SEPARATING WEAK α-CHANGE AND α-CHANGE GENERICITY

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ABSTRACT. In [13], a transfinite hierarchy of genericity notions stronger than 1-genericity and weaker than 2-genericity was introduced. We close a line of questioning begun there by showing that for every $\alpha \leqslant \varepsilon_0$ which is a power of ω, there is a Δ^0_2 Turing degree $\mathbf{a}$ which is weakly α-change generic, but not α-change generic.

1. INTRODUCTION

Forcing, in its many guises and variations, is pervasive in computability theory. The (weak) n-random reals, the most important notions in algorithmic randomness, may be regarded as those reals which are generic with respect to a version of Solovay forcing (see [9]). Recently, Mathias forcing has been used to great effect to establish central theorems in reverse mathematics (see [12] and [14]).

Our focus in the present article is on Cohen forcing and its variations. The reals generic with respect to this notion of forcing have an especially long history in the subject. Named after Cohen, who used this method in his proof of the independence of the continuum hypothesis from ZFC set theory, many consider the *finite extension method*, used by Kleene and Post ([10]) in their construction of a pair of incomparable Δ^0_2 degrees, to be a precursor.

It is instructive to briefly recall the finite extension method. We wish to construct a real A which satisfies a countable collection $\langle \psi_i \rangle_{i<\omega}$ of conditions. We do so by constructing A in stages: at stage s we define σ_s such that $\kappa_{s-1} \prec \sigma_s$, and then let $A = \bigcup \sigma_s$. The idea is that at stage s, having already constructed the string κ_{s-1}, we seek a string $\sigma_s \succ \kappa_{s-1}$ which "forces" ψ_s to be satisfied, in that every real which extends σ_s satisfies ψ_s. A real which is Cohen generic is then one which satisfies all properties which can be obtain via a finite extension method; we shall say more on this below, but see [9] for a more precise discussion of this connection.

The notions of n-genericity, for $n \in \omega$, are the most commonly used in computability theory. Though originally introduced by Feferman ([6]) using the language of forcing sentences of arithmetic, the more frequently used definition, due to Jockusch and Posner (see [8]), is as follows. Let S be a set of finite binary strings, and A a subset of natural numbers, which we think of as an infinite binary sequence. We say that A meets S if there is some $\sigma \prec A$ such that $\sigma \in S$, and A avoids S if there is some $\sigma \prec S$ such that no extension of σ is in S. Then A is said to be n-generic if for every Σ^0_n set of strings S, A either meets or avoids S. Equivalently, A is n-generic if it satisfies every property that can be obtained via a finite extension argument carried out with the assistance of $\varnothing^{(n)}$ as an oracle.

These classes form a proper hierarchy: for all $n \geqslant 1$, the n-generic sets properly contain the $n+1$-generic sets. This was later refined by Kurtz [11] who introduced the weakly n-generic sets. We say that a set of strings S is dense if every finite binary string has an extension in S. Then a set is weakly n-generic if it meets every dense Σ^0_n set of string. Kurtz [11] showed that for all $n \geqslant 1$, the n-generic sets properly contain the weakly $n+1$-generic sets, which properly contain the $n+1$-generic sets. These proper containments even hold for Turing degrees, where a Turing degree is (weakly) n-generic if it contains a (weakly) n-generic set.

Most of the interest in genericity occurs at the levels $n = 1$ and $n = 2$. Even at these low levels, there is a large difference in the behaviour of 1-generic and 2-generic sets. For example, the class of 1-generic sets has measure 1, whereas the class of 2-generic sets has measure 0. As another example, the 2-generic sets are downward dense below 2-generic sets, whereas this fails in general for 1-generics ([8], [7], [3]). Intuitively, typical behaviour seems to start with 2-genericity, but can sometimes fail at the level of 1-genericity. Many such results for genericity, as well as randomness, are given in [1], and the survey article [2].

Wishing to explore the landscape between 1-genericity and 2-genericity, the authors in [13] introduced a transfinite hierarchy of genericity notions. Our main inspiration was Downey and Greenberg's [5]. There, the notion of an α-computably approximable function, for α a suitably small ordinal, is used to give a refinement of Δ^0_2 functions. Rougly speaking, an approximation to a function may change its mind only when a *marker* in the ordinal α decreases. A c.e. Turing degree is then said to be *totally α-c.a.* if every function computable in the degree is α-computably approximable. This class, together with several related ones, form a transfinite hierarchy in the low_2 c.e. degrees. Some levels in this hierarchy give characterisations of those degrees below which certain lattice may be embedded. For example, a c.e degree bounds a critical triple in the c.e. degrees if and only if it is not totally ω-c.a. As another example, a c.e. degree bounds a copy of the 1-3-1 lattice if and only if it is not totally $< \omega^\omega$-c.a. More generally, these degree seem to capture the computational strength required to perform various natural classes of constructions in computability theory. It is expected that these ideas will play an important role in classical computability theory.

Making use of this approach, we consider finite extension arguments where *multiple* attempts may be required in order to satisfy a single condition. The ordinal α then keeps track of how many times we may need to act; every time we must act again for some condition, a marker in the ordinal α decreases.

The hierarchy of multiple genericity notions shares several features with the hierarchies defined in [5]. Firstly, it is transfinite, with levels at each power of ω below ε_0. In addition, we are able to define *uniform* levels of the hierarchy, in analogy with the uniformly totally α-c.a. degrees from [5]. We also define *weak* levels of the hierarchy, in analogy with the notions of weak n-genericity for $n \in \omega$.

$$
\begin{array}{c}
\text{weakly 2-generic} \\
\Downarrow \\
\vdots \\
\Downarrow \\
\omega^2\text{-change generic} \\
\Downarrow \\
\text{weakly } \omega^2\text{-change generic} \\
\Downarrow \\
\text{uniformly } \omega^2\text{-change generic} \\
\Downarrow \\
\omega\text{-change generic} \\
\Downarrow \\
\text{weakly } \omega\text{-change generic} \\
\Downarrow \\
\text{uniformly } \omega\text{-change generic} \equiv \text{pb-generic} \\
\Downarrow \\
\text{1-change generic} \equiv \text{1-generic}
\end{array}
$$

$$\Downarrow$$

weakly 1-generic

As with any hierarchy in computability theory, an important question is whether the levels of the hierarchy are proper, and whether this can be witnessed by a Turing degree. In [13], we showed that for every ordinal β with $\omega^\beta \leqslant \varepsilon_0$, there is a Turing degree which is uniformly ω^β-change generic but not weakly ω^β-change generic, and that there is a Turing degree which is ω^β-change generic but not uniformly $\omega^{\beta+1}$-change generic. We show in our main theorem, Theorem 4.64, that there is a Turing degree which is weakly ω^β-change generic but not ω^β-change generic.

In the following section, we briefly revise the definitions of the multiple genericity notions. In Section 3, we explore the relationship between the domination properties from [5] and multiple genericity. In the final section, we give a construction, leading to our main theorem.

2. Definitions

We briefly recall the definitions of the multiple genericity notions from [13]. We refer the reader to section 2 of [13], and also [5], for some subtleties and details concerning computable ordinals and their presentations. Further discussion of the legitimacy of these definitions, and the relationship between these notions and the (weakly) n-generics, is given in [13].

Definition 2.1. Let $\alpha \leqslant \varepsilon_0$, and let ∞ denote the greatest element of the linear ordering $\alpha+1$. An *α-change test* is a sequence $\langle f_s, o_s\rangle_{s<\omega}$ of pairs of uniformly computable functions $f_s : 2^{<\omega} \to 2^{<\omega}$ and $o_s : 2^{<\omega} \to \alpha + 1$ such that for all $\sigma \in 2^{<\omega}$ and $s \in \omega$,

- $f_s(\sigma) \succcurlyeq \sigma$,
- $o_{s+1}(\sigma) \leqslant o_s(\sigma)$, and
- if $f_{s+1}(\sigma) \neq f_s(\sigma)$, then $o_{s+1}(\sigma) < o_s(\sigma)$.

For a test $\langle f_s, o_s\rangle_{s<\omega}$, we let the range of the test be

$$\operatorname{range} \langle f_s, o_s\rangle_{s<\omega} = \{\lim_s f_s(\sigma) : \lim_s o_s(\sigma) < \infty\}.$$

Definition 2.2. Let $\alpha \leqslant \varepsilon_0$ and let $\mathbf{a}$ be a Turing degree. We say that $\mathbf{a}$ is *α-change generic* if there is a set $A \in \mathbf{a}$ which meets or avoids the range of all α-change tests.

Definition 2.3. Let $\langle f_s, o_s\rangle_{s<\omega}$ be an α-change test. We say that $\langle f_s, o_s\rangle_{s<\omega}$ is *total* if for all $\sigma \in 2^{<\omega}$, $\lim_s o_s(\sigma) < \infty$. We say that $\mathbf{a}$ is *weakly α-change generic* if there is a set $A \in \mathbf{a}$ which meets the range of all total α-change tests.

Definition 2.4. Let $h : 2^{<\omega} \to \alpha$ be a total computable function. We say that the α-change test $\langle f_s, o_s\rangle_{s<\omega}$ is *h-bounded* if for all $\sigma \in 2^{<\omega}$ and $s \in \omega$, if $o_s(\sigma) < \infty$, then $o_s(\sigma) < h(\sigma)$.

Let $\leqslant_{\mathrm{L}}$ denote the usual length-lexicographic ordering on $2^{<\omega}$. We say that $h : 2^{<\omega} \to \alpha$ is an *α-order function* if h is total and computable with range cofinal in α, and such that if $\sigma \leqslant_{\mathrm{L}} \nu$, then $h(\sigma) \leqslant h(\nu)$.

We say that $\mathbf{a}$ is *uniformly α-change generic* if for all (some) α-order functions h, there is a set $A \in \mathbf{a}$ such that A meets the range of all total h-bounded α-change tests.

Up to Turing degree, the notion of pb-genericity from [4] is equivalent to uniform ω-change genericity (see Theorem 3.2 of [13]).

3. Domination and genericity

In sections 4 and 5 of [13], a close connection between these genericity notions and the domination properties introduced in [5] was explored. We have the following pleasing characterisations.

Theorem 3.1 (Theorem 4.15 of [13]). *Every Turing degree computes a uniformly α-change generic degree if and only if it is not uniformly α-c.a. dominated.*

Theorem 3.2 (Theorem 4.13 of [13]). *Every Turing degree computes a weakly α-change generic degree if and only if it is not α-c.a. dominated.*

Note in particular that the ordinal levels in the two hierarchies coincide for these notions. It was shown that a similar result holds for the α-change generic degrees under the additional assumption of computable enumerability.

