *limitwise monotonic* if it total and can be approximated from below by a non-decreasing computable function [Khi98, KNS97]. The function s defined above is clearly limitwise monotonic, with all possible uniformity. Initiate the enumeration of an equivalence structure U in which $\#U_i = s(i)$. Note that we never refer to the Ulm type of the input group.

We now check that the procedure described above satisfies the desired properties. Clearly, it is uniformly effective. Furthermore, regardless of the Ulm type of G, the function $i \to s_i$ is limitwise monotonic and thus U is well-defined. We claim that if the Ulm type of G is 1 then $U \cong E_G$. For this purpose we define a full decomposition of G induced by the definition of s_i and the choice of the basis $x_0, x_1, \dots$, as follows.

Note that $s_0 = h_p(x_0)$. Fix a (maximal) chain of p-divisions below x_0 that witnesses $h_p(x_0)$, and let C_0 be the subgroup of G generated by C_0. Then C_0 is either a pure cyclic or a quasi-cyclic subgroup of G. Since C_0 is either pure cyclic or divisible, it detaches as a direct summand of G,

$$G = C_0 \oplus A_1.$$

Fix the projection π_1 onto A_1. We claim that

$$h_p^{A_1}(\pi_1(x_1)) = h_p^{G/C_0}(x_1) = \sup_{m_0 \in \mathbb{Z}_p} h_p(x_1 - m_0 x_0) = s_1.$$

Fix a full decomposition of A_1 which clearly exists since the Ulm type of A_1 is 1 (note A_1 could be not reduced). Then

$$x_1 = n_0 x_0 + \sum n_i y_i,$$

where the y_i come from distinct summands in the induced full decomposition of the socle of A_1. Note that $h_p^{G/C_0}(x_1) = h_p(\sum n_i y_i)$ and $h_p(m x_0 + \sum n_i y_i) \leq h_p(\sum n_i y_i)$ for any m. It follows from the definition of s_1 that $h_p^{A_1}(\pi_1(x_1)) = s_1$, as claimed. Fix a chain of p-divisions in A_1 that witnesses $h_p^{A_1}(\pi_1(x_1)) = s_1$, and let C_1 be the subgroup of A_1 generated by this chain. Similarly to C_0, it must be the case that C_1 detaches in A_1.

Suppose we have defined $C_0, \dots, C_n$ and A_{n+1} where the C_i are either cyclic or quasicyclic, and

$$G = \left(\bigoplus_{i=0}^{n} C_i \right) \oplus A^{n+1}.$$

As above, we can choose C_{n+1} that witnesses

$$h_p^{A_1}(\pi_{n+1}(x_{n+1})) = h_p^{G/\sum_i C_i}(x_{n+1}) = \sup_{m_0,\dots,m_{i-1} \in \mathbb{Z}_p} h_p(x_i - \sum_{j=0}^{i-1} m_j x_j) = s_{n+1},$$

the proof of which is almost identical to the case $n = 1$ (here π_{n+1} is the projection onto A_{n+1}). As before, we get that C_{n+1} detaches within A_{n+1} to form A_{n+2}.

This way we produce a subgroup B of G that satisfies the properties resembling those of p-basic subgroups of reduced p-groups ([Fuc70]):

i. $B = \bigoplus_i C_i$, where the C_i are cyclic or quasi-cyclic subgroups,
ii. $[B]_p = [G]_p$, i.e. the socle of B is equal to the socle of G.

Property i. follows from the definition of $C_0, C_1, \dots$. To see why ii. holds, recall that $x_0, x_1, \dots$ is a basis of $[G]_p$, and

$$Span_{\mathbb{Z}_p}\{x_0,\, x_1 - n_{0,0}x_0,\, x_2 - n_{1,0}x_0 - n_{1,1}x_1,\, \dots\} = Span_{\mathbb{Z}_p}\{x_0, x_1, \dots\} = [G]_p$$

for any choice of $n_{i,j} \in \mathbb{Z}_p$. The generators of $[C_i]_p$ are of the form $x_i - \sum_{j<i} n_{i,j} x_j$, thus ii. holds.

Recall that by our assumption G is itself a direct sum of cyclic and quasi-cyclic p-groups. We claim that i. and ii. together imply that $B = G$. Aiming for a contradiction, assume $\alpha \in G \setminus B$. Suppose also that $p^n\alpha \in B$ while $p^{n-1}\alpha \notin B$. Note such an n exists since $[B]_p = [G]_p$ (by ii. above). Without loss of generality, we can assume that $n = 1$. We arrive at

$$p\alpha = \sum_{i \le k} d_i,$$

where $d_i \in C_i$ for each $i = 0, \dots, k$. We may assume that each $d_i \neq 0$, otherwise we re-arrange the indexing of the C_i. This assumption is used throughout the proof of the claim below.

Claim 3.5. *In the notation as above, for each $i \le k$ there exists $d_i' \in C_i$ such that $pd_i' = d_i$.*

Proof of Claim. Suppose such a d_k' does not exist (the case when $i = k$). But then the chain that generates C_k is not maximal in $G/(\sum_{j<k} C_j)$ as witnessed by the projection of a suitably chosen $\mathbb{Z}_p$-multiple of the coset of α. Thus, $d_k = pd_k'$ for some $d_k' \in C_k$.

To see why d_{k-1}' exists, consider $p(\alpha - d_k') = p\alpha - d_k \in \bigoplus_{j<k} C_j$. Just as we had above with α and d_k, the C_{k-1}-projection of the element $(\alpha - d_k')$ will witness the failure of maximality (in $G/\sum_{j<k-1} C_j$) of the chain used to define C_{k-1}, unless d_{k-1}' exists.

We proceed in this manner to find $d_{k-2}', \dots, d_0'$. □

We conclude that $p\alpha = \sum_{i \le k} pd_i'$. Then $p(\alpha - \sum_{i \le k} d_i') = 0$ and thus

$$\alpha - \sum_{i \le k} d_i' \in [G]_p = [B]_p \subseteq B.$$

Together with $\sum_{i \le k} d_i' \in B$ this gives $\alpha \in B$, contradicting the choice of α.

Finally, since all full decompositions of G are isomorphic (as decompositions), we have that $U \cong E_G$. □

Corollary 3.6. Given a computable p-group A of Ulm type 1 we can *uniformly* pass to a computable presentation H of A that admits an effective full decomposition into cyclic and quasi-cyclic subgroups. Furthermore, if the input is an abelian p-group whose Ulm type is not necessarily equal to 1, then the output is an effectively decomposed abelian p-group of Ulm type 1.

We apply Proposition 3.4 to study the weak homogeneity property. The next lemma will be very useful in guessing the isomorphism between two groups satisfying the WHP.

Lemma 3.7. *The isomorphism type of a countable abelian p-group X satisfying the WHP is completely determined by the collection of finite substructures of E_X.*

Proof. Let X and Y be two abelian groups satisfying the WHP. We uniformly pass to the respective equivalence structures E_X and E_Y. For an arbitrary algebraic structure A, let $Age(A)$ be the collection of all finite substructures embeddable into A. We claim that $X \cong Y$ iff $Age(E_X) = Age(E_Y)$. One implication is trivial.

Suppose $Age(E_X) = Age(E_Y)$. Recall that $X \cong Y$ iff $E_X \cong E_Y$. Both E_X and E_Y must have the same number of classes. We first consider the case when all classes in E_X are infinite, i.e when X is divisible. If E_X has only finitely many classes, and all these classes are infinite, then E_Y also must be like that. Suppose both E_X and E_Y have infinitely many classes, and all classes in E_X are infinite. Recall that X and Y both satisfy the WHP. In particular, almost every class in E_Y must be of some fixed size, thus it must be infinite. Also, it cannot have any finite class since it would witness the failure of the WHP. Thus, X and Y are simultaneously divisible and of the same rank, and in this case $X \cong Y$.

Suppose X is not divisible, and assume the rank of the socle is finite. As noted above, both E_X and E_Y must have some fixed finite number of equivalence classes. It is well-known that equivalence structures having finitely many equivalence classes are completely described by their finite substructures, up to isomorphism (see e.g. [ADH$^+$] for a recent application).

Suppose now E_X has infinitely many classes, and let m be the size of almost every class of E_X. Note such an m exists and must be a natural number, by the WHP and by our assumption. Then E_Y must also be of the same form, with a.e. class of size m_1 for some m_1. We first argue that $m_1 = m$. If $m > m_1$ then a sufficiently large finite substructure of E_X would not be embeddable into E_Y.

Recall that both X and Y satisfy the WHP, and therefore m must be no greater than the sizes of the other finitely many classes in E_X (same in E_Y), and without loss of generality it is smaller than these sizes. If E_X has k exceptional classes that were not of size $> m$, then we claim that E_Y has at least k such exceptional classes. Indeed, these exceptional classes (if there are any) must have size greater than m, and thus a large enough finite substructure of such classes in E_X can be embedded only into the part consisting of exceptional classes in E_Y. Thus, the number of exceptional classes k is the same for both E_X and E_Y. A large enough finite substructure with k classes may be embedded only into the exceptional part. It follows that the collection of all finite structures embeddable into E_X determines the isomorphism type of the exceptional part, and thus of the whole equivalence structure and of the respective group. □

3.4. **Relaxing isomorphism between equivalence structures.** All groups in this subsection are countable abelian p-groups of Ulm type 1. The information contained in this section will allow us to completely remove groups from all our arguments and work only with equivalence structures.

Suppose $\phi : A \to G$ is an isomorphism between two (computable) abelian p-groups of Ulm type 1 with some fixed (effective) full decompositions, $A = \bigoplus_i A_i$ and $G = \bigoplus_i G_i$. Note that

ϕ does not have to agree with the fixed decompositions. For each non-zero $\alpha_i \in [A_i]_p$ (i.e., $\alpha_i \in A$ of order p) its image $\phi(\alpha_i)$ will be expressed as a linear combination of elements β_j coming from various $[G_j]_p$,

$$\phi(\alpha_i) = \sum_j m_j \beta_j, \quad \beta_j \in [G_j]_p,$$

where the sum is finite. Having in mind the natural correspondence between each A_i and the respective equivalence class E_{A_i} of E_A (and similarly for G_i and E_G), we define ϕ^* that maps a class of E to a finite set of classes in E_G by the rule

$$\phi^*(E_{A_i}) = \{E_{G_j} : m_j \neq 0 \text{ in } \phi(\alpha_i) = \sum_j m_j \beta_j\}.$$

We define the size of a finite collection $F = \{S_0, \ldots, S_k\}$ of equivalence classes S_i in E to be equal to the minimum among the sizes of its members $S_0, \ldots, S_k$. We write $h(F)$ to denote the size of F. If I is a class then we write $h(I)$ for $h(\{I\}) = \#I$. The size can either be a natural number or the symbol ∞. This definition agrees with our definition of p-height.