Theorem 3.3 (Theorem 5.3 of [13]). *A c.e. Turing degree computes an α-change generic degree if and only if it is not α-c.a. dominated.*

However, what can be said about the relationship between computing an α-change generic degree and these domination notions in the *global* Turing degrees? Theorems 4.15 and 4.13 of [13] were established using the technique of forcing. With this approach, the following theorem seems optimal.

Theorem 3.4. *Suppose $\alpha \leqslant \varepsilon_0$ is a power of ω and let β be such that $\alpha = \omega^\beta$. Then every not uniformly $\omega^{\beta+1}$-c.a. dominated degree computes a ω^β-change generic degree.*

Proof. We define a total function $q : \omega \times 2^{<\omega} \to \omega$ as follows. Let $q(e, \sigma) = 0$ if there is no ν with $\lim_s f_{e,s}(\nu) \succcurlyeq \sigma$ and $\lim_s o_{e,s}(\nu) < \infty$. Otherwise, let s be least such that there is some ν with $f_{e,s}(\nu) \succcurlyeq \sigma$ and $o_{e,s}(\nu) < \infty$, and let ν be the least such string. Let $q(e, \sigma)$ be the least $t \geqslant s$ such that $o_{e,u}(\nu) = o_{e,t}(\nu)$ for all $u \geqslant t$.

We claim that q is $(\alpha+1)$-computably approximable. We let $q_s(e, \sigma) = 0$ and $v_s(e, \sigma) = \alpha$ if there is no ν with $f_{e,s}(\nu) \succcurlyeq \sigma$ and $o_{e,s}(\nu) < \infty$. Otherwise, let $s_0 \leqslant s$ be least such that there is some ν with $f_{e,s}(\nu) \succcurlyeq \sigma$ and $o_{e,s_0}(\nu) < \infty$, and let ν be the least such string. Let $q_s(e, \sigma)$ be the least $t \leqslant s$ such that $o_{e,t}(\nu) = o_{e,s}(\nu)$, and let $v_s(e, \sigma) = o_{e,s}(\nu)$. Then $\langle q_s, v_s \rangle_{s<\omega}$ is an $(\alpha + 1)$-computable approximation for q.

Let $q'(n) = \max \{ q(e, \sigma) : e \leqslant n, |\sigma| \leqslant n \}$. We claim that q' is $\omega^{\beta+1}$-computably approximable. Let $q'_s(n) = \max \{ q_s(e, \sigma) : e \leqslant n, |\sigma| \leqslant n \}$ and let $v'_s(n) = \bigoplus \{ v_s(e, \sigma) : e \leqslant n, |\sigma| \leqslant n \}$. It is straightforward to verify that $\langle q'_s, v'_s \rangle_{s<\omega}$ is an $\omega^{\beta+1}$-computable approximation for q'.

Let $\mathbf{a}$ be a not uniformly $\omega^{\beta+1}$-c.a. dominated degree, and let $g \leqslant_{\mathrm{T}} \mathbf{a}$ be a function which is not dominated by q'. We now carry out the construction from the previous theorem using this function g. Let G be the set constructed. As before, $G \leqslant_{\mathrm{T}} \mathbf{a}$.

Lemma 3.5. *G either meets or avoids the range of every ω^β-change test.*

Proof. Assume by induction that we do not act for any $d < e$ after stage s. Let $t > s$ be least such that $t \geqslant e$ and $g(t) > q'(t)$. There are two possibilities. First, e does not require attention at stage t. Then there is no string in range f_e which extends κ_{t-1}. Therefore, because $G \succ \kappa_{t-1}$, G avoids the range of range f_e. The other possibility is that e does require attention at stage t. Then because $t > s$, we will act for e at stage t, and we will continue to act for e at all subsequent stages until some stage $t' \geqslant t$ where $\kappa_{t'}$ meets range f_e. □

□

4. The main theorem

We now have the following interesting situation. Let $\alpha \leqslant \varepsilon_0$ be a power of ω, and let β be such that $\alpha = \omega^\beta$. By Theorem 3.4, a general degree that is not uniformly $\omega^{\beta+1}$-c.a. dominated can compute an ω^β-change generic degree, whereas a c.e. degree need only be not ω^β-c.a. dominated in order to compute an ω^β-change generic degree. We might wonder whether this gap is necessary. The following theorem shows that it is. Our main theorem, Theorem 4.64, is a straightforward consequence, and gives the result claimed in the abstract.

Theorem 4.1. *Let $\alpha \leqslant \varepsilon_0$ be a power of ω. Then there is a not α-c.a. dominated Δ^0_2 Turing degree which does not compute an α-change generic degree.*

Proof. We construct a set A as the limit of a uniformly computable sequence of sets $\langle A_s\rangle_{s<\omega}$. The Turing degree of A will be as required.

We first agree on some definitions.

4.1. **Functionals.** Fix primitive recursive bijections $\langle\cdot\rangle : \omega^{<\omega} \to \omega$ and $\ulcorner\cdot\urcorner : 2^{<\omega} \to \omega$. An *axiom* is some natural number of the form $\langle\ulcorner\sigma\urcorner, n, a\rangle$ where $\sigma \in 2^{<\omega}$, and $n, a \in \omega$. We shall often write $\langle\sigma, n, a\rangle$ for $\langle\ulcorner\sigma\urcorner, n, a\rangle$. We may say that the axiom $\langle\sigma, n, a\rangle$ has *use* σ, and that it is *for* n.

For $n \in \omega$, we say that a set Γ of axioms is *consistent for* n if for all $\sigma, \tau \in 2^{<\omega}$ and all $a, b \in \omega$, if $\langle\sigma, n, a\rangle, \langle\tau, n, b\rangle \in \Gamma$, and σ and τ are comparable, then $a = b$. We say that a set of axioms is *consistent* if it is consistent for all natural numbers. For $n \in \omega$, we let $\operatorname{dom}\Gamma(n) = \{\gamma : \exists a. \langle\gamma, n, a\rangle \in \Gamma\}$.

Let Γ be a consistent set of axioms, $\nu \in 2^{\leqslant\omega}$, and $n \in \omega$. We write $\Gamma(\nu, n)\downarrow$ if there is some $\sigma \preccurlyeq \nu$ and some a such that $\langle\sigma, n, a\rangle \in \Gamma$. In this case, we say that σ is the *use* of $\Gamma(\nu, n)$, and we write $\Gamma(\nu, n) = a$. Otherwise, we write $\Gamma(\nu, n)\uparrow$. By the consistency of Γ, this is well-defined. A *Turing functional* is a computably enumerable and consistent set of axioms.

Let Γ be a consistent set of axioms and $A \in 2^\omega$. We say that $\Gamma(A)$ is *total* if $\Gamma(A, n)\downarrow$ for all $n \in \omega$. In this case, we define $\Gamma(A) \in \omega^\omega$ by letting $\Gamma(A)(n) = \Gamma(A, n)$ for all $n \in \omega$.

Suppose that $\langle\Phi_s\rangle_{s<\omega}$ is a computable enumeration of axioms. We say that $\langle\Phi_s\rangle_{s<\omega}$ is *length increasing* if for all $s, t \in \omega$ with $s < t$, if $\langle\sigma_s, n_s, a_s\rangle$ is enumerated into Φ at stage s and $\langle\sigma_t, n_t, a_t\rangle$ is enumerated into Φ at stage t, then $|\sigma_s| < |\sigma_t|$.

Given a computable enumeration $\langle\Phi_s\rangle_{s<\omega}$ of the Turing functional Φ, we can produce a length increasing computable enumeration $\langle\Psi_s\rangle_{s<\omega}$ of axioms via a simple construction. We may assume that for all $s \in \omega$, at most one axiom is enumerated into Φ at stage s. At stage s of the construction, if the axiom $\langle\sigma, n, a\rangle$ is enumerated into Φ, then we do the following. If the length of σ is greater than the length of the use of any axiom in $\bigcup\{\Psi_r : r < s\}$, then we let $\Psi_s = \Psi_{s-1} \cup \{\langle\sigma, n, a\rangle\}$, and otherwise, we choose some large natural number l and let

$$\Psi_s = \Psi_{s-1} \cup \{\langle\tau, n, a\rangle : \tau \succ \sigma \text{ and } |\tau| = l\}.$$

It is straightforward to show that $\langle\Psi_s\rangle_{s<\omega}$ is a computable enumeration of a Turing functional Ψ, and for all $A \in 2^\omega$, $\Psi(A)$ is total if and only if $\Phi(A)$ is, and if $\Phi(A)$ is total, then $\Psi(A) = \Phi(A)$.

Suppose that Φ is a Turing functional with length increasing computable enumeration $\langle\Phi_s\rangle_{s<\omega}$. We define the partial computable function $\Phi^\circ : 2^{<\omega} \to \omega^{<\omega}$ as follows. Suppose that $\langle\sigma, n, a\rangle$ is enumerated into Φ at stage s. If $\Phi_s(\sigma, 0)\downarrow$, then let n be greatest such that for all $m \leqslant n$, $\Phi_s(\sigma, m)\downarrow$. We define $\Phi^\circ(\sigma)$ to be the string of length $n + 1$ where for all $m \leqslant n$, $\Phi^\circ(\sigma)(m) = \Phi_s(\sigma, m)$.

Since the enumeration is length increasing, for all strings σ and τ in the domain of Φ°, if $\sigma \prec \tau$, then $\Phi^\circ(\sigma) \preccurlyeq \Phi^\circ(\tau)$. We may define Φ° in a way (consistent with the above definition) that ensures its domain is downward closed under the prefix relation, though this not necessary for our purposes. For a S a set of strings in the domain of Φ°, we write $\Phi^\circ(S)$ for $\{\Phi^\circ(\sigma) : \sigma \in S\}$. We shall often simply write Φ for Φ°.

Let $\langle\Phi_e\rangle_{e<\omega}$ be an acceptable enumeration of all Turing functionals such that for all $e \in \omega$ and all axioms $\langle\sigma, n, a\rangle \in \Phi_e$, $a \in \{0, 1\}$. For all $e \in \omega$, let $\langle\Phi_{e,s}\rangle_{s<\omega}$ be a computable enumeration of Φ_e. Using the procedure above if necessary, we may assume that for all $e \in \omega$, $\langle\Phi_{e,s}\rangle_{s<\omega}$ is a length increasing enumeration.

Suppose that Φ is a Turing functional with length increasing computable enumeration $\langle\Phi_s\rangle_{s<\omega}$. Two strings σ_0 and σ_1 in the domain of the partial computable function Φ are said to Φ-*split* if $\Phi(\sigma_0)$ and $\Phi(\sigma_1)$ are incomparable; note that by the consistency of Φ,

σ_0 and σ_1 must be incomparable. A set S is a *set of* Φ*-splits* if the strings in S pairwise Φ-split.

We say that a set $S \subset 2^{<\omega}$ is *sparse* if for all $l \in \omega$, there is at most one string in S of length l.

4.2. **Tests.** Some terminology will help in the informal discussion, as well as in the verification. For some α-change test $t = \langle a_s, b_s\rangle_{s<\omega}$, we may refer to $a_s(\rho)$ as the *arrow for* ρ in the test t at stage s, and we may refer to $b_s(\rho)$ as the *bound for* ρ in the test t at stage s. We may say that we *update the arrow for* ρ in t at stage s if $a_s(\rho) \neq a_{s-1}(\rho)$, and similarly, we may say that we *update the bound for* ρ in t at stage s if $b_s(\rho) \neq b_{s-1}(\rho)$. We may say that ρ *is a base point* in t at stage s if $b_s(\rho) < \infty$, and that ρ *is a base point* in t if $\lim_s b_s(\rho) < \infty$.

4.3. **The requirements.** Let $\langle\langle f_{i,s}, o_{i,s}\rangle_{s<\omega}\rangle_{i<\omega}$ be an effective list of all tidy $(\alpha + 1)$-computable approximations whose limits $f_i = \lim_s f_{i,s}$ consist of all α-c.a. functions. So that A is of not α-c.a. dominated degree, we construct a Turing functional Γ such that $\Gamma(A)$ is total, and meet for every $i \in \omega$ the requirement

$$\begin{gathered} P_i : \text{ if } \langle f_{i,s}, o_{i,s}\rangle_{s<\omega} \text{ is eventually } \alpha\text{-computable, then} \\ \text{there is some } n \in \omega \text{ such that } \Gamma(A, n) > f_i(n). \end{gathered}$$

It follows straightforwardly from the definitions in Section 2 that every α-change generic degree is 1-generic. Since the degree of the computable sets is not 1-generic, to ensure that A does not compute an α-change generic degree, it suffices to meet for every $e \in \omega$ the requirement

$$\begin{gathered} Q_e : \text{if } \Phi_e(A) \text{ is total and noncomputable, then } \Phi_e(A) \\ \text{does not meet or avoid the range of every } \alpha\text{-change test.} \end{gathered}$$

4.4. **The basic strategies.** First consider the P-requirement P_i working in isolation. We begin at stage s by choosing a large follower n for P_i. We let $A_s = A_{s-1}$. We choose some string $\gamma_s \prec A_s$ and define $\Gamma_s(\gamma_s, n) = 0$. We then wait until a stage t where we see that $o_{i,t}(n) < \infty$. If we never see such a stage t, then no further action for P_i is necessary, and P_i is met. So suppose that we do. We say that P_i *is realised with* n *at stage* t. We choose some string γ_t that is incomparable with γ_s. We let A_t extend γ_t and define $\Gamma_t(\gamma_t, n) > f_{i,t}(n)$. If at any later stage u we see that $\Gamma_{u-1}(A_{u-1}, n) \leqslant f_{i,u}(n)$, then we do the following. We choose some string γ_u which is incomparable with all strings in the domain of $\Gamma_{u-1}(n)$. We let A_u extend γ_u and define $\Gamma_u(\gamma_u, n) > f_{i,u}(n)$.

Since $\langle f_{i,s}, o_{i,s}\rangle_{s<\omega}$ is an $(\alpha + 1)$-computable approximation, this strategy is finitary. It is straightforward to show that this strategy can be carried out in a way that ensures that Γ is consistent for n. Furthermore, we can easily make $\Gamma(A)$ total.

Now consider the Q-requirement Q_e working in isolation. We build for the sake of Q_e an α-change test $t_e = \langle a_{e,s}, b_{e,s}\rangle_{s<\omega}$ such that if $\Phi_e(A)$ is total and noncomputable, then $\Phi_e(A)$ neither meets nor avoids range t_e. Since the range of t_e must be infinite if $\Phi_e(A)$ is total and noncomputable, the action for Q_e is infinitary in general.

We break Q_e into infinitely many Q_e-subrequirements $Q_{e,i}$ for all $i \geqslant e$. The Q_e-subrequirements will work together to define the test t_e. They do this by, roughly speaking, partitioning $\Phi_e(A)$ into finite segments. Each Q_e-subrequirement will be responsible for defining a base point in t_e along its segment, and ensuring that the arrow for any base point along its segment is not an initial segment of $\Phi_e(A)$.

We develop a finitary strategy for each $Q_{e,i}$ working with the other Q_e-subrequirements. Although this strategy is more complicated than is currently necessary, the additional structure will be of benefit later on. If $i, j \geqslant e$ and $i < j$, then we let $Q_{e,i}$ be of stronger priority than $Q_{e,j}$.

At each stage s and for each $Q_{e,i}$, we may define a string $\varphi_{e,i,s}$. We also define $\varphi_{e,e-1,s}$ to be the empty string for all $s \in \omega$ and may assume that $\Phi_e(\langle\rangle) = \langle\rangle$.

We say that $Q_{e,i}$ is satisfied at stage s if

- $\varphi_{e,i,s}$ is defined and $\varphi_{e,i,s} \prec A_s$,
- $\Phi_e(\varphi_{e,i-1,s}) \prec \Phi_e(\varphi_{e,i,s})$,
- there is some ρ with $\Phi_e(\varphi_{e,i-1,s}) \prec \rho \preccurlyeq \Phi_e(\varphi_{e,i,s})$ such that $b_{e,s}(\rho) < \infty$, and
- for all ρ with $\Phi_e(\varphi_{e,i-1,s}) \prec \rho \preccurlyeq \Phi_e(\varphi_{e,i,s})$, if $b_{e,s}(\rho) < \infty$ then $a_{e,s}(\rho) \not\prec \Phi_{e,s}(A_s)$.

If all Q_e-subrequirements are satisfied at all but finitely many stages, and t_e is indeed an α-change test, then Q_e is met.

Since we currently do not need to meet any of the P-requirements, we may let $A_s = 0^\omega$ for all $s \in \omega$.

Suppose that all Q_e-subrequirements of stronger priority than $Q_{e,i}$ are satisfied at all stages after stage s. At some later stage t, the strategy for $Q_{e,i}$ will look for strings δ and φ in the domain of the partial computable function Φ_e such that $\varphi_{e,i-1,t} \prec \delta \prec \varphi \prec A_t$ and $\Phi_e(\delta) \prec \Phi_e(\varphi)$. If we never see such a stage t, then we will be able to show that $\Phi_e(A)$ is not total, so Q_e is met, and no further action for $Q_{e,i}$ need be taken. So suppose that we do. We define the base point $\Phi_e(\delta)$ in the test t_e as follows. We define the bound for $\Phi_e(\delta)$ in t_e at stage t to be 0, and we define the arrow for $\Phi_e(\delta)$ in t_e at stage t to be some string which properly extends $\Phi_e(\delta)$ and which is incomparable with $\Phi_e(\varphi)$. We define $\varphi_{e,i,t} = \varphi$. Then $Q_{e,i}$ is satisfied at stage t, and since we never move the approximation to A, $Q_{e,i}$ is satisfied at all later stages.

4.5. **Splits.** We now consider the interaction of the P- and Q-requirements. First suppose that we have the P-requirement P_i working together with the Q-requirement Q_e. Then since the basic strategy for the P-requirements is finitary, we may satisfy Q_e by simply resetting t_e whenever we act for P_i, and following the basic strategy for the Q_e-subrequirements as above. To be slightly more precise, to reset the test t_e at stage s, we abandon the definitions made in the test so far, and will at later stages define a new α-change test for the sake of Q_e.

However, as there are infinitely many P-requirements, we will not always be able to simply reset the test t_e whenever we act for some P-requirement. Therefore, we must develop some alternate strategy. We do so now.

Consider P_i working together with $Q_{e,i}$, of equally strong priority. Suppose that we begin the basic strategy for P_i at stage s. As above, we pick a large follower n for P_i, let $A_s = A_{s-1}$, choose some string $\gamma_s \prec A_s$, and define $\Gamma_s(\gamma_s, n) = 0$. Suppose that at some later stage t, we see strings δ and φ and define the bound and arrow for the base point $\Phi_e(\delta)$ in t_e, as above.

There are two cases to consider. First suppose that $\varphi \prec \gamma_s$. Suppose that we wish to act for P_i at some stage u with $u > t$. Then we will be able to choose some string γ_u extending φ and incomparable with all strings in the domain of $\Gamma_{u-1}(n)$ to be the use of the computation we define at stage u. In particular, for all $u > t$, we will have that $\Phi_e(\varphi) \preccurlyeq \Phi_{e,u}(A_u)$. Therefore, we will never need to update the arrow for $\Phi_e(\delta)$ in the test t_e at any stage after stage t.

Now suppose that $\gamma_s \preccurlyeq \varphi$. Suppose that at some stage u with $u > t$, we will see that $o_{i,u}(n) < \infty$. We choose some string γ_u which is incomparable with γ_s, let A_u extend γ_u, and define $\Gamma_u(\gamma_u, n) = f_{i,u}(n) + 1$. However, since A_u no longer extends φ, the opponent is free to define Φ_e in such a way that $\Phi_{e,u}(A_u) \succcurlyeq a_{e,u-1}(\Phi_e(\delta))$. Since the bound for $\Phi_e(\delta)$ is 0, we are not able to update the arrow for $\Phi_e(\delta)$ at any stage after stage t. Then $\Phi_e(A)$ would meet the range of t_e, and Q_e would not be met.

Indeed, even if we had chosen a larger value for the bound for $\Phi_e(\delta)$, the opponent is able to declare $o_{i,u}(n)$ to be even larger. The current strategies could force us to move the approximation to A through many different strings, and therefore force us to update the arrow for $\Phi_e(\delta)$ in t_e more than its bound can accommodate.