Fix some isomorphism $\phi : A \to G$. Is it true that any class in E_G is realized as the least class in some $\phi^*(E_{A_i})$? Although the question seems somewhat arbitrary, the affirmative answer that we establish below will be very useful in the next subsection.

Lemma 3.8. *In the notation above (assuming ϕ is an isomorphism), for every class I in E_G there exists a class J in E_A such that $I \in \phi^*(J)$ and $h(I) = h(\phi^*(J))$.*

Proof of Lemma. Recall that we fixed full decompositions $A = \bigoplus_i A_i$ and $G = \bigoplus_i G_i$, and let $B = \{x_0, x_1, \ldots\}$ and $B' = \{z_0, z_1, \ldots\}$ be bases of the socles of A and G (respectively) that agree with these decompositions, i.e., $x_i \in A_i$ and $z_i \in G_i$ for each i. According to our conventions, the p-height of each x_i [and z_i] is equal to λ such that $x_i \in A_i \cong \mathbb{Z}_{p^\lambda}$ [respectively, $z_i \in G_i \cong \mathbb{Z}_{p^\lambda}$].

It is sufficient to prove that for each $z_i \in B'$ there exists an $x_k \in B$ such that $\phi(x_k) = \sum_j m_j z_j$ mentions z_i with a non-zero $m_i \in \mathbb{Z}_p$, and furthermore

$$h_p(x_k) = h_p(z_i) = \min\{h_p(z_j) : m_j \neq 0 \text{ in } \sum_j m_j z_j\}.$$

Aiming for a contradiction, assume that each $\phi(x_k)$ that mentions z_i in its decomposition $\phi(x_k) = \sum_j m_j z_j$ also mentions some $z_l \neq z_i$ such that $h_p(z_l) < h_p(z_j)$ and with a non-zero coefficient. Since the z_i come from distinct direct summands, the minimum of the heights of the z_j that are mentioned in $\phi(x_i)$ with $m_i \neq 0$ is equal to $h_p(\phi(x_k)) = h_p(x_k)$. Thus, in particular, $h_p(\phi(x_k)) = h_p(x_k)$ is smaller than $h_p(z_i)$. Since ϕ is an isomorphism, the elements $\phi(x_0), \phi(x_1), \ldots$ form a basis of the socle of G. For some coefficients n_k, we have

$$z_i = \sum_k n_k \phi(x_k).$$

According to the above assumption, each of the $\phi(x_k)$ that mention z_i non-trivially in their decomposition must also non-trivially mention some z_l of a smaller height.

In the notation as above, let a be the sum of all $n_k \phi(x_k)$ such that $n_k \neq 0$ and $h_p(\phi(x_k)) \geq h_p(z_i)$. It follows that a does not mention z_i in its decomposition, as each such $\phi(x_k)$ does not. (If they did then they'd have a smaller p-height, see above.) Then the element $z_i - a$ is non-zero (since a does not mention z_i at all) and also must satisfy:

(1.) $h_p(z_i - a)$ is at least $h_p(z_i)$, because $h_p(a)$ is at least $h_p(z_i)$

(2.) $z_i - a = \sum_k n'_k \phi(x_k)$, where each x_k (equivalently, $\phi(x_k)$) has height $< h_p(z_i)$, by our choice of the $\phi(x_k)$.

Now recall ϕ is an isomorphism, so we have $\phi^{-1}(z_i - a)$ is not zero and

$$\phi^{-1}(z_i - a) = \sum_k n'_k x_k,$$

where each x_k with $n'_k \neq 0$ has height $< h_p(z_i)$. But these x_k come from distinct direct components of A, and thus

$$h_p(\phi^{-1}(z_i - a)) = h_p(\sum_k n'_k x_k) = \inf_k h_p(x_k) = h_p(x_s)$$

for one such x_s. But we have $h_p(x_s) = h_p(\phi(x_s)) < h_p(z_i)$ (see condition (2.) above), while condition (1.) gives:

$$h_p(\phi^{-1}(z_i - a)) = h_p(z_i - a) \geq h_p(z_i).$$

So we conclude that $h_p(z_i) \leq h_p(\phi^{-1}(z_i - a)) < h_p(z_i)$, a contradiction. □

3.5. **A construction that must fail.** In this subsection we describe an effective priority construction which, if successful, will build $M \cong A$ such that $M \not\cong_c A$. If A is c.c., then the construction must *fail* to satisfy its requirements. Since the construction will be uniform in the diagram of A and will be a finite injury one, this fact can be expressed by a Σ^0_3-predicate which says that for some $e \geq 0$ there are infinitely many expansionary stages (to be clarified). The construction will be used to "safe" one quantifier in the description of a c.c. TAGs, in the following sense. The existence of infinitely many e-expansionary stages (which is Σ^0_3) can be used to show that almost every p-component of A satisfies the WHP, while the latter statement is Σ^c_4 in general.

Remark 3.9. We invite the reader to verify that the index set of TAGs A such that a.e. A_p satisfies the WHP forms a Σ^0_4-complete set (within all computable TAGs whose p-components have Ulm type 1). The proof is fairly straightforward. Thus, the complexity Σ^c_4 of "a.e. A_p satisfies the WHP" is optimal, and this is one of the major obstacles in producing a Π^0_4-definition of computable categoricity.

We identify a computable group with its index.

Proposition 3.10. *There exists Σ^0_3 predicate Ξ such that for any computable TAG A whose p-components are all relatively c.c.,*

$$\neg\Xi(A) \implies A \text{ is not c.c.},$$

and

$$\Xi(A) \implies \text{a.e. } A_p \text{ satisfies the WHP.}$$

Corollary 3.11. *If a TAG A is c.c. then A_p satisfies the WHP for almost every p.*

Proof of Proposition. Suppose A is a computable TAG each p-component A_p of which is (relatively) c.c. By Proposition 3.4, we can uniformly pass to $\bigoplus_p G_{E_{A_p}} \cong A$ which possesses a computable complete decomposition into cyclic and quasi-cyclic summands, for various p. Therefore, w.l.o.g. we may assume that A has a computable full decomposition.

3.5.1. *Informal description.* Since A has an effective full decomposition, it will be convenient to identify E_A with A. Indeed, all that will matter in the construction is the sizes of the cyclic summands in the decomposition. We will construct a computable $M \cong A$ and attempt to diagonalise against each potential isomorphism $\varphi_e : M \to A$. The group M will also be given together with an effective decomposition. Since we will be working mainly with E_A and E_M, it will be sufficient to diagonalise against all φ^*_e that satisfy the property from Lemma 3.8 (saying that each class in E_A is realized as a minimal class in the φ^*_e-image of some E_A-class). This property will be the only explicit trace of group theory in the construction, the rest will be purely combinatorial. Nonetheless, sorting out this combinatorics will be a rather non-trivial task.

We will explicitly construct $\psi : M \to A$ and will attempt to make it a Δ^0_2 isomorphism. At every stage the map ψ will agree with the full decompositions of A and M thus ψ^* will map classes to classes (not merely to finite sets of classes). Although ψ will perhaps fail to be an isomorphism or perhaps might even fail to be total, we will use the partial ψ to illustrate that $A \cong M$.

The basic idea that we try to implement is rather brute-force. We restrict ourselves to some fixed prime p and the respective p-components in both M and A. For simplicity, we first describe the situation when $\varphi_e : M \to A$ happens to agree with the full decompositions of M and A, and thus φ_e^* maps classes to classes. Suppose $x \in E_A$ is within a class currently of size k, and assume at a stage some other class y is ready to outgrow the class of x in size, according to the enumeration of E_A. We may assume that φ_e^* has already provided us with the pre-images of z and y. Furthermore, we may adjust ψ so that these pre-images agree with their ψ^*-preimages (to be clarified in the formal proof). After the necessary adjustment is done, we *first* "swap" the ψ^*-images of these pre-images, and *then* grow the class of y and the class of its new ψ^*-preimage in M. If the size of the class of x was final, we guarantee that φ_e is not an isomorphism since φ_e^* does not preserve # (equivalently, φ_e it does not preserve p-heights).

Before we informally explain what can go wrong with this naive strategy, the reader should pause and convince themselves that *something along these lines perhaps should work* if A_p does not have the WHP. Indeed, in this case there will be infinitely many classes y that will attempt to pass some fixed x in size, and thus hopefully we will eventually succeed in our diagonalisation. On the other hand, if we fail then some strategy will eventually control almost all A_p, and thus we can hopefully argue that all these A_p must satisfy the WHP.

Unfortunately, the naive strategy above is quite different from the actual strategy we will have to implement. First of all, φ_e does not have to agree with the fixed decompositions/classes, but Lemma 3.8 will be particularly helpful here. Second, making sure that the ψ^*-preimages line up nicely with φ_e will introduce some extra noise to the construction, and this will need to be addressed carefully. Finally, even the naive strategy above can be *iterated* as follows. In the notation above, suppose we have swapped ψ^* on x and y according to the basic naive strategy. Now the class of x may try to pass some other class x' in size, and surely the priority of x' will be very high. In this case we will have to "swap" ψ on x and x', which will result in x, y and x' now forming a *tangle* with three respective classes in M. These *tangles* (to be defined) will significantly influence the construction. For example, if the classes keep growing, under which conditions can they leave the tangle? Can a class be contained in more than one tangle? Questions of this sort need to be answered explicitly.

As we see, there is quite a bit of work to be done. Since the construction is a finite injury one, we will be able to produce the desired Σ^c_3 formula Ξ.

3.5.2. *The requirements.* So we have a TAG A which has a computable full decomposition, and in which every p-component is (relatively) c.c. We build a computable $M \cong A$ and attempt to meet, for every e, the requirement:

R_e : $\varphi_e : M \to A$ is not an isomorphism.