This problem is rather serious. In order for $\Phi_e(\delta)$ to be a base point in the test t_e, we must define the bound for $\Phi_e(\delta)$ to be some ordinal less than ∞. However, there is no way of knowing whether $\lim_s o_{i,s}(n) < \infty$, so we cannot simply wait until a stage u where we see the value of $o_{i,u}(n) < \infty$.

If we were somehow able to define A_u in such a way that $\Phi_{e,u}(A_u)$ is incomparable with $\Phi_e(\delta)$, then we would no longer be concerned with updating the arrow for $\Phi_e(\delta)$. We could define a new base point ρ in the test t_e, and moreover, since the value $o_{i,u}(n)$ has now been revealed by the opponent, we could incorporate this into the value of the bound for ρ.

What we would like is a pair of Φ_e-splits. Since we may assume that $\Phi_e(A)$ is total and noncomputable, there are indeed infinitely many Φ_e-splits along A, and we may use such splits in our strategy for the Q_e-subrequirements.

4.6. **Working with splits.** We revise the strategies for the P-requirements and the Q_e-subrequirements to work with splits.

Again suppose that we begin the strategy for P_i at stage s. While waiting for P_i to be realised, the strategy for $Q_{e,i}$ will look for a pair of Φ_e-splits. If we never see a pair of Φ_e-splits, then $\Phi_e(A)$ must computable, so Q_e is met, and no further action for $Q_{e,i}$ need be taken. So suppose that we see the Φ_e-splits δ_0 and δ_1 at stage t. We arbitrarily choose one of the splits, δ_0, say, and let A_t extend δ_0.

At all stages u after stage t, we would like for A_u to extend either δ_0 or δ_1, so that $\Phi_{e,u}(A_u)$ extends either $\Phi_e(\delta_0)$ or $\Phi_e(\delta_1)$. Suppose that P_i has follower n at stage t, and that γ_t is the use of the computation $\Gamma_t(A_t, n)$. It is possible that both $\delta_0 \succ \gamma_t$ and $\delta_1 \succ \gamma_t$. Then, if we were to act for P_i at some later stage, we would need to move the approximation to A permanently away from γ_t, and therefore away from δ_0 and δ_1. We can avoid this situation by simply initialising P_i at stage t. We will wait until P_i has chosen a new follower before proceeding. We are then free to define axioms which have use that properly extends one of the splits. This has no detrimental effect on the rest of the construction.

We would like to define $\Phi_e(\delta_0)$ as a base point in the test t_e, but we will need an analogue of the string φ from above before doing so. Therefore, we wait for a later stage u where there is some φ such that $\delta_0 \prec \varphi \prec A_u$ and $\Phi_e(\varphi) \succ \Phi_e(\delta_0)$. If we never see such a stage u, then Φ_e is not total, so we may assume that we do. At stage u, we define the base point $\Phi_e(\delta_0)$ in t_e as above, as well as $\varphi_{e,i,u}$.

Suppose that P_i has follower n at some stage u with $u > t$, and that P_i is realised with n at stage u. Then regardless of the action that we have taken for $Q_{e,i}$ since stage t, we let A_u extend σ_1. We now choose γ_u to be some suitable string extending δ_1, and define $\Gamma_u(\gamma_u, n) = f_{i,u}(n) + 1$.

$Q_{e,i}$ again waits for some stage v where we see some string φ such that $\delta_1 \prec \varphi \prec A_v$ and $\Phi_e(\varphi) \succ \Phi_e(\delta_1)$. Suppose that we do see such a stage v. We define the base point $\Phi_e(\delta_1)$ in t_e as follows. We define the bound for $\Phi_e(\delta_1)$ in t_e at stage v to be $o_{i,u}(n)$, and we define the arrow for $\Phi_e(\delta_1)$ in t_e at stage v to be some string which properly extends $\Phi_e(\delta_1)$ and which is incomparable with $\Phi_e(\varphi)$.

Suppose that at some later stage w, we see that $\Gamma_{w-1}(A_{w-1}, n) \leqslant f_{i,w}(n)$. As above, the action that we take for P_i may then result in us needing to update the arrow for $\Phi_e(\delta_1)$ in t_e at some later stage. However, we will be able to show that that $f_{i,w}(n) \neq f_{i,u}(n)$, and therefore $o_{i,w}(n) < o_{i,u}(n)$. Then when we update the bound for $\Phi_e(\delta_1)$, its value will be at most $o_{i,w}(n)$, which suffices for showing that t_e is an α-change test.

4.7. **Common splits.** We now consider P_i, $Q_{d,i}$, and $Q_{e,i}$ working together, all of equally strong priority.

Suppose that we begin the strategy for P_i at stage s as usual, and that we begin the strategies for $Q_{d,i}$ and $Q_{e,i}$ as above. Suppose that at some later stage t, we see the strings δ_0 and δ_1 which, by some miracle, both Φ_d-split *and* Φ_e-split. Then the strategies for the Q-subrequiments are able to work independently, with $Q_{d,i}$ defining base points in the test t_d and $Q_{e,i}$ defining base points in the test t_e. In fact, if we were always fortunate enough to find common splits, then with minimal modification, the strategies we have introduced would be sufficient to meet all requirements.

4.8. **Splits above splits.** We will of course not always be so lucky as to discover common splits. We would like to develop a way to *produce* common splits in a stepwise fashion.

Let us put aside the consideration of the requirements for the moment, and focus on how we may go about producing common splits. Suppose instead that δ_0 and δ_1 only Φ_e-split. We would like to extend δ_0 and δ_1 to a pair of common splits. Suppose that we later find the Φ_d-splits δ_{00} and δ_{01} above δ_0, and the Φ_d-splits δ_{10} and δ_{11} above δ_1. Then it is straightfoward to show that there is some pair σ_0 and σ_1 chosen from the set $\{\delta_{00}, \delta_{01}, \delta_{10}, \delta_{11}\}$ which Φ_d-split, and moreover, such that $\sigma_0 \succ \delta_0$ and $\sigma_1 \succ \delta_1$.

This can be carried out quite generally; Lemma 4.5 below gives the formal statement and proof of the result we use.

Looking ahead to when we consider the simultaneous action of many functionals, we will build at every stage s a set $T_{i,s}$ of splits. Each split in $T_{i,s}$ is said to be of a certain *level*. We will also define a set $E_{i,s}$ of natural numbers. The intention is that $E_{i,s}$ contains the indices e such that we are ready to define a base point corresponding to some string in $T_{i,s}$ in some test for Q_e.

In our example, if δ_0 and δ_1 were found at stage t, we will let $T_{i,t} = \{\delta_0, \delta_1\}$. We say that δ_0 and δ_1 are of level 1, and we enumerate d into E_i. If the strings σ_0 and σ_1 were found at stage u, then we enumerate σ_0 and σ_1 into T_i at stage u. We say that σ_0 and σ_1 are of level 2, and we enumerate e into E_i.

4.9. **Forbidden strings.** We again revise the strategies to incorporate the search for common splits.

Suppose that we find a pair of strings δ_0 and δ_1 which Φ_e-split at stage t. Suppose that we let A_t extend δ_0. We initialise P_i, and so choose a large follower n for P_i. We would like to be able to produce a pair of common splits, and so the strategy for $Q_{d,i}$ now looks for a pair of Φ_d-splits above δ_0. Meanwhile, the strategy for $Q_{e,i}$ will wait for convergence, and may later define the base point $\Phi_e(\delta_0)$ in the test t_e.

Suppose that P_i is realised with n at some later stage u, and that we have *not* yet seen a pair of Φ_d-splits above δ_0. For the sake of $Q_{e,i}$, we let A_u extend δ_1.

This action, however, has interfered with the search for common splits. If we were to find a pair of Φ_d-splits above δ_1, we would not yet be in a position to produce common splits: we would need to find a *further* Φ_e-split above which there is a pair of Φ_d-splits. We may however have defined the base point $\Phi_e(\delta_0)$ when P_i had *not* been realised with n. Therefore, we could not return to δ_0, since the bound for $\Phi_e(\delta_0)$ would not be able to accommodate the changes necessary for the strategy for P_i.

This problem is easily fixed. The strategy for $Q_{e,i}$ instead looks for *three* Φ_e-splits. This is no more strenuous a test on the hypothesis that $\Phi_e(A)$ is total and noncomputable. If we do find three Φ_e-splits at some stage, then we will be able to move to a new split when P_i is realised with its follower, but will have sufficient room to move in the search for common splits. We note that the precise number of splits which must be found at various stages is more complicated in general, though not difficult. We discuss this further in Section 4.13 below.

Looking ahead to when we handle all the P-requirements, it will be convenient to keep track of those strings to which we may not later return. We do so by defining at every stage s a set $F_{i,s}$ of *forbidden* strings. We will have that $F_{i,s} \subseteq T_{i,s}$.

4.10. **Guessing.** We rewind, and see how the construction may play out from the beginning. Suppose that we first begin the strategy for P_i at stage s. Then $Q_{d,i}$ will look for three Φ_d-splits, and similarly for $Q_{e,i}$. Suppose that we see the Φ_e-splits δ_0, δ_1, and δ_2 at some later stage t. We enumerate these splits into T_i, let each be of level 1, and enumerate e into E_i. We let A_t extend δ_0. We initialise P_i, and choose a large follower n

for P_i. Next, suppose that P_i is realised with n at some later stage u. We enumerate δ_0 into F_i, and let A_u extend δ_1.

As always, $Q_{e,i}$ will be looking to define base points in the test t_e. Suppose that we define the base point $\Phi_e(\delta_1)$ in the test t_e at some later stage v. Then since P_i has been realised with n by stage v, we define the bound for $\Phi_e(\delta_1)$ in t_e to be $o_{i,v}(n)$.

Suppose that we later find a pair of Φ_d-splits above δ_1. We then move the approximation to A to extend δ_2 in the hopes of finding a common split. Suppose that at some later stage w, we do in fact find another pair of Φ_d-splits above δ_2, and then produce the pair of common splits σ_0 and σ_1. We enumerate σ_0 and σ_1 into T_i, let σ_0 and σ_1 be of level 2, and enumerate e into E_i. As we saw above, we must again initialise P_i. Suppose that we choose the large follower n' for P_i.

We must ensure that the base points in the test t_e are able to cope with the action that we may later take for P_i. There is a problem. It is possible for $\Phi_e(\sigma_0)$ to extend some base point in t_e at stage w, and also for $\Phi_e(\sigma_1)$ to extend some base point in t_e at stage w. Then if P_i were realised with n' at some later stage, regardless of which common split we let the approximation to A extend, the approximation to $\Phi_e(A)$ will extend some base point in t_e which was defined before the follower n' was realised.

To solve this problem, we use *guessing*, a standard feature of infinite injury priority arguments. Each Q-requirement has two *outcomes*: the outcome `inf`, which holds if $\Phi_e(A)$ is total and noncomputable, and the outcome `fin`, which holds otherwise. Our priority ordering will order the Q-requirements. Suppose that Q_d is of stronger priority than Q_e. Then a strategy working for Q_e-subrequirement will guess as to the outcome of Q_d. A strategy for a Q_e-subrequirement which guesses that Q_d has the outcome `inf` will always wait until we see common splits, and will define base points in the test $t_{\texttt{inf}}$, whereas a strategy for a Q_e-subrequirement which guesses that Q_d has the outcome `fin` assumes that we will not find any further Φ_d-splits, and will define base points in the test $t_{\texttt{fin}}$. Whenever we receive further evidence that the outcome `inf` is incorrect, namely when we *do* find more Φ_d-splits, the test $t_{\texttt{fin}}$ will be reset.

Guessing completely solves our problem. Before we see common splits, the strategy for $Q_{e,i}$ which guesses that Q_d has the `fin` outcome will define base points in the test $t_{\texttt{fin}}$. We then reset $t_{\texttt{fin}}$ at stage w when common splits are found. The strategy for $Q_{e,i}$ which guesses that Q_d has the `inf` outcome will then define base points in the test $t_{\texttt{inf}}$.

4.11. **Freezing base points.** The guessing is, as usual, asymmetric: while the strategies working for Q_e-subrequirements are able to guess about the outcome of Q_d, we cannot allow the strategies working for Q_d-subrequirements to guess about the outcome of Q_e.

Therefore, we must consider the sequence of events from the previous section, but with d and e interchanged. So suppose that we find the Φ_d-splits δ_0, δ_1, and δ_2 at stage t. We enumerate these splits into T_i, let each be of level 1, and enumerate d into E_i. Suppose that we later produce the common splits σ_0 and σ_1 at some later stage u. We enumerate these splits into T_i, let each be of level 2, and enumerate e into E_i. However, we are not able to simply reset the test for t_d, as we did for $t_{\texttt{fin}}$ above. We would like some way to limit the number of times that we update the arrow for base points that were defined in t_d before we found common splits.

Suppose that we defined the base point $\Phi_d(\delta_1)$ in the test t_d before stage u. We would like to "freeze" the base point $\Phi_d(\delta_1)$ by updating the arrow for $\Phi_d(\delta_1)$ at some later stage to be some string which the approximation to $\Phi_e(A)$ *never* extends. Then we would never need to update the arrow for $\Phi_d(\delta_1)$ again.

Recall that a set $S \subset 2^{<\omega}$ is *sparse* if for all $l \in \omega$, there is at most one string in S of length l. Suppose that we found at some stage v with $v > u$ some finite set X such that

- X is sparse,
- for all $\chi \in X$, $|\chi| > |\Phi_d(\delta_1)|$, and
- for all $w \geqslant v$, $\Phi_{e,w}(A_w)$ extends some string in X.

Then we can freeze $\Phi_d(\delta_1)$ by defining the arrow for $\Phi_d(\delta_1)$ to be some string which properly extends $\Phi_d(\delta_1)$ and which is incomparable with every string in X.

Suppose that we are able to extend the tree of splits to a third level, where the strings of level 3 pairwise Φ_d-splits. Then if M is the set of strings of level 3, we would like for the set $\Phi_e(M)$ to play the role of X. It is relatively straightforward to show that if $\Phi_d(A)$ is total and noncomputable, then we can find splits which guarantee that $\Phi_e(M)$ has the properties listed above.

We must consider how this feature interacts with the guessing. When the common splits of level 2 are found, we immediately pause the definition of the test t_d, and will wait until the further level of Φ_d-splits is found. We therefore *remove* d from E_i. The strategy for $Q_{e,i}$ will begin to define base points in the test $t_{\mathtt{fin}}$. If the further level of Φ_d-splits is later found, then the strategy for $Q_{d,i}$ will freeze base points in the test t_d if necessary, and define new base points in t_d corresponding to strings of level 3, and $Q_{e,i}$ will begin to define base points in the test $t_{\mathtt{inf}}$.

We must make a small alteration to the definition of the bound of the base points in the test t_d. Suppose that we wish to define the bound for some base point $\Phi_d(\delta_i)$ in t_d at some stage t before common splits are found. Suppose that P_i has follower n at stage t. If P_i has not been realised with n by stage t, we define the bound to be 1, and otherwise, we define the bound to be $o_{i,t}(n) + 1$. Then the bound is able to accommodate a further update of the arrow as a result of freezing.

4.12. **General remarks about the construction.** It may be helpful to make some general remarks about the way in which the construction is organised. As we saw above, the strategy for the P-requirements is finitary. While the Q-requirements are infinitary, we are able to break each Q-requirement Q_e into infinitely many finitary Q_e-subrequirements.

Our priority ordering will include nodes working for P-requirements, Q-requirements, and also Q-subrequirements. The node working for the Q-requirement Q_e is responsible for resetting the tests that we build for the sake of Q_e. The nodes working for Q_e-subrequirements will be responsible for defining base points in the tests and updating the arrows for these base points.

Of equally strong priority as P_i, we have the Q-requirement Q_i, as well as the Q-subrequirements $Q_{e,i}$ for all $e \leqslant i$. As we have seen, much of the complexity of the construction is in coordinating the interaction of the strategy for P_i with the strategies working for Q-subrequirements of equally strong priority as P_i. The tree of splits is built in order to facilitate this. The coordinating action is rather significant, and it will be convenient to introduce for every $i \in \omega$ the quasirequirement T_i, of equally strong priority as P_i, to manage this.

The mechanism of freezing can be viewed as a way for strategies to deal with injury due to the action of strategies working for Q-requirements of weaker global priority, but of equally strong local priority.

4.13. **Combinatorics.** We now give some brief comments about the combinatorics required for the general construction of the trees of splits.

For all $i \in \omega$, and at every stage s, the tree $T_{i,s}$ must handle splits for all functionals Φ_e with $e \leqslant i$.

We define in Section 4.15.2 a sequence $\langle h_i \rangle_{i<\omega}$ with the intention that h_i bounds the number of levels that $T_{i,s}$ has. We also define the sequence $\lambda_{i,s} \in \omega^{<\omega}$. With this sequence, it will be straightforward to see that $T_{i,s}$ does indeed have at most h_i many levels. This fact is recorded in part (3) of Lemma 4.2.

For all $i \in \omega$ and all $l \in [1, h_i]$, we also define the natural number $m_{i,l}$, which is intended to give the number of strings of level l in $T_{i,s}$. Lemma 4.2 and Lemma 4.6 give careful formal arguments that these definitions suffice.

4.14. **Final notes.** The fact that A is Δ^0_2, and in particular, not computably enumerable, together with the infinitary nature of the Q-requirements, means that the construction and its verification are rather involved. We choose to take a rather cautious approach.

We define several other objects during the construction which will be of use in the verification. We give some brief remarks which will hopefully help to orient the reader.

For every (quasi)requirement T_i, and at every stage s, we may define a string $\varepsilon_{i,s}$. If the set of splits $T_{i,s}$ is nonempty, then $\varepsilon_{i,s}$ will in fact be defined, and every string in $T_{i,s}$ will extend $\varepsilon_{i,s}$. The main purpose of this string is to serve as a restraint for the action of T_i and P_i.

As mentioned above, Lemma 4.5 will play a crucial role in defining the trees of splits. We must be rather careful though to define the tree of splits from suitable collections of splits that we find during the construction. Therefore, for every $i \in \omega$, for every $e \leqslant i$, and at every stage s, we define a set of $Y_{i,e,s}$ which will contain Φ_e-splits that may later be used in the definition of the tree of splits T_i. These sets play their most prominent role in the proof of Claim 4.39 below.

For various reasons, we will often wish for the Φ_e-splits that we find to have image under Φ_e of at least a certain length. Therefore, for every $e \in \omega$ and at every stage s, we define a set $Z_{e,s}$ of strings. This will be important in showing, for example, that the set of Φ_e-splits of a certain level has sparse image under Φ_e.

For every Q-subrequirement $Q_{e,i}$, and at every stage s, we will define a collection $N_{e,i,s}$ of guesses $\nu \in \{\mathtt{inf}, \mathtt{fin}\}^e$ which $Q_{e,i}$ must consider when defining base points. The only important point to note here is that while we may see that for some $d < e$, $Q_{d,i}$ is satisfied at stage s, some weaker priority Q_d-subrequirement may never become satisfied, and so the "true outcome" for Q_d is in fact $\mathtt{fin}$. The set $N_{e,i,s}$ is defined so that we may properly maintain the base points in the tests which guess incorrectly about the local outcome, but which guess correctly about the global outcome.

We now proceed to the formal part of the argument.

4.15. **Further definitions and conventions for the construction.** For convenience and concreteness, we collect all definitions and conventions which will be required for the formal construction which have not already been given in Section 4.1, Section 4.2, or Section 4.3. As usual, all objects that we define during the construction will keep their values at the following stage unless otherwise specified.

4.15.1. *The priority ordering.* We let

$$\mathcal{R} = \{T_i : i \in \omega\} \cup \{P_i : i \in \omega\} \cup \{Q_e : e \in \omega\} \cup \{Q_{e,i} : e, i \in \omega, e \leqslant i\}.$$

We define a priority ordering $\leqslant_{\mathcal{R}}$ on $\mathcal{R}$ as follows. We first define $o : \mathcal{R} \to \omega$. For all $i \in \omega$, we let $o(T_i) = o(P_i) = i$, and for all $e \in \omega$, we let $o(Q_e) = e$. For all $e, i \in \omega$ with $e \leqslant i$, we let $o(Q_{e,i}) = i$. Then we let $R_1 \leqslant_{\mathcal{R}} R_2$ if and only if $o(R_1) \leqslant o(R_2)$. We define $<_{\mathcal{R}}$ as usual.

4.15.2. *The combinatorics.* For all $i \in \omega$, let H_i be the set of all sequences $\sigma \in \omega^{<\omega}$ such that

- for all $k < |\sigma|$, $\sigma(k) \leqslant i$, and
- for all $j < l < |\sigma|$, if $\sigma(j) = \sigma(l)$, then there is some k with $j < k < l$ such that $\sigma(k) > \sigma(j)$.