Regardless of the outcomes, we will build M total and isomorphic to A (this is a global requirement). Furthermore, we will also explicitly and effectively construct a full decomposition of M into cyclic and quasi-cyclic summands. The full decompositions will never be re-arranged, i.e. the full decomposition of $M[s+1]$ will be naturally extending the full decomposition of $M[s]$ component-wise (the same for A). We also attempt to build a Δ^0_2 isomorphism $\psi : M \to A$. At every stage s the map $\psi[s]$ will be a true isomorphism from $M[s]$ onto $A[s]$ which furthermore respects the full decompositions, i.e., maps distinct cyclic summands of the fixed decomposition of $M[s]$ onto distinct cyclic summands of the fixed decomposition of $A[s]$. Although we may fail to make ψ total at the end, the partial Δ^0_2 map ψ will be used to prove that $M \cong A$.

3.5.3. *Notation and conventions.* At every stage of the construction, R_e will have several p-components of A assigned to it. Each such p-component (at every stage) will be effectively split into a number of cyclic finite p-groups, and these cyclic summands can be equivalently thought of as distinct equivalence classes of E_{A_p}. More specifically, every cyclic C in the fixed decomposition of $A_p[s]$ is in the natural 1-1 correspondence with the respective equivalence class in $E_{A_p}[s]$, and it will be very useful to identify these objects. The decompositions of neither A nor M will never be re-arranged (as noted above), and the construction will only refer to the sizes of various cyclic summands in the decomposition. Thus most of group theory can be completely stripped away (to be clarified in the convention below).

Convention 3.12. From this point on, and throughout the proof, we will identify a class x in E_{A_p} with the respective cyclic C_x in the fixed decomposition of A_p, *and also with some element in $[C_x]_p$ whose p-height is equal to $\#C_x$.* In fact, the p-1 distinct non-zero elements in the socle of C_x will also be all identified. We will call such elements *class-elements*. All that matters for the construction is the size of the respective class/summand that contains the class-element. The same convention will be used in M. We will also identify the maps ψ and ψ^*, and also φ_e with the respective φ^*_e.

Recall that we are using a non-standard definition of the p-height (which is equal to the standard plus 1).

Definition 3.13. For a class-element x, we write $h(x)$ to denote the size of C_x, which is equal to $h_p(x)$ for the respective p and x (according to our conventions).

As we already noted above, for an isomorphism

$$\phi : M \to A,$$

the ϕ-image of a class-element x could be *not* among the class-elements of the fixed above decomposition of A. It will be sufficient for us to identify the image with the finite collection of the class-elements in A that generate $\phi(x)$ with non-zero $\mathbb{Z}_p$-coefficients.

Definition 3.14. For a finite set X of class-elements, we define $h(X)$ equal to the least $h(y)$ among all members y of X.

The above definition is consistent with the properties of p-height, with the properties of ϕ^* which is identified with ϕ. Thus in particular $h(z) = h(\phi(z))$ for any isomorphism $\phi : M \to A$ and each class-element $z \in M$. It also makes sense to write $x \in \phi(z)$ for class-elements $x \in A$ and $z \in M$. By Lemma 3.8, if ψ is an isomorphism then each class-element $x \in A$ is contained in $\psi(z)$ for some class-element $z \in M$ which satisfies $h(x) = h(\phi(z))$.

From this point on, we completely reduce the situation to the case when both A and M are viewed as cardinal (i.e., disjoined and ordered) sums of uniformly computable equivalence structures E_{A_p} and (respectively) E_{M_p}. To illustrate that $M \not\cong_c A$, it is sufficient to to diagonalise against each computable total and injective ϕ that satisfies $h(x) = h(\phi(x))$ and also has the property $(\forall x \in E_{A_p})(\exists z \in E_{M_p})[x \in \phi(z) \,\&\, h(x) = h(\phi(z))]$ for each p (see above, and see Lemma 3.8).

3.5.4. *The subrequirements.* For each individual class-element x in E_{A_p} (or/and the corresponding cyclic C_x), we will attempt to satisfy:

$$R_{e,x}: \ [\exists^\infty y\,(h(y) > h(x))] \implies [(\exists z \in M_p)\, h(z) \neq h(\varphi_e(z))],$$

unless φ_e proves that it is not an isomorphism for some global reason (such as non-totality, to be clarified).

3.5.5. *Priority.* In both M and A, the class-elements will be ordered according to the index (number) that represents the respective class/summand in the presentation of M or A, respectively. We assume that class-elements having smaller indices appear earlier in the construction. If a class-element x has a smaller index than a class-element x' of the same p-component, then we say that x has a higher priority than x', or *x is on the left of x'*. The priority order on the requirements R_e is standard for finite injury constructions, while for the fixed e the subequirements $R_{e,x}$ are ordered according to the priority of their witnesses x. No priority tree will be necessary.

3.5.6. *Informal diagrams.* We will be using informal diagrams to represent parts of M and A. In such diagrams, line segments represent class-elements, and the length of the segment reflects the height/size of the respective class-element with longer segments representing class-elements of greater heights. Recall that $\varphi_e(x)$ (which is identified with φ_e^*) is equal to a finite collection of class-elements in A. This situation will be reflected on the diagrams as well, and for this we group the respective class-elements of A into a *clique* (the technique of cliques was introduced in [DMN15]). In many occasions we will suppress cliques to simplify a diagram. Then the clique is replaced with a class (typically of the highest priority among those) having the height equal to the height of the whole clique (i.e., the smallest in size). The structure M (identified with E_M) will always be at the bottom of any diagram, and A (or E_A) will be at the top. Classes of a higher priority will be to the left of classes having lower priority. We will define ψ so that at every stage it matches a class-element in A with a single class-element in M (and not with merely a finite set of those). We use dashed lines to represent ψ (which is identified with ψ^*). And we use arrows to represent computations of φ_e, and we circle the class-elements that share the same pre-image under φ_e. See an example below.

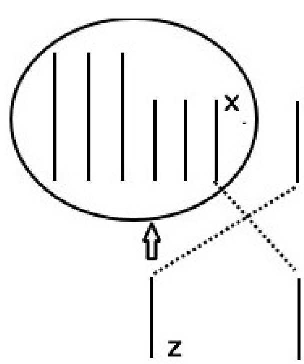

FIGURE 1. Here $z \in M$ and $\varphi_e(z)$ contains $x \in A$, but $\psi(z)$ is outside $\varphi_e(x)$.

> *Disclaimer*: Although such informal diagrams may help the reader to understand the construction, the formal proof does not rely on the diagrams. The reader should not completely rely on the suggested graphical intuition which is often misleading.

3.5.7. *Tangles.* At every stage of the construction a class x of A_p or a class z of M_p can be put into a tangle. The tangles appear from several repeated applications of the basic diagonalisation strategy. We will define tangles formally later (by recursion in the construction). At this point we note that a "typical" configuration of classes and maps that form a tangle can be (informally) described by the diagram below.

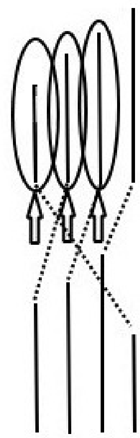

FIGURE 2. A "typical" tangle.

Note that the heights/sizes of class-elements increase from left to right in A_p (top), while the sizes/heights are shifted in M_p (bottom). Recall the dashed lines stand for ψ and the arrows refer to φ_e. The diagram is quite informal, since the φ_e-image of a class-element of M_p is in general a linear combination of class-elements in A_p. In the tangle, we show only the class-element of the highest priority in $\varphi_e(z)$ that realizes its height. Note also that we require $h(\varphi_e(z)) = h(z)$ for any z in the bottom, unless $\varphi_e(z)$ is undefined (this could be the case for the right-most class-element on the diagram). All tangles will necessarily have at least two distinct class-elements of A in it. Several elementary properties of tangles will be stated later in the verification.

3.5.8. *Condition for $R_{e,x}$ to be eligible to act.* Let x be a class-element of A_p. The class-element will be permanently assigned to $R_{e,x}$ and will be assumed to be currently *not* a part of any *tangle*. Recall that at most one class of E_A may grow at any stage. We say that $R_{e,x}$ is eligible to act if:

(1) There is some class-element z in M for which $x \in \varphi_e(z)\downarrow$ and $h(x) = h(\varphi_e(z))$.
(2) For every class-element $z \in M_p$ with index less than the index of x we have $\varphi(z)\downarrow$, and furthermore for each such z either $h(\varphi(z)) = h(z)$ or alternatively both $h(\varphi(z))$ and $h(z)$ have grown larger than $h(x)$.
(3) For every class-element $x' < x$ either $h(x') > h(x)$ or there is some $z' \in M$ such that:
    (a) $x' \in \varphi_e(z')\downarrow$,
    (b) $h(x') = h(z')$ (see Lemma 3.8),
    (c) $\varphi_e(z')$ does not include the unique class-element y of A_p that is ready to increase its height height since the previous stage (it follows from (4) that there is such a class).

(4) There exists a unique class-element $y \in A_p[s-1]$ that is now ready to grow its height since the previous stage (according to the fixed enumeration of A) and satisfies:
   (a) the index of y is greater than the index of x,
   (b) y has never been *declared used* with respect to x (to be defined)
   (c) y tries to pass x in height. That is, $h(y)[s-1] = h(x)[s-1]$.

(Note that (c) of (3) and (4) share the same class-element y.)

3.5.9. *Action at stage s.* Assume a class $y \in A$ is ready to grow in size (by 1), according to the enumeration of A. Before we let it grow, we attempt to meet R_e (which controls the respective p-component $A_p \ni x$). Pick the highest priority $R_{e,x}$ eligible to act at the stage. Then act for $R_{e,x}$ as follows:

(1) Extending a tangle. If y is not currently a part of any tangle, or is the left-most element of a tangle, then first perform the line-up and then the swap substeps as described below (do nothing otherwise). This way we either adjoin x to the tangle of y or will create a new tangle with y.

(1.1): Line-up. Let z be such that $\varphi_s(z) \ni x$ and $h(z) = h(x)$. If $\psi(z) = x$ then do nothing. Otherwise, let $a = \psi^{-1}(x)$ and $x' = \psi(z)$.

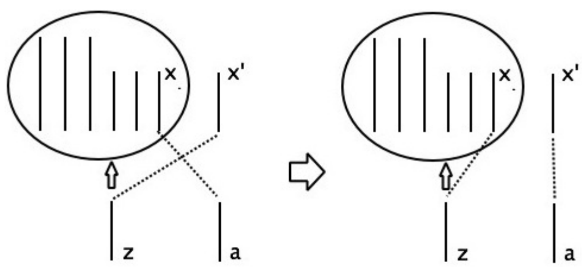

FIGURE 3. A "typical" line-up.

Note $h(z) = h(x)$ by assumption on x (it realizes the minimum height among all members of z). Reset ψ on z and a by swapping their images:

$$\psi(z) = x \text{ and } \psi(a) = x'.$$

(1.2): Swap. Suppose y is the left-most A-class of a tangle T, and let $b = \psi^{-1}(y)$. Then b is the right-most M-class of the same tangle T. The swapping substage is performed in two phases:

Phaze 1. Interchange the ψ-images of z and b by setting $\psi(z) = y$ and $\psi(b) = x$.

Phaze 2. Grow y in size thus increasing $h(y)$ by one, and grow z accordingly, to maintain $h(z) = h(\psi(z)) = h(y)$. *Declare y used with respect to x.*

(2) Refining the tangle T. If the class y has become equal in size/height to another element u in its tangle T (in which case it is necessarily the element right-adjacent to u in T), then swap ψ^{-1} between y and u by interchanging their ψ-preimages, and then remove the pair $(y, \psi^{-1}(y))$ from the tangle T. Note in this case the height/size the new ψ-preimage of the class u (that stays in T) must be increased.

3.5.10. *Initialisation and e-expansionary stages.* There exists a Π^0_2 predicate that holds on $e \in \omega$ and two infinite structures $\mathcal{A}$ and $\mathcal{B}$ (the latter two taken as oracles) iff $\varphi_e : \mathcal{A} \to \mathcal{B}$ is an onto-isomorphism.

Declare a stage *e-expansionary* if the predicate "fires" on A and M but does not "fire" on any $e' < e$. Then initialize all $R_{e''}$ with $e'' > e$ by setting all their parameters undefined, and

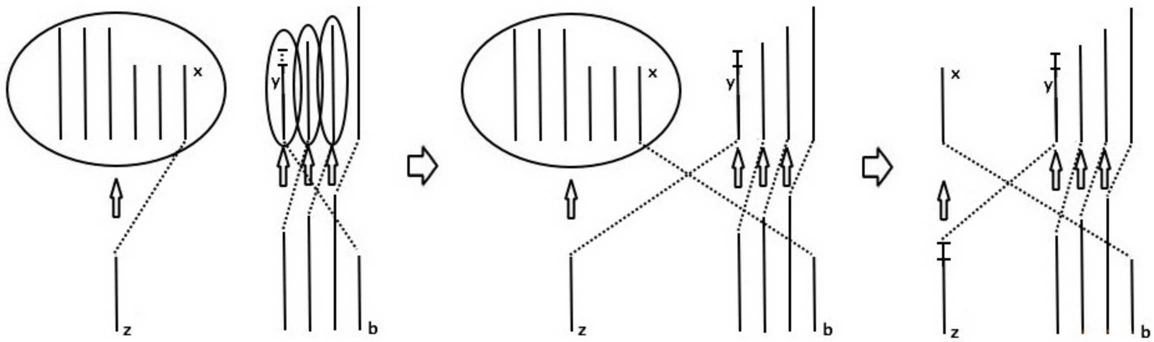

FIGURE 4. A "typical" swap. In the left-most part of the diagram, y is ready to increase its height (as indicated by the extra segment on top of y).

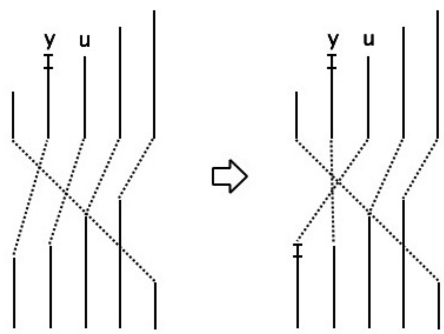

FIGURE 5. A "typical" refining operation.

assign all the p-components previously used by the $R_{e''}$ (plus at least one extra p-component on top of these) to R_e. From these point onwards, unless R_e gets initialized, these p-boxes will be permanently associated with R_e.

Remark 3.15. *One may come up with a detailed careful definition of what it means for φ_e to look more like an isomorphism at a given stage, but it is clearly a Π^0_2-condition. For our purposes the above crude approach is equally fine. Nonetheless, two such carefully designed properties have already appeared in (2) and (3) for eligibility of $R_{e,x}$-action. These properties will be used in the verification.*

3.5.11. *Construction.* In the construction, we assume that A has a complete effective decomposition (otherwise, uniformly pass to $\bigoplus_p G_{E_{A_p}}$). We do not distinguish between the elementary summands (in the fixed decomposition of A) and the respective equivalence classes in E_A. Under this identification, we adjust the enumeration of A so that at most one class of E_{A_p} grows by at most one element at every stage, and furthermore this happens for at most one prime p.

We follow the definition of an e-expansionary stage to see which p-components are controlled by which requirements. If at the beginning of stage s the structure $A_p[s-1]$ is ready to increase the height/size of one of its class-elements (according to its enumeration), we *first* let the respective requirement act according to its instructions (see action at stage s), and *then* we resume the enumeration of A_p. If a new class-element c is introduced to A, we also introduce a new class-element u to M_p and set $\psi(u) = c$.

3.5.12. *Verification.* The following properties of tangles follow (by induction) from the description of the construction.

(1) *Each class-element x of A can be a part of at most one tangle at any stage of the construction.* Indeed, to enter a new tangle, a class-element must be currently not in any tangle. Since ψ is an injection at any stage, and furthermore if $x \in A$ is in a tangle then $\psi(x)$ must also be in the same tangle, we also conclude that every $z \in M$ can be a member of at most one tangle at any stage.

(2) *A class-element $x \in A_p$ the height of which is not maximal in its tangle leaves the tangle only if $h(x)$ increases in the construction.* Indeed, according to *refining the tangle* substep, the only possibility for a class-element in A_p to leave its tangle is to increase in size, unless the class-element has the largest height among all elements in the tangle.

(3) *If $z \in M_p$ is in a tangle, and $\psi(z)$ changes due to some other classes leaving the tangle, then $h(z)$ must have increased.* This follows from the refining the tangle substage (see the last line of its description).

(4) *If $x \in A_p$, and $h(x) < \infty$ has reached its maximum at stage s, then either x will never enter any tangle after stage s, or it will be eventually in a stable tangle that it will never leave.* Indeed, if x is currently in some tangle but its height is not the largest among the other class-elements in the tangle, then there will always be at least one element in the same tangle having its height greater than $h(x)$. In this case the tangle cannot be completely dissolved (follows from refining the tangle substage). If $h(x)$ is the largest among the other elements in the tangle, then it could leave the tangle. In any case, if x ever finds itself outside any tangle, it cannot enter a new tangle as the right-most class (see extending the tangle substage), for it would have to increase its height. Then it either stays forever untangled, or it enters some new tangle (but not as the largest-height class-element), in the latter case it will stay there forever.

Lemma 3.16. *For any p, $A_p \cong M_p$.*

Proof. Recall at every stage of the construction we had an isomorphism $\psi[s] : M[s] \to A[s]$ that agree with the full decompositions. Thus, regardless of the true outcomes, we always build M such that each M_p has Ulm type 1. We split the lemma into several claims. Although we usually suppress index s, all our considerations refer to a situation at some stage of a construction (unless specifically said otherwise).

Claim 3.17. *Suppose a class-element $x \in A_p$ has finite height $h(x) = k$. Then $\psi^{-1}(x)$ will be redefined at most finitely often in the construction.*

Proof of Claim. Note that, regardless of the outcomes, each particular A_p can be controlled by at most finitely many R_e (one after another) during the construction, and eventually A_p is stably associated with some R_e. In the cases below we assume that the stage is large enough so that A_p is always controlled by some fixed R_e. The case when $A_p = 0$ gives $M_p = 0$ for free, since according to the construction we never get to start enumerating M_p. So we assume that $x \in A_p$ and $h(x) > 0$ is finite. We also assume that $h(x)$ has reached its stable finite value, i.e. x will never "grow". There could be several reasons for $\psi(x)$ to be redefined during the construction, these are exhausted by x being potentially involved in either the line-up procedure, or the swap, or the refining a tangle. Each of these possibilities has several possible subcases, we go through these subcases carefully.

Case 1. *Line-up in which x is involved as a class "on the left".* That is, x is in $\varphi_e(z)$ but $\psi(z)[s] \notin \varphi_e(z)$. Then we set $\psi(z) = x$. Once we perform this particular line-up procedure, x will join (start) a new tangle in which x will become the left-most element. Note Case 1 cannot occur again before x leaves the tangle, because $R_{e,x}$ cannot be active again while x is a part of any tangle. (But Case 2 might hold, see below.) Note, however, that x leaves the tangle only if $h(x)$ increases (see properties of tangles).

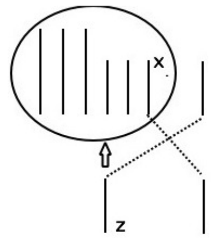

FIGURE 6. A "typical" Case 1.

Case 2. *Line-up in which x is involved as a class "on the right".* In this case, for some $x' \in$

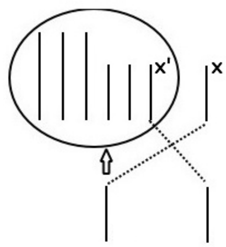

FIGURE 7. A "typical" Case 2.

$\varphi_e(\psi^{-1}(x))$ the requirement $R_{e,x'}$ initiates the line-up and asks for ψ to be redefined on $z = \psi^{-1}(x)$ and to be set equal to x' (on z). Note that x' must be equal to x in size, and also x' must be of a higher priority than x. Case 2 cannot happen again to x with the same x' because x' will now be placed in some tangle, and $h(x')$ will have to be increased before $R_{e,x'}$ acts again (if ever).