Note that for all $i \in \omega$, $|H_i|$ is finite. For all $i \in \omega$, let h_i be the length of the longest sequence in H_i. The function $i \mapsto h_i$ is primitive recursive, and for all $i \in \omega$, $h_i > 0$. In fact, it is straightforward to show that for all $i \in \omega$, h_i is the i^{th} triangular number, starting with $h_0 = 1$.

For all pairs $(i, l) \in \omega \times \omega$ with $l \in [1, h_i]$, we define $m_{i,l} \in \omega$ as follows. It is clear that $h_0 = 1$. We define $m_{0,1} = 1$. Now fix some $i \geqslant 1$. We define $m_{i,h_i} = 2$. For all $l \in [1, h_i)$, given $m_{i,l+1}$, we define

$$m_{i,l} = 2(i+1).(m_{i,l+1} - 1) + 1.$$

4.15.3. *The trees.* For every $i \in \omega$ and at every stage $s \in \omega$, we will define a set $T_{i,s} \subset 2^{<\omega}$, a set $F_{i,s} \subseteq T_{i,s}$, a set $E_{i,s} \subset \omega$, and a sequence $\lambda_{i,s} \in \omega^{<\omega}$. We may also define a string $\varepsilon_{i,s}$. For every $e \leqslant i$, we will also define a set $Y_{i,e,s} \subset 2^{<\omega}$.

We view $T_{i,s}, F_{i,s}$, $Y_{i,e,s}$, and $E_{i,s}$ as the computable approximations at stage s to sets T_i, F_i, $Y_{i,e}$, and E_i, respectively. Therefore, we may say that *we enumerate σ into T_i at stage s*, or that *we remove σ from T_i at stage s*, with the obvious meaning, and similarly for F_i, $Y_{i,e}$ and E_i.

If we say at stage s that we *empty* $Y_{i,e}$, then we set $Y_{i,e,s} = \varnothing$. If we say at stage s that we *initialise* T_i, then we set $T_{i,s} = \varnothing$, $F_{i,s} = \varnothing$, $E_{i,s} = \varnothing$, $\lambda_{i,s} = \langle\rangle$, we let $\varepsilon_{i,s}$ be undefined, and for all $e \leqslant i$, we empty $Y_{i,e}$.

For some $e \in \omega$, we may say at some stage that *we have found Φ_e-splits above δ*, for some string δ, that some strings are *Φ_e-splits*, or that some strings are *of level l*, for some $l \in \omega$. These statements will hold at all subsequent stages.

4.15.4. *The P-requirements.* For every $i \in \omega$, and at every stage $s \in \omega$, we may define for P_i a *follower*, which will be some natural number. We may say that *P_i has been realised with n* at stage s, for some $n \in \omega$; this statement will hold at all subsequent stages. We may also say that *P_i is waiting to define an axiom at stage s*.

We say that P_i *requires attention at stage s* if

- all P-requirements P with $P <_{\mathcal{R}} P_i$ are satisfied at stage s,
- for all $j \leqslant i$, $\varepsilon_{j,s-1}$ is defined

and for $l = |\lambda_{i,s-1}|$, either

(1) P_i does not have a follower at stage s,
(2) P_i has follower n at stage s, P_i has not been realised with n by stage s, but $o_{i,s}(n) < \infty$,
(3) A_{s-1} extends some string which is in $T_{i,s-1}$ and of level l, and P_i is waiting to define an axiom at stage s, or
(4) P_i has follower n at stage s, P_i has been realised with n by stage s, $\Gamma_{s-1}(A_{s-1}, n)\downarrow$, but $\Gamma_{s-1}(A_{s-1}, n) \leqslant f_{i,s}(n)$.

We say that P_i *is satisfied at stage s* if P_i has a some follower n at stage s, $\Gamma_{s-1}(A_{s-1}, n)\downarrow$, and if $o_{i,s}(n) < \infty$ then $\Gamma_{s-1}(A_{s-1}, n) > f_{i,s}(n)$.

To initialise P_i at stage s, we remove its follower, and we do not say that P_i is waiting to define an axiom at stage $s+1$.

4.15.5. *The Q-requirements.* For every $\nu \in \{\mathtt{inf}, \mathtt{fin}\}^{<\omega}$, we will build an α-change test t_ν. If we say at stage s that we *reset* t_ν, then we abandon the test t_ν, and will at later stages build a new test, which, abusing notation, we will also call t_ν. To reset the test $t_\nu = \langle a_{\nu,s}, b_{\nu,s}\rangle_{s<\omega}$ at stage s, we set $a_{\nu,r}(\sigma) = \sigma$ and $b_{\nu,r}(\sigma) = \infty$ for all $r \leqslant s$ and all $\sigma \in 2^{<\omega}$.

For all $e, s \in \omega$, we will define a set of strings $Z_{e,s}$. As above, we may say that we *enumerate* elements into Z_e at stage s, but note that we will not remove elements from Z_e at any stage.

For every Q-subrequirement $Q_{e,i}$ and every $s \in \omega$, we may define a string $\varphi_{e,i,s}$. For all $e, s \in \omega$, we let $\varphi_{e,e-1,s} = \langle\rangle$, and we may assume that $\Phi_e(\langle\rangle) = \langle\rangle$.

We say that $Q_{e,i}$ *is satisfied at stage s* if $e \in E_{i,s-1}$ and $\varphi_{e,i,s-1}$ is defined.

We say that $Q_{e,i}$ *requires attention at stage s* if

- all P-requirements P with $P \leqslant_{\mathcal{R}} Q_{e,i}$ are satisfied at stage s,
- for all $j \leqslant i$, $\varepsilon_{j,s-1}$ is defined,
- all Q_e-subrequirements Q with $Q <_{\mathcal{R}} Q_{e,i}$ are satisfied at stage s,
- $e \in E_{i,s-1}$,

- $\varphi_{e,i,s-1}$ is not defined, and
- if stage r is the last stage before stage s at which e was enumerated into E_i, then there is some Φ_e-split δ in $T_{i,s-1}$ of level $l_e = |\lambda_{i,r}|$ and some string φ in the domain of Φ_e at stage s such that
  - $\delta \prec \varphi \prec A_{s-1}$,
  - $|\Phi_e(\varphi)| > |\Phi_e(\zeta)|$ for all $\zeta \in Z_{e,s-1}$,
  - $|\Phi_e(\sigma)| > |\Phi_e(\varphi_{e,j,s-1})|$ for all $j \in [e, i)$, and
  - $|\varphi| < s$.

For every $e, i, s \in \omega$, we let $N_{e,i,s}$ be the set of all $\nu \in \{\mathtt{inf}, \mathtt{fin}\}^e$ such that for all $d < e$, if $d \notin E_{i,s-1}$, then $\nu(d) = \mathtt{fin}$.

To initialise the Q-requirement Q_e at stage s, we reset all $\nu \in \{\mathtt{inf}, \mathtt{fin}\}^e$. To initialise the Q-subrequirement $Q_{e,i}$ at stage s, we let $\varphi_{e,i,s}$ be undefined.

4.15.6. *The T-(quasi)requirements.* For all $i \in \omega$ and all $e \leqslant i$, we say that T_i *requires attention via e at stage s* if

- for all $j \leqslant i$, $\varepsilon_{j,s-1}$ is defined,
- all Q_e-subrequirements Q with $Q <_{\mathcal{R}} Q_{e,i}$ are satisfied at stage s,
- $e \notin E_{i,s-1}$,
- δ is a string in $T_{i,s-1}$ of level $l = |\lambda_{i,s-1}|$ which A_{s-1} extends,
- for all $d \leqslant e$, we have not found Φ_d-splits above δ, and
- there is some set S of Φ_e-splits of cardinality $m_{i,l+1}$ such that for all $\sigma \in S$,
  - $\delta \prec \sigma$,
  - $\Phi_e(S)$ is sparse,
  - $|\Phi_e(\sigma)| > |\Phi_e(\zeta)|$ for all $\zeta \in Z_{e,s-1}$,
  - $|\Phi_e(\sigma)| > |\Phi_e(\varphi_{e,j,s-1})|$ for all $j \in [e, i)$, and
  - $|\sigma| < s$.

We show in Lemma 4.7 below that the construction can be carried out as described.

4.16. **The construction.**

Stage 0: We set $A_0 = 0^\omega$ and $\Gamma_0 = \varnothing$. For all $e \in \omega$, we set $Z_{e,0} = \varnothing$. We initialise all $R \in \mathcal{R}$ and reset t_ν for all $\nu \in \{\mathtt{inf}, \mathtt{fin}\}^{<\omega}$. We proceed to stage 1.

Stage s, $s \geqslant 1$: we follow the instructions below. Then if s is not currently the follower of some P-requirement and $\Gamma_s(A_s, s)\uparrow$, we enumerate $\langle\langle\rangle, s, 0\rangle$ into Γ. We proceed to the next stage.

$s = 4n + 1$ *for some* $n \in \omega$

Let i be least such that $T_{i,s-1}$ is empty. We define $\varepsilon_{i,s}$ to be some initial segment of A_{s-1} of a large length, and let $T_{i,s} = \{\varepsilon_{i,s}\}$. We say that $\varepsilon_{i,s}$ is of level 0. We initialise all $R \in \mathcal{R}$ with $T_i <_{\mathcal{R}} R$.

$s = 4n + 2$ *for some* $n \in \omega$

If there is some P-requirement which requires attention at stage s, then we follow the instructions below.

Let P_i be the P-requirement of strongest priority which requires attention at stage s. Let $l = |\lambda_{i,s-1}|$. We follow the instructions of the first case below which pertains. We say that we act for P_i at stage s. We then initialise all $R \in \mathcal{R}$ with $P_i <_{\mathcal{R}} R$.

Case 1a: P_i requires attention at stage s via (1). We choose a large follower for P_i. We say that P_i is waiting to define an axiom at stage $s + 1$.

Case 1b: P_i requires attention at stage s via (2). Suppose that P_i has follower n at stage s. We say that P_i is realised with n. We say that P_i is waiting to define an axiom at stage $s + 1$.