Remark 3.18. *Note that Case 2 can potentially occur even if x is currently involved in a tangle, but only if x is the left-most class in its tangle. To see why x has to be left-most in its tangle, note that otherwise $h(\varphi_e(\psi^{-1}(x))) < h(x)$ contrary to the necessary condition $h(x') = h(x)$. Therefore, since x is left-most in its tangle, the swap will result in particular in swapping the ψ-preimages of x and x' which have equal height/sizes, and it will be consistent with our definition of a tangle.*

Case 3. *Swap in which x is involved as a class "on the left".* Once this swap is done, x will become the left-most element of a tangle, and $R_{e,x}$ will not act again unless the class leaves the tangle (and thus $h(x)$ increases). Thus, this case cannot hold again.

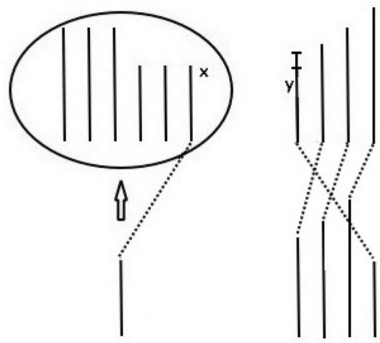

FIGURE 8. A "typical" Case 3.

Case 4. *Swap in which x is involved as a class "on the right".* Note in this case $h(x)$ must be ready to grow, contrary to the hypothesis.

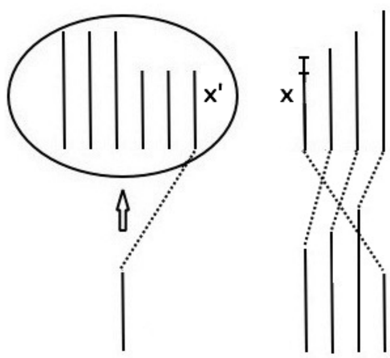

FIGURE 9. A "typical" Case 4.

Case 5. *x is involved in refining a tangle.* By our assumption, $h(x)$ has reached its maximum value. Therefore, there are only two possible subcases in which x could be involved in a refining a tangle substep. In both subcases below we may assume that the cases discussed above (such as line-up involving x) no longer apply to x.

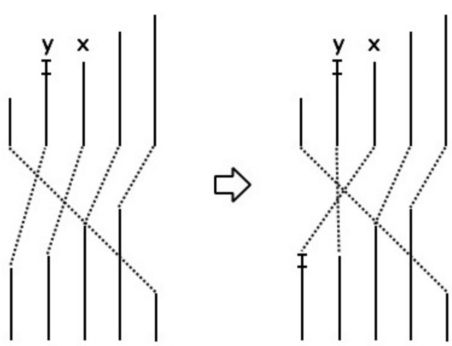

FIGURE 10. A "typical" refining operation in the case when x is not the right-most class.

(1) x is the right-most class in its current tangle. In this case x leaves the tangle. If it never enters any other tangle, then $\psi^{-1}(x)$ is stable. It can later be involved in some other tangle, due to $R_{e,x}$ acting, but only as the left-most class. If it does enter some tangle, then it will become the stable left-most class of the tangle (since $h(x)$ is stable). In the latter case ψ^{-1} will never have to be changed either.
(2) Otherwise, i.e. x is not the right-most class in its tangle. Then the only possibility for x to leave the tangle is that $h(x)$ increases, thus it stays. But $\psi^{-1}(x)$ may be changed due to some y to the left of x leaving the tangle, in which case we have $h(y) \geq h(x)$ and y can no longer be involved in such action again. Since there are only finitely many $y < x$, the case will eventually no longer apply.

Note that we were excluding the possibility of Case j, $j < k$ when we were looking at Case k. It is crucial that there is no circularity in this assumption in this particular proof. Thus, regardless in which order the cases discussed above occur (if this happens at all), we eventually have $\psi^{-1}(x)$ stable. □

Claim 3.19. *Let $z \in M_p$ be a class-element such that $h(z)$ is eventually stably finite. Then $\psi(x)$ eventually gets a stable definition that will never be changed in the construction.*

Proof. Unfortunately, we cannot use the previous claim. We have to go through similar cases here, but with z being in M rather than in A. The cases are quite similar to those in the previous claim, but they are more subtle. For instance, some cases will be using the conditions for $R_{e,x}$ to be eligible to act. We assume that $h(z)$ has reached its stable value.

Case 1. *$\psi(z)$ is re-defined during a line-up.* Suppose z is involved in the line-up substep of $R_{e,x}$, for some x. There are several subcases to consider.

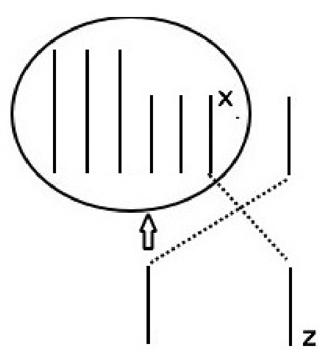

Figure 11. A "typical" Case 1.

(1) The subcase when $x \in \varphi_e(z)$ and consequently z needs to be lined-up with x, is impossible for the following reason. After such a line-up is finished, we would have lined-up z with the class-element which tries to pass x in height/size, in particular increasing $h(z)$ which is impossible by our assumption on the stage.
(2) The subcase when z satisfies $\psi(z) = x \in \varphi(y)$, but $\psi(y) \neq x$ (where x is the witness of $R_{e,x}$ which is eligible to act). Each R-subrequirement can possibly involve z into its line-up at most once, for the following reason. Indeed, each time $R_{e,w}$ acts its witness w enters a tangle as the left-most class. It will not leave the tangle unless $h(w)$ increases (see properties of tangles). However, for z to be involved in a line-up with w me must have $h(w) = h(z)$, and we assumed $h(z)$ is stable.

Recall that we identify class-elements with their indices in the construction. Note that there are only finitely many subrequirements that have witnesses with smaller indices than the index of z, and each such subrequirement can swap z at most once. We assume this has already happened, and thus it remains to see what happens with $R_{e,x}$ for $x > z$.

If $\varphi_e(z) \uparrow$ then $R_{e,x}$ is forbidden from acting (see the conditions for $R_{e,x}$ to be eligible to act). If $\varphi_e(z) \downarrow$ but $h(z) \neq h(\varphi(z))$ then again $R_{e,x}$ is not allowed to act. Thus, we assume that $\varphi_e(z) \downarrow$ and $h(\varphi_e(z)) = h(z)$ (at a stage). In fact, we can assume that the equality $h(\varphi_e(z)) = h(z)$ is stable, for the following reason. If it is not stable, then we'd have an evidence that φ_e cannot be an isomorphism (Lemma 3.8) and R_e would eventually never act again on M_p and A_p (see the expansionary stage definition). In particular, ψ would be eventually stable on $M_p \ni z$.

So let $x_0 \in \varphi_e(z)$ be a class-element with $h(x_0) = h(z)$. In fact, we can assume that $h(x_0)$ is stable (if such an x_0 does not exist, then $h(\varphi_e(z)) \neq h(z)$). There are only finitely many finitely many subrequirements $R_{e,x'}$ with $x' < x_0$ that can only use z in their line-up. As we have noted above, each of these $R_{e,x'}$ will line-up with z at most once. If $x > x_0$ then $R_{e,x}$ would not act since R_{e,x_0} would be acting instead. This means that x_0 will be put into a tangle as the left-most class. But since we have agreed that $h(x_0)$ is stable, this means that x_0 will never leave its tangle (as we have already discussed above). This means all $R_{e,w}$ of lower priority than R_{e,x_0} (including $R_{e,x}$) *will be permanently blocked from acting.*

Case 2. z is involved in a swap. Without loss of generality, we have reached the stage after which z is never involved in any line-up (Case 1). Since there are exactly 2 classes of M that are involved in a swap, there are two possibilities.

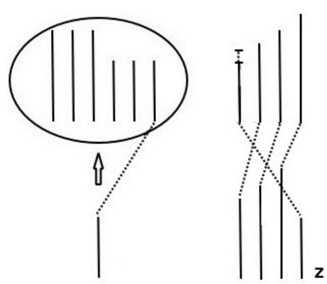

FIGURE 12. A "typical" Case 2.

(1) $\psi(z) = x \in \varphi_e(z)$ and $h(\psi(z)) = h(z)$, and consequently the new ψ-image of z will have to be the class-element y trying to outgrow x of $R_{e,x}$. But this situation is impossible, since $h(z)$ will have to be increased, contrary to the hypothesis on the stage ($h(z)$ is stable).

(2) $\psi(z) = y$, where y is ready to outgrow x of $R_{e,x}$ in height. In this case, since $R_{e,x}$ is assumed to act on y, the class-element x must be of a higher priority than y. Note that z could be involved in a tangle, but then z must be the right-most class in its tangle. (In fact, assuming z is no longer involved in any line-up procedure, z can leave its tangle only if z increases its height.) According to the swap procedure, $\psi^{-1}(x)$ and z will exchange their ψ-images, which will necessarily result in moving

$\psi(z)$ to a higher priority class (namely, from y to x). We conclude that in this scenario $\psi(z)$ can change only to a higher priority class due to a swap.

Case 3. $\psi^{-1}(z)$ is reassigned due to refining a tangle. We also assume that Cases 1 and 2 no longer apply. If z stays in the tangle but is involved in re-arranging of ψ due to some other class leaving, then $h(z)$ necessarily increases, contrary to our assumption. Thus, for Case 3 to be applicable z must leave the tangle. By our assumption Cases 1 and 2 no longer apply, thus if z leaves the tangle then it will never enter any other tangle again.

□

The two claims above imply that all class-elements of finite sizes in A_p and M_p are eventually stably and bijectively matched by ψ. (Bijectivity follows from $\psi[s]$ being bijective at every stage.) It follows also that ψ restricted to finite classes preserves height/size.

When it comes to infinite classes, ψ may have no stable definition (we leave an example to the reader). Nonetheless, using that A_p is relatively c.c. and the properties of ψ established above, we demonstrate that $A_p \cong M_p$.

First, assume that the rank of the socle of A_p is finite. Equivalently, E_{A_p} has finitely many classes. At every stage the number of classes in both E_{A_p} and E_{M_p} is the same. By the property of ψ demonstrated above, the number of finite classes in both E_{A_p} and E_{M_p} is the same. Thus, the number of infinite classes in both E_{A_p} and E_{M_p} is the same as well, and consequently $A_p \cong M_p$.

If E_{A_p} has infinitely many classes, then A_p being r.c.c. implies that almost all classes in E_{A_p} are equal in size. Assume that this size is finite and is equal to m. The images of all finite class-elements (in both A_p and M_p) will eventually match both ways. Then A_p will have at most finitely many class-elements of infinite height. Go to a stage s after which the height of all these finitely many elements is greater than m. Then $M_p[s]$ will also have the same number of class-elements to match these finitely many class-elements of $A_p[s]$. Although ψ can keep changing between these finitely many class-elements, M_p will end up with the same number of class-elements having infinite height.

Finally, suppose E_{A_p} has only finitely many finite classes and infinitely many infinite classes. Since the finite class-elements must match, it is sufficient to observe that both E_{A_p} and E_{M_p} must have infinitely many classes. Indeed, there will be infinitely many stages at which new class-element are introduced to both A_p and M_p. Since the finite-height ones must match bijectively, the rest must be infinite-height class-elements. We conclude that $A_p \cong M_p$ in this case as well. □

Note that there exists a uniform Σ^0_3 predicate Ξ such that Ξ^A holds if and only if the construction described above has infinitary many e-expansionary stages for some e. Although the construction refers to $\bigoplus_p E_{A_p}$ (and not to A itself) with all possible uniformity, Proposition 3.4 ensures that it could instead refer to A itself without any loss of uniformity. The construction does not really use the fact that A_p is c.c. for every p, we use this property only to illustrate that under this hypothesis the construction has nice properties.

Now suppose Ξ holds on A.

Lemma 3.20. *If there exist infinitely many e-expansionary stages for some e, then A_p satisfies the WHP for almost every p.*

Proof. Fix e least such. It is sufficient to consider any A_p which is eventually controlled by R_e, say after stage s. Assume A_p is does not satisfy the WHP. This means that for some class-element x of A_p with $h(x) = k < \infty$ there exist infinitely many class-elements $y \in A_p$ such that $h(x) < h(y)$. Without loss of generality, k is the least possible and x is the highest priority class with $h(x) = k$. Recall that A_p is relatively c.c. by assumption. Fix a stage $s_0 > s$ large enough such that:

I. Each of the finitely many class-elements x' of A_p with the property $h(x') = k$ (including x itself) has reached its final height. (Note A_p being relatively c.c. combined with the assumption on x imples there could be at most finitely many such x'.)
II. Each of the finitely many x' (as above) with $h(x') = k$ are either stably involved in some tangle, or will never be involved in any tangle. (Note that x has stable height.)
III. For each $y \in A_p$ of priority higher than x, $h(y)[s_0] > k$. (Each such y must have height larger than k, by the choice of x and k.)

If x is involved in a tangle itself, then the tangle will necessarily be stable. Indeed, all y on the left of x are large, and therefore x must be *not the right-most class-element in its tangle* (indeed, A-elements increase in height from left to right in any tangle). In this case there will be at least one class-element permanently involved in the tangle of x which will be of a greater height than x. This tangle will witness that R_e is met, contrary to the choice of e. The same can be said about any tangle which will ever be formed by $R_{e,x}$ after stage s_0.

Aiming for a contradiction, assume x is not in any tangle at and after stage s_0. First of all, note that there are only finitely many class-elements $y \in A_p$ such that y is ever declared x-used. Furthermore, there will be only finitely many class-elements in A_p that had been declared used by $x' < x$ before they try to grow their height larger than $k = h(x)$ (recall all such x' are eventually very large). Fix $s_1 \geq s_0$ at which all these finitely many classes listed above have size at least k. Since φ_e is an isomorphism, we may assume that for some z, $\varphi_{e,s_1}(z) \downarrow\ni x$ and $k = h(x) = h(\varphi_e(z)) = h(z)$, and that $R_{e,x}$ is eligible to act when necessary.

Take the highest priority class-element z^* which is not in $\varphi_e(z)$ (and not in $\varphi_e(z') \ni x'$ for all $x' < x$) and which attempts to grow its height greater than $h(x)$ at a stage $t > s_1$. By the choice of z^*, each class-element to the left of z^* which is not declared used by any $x' \leq x$ must have already demonstrated that its height is greater than $h(x^*)$ (and w.l.o.g. than $h(x)$). This means that none of the classes to the left of z^* can force z^* to be tangled with them, since all these classes are too large. Although z^* can still be tangled with some classes on its right, $R_{e,x}$ will be eligible to act with z^* and thus R_e will be permanently met. □

This gives the second half of the proposition. □

3.6. **A predicate Ψ describing categoricity.** The property Ψ is the conjunction of the following:

a. A is a TAG.
b. For very p, A_p is relatively c.c.
c. $\Xi(A)$ from Proposition 3.10.
d. For every computable G (identified with its index),
   if G is a TAG and there exists $k \in \omega$ such that:
   i. $\forall p > k$ G_p and A_p satisfy the WHP, and
   ii. $\forall p \leq k$ G_p and A_p are relatively c.c., and
   iii. $\forall p > k$ $E_{G_p} \cong E_{A_p}$, and
   iv. $\forall p \leq k$ $G_p \cong A_p$,
   then $G \cong_c A$.

We view predicate Ψ as both a statement in a meta-language and a predicate on ω. Its computability-theoretic complexity will be analysed shortly. Regardless of the complexity of Ψ, we first prove the most important lemma below.

Claim 3.21. *A satisfies Ψ iff A is a c.c. TAG.*

Proof. Suppose A satisfies the predicate. Then the first three conjuncts and Proposition 3.10 imply that for every p the p-component A_p has Ulm type 1, and furthermore almost all p-components of A satisfy the WHP. Now let $G \cong A$, also computable. Then, since A has the listed above properties, so does G. Thus, there exists a k with the properties listed in the conditions i.-iv. that works for both A and G. Since A satisfies the predicate, we must have $A \cong_c G$.

Now suppose A is a c.c. TAG. Then *a.* and *b.* are satisfied trivially. Since $\neg\Xi(A)$ is impossible (see Proposition 3.10), *c.* must hold as well. Proposition 3.10 in conjunction with *a.* and *b.* implies that a.e. A_p satisfies the WHP. Pick any G so that for some k conditions i.-iv. hold. It is clear that i.-iv. together imply that $A \cong G$. (Use Lemma 3.2 to see that iii. implies isomorphism of the respective p-components). Now, since A is c.c., $G \cong_c A$ holds. □

Claim 3.22. *The complexity of* Ψ *is* Π^0_4.

Proof. Property a. of being a TAG can be expressed by a Π^c_2-formula. Property *b.* saying that for very p, A_p is relatively c.c. is Π^0_4 since being a relatively c.c. TAG is a (uniformly) Σ^0_3-property (Prop. 6.4). From Proposition 3.10 we know that Ξ has complexity Σ^0_3. Property *d.* requires more care.

Note that the WHP is a Π^c_3-property (Lemma 3.3), thus i. is Π^0_3. As we noted above, being relatively c.c. is Σ^0_3 in general, with all possible uniformity. Since the quantifier $\forall p \leq k$ is bounded, the property ii. is Σ^0_3. Before we look at iii., note that in iv. isomorphism can be equivalently replaced with a computable isomorphism, and existence of a computable isomorphism is Σ^0_3. Although iii. looks like a Π^0_4-property, we claim that iii. is in fact Π^0_2. Indeed, by Lemma 3.7 to express that $E_{G_p} \cong E_{A_p}$ it is sufficient to say that, for each finite equivalence structure D, $D \subseteq E_{G_p}$ (a uniformly Σ^0_1-fact) iff $D \subseteq E_{A_p}$. Thus, iii. is indeed Π^0_2 as claimed. Finally, existence of a computable isomorphism between G and A is Σ^0_3. We now collect the complexities:

$$\Pi^0_2 \,\&\, \Pi^0_4 \,\&\, \Sigma^0_3 \,\&\, \forall(\Pi^0_2 \,\&\, [\exists\ (\Pi^0_3 \,\&\, \Sigma^0_3 \,\&\, \Pi^0_2 \,\&\, \Sigma^0_3)] \implies \Sigma^0_3)$$

which simplifies to $\Pi^0_4 \,\&\, \forall(\Sigma^0_4 \implies \Sigma^0_3)$ and finally boils down to Π^0_4. □

4. Proving Π^0_4-completeness

Throughout the proof, we fix some Π^0_4-complete set S. For any e we will uniformly produce a computable torsion abelian group M^e which satisfies:

$e \in S \iff M^e$ is computably categorical.

Given any e, we can uniformly produce a double array of uniformly c.e. sets $\{V^x_y\}$ such that:

$e \notin S \implies (\exists x)(\forall y)\ V^x_y$ is finite,

$e \in S \implies (\forall x)(a.e.\ y)\ V^x_y$ is infinite.

Indeed, using a standard technique we may guarantee that if $e \notin S$ then for x and y that witness V^x_y is infinite, all sets $V^x_{y'}$ with $y' > y$ are also infinite.

4.0.1. *The basic diagonalisation strategy.* We suppress e and write M^e. Suppose we are building a TAG M and an auxiliary TAG T and want to make sure that $\varphi_e : M \to T$ is not an isomorphism. We use the following brute-force diagonalisation against φ_e:

(1) Reserve a prime p which will be used only by the strategy.
(2) Keep both A_p and M_p isomorphic to $\mathbb{Z}_p$ and wait for φ_e to converge on some generator x of M_p.
(3) Extend A_p to $\mathbb{Z}_{p^2}$. (In the construction, we will also be waiting for certain higher priority strategies to respond, this is why we made (3) a separate substep).
(4) Make both A_p and M_p isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$, but so that $x \in \mathbb{Z}_p$ in M_p while $\varphi_e(x) \in \mathbb{Z}_{p^2}$ in A_p.

It is clear that we have diagonalised against φ_e.