If $E_{i,s-1} \neq \varnothing$, then we do the following. Suppose that n was chosen at stage q. Let F be the set of all $\delta \in T_{i,s-1}$ of level l such that for some $r \in [q, s)$, $A_r \succ \delta$. We enumerate the elements of F into F_i. We choose some string $\alpha \in T_{i,s-1}$ which is of level l and which is not in $F_{i,s}$, and let $A_s = \alpha\,\hat{}\,0^\omega$.

Case 1c: P_i requires attention at stage s via (3). Let δ be the string in $T_{i,s-1}$ of level l which A_{s-1} extends. Suppose that P_i has follower n at stage s. We choose some string γ of a large length, which extends δ, and such that there is no $\iota \in \operatorname{dom}\Gamma_{s-1}(n)$ with $\delta \prec \iota \preccurlyeq \gamma$. We let $A_s = \gamma\,\hat{}\,0^\omega$. If P_i has been realised with n, then we enumerate $\langle \gamma, n, f_{i,s}(n) + 1\rangle$ into Γ, and otherwise, we enumerate $\langle \gamma, n, 0\rangle$ into Γ. We do not say that P_i is waiting to define an axiom at stage $s + 1$. We initialise $Q_{d,i}$ for all $d \in [0, i]$.

Case 1d: P_i requires attention at stage s via (4). We say that P_i is waiting to define an axiom at stage $s + 1$.

<u>$s = 4n + 3$ *for some* $n \in \omega$</u>

If there is some i for which there is some e such that T_i requires attention via e at stage s, then we follow the instructions below.

Let i be least as above, and let e be greatest for this i. Let S_s be the set S as in the definition of requires attention for this choice of i and e. We say that we have found Φ_e-splits above δ, and enumerate the elements of S_s into $Y_{i,e}$ and Z_e. Let $l = |\lambda_{i,s-1}|$.

We act according to the following cases. We say that we act for T_i with e at stage s. We initialise $Q_{d,i}$ for all $d \in [0, i]$ and all $R \in \mathcal{R}$ with $T_i <_{\mathcal{R}} R$. We reset all t_ν with $|\nu| > e$ and $\nu(e) = \texttt{fin}$.

Case 2a: either $l = 0$, or there is some $e \leqslant i$ for which there are $m_{i,l+1}$ many strings in $T_{i,s-1}$ of level l which are not in $F_{i,s-1}$, and above which we have found Φ_e-splits. We choose the greatest such e.

Let $T \subset Y_{i,e,s}$ be a set of Φ_e-splits of cardinality $m_{i,l+1}$ such that every string in T properly extends some unique string of level l in $T_{i,s-1}$ which is not in $F_{i,s-1}$, and such that $\Phi_e(T)$ is sparse. We enumerate the elements of T into T_i.

We choose some $\alpha \in T$ and let $A_s = \alpha\,\hat{}\,0^\omega$. We enumerate e into E_i, and for all $d < e$, we remove d from E_i. For all $d < e$, we empty $Y_{i,d}$. We let $\lambda_{i,s} = \lambda_{i,s-1}\,\hat{}\,\langle e\rangle$. We say that every string in T is a Φ_e-split, and is of level $l + 1$. We initialise P_i.

Case 2b: otherwise. Choose some string α in $T_{i,s-1}$ which is of level l and not in $F_{i,s-1}$, and above which we have not found Φ_d-splits for any d. We let $A_s = \alpha\,\hat{}\,0^\omega$. We say that P_i is waiting to define an axiom at stage $s + 1$.

<u>$s = 4n$ *for some* $n \geqslant 1$</u>

If there is some Q-subrequirement which requires attention at stage s, then we proceed as follows. Let i be least such that there is some e such that $Q_{e,i}$ requires attention at stage s, and choose the least such e for this i.

Let δ, l_e, and φ, be as in the definition of requires attention. Let M be the set of all strings in $T_{i,s-1}$ of level l_e. We define $\varphi_{e,i,s} = \varphi$. Let P_s^+ be the set of all $j < s$ such that P_j has some follower $n_{j,s}$ at stage s, and P_j has been realised with $n_{j,s}$ by stage s.

We carry out the instructions in step 1 and step 2. We then initialise all $R \in \mathcal{R}$ with $Q_{e,i} <_{\mathcal{R}} R$.

<u>*Step 1*</u>

Let $\rho = \Phi_e(\delta)$. If $b_{\nu,s-1}(\rho) = \infty$ for some $\nu \in N_{e,i,s}$, then for all such ν, we do the following. Let $I_{\nu,\rho,s} = [e, i]$ and let $I^+_{\nu,\rho,s} = I_{\nu,\rho,s} \cap P^+_s$. We define $a_{\nu,s}(\rho)$ to be some string which properly extends ρ and which is incomparable with $\Phi_e(\varphi)$, and we define

$$b_{\nu,s}(\rho) = \left(\bigoplus_{j \in I^+_{\nu,\rho,s}} f_{j,s}(n_{j,s})\right) \oplus |I_{\nu,\rho,s}|.$$

<u>*Step 2*</u>

For all $\nu \in N_{e,i,s}$ and all ρ such that $\Phi_e(\varphi_{e,i-1,s}) \prec \rho \preccurlyeq \Phi_e(\varphi)$, $b_{\nu,s-1}(\rho) < \infty$, and $a_{\nu,s-1}(\rho) \prec \Phi_e(\varphi)$, we act according to the subcases below.

Subcase 3a: $\rho \prec \Phi_e(\delta)$. We define $a_{\nu,s}(\rho)$ to be some string which properly extends ρ and which is incomparable with $\Phi_e(\mu)$ for all $\mu \in M$. Let $I_{\nu,\rho,s} = [e,i)$ and let $I^+_{\nu,\rho,s} = I_{\nu,\rho,s} \cap P^+_s$. We define $b_{\nu,s}(\rho)$ as above.

Subcase 3b: otherwise. We define $a_{\nu,s}(\rho)$ to be some string which properly extends ρ and which is incomparable with $\Phi_e(\varphi)$. Let $I_{\nu,\rho,s} = [e,i]$ and let $I^+_{\nu,\rho,s} = I_{\nu,\rho,s} \cap P^+_s$. We define $b_{\nu,s}(\rho)$ as above.

4.17. **The verification.** We first work towards Lemma 4.7, which says that the construction can be carried out as described. The following lemma, while highly technical, summarises many important properties of the construction.

Lemma 4.2. *For all $s \in \omega$, if all stages up to and including stage s of the construction can be carried out as described, then for all $i \in \omega$, for $l = |\lambda_{i,s}|$,*

(1) if $E_{i,s} = \varnothing$ then $F_{i,s} = \varnothing$.
(2) if $T_{i,s} \neq \varnothing$ then A_s extends some string in $T_{i,s}$ of level l which is not in $F_{i,s}$.
(3) $\lambda_{i,s} \in H_i$.
(4) $E_{i,s} = \varnothing$ if and only if $\lambda_{i,s} = \langle\rangle$.
(5) if $E_{i,s} \neq \varnothing$ then there are $m_{i,l}$ many strings in $T_{i,s}$ of level l.
(6) if $E_{i,s} \neq [0,i]$ then $l < h_i$.
(7) if P_i has follower n at the end of stage s, and P_i has not been realised with n by the end of stage s, then no string in $T_{i,s}$ of level l is in $F_{i,s}$.
(8) if A_{s-1} extends some string $\delta \in T_{i,s-1}$ but $\delta \not\prec A_s$, then if we do not initialise T_i at stage s, we act either for P_i in case 1b or for T_i at stage s.
(9) if $0 < l < h_i$, then for every $e \leqslant i$, there are at most $m_{i,l+1} - 1$ many strings in $T_{i,s}$ which are of level l and not in $F_{i,s}$, and above which we have found Φ_e-splits.
(10) if $\delta \in T_{i,s}$ is a Φ_e-split, then there is some unique stage r with $r \leqslant s$ at which δ was enumerated into T_i, and some unique stage q with $q \leqslant r$ at which δ was enumerated into $Y_{i,e}$ and Z_e.
(11) if $e \in E_{i,s}$, stage r is the last stage before stage s at which e was enumerated into E_i, $l_e = |\lambda_{i,r}|$, and M is the set of strings in $T_{i,s}$ of level l_e, then M is a set of Φ_e-splits of cardinality m_{i,l_e}, and $\Phi_e(M)$ is sparse.
(12) if $T_{i,s} \neq \varnothing$, then $\varepsilon_{i,s}$ is defined, and if $\lambda_{i,s} = \langle\rangle$ then $T_{i,s} = \{\varepsilon_{i,s}\}$.

Proof. By induction on s, often together with a straightforward but tedious case analysis. □

Lemma 4.3. *Suppose that $\sigma \in 2^{<\omega}$ and that $S \subset 2^{<\omega}$ is sparse. Then for all $l > |\sigma|$ there is some string ν of length l which extends σ, and such that there is no string $\iota \in S$ with $\sigma \prec \iota \preccurlyeq \nu$.*

Proof. By induction on l. □

Lemma 4.4. *For all $s \in \omega$, if all stages up to and including stage s of the construction can be carried out as described, then for all $n \in \omega$ and all $l \in \omega$, $\operatorname{dom}\Gamma_s(n)$ is sparse.*

Proof. By induction, as in the proof of Lemma 4.2. □

Lemma 4.5. *Let Φ be a Turing functional with a length increasing computable enumeration. Suppose that $m \geqslant 2$, and that $\{S_i\}_{i<m}$ is a collection of pairwise disjoint sets of Φ-splits of cardinality m. Then there is a set S of Φ-splits of cardinality m, where for each $i < m$, there is some $\delta \in S_i$ with $\delta \in S$.*

Proof. We show this by induction on m, starting with $m = 2$. For $m = 2$, let

$$l = \max\{|\Phi(\sigma)| : \sigma \in S_0 \cup S_1\}.$$

Without loss of generality, we may assume that $\sigma_0 \in S_0$ is such that $|\Phi(\sigma_0)| = l$. Now, by the choice of σ_0, for each $\sigma \in S_1$, either $\Phi(\sigma) \preccurlyeq \Phi(\sigma_0)$, or $\Phi(\sigma)$ is incomparable with $\Phi(\sigma_0)$. Since S_1 is a set of Φ-splits, we may choose some string from S_1 which, together with σ_0, forms a set S of Φ-splits as required.