4.0.2. *Guessing combined with the diagonalisation strategy.* According to the notation introduced above, if $e \notin S \implies (\exists x)(\forall y)\ V^x_y$ is finite, then we need to produce an isomorphic computable copy of M that is not computably isomorphic to M.

Each potential existential witness x will be associated with its own substrategy responsible for building T_x and guessing whether $(\forall y)\ V^x_y$ is finite. Each set V^x_y will be associated with infinitely many primes $p_{\langle x,y,n\rangle}$, $n \in \omega$, and thus with the respective p-components of T_x and M. Although T_x will contain all $p_{\langle x',y,n\rangle}$-components with $x' \neq x$, the strategy guessing x will be acting non-trivially only on the $p_{\langle x,y,n\rangle}$-components of T_x and A, while the other primary components of T_x will be simply copying the respective primary components of A.

Initially, we keep all $p_{x,e,k}$-components equal 0. At stage s, the action of the guessing-diagonalisation strategy depends on whether V^x_e has grown ($e \leq s$):

Case 1: If $|V^x_y[s]|$ (the cardinality) has not increased from the previous stage, then set $n = |V^x_e[s]|$ and $p = p_{\langle x,e,n\rangle}$ and implement (or continue implementing) the basic diagonalisation strategy against φ_e within the p-components of T_x and M.

Case 2: Otherwise (if $|V^x_y[s]| = |V^x_y[s-1]| + 1$), make the p-components of both T_x and M isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$, where $p = p_{\langle x,e,n\rangle}$ and $n = |V^x_e[s-1]|$.

If it is indeed the case that $(\forall y)\ V^x_y$ is finite, then for every e there will be a stage at which the basic strategy will get a stable control over some p-component, and thus we will diagonalise against φ_e. The only difference with the basic diagonalisation strategy is that we don't know in advance in which p-component the diagonalisation will be successful (if ever).

On the other hand, if $(a.e.\ y)\ V^x_y$ is infinite, then we are left with at most finitely many distinct primary components that are not isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$ for the respective p. This feature will be used to non-uniformly build a computable isomorphism with the e'th potential isomorphic copy of M in the case when M needs to be c.c. (to be explained later).

The strategy will not have any meaningful outcomes useful for the other strategies in the construction (although its success or failure can be measured in a Π^0_3 way). We introduce one neutral outcome just for the sake the tree of strategies.

4.0.3. *The basic pressing strategy.* Fix an effecive enumeration $A_1, A_2, \dots$ of all partial computable structures in the language of additive groups. The basic pressing strategy associated with e will attempt to satisfy

$$A_e \cong M \implies A_e \cong_c M.$$

The strategy will attempt to build a computable isomorphism $\psi_e : M \to A_e$. Even if it fails to build an isomorphism, in certain cases the map ψ_e can be adjusted to a computable isomorphism (to be explained).

In the construction, the strategy will be given a priority. The definition of ψ_e will be different within different p-components, depending on whether the p-component is controlled by a higher or lower priority strategy.

Case 1: If M_p is controlled by a higher-priority guessing-diagonalisation strategy, then we map an arbitrarily chosen basis of the socle of M_p onto the first found basis of the socle of $[A_e]_p$. If $x \in [M]_p$ is an element of such a basis and there is a y such that $py = x$, then we map y to a $z \in A_e$ such that $pz = \psi_e(x)$.

Case 2: Otherwise, if M_p is controlled by a lower-priority guessing-diagonalisation strategy σ, at its substage (3) σ will wait for a confirmation that the p-component of A_e has also grown from $\mathbb{Z}_p$ to $\mathbb{Z}_{p^2}$. At this point we would have already mapped the generator of $M_p \cong \mathbb{Z}_p$ to the generator of the respective $\mathbb{Z}_p$ in A_e, and once A_e responds by growing we extend ψ_e naturally.

If the lower priority σ initializes the p-component due to some V^x_y increasing in size, we immediately stop waiting and treat the p-component as if it had a higher priority (see above).

If the p-component of A_e enumerates itself too quickly, or proves that it is not isomorphic to M_p by giving a finite substructure not embeddable into M_p, we *freeze* M_p and restart the lower priority diagonalisation strategies by forcing them to use new fresh primary components.

Note that our definition of ψ will be effective on a p-component controlled by a lower priority strategy. The verification of this fact is trivial and is left to the reader[2].

Also note that if all the higher priority strategies end up failing to diagonalise, then almost all of their p-components will be homogeneous (i.e., of the form $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$ for the respective p). This means that our naive definition of ψ_e will be correct for almost all such components. On the other hand, if one such higher-priority strategy succeeds in its diagonalisation, it means that M is not c.c. and it is quite natural that ψ_e is not an isomorphism.

The outcomes are ∞ and w, the former corresponds to A_e responding and following M, the latter incorporates the finitary winning and the waiting outcomes.

4.0.4. *The tree of strategies, and initialisation.* The priority ordering is standard, and the tree of strategies is usual for infinite injury constructions, thus we skip their formal definitions. We note that the guessing-diagonalisation strategies will be cloned and will act according to their guesses, and will use distinct arrays of primes. In contrast, the basic pressing strategies will not be cloned, and one level of the tree will work with exactly the same pressing strategy.

The definition of the current true path is usual. We initialize all guessing-diagonalisation strategies to the right of the current true path by instanteneously making their p-components isomorphic to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$. We also assign the strategies with new fresh p-components. From this point on, the homogenized p-components will be treated by pressing strategies via thier Case 1, i.e. as higher priority p-components. The pressing strategies are never initialized.

4.0.5. *Finalizing the proof.* The construction is standard, and the most significant part of its verification was incorporated into the description of strategies. We put the pieces together. If $(\exists x)(\forall y)$ [V^x_y is finite] then the clone of the guessing-diagonalisation strategy associated with x and along the true path will build $T_x \cong M$ but $T_x \not\cong_c M$, thus M is not c.c. On the other hand, if $(\forall x)(a.e.\ y)$ [V^x_y is infinite], as we have noted above each pressing strategy assigned with $A_e \cong M$ will end up building a computable ψ_e which will be different from an actual

[2]Note that any basis of the socle of $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$ induces its full decomposition. Also note that any element of the socle of $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ not divisible by p generates a $\mathbb{Z}_p$-direct summand of the group. Thus, our actions in Case 2 always lead to an isomorphism.

isomorphism at only finitely many inputs. We can non-uniformly correct ψ_e and see that $A_e \cong_c M$, as desired.

5. Proof of Theorem 1.5

Suppose $A = \bigoplus_p A_p$ is a torsion abelian group which is not computably categorical. Our task it to produce infinitely many computable copies of A that are not pairwise computably isomorphic. If for some p the p-component A_p of A is not c.c., then we can produce infinitely many computable copies of A_p which are not computably isomorphic [Gon80]. Then we can copy the other primary components trivially to get infinitely many copies of the whole A that are not pairwise computably isomorphic.

Now suppose each of the A_p is computably categorical, but A is not. Then fix some other computable copy B of A that is not computably isomorphic to A. Unfortunately, it could be the case that $B \not\cong_{\Delta^0_2} A$, thus we cannot simply refer to a meta-theorem of Goncharov (see, e.g., [EG00]) saying that in this case there exist infinitely many computable copies of A up to computable isomorphism. The other general methods (e.g. [EG00]) seem to be of no significant help either. A new idea is required to prove the theorem.

For each $j = 0, 1, 2, \dots$ we build a computable copy

$$C^j = \bigoplus_p C^j_p,$$

of A. We need to satisfy

$$R_{e,j,k} : \varphi_e \text{ is not and isomorphism of } C^j \text{ onto } C^k.$$

We will also define (computable) isomorphic maps ϕ^j_p from C^j_p onto *either* A_p *or* B_p, the final choice will be determined by the construction. In fact, our guess on the range of ϕ^j_i can be changed (at most) finitely many times, but we will see that it will not effect the isomorphism type of C^j.

5.0.1. *Expansionary stages and initialisation.* Recall that "φ_e is an onto isomorphism of C_i onto C_k" is a uniformly Π^0_2-property. Using the (uniform) recursion theorem, we assume that the correctness of this statement will be measured by a Π^0_2 predicate $\exists^\infty x R(e, i, j; x)$, and at every stage the *length of agreement* $l_{e,i,j}$ will be set equal to the maximal number n such that we have seen $\exists^n x R(e, i, j; x)$ at stage s. (In fact, we could avoid using the recursion theorem here by allowing R to use the copies as oracles.) A stage is (e, i, j)-expansionary if $l_{e,i,j}$ has been increased from the previous stage. We note that regardless of the outcomes, each $C_i \cong A$.

At every stage each active $R_{e,i,j}$-requirement will be given control over at most $(l_{e,i,j} + 1)$-many distinct primes p, and thus over the respective p-components of C^i and C^j. (If $1, \dots, p_i$ are the primes currently controlled by the higher priority R-strategies, then $R_{e,i,j}$ will be given control over $p_{i+1}, \dots, p_{i+l_{e,i,j}+1}$.) If some higher priority strategy increases its respective $l_{e',i',j'}$ then we initialize $R_{e,i,j}$ by (perhaps) giving the control over ϕ^j_p and ϕ^i_p to the next highest priority strategy (if its l-parameter is large enough). In this case the current definitions of ϕ^i_p and ϕ^j_p may be changed, according to the instructions of the higher priority strategy.

5.0.2. *The strategy.* Fix a prime p controlled by $R_{e,i,j}$ at stage s. The strategy for $R_{e,i,j}$ is rather straightforward:

Case 1. Suppose ϕ^p_i and ϕ^p_j has never been defined on any part of C^i_p and C^j_p, respectively. Then define ϕ^p_i and ϕ^p_j so that C^i_p and C^j_p copy A_p and B_p, as follows. Declare the ranges of $\phi^i_p[s]$ and $\phi^j_p[s]$ to be $A_p[s]$ and $B_p[s]$, respectively, and enumerate C^i_p and C^j_p

so that the group operation is obtained by the straigthforward pull-back along $\phi_p^i[s]$ and $\phi_p^j[s]$.