Now let $m > 2$, and suppose by induction that the statement of the lemma holds for $m - 1$. Let

$$l = \max\{|\Phi(\sigma)| : \sigma \in \bigcup_{i<m} S_i\}.$$

Without loss of generality, we may assume that $\sigma_0 \in S_0$ is such that $|\Phi(\sigma_0)| = l$. Now, by the choice of σ_0, for all $i \in [1, m]$ and all $\sigma \in S_i$, either $\Phi(\sigma) \preccurlyeq \Phi(\sigma_0)$, or $\Phi(\sigma)$ is incomparable with $\Phi(\sigma_0)$. Since S_i is a set of Φ-splits, there is at most one string $\sigma \in S_i$ such that $\Phi(\sigma) \preccurlyeq \Phi(\sigma_0)$. So for all $i \in [1, m]$, there is a set $R_i \subset S_i$ of cardinality $m - 1$ such that for all $\sigma \in R_i$, σ and σ_0 Φ-split. By the inductive hypothesis, there is a set R of Φ-splits of cardinality $m - 1$, where for each $i \in [1, m]$, there is some $\delta \in R_i$ with $\delta \in R$. Since $\{S_i\}_{i \in [1,m]}$ is a collection of pairwise disjoint sets, the set $\{\sigma_0\} \cup R$ is a set of cardinality m, and is as required. $\square$

Lemma 4.6. *Suppose that $s \in \omega$, and that all stages before stage s can be carried out as described. Let $i \in \omega$, $l = |\lambda_{i,s}|$, and suppose that $E_{i,s-1} \neq \varnothing$. Then*

(1) if $E_{i,s-1} = [0, i]$, then at most 1 string in $T_{i,s}$ which is of level l is in $F_{i,s}$.
(2) if $E_{i,s-1} \neq [0, i]$, then at most $(i + 1).(m_{i,l+1} - 1) + 1$ many strings in $T_{i,s}$ which are of level l are in $F_{i,s}$.

Proof. We show this by induction on s. The lemma is clear for $s = 0$, so assume that $s > 0$, and that all stages before stage s can be carried out as described. The only difficulty is if we act in case 2a or in case 1b at stage s.

First suppose that we act for T_i in case 2a at stage s. Then no string in $T_{i,s}$ of level l is in $F_{i,s}$, and the lemma holds.

Now suppose that we act for P_i in case 1b at stage s. Suppose that P_i has follower n at stage s, and that n was chosen at stage q. Let F be the set of all strings $\delta \in T_{i,s-1}$ of level l such that for some $r \in [q, s)$, $A_r \succ \delta$.

Below, we shall frequently refer to the various parts of Lemma 4.2. Note that by the choice of q, $\lambda_{i,r} = \lambda_{i,q}$ for all $r \in [q, s]$. Since part (2) holds at stage $s - 1$, A_{s-1} extends some string δ_{s-1} in $T_{i,s-1}$ of level l. Then $\delta_{s-1} \in F$. By the choice of q, we cannot act for T_i in case 2a at any stage after stage q and before stage s, since doing so would initialise P_i. We also cannot act for P_i in case 1b at any stage after stage q and before stage s, since P_i has not been realised with n by stage s. Now since part (8) holds at all stages before stage s, the cardinality of F is one greater than the number of stages after stage q and before stage s at which we act for T_i in case 2b.

First suppose that $E_{i,s-1} = [0, i]$. By the choice of q, $E_{i,r} = E_{i,q}$ for all $r \in [q, s]$. Then by the definition of requires attention, we cannot act for T_i with e at any stage r with $r \in (q, s)$.

Now suppose that $E_{i,s-1} \neq [0, i]$. Suppose that we act for T_i in case 2b at some stage r with $r \in (q, s)$. Then for some $e \leqslant i$, we say that we have found Φ_e-splits above δ_{r-1} at stage r. Since part (9) holds at stage $s - 1$, there are at most $(i + 1).(m_{i,l+1} - 1)$ many stages r with $r \in (q, s)$ at which we act for T_i in case 2b. $\square$

Lemma 4.7. *The construction can be carried out as described.*

Proof. We show this by induction on the stage number. It is clear that stage 0 of the construction can be carried out as described. So let $s > 0$, and suppose by induction that all stages before stage s can be carried out as described. Let $l = |\lambda_{i,s-1}|$. Below, we shall frequently refer to the various parts of Lemma 4.2.

First suppose that $s = 4n + 1$ for some $n \in \omega$. It is clear that stage s can be carried out as described.

Now suppose that $s = 4n + 2$ for some $n \in \omega$. Let P_i be as at stage s. It is clear that if case 1a or case 1d applies at stage s, then stage s can be carried out as described.

Suppose that case 1b applies at stage s. The only difficulty is if $E_{i,s-1} \neq \emptyset$, so assume this. We must show that there is some string α in $T_{i,s}$ which is of level l and not in $F_{i,s}$. We have that $\lambda_{i,s-1} = \lambda_{i,s}$. Since part (5) hold at stage $s-1$, there are $m_{i,l}$ many strings in $T_{i,s-1}$ of level l. If $E_{i,s-1} = [0, i]$, then by Lemma 4.6, and the fact that $m_{i,l} \geqslant 2$ for all $i \in \omega$ and all $l \in [1, h_i]$, there is some string α as required, and if $E_{i,s-1} \neq [0, i]$, then by Lemma 4.6 and the definition of $m_{i,l}$, there is some string α as required.

Suppose that case 1c applies at stage s. The only difficulty is showing that a string γ as in the construction can be chosen. This follows from Lemma 4.4 and Lemma 4.3.

Now suppose that $s = 4n + 3$ for some $n \in \omega$. First suppose that case 2a applies at stage s. Let i, e, and l be as at stage s. We must show that we there is some set T as in the construction at stage s.

Suppose that $l = 0$. Then since T_i requires attention via e at stage s, we have that $T_{i,s-1} \neq \emptyset$, and since part (12) holds at stage $s-1$, we have that $T_{i,s-1} = \{\varepsilon_{i,s-1}\}$. Now since parts (1) and (4) hold at stage $s-1$, $\varepsilon_{i,s-1} \notin F_{i,s-1}$. Then we may take T to be S_s.

Now suppose that $l \neq 0$. Then there are $m_{i,l+1}$ many strings in $T_{i,s-1}$ of level l which are not in $F_{i,s-1}$, and above which we have found Φ_e-splits. Let Δ be the set of all such strings. For all $\delta \in \Delta$, suppose that we found Φ_e-splits above δ at stage s_δ. We write S_δ for S_{s_δ}. Given $\{S_\delta\}_{\delta \in \Delta}$, let S_Δ be the set produced by Lemma 4.5. It is clear that S_Δ is a set of Φ_e-splits of cardinality $m_{i,l+1}$ where every string in S_Δ extends some unique string in Δ. It suffices to show the following.

Claim 4.8. *$\Phi_e(S_\Delta)$ is sparse.*

Proof. Let $\sigma_0, \sigma_1 \in S_\Delta$. So for $i < 2$, suppose that $\delta_i \in \Delta$ is such that $\sigma_i \in S_{\delta_i}$. By uniqueness, we have that $\delta_0 \neq \delta_1$. Without loss of generality, we may assume that $s_{\delta_0} < s_{\delta_1}$. Then σ_1 was enumerated into Z_e *after* σ_0 was enumerated into Z_e. Now by the definition of requires attention at stage s_{δ_1}, we have that $|\Phi_e(\sigma_1)| > |\Phi_e(\sigma_0)|$, which suffices to establish the claim. □

Now suppose that case 2b applies at stage s. Then $l \neq 0$, and since part (4) holds at stage $s-1$, there are $m_{i,l}$ many strings in $T_{i,s-1}$ of level l. Since we act for T_i with e at stage s, by the definition of requires attention, $e \notin E_{i,s-1}$. So $E_{i,s-1} \neq [0, i]$, and by Lemma 4.6, there are at most $(i+1).(m_{i,l+1} - 1) + 1$ many strings in $T_{i,s-1}$ which are of level l and in $F_{i,s-1}$. Then since part (9) holds at stage $s-1$, and by the definition of $m_{i,l}$, we may choose a string α as in the construction at stage s.

Finally, suppose that $s = 4n$ for some $n \geqslant 1$. Let δ and M be as at stage s. Then $\delta \in M$. Suppose that we wish to act for $Q_{e,i}$ at stage s. The only difficulty is if for some $\nu \in N_{e,i,s}$ and for some string ρ, we wish to update the arrow for ρ in t_ν in case 3a at stage s. So assume this. Then $\rho \prec \Phi_e(\delta)$. Since part (11) holds at stage $s-1$, M is a set of Φ_e-splits, and $\Phi_e(M)$ is sparse. Therefore, no initial segment of $\Phi_e(\delta)$ is equal to $\Phi_e(\mu)$ for some $\mu \in M$, and now by Lemma 4.3, we can define $a_{\tau,s}(\rho)$ as required. □

Lemma 4.9. *Suppose that P_i is initialised for the final time at stage s^*. Then we may act for P_i at at most finitely stages after stage s^*.*

Proof. We may act at most once for P_i in case 1a after stage s^*, and at most once for P_i in case 1b after stage s^*. As $\langle f_{i,s}, o_{i,s}\rangle_{s<\omega}$ is an $(\alpha+1)$-computable approximation, we may act at most finitely many times for P_i in case 1d after stage s^*. For any $t \in \omega$, if we do not act for P_i in case 1d at any stage after stage t, then we may act at most once for P_i in case 1c after stage t. □

Lemma 4.10. *Suppose that T_i is initialised for the final time at stage s^*. Then*

- *we may act for T_i in case 2a at at most h_i many stages after stage s^*, and*
- *we may act for T_i in case 2b at at most finitely many stages after stage s^*.*

Proof. The first part follows from part (3) of Lemma 4.2. Now since the first part holds, to show that the second part holds, it suffices to show that if $s > s^*$ is such that we do not act for T_i in case 2a at any stage after stage s, then we act for T_i in case 2b at at most finitely many stages after stage s. So let s be as above. By assumption $\lambda_{i,t} = \lambda_{i,s}$ for all $t \geqslant s$. Therefore, if $\lambda_{i,s} = \langle\rangle$, we cannot act for T_i in case 2b at any stage after stage s. So suppose that $\lambda_{i,s} \neq \langle\rangle$. Let $l = |\lambda_{i,s}|$. For all $t \geqslant s$, by part (5) of Lemma 4.2, there are $m_{i,l}$ many maxmal strings in $T_{i,t}$. Then since the construction can be carried out as described, we may act for T_i in case 2b at at most $m_{i,l}$ many stages after stage s. □

Lemma 4.11. *