Case 2. Suppose ϕ_i^p and ϕ_j^p has already been defined on $C_p^i[s]$ and $C_p^j[s]$, respectively. If currently $\phi_i^p : C_p^i \to A_p$ and $\phi_j^p : C_p^j \to B_p$, then keep extending the definition just as in Case 1. Otherwise, if currently $\phi_i^p : C_p^i \to B_p$ according to the instructions of the lower priority strategy that has now been initialized, then search for a finite substructure $F \cong C_p^i[s]$ of A_p (this exists since $A_p \cong B_p$). Then redefine $\phi_i^p : C_p^i \to F \leqq A_p$ and from now on extend this definition naturally as in Case 1, unless later initialized. (The adjustment of ϕ_j^p is similar).

5.0.3. *Construction.* We order the strategies linearly according to some natural effective priority order. Initially, we set the length of agreement $l_{e,i,j} = 0$ for every e, i and j. At stage s we let the first s strategies (in the order of decreasing priority) act according to their instructions.

5.0.4. *Verification.* We prove that one requirement in isolation is met. Indeed, suppose φ_e is an isomorphism from C_i onto C_j, then we would have no more expansionary stages. However, we would have effectively copied A and B into C_i and C_j, and thus the composition $\phi_j \circ \varphi_e \circ \phi_i^{-1} : A \to B$ would contradict the assumption $A \ncong_c B$.

Now, in the general case, suppose that a stage is large enough that all the higher priority requirements have no expansionary stages. There will be only finitely many p in which ϕ_i^p permanently copies B_p (and not A_p as the requirement $R_{e,i,j}$ prescribes), and similarly for ϕ_j^p. Those finitely many p are controlled by the higher priority requirements.

But recall that each of the A_p (or B_p) is computably categorical. Therefore, as above, if $R_{e,i,j}$ had infinitely many expansionary stages then we would produce a computable map $\phi_j \circ \varphi_e \circ \phi_i^{-1} : A_p \to B_p$ for every p eventually controlled by the strategy. Although $\phi_j \circ \varphi_e \circ \phi_i^{-1}$ would be wrong on the finitely many p that are controlled by the higher priority strategies, we would be able to non-uniformly reconstruct the finitely many indices of actual computable isomorphisms between these C_p^i and C_p^i thus producing a computable map between A and B, contrary to the hypothesis.

Finally, $C_i \cong A$ because we let each strategy permanently copy at least one p-component of either A of B. Inedeed, recall we used $l_{e,i,j} + 1$, so even if in the limit $l_{e,i,j} = 0$ at least one p-component is permanently copied by the strategy.

6. Relative categoricity

All known algebraic descriptions of computable categoricity in natural classes are also descriptions of relative computable categoricity in these classes. Of course, relative computable categoricity does not have to have an algebraic description in a class. For example, there is no reasonable algebraic characterisation of relatively c.c. graphs. The best we can say is that a relatively c.c. graph has a c.e. $\exists$-Scott family. Although relatively c.c. TAGs possess many nice uniform properties, they are not as nicely behaved as one would hope for. We open the section with a non-trivial example of a relatively c.c. TAG.

Example 6.1. Let $(p_i)_{i\in\omega}$ be an effective enumeration of all primes. Suppose the ith primary component of A has the form:

$$A_{p_i} = (\mathbb{Z}_{p_i^\infty} \oplus \mathbb{Z}_{p_i^\infty} \oplus \mathbb{Z}_{p_i^i} \oplus \mathbb{Z}_{p_i^{i-1}} \oplus \ldots \oplus \mathbb{Z}_{p_i^2}) \oplus \bigoplus_{j\in\omega} \mathbb{Z}_{p_i}, \quad i = 0, 1, \ldots.$$

We claim that A is relatively c.c. Suppose B is some other copy. To match cyclic and quasi-cyclic elementary summands in A_{p_i} and B_{p_i}, wait until some cyclic subgroup of size $> i$ appears in A_{p_i} and in B_{p_i}. As soon as they appear, it is safe to map them to each other and then extend this mapping at later stages naturally. The same applies to the second quasi-cyclic summand. Clearly, we cannot assume that B has a computable full decomposition into elementary summands, but it is not a problem here. We run a back-and-forth on the divisible parts. For that, it is sufficient to search for an element b below the given element x in the divisible part (i.e., $pb = x$) whose height is great enough, in this case greater than p^i. Similarly, we can match the cyclic summands of sizes $\geq p_i^2$, as follows. We first wait for long enough initial segments of $\mathbb{Z}_{p_i^\infty} \oplus \mathbb{Z}_{p_i^\infty}$ to appear in both A_{p_i} and B_{p_i}, and then we wait for (an independent) $\mathbb{Z}_{p_i^i}$ to be enumerated in both groups. Once this happens it is safe to match them. Then we look for $\mathbb{Z}_{p_i^{i-1}}$, etc. As soon as we are done with the cyclic summands of sizes $\geq p_i^2$, we run a back-and-forth on $\bigoplus_{j\in\omega} \mathbb{Z}_{p_i}$.

Remark 6.2. In the example above, we could define the isomorphism type of a relatively c.c. group *dynamically* instead of using a nice formula. For example, once the initial segments of the two quasi-cyclic summands of the pth component have grown large enough, we can safely change our mind and declare them to be large primary cyclic of equal size. It won't change the property of being relatively c.c., but it will make the isomorphism type fairly unpredictable. For example, we could code a Π^0_1 set into the primes p_i for which G_p is reduced (i.e., has no quasi-cyclic summands).

Relative computable categoricity in the class of TAGs is a property of an effective enumeration of the group, and not of its algebraic isomorphism type. The nature of this combinatorial complexity is best illustrated by the rather unexpected Proposition 6.3 below. To state the proposition, we need a notation.

Suppose G is a torsion abelian group which splits into a direct sum of primary cyclic and quasi-cyclic groups. Define E_G to be the cardinal sum of the equivalence structures E_{G_p} over all primes p. We know that the functor $G \to E_G$ is uniformly effective. From the purely algebraic standpoint, the isomorphism invariants of G and E_G are essentially the same. From the categoricity standpoint, the only essential difference between G and E_G is that G does not necessarily have a computable full direct decomposition into elementary summands. However, the functor preserves relative computable categoricity for abelian p-groups. It is bizarre that the functor does not preserve relative computable categoricity for torsion abelian groups.

Proposition 6.3. *There exists a computable TAG G which splits into a direct sum of cyclic and quasi-cyclic p-groups, such that E_G is relatively c.c. but G is not even c.c.*

Proof. We will construct a computable TAG G such that for every p,

$$G_p = B_p \oplus \bigoplus_{i\in\omega} \mathbb{Z}_p,$$

where B_p will be either $\mathbb{Z}_{p^2}$ or $\mathbb{Z}_{p^3}$. We always start with $B_p = \mathbb{Z}_{p^2}$, and we may grow it later to $\mathbb{Z}_{p^3}$ if necessary. No matter what our uniformly computable choice on the isomorphism type of B_p will be, E_G is clearly relatively c.c. To see why, note that there is exactly one class of size > 1 in E_{G_p}. We can wait for a class of size at most 2 within the pth summands of both copies and safely match them. We then extend the isomorphism to the rest of the pth summands naturally.

However, it is not hard to ensure that G is not even computably categorical, as follows. We are constructing G and another copy A of G. We diagonalise against $\varphi_e : G \to A$ using the primary component G_{p_e}, as follows. Initially, A_{p_e} copies G_{p_e}.

(1) Wait for φ_e to converge on a generator x_e of $B_{p_e} \cong \mathbb{Z}_{p_e^2}$ in G_{p_e}.
(2) Assuming $y = \varphi_e(x_e)$ has order p_e^2 in A_{p_e}, introduce a new $z \in A_{p_e}$ and declare $p_e^2 z = p_e y$.
(3) Grow B_{p_e} to $\mathbb{Z}_{p_e^3}$ in G_{p_e} by making x_e divisible by p_e.

If z is never created, then φ_e cannot be an isomorphism, becasue any isomorphism must preserve orders of elements. Otherwise, if we eventually introduce z, then $p_e|x_e$ but $p_e \not| \varphi_e(x_e)$. It is not hard to see that z (if it is ever created) generates a pure cyclic subgroup of A_{p_e} of order p_e^3. The element $(p_e z\text{-}y)$ has order p_e, is not divisible by p_e, and thus contributes to the infinite direct sum of $\mathbb{Z}_{p_e}$. We conclude that $A_{p_e} \cong G_{p_e}$ for all e. □

We discuss the meaning of Proposition 6.3 a bit more. We strongly conjecture that relatively c.c. cardinal sums of equivalence structures admit a rather tedious and seemingly useless combinatorial "description" in terms of settling stages ([DM13]); we omit the exact long formulation and only note that it is *not* purely algebraic. Proposition 6.3 says that any description of relative computable categoricity of TAGs must *additionally* respect the dynamic process of finding a complete decomposition of the group, thus making any potential combinatorial description of relative categoricity unbearable.

It seems the existence of a c.e. $\exists$-Scott family is the most convenient (definitely the most compact) local criterion of relative computable categoricity for the class of TAGs. Although we are sceptical, we of course encourage the reader to try finding a local combinatorial description of relative categoricity for TAGs that takes less than a page to state. However, some of the many nice uniform properties that relatively c.c. TAGs possess might be interesting on their own right, but we leave this direction open. Instead, to finish the paper we outline the proof of the easy:

Proposition 6.4. *The index set of relatively c.c. TAGs is Σ^0_3-complete.*

Proof. It is known that the index set of all relatively c.c. structures in a given computable language is Σ^0_3 [DALD]. Since the class of torsion abelian groups is Π^c_2-axiomatisable, the upper bound remains Σ^0_3 when the index set is restricted to the class.

For completeness, combine the main diagonalisation strategy from the proof of the Π^0_4-completeness result in Section 4 with a Σ^0_3-guessing, as follows. As before, every local diagonalisation strategy will be working within its own primary summand, and the location can be abandoned. Whenever it is abandoned due to an action of the Σ^0_3-guessing, it is instantly homogenised: it is set equal to $\mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2}$.

The Σ^0_3-guessing can be organised in a movable markers fashion, so that in the Π^0_3-case every strategy gets to act within a stable component, thus producing a TAG that is not c.c. In the Σ^0_3-case all except for finitely many primary components will be homogenised. Since those which are not homogenised are actually finite, the group is relatively c.c. We leave the elementary details to the reader. □

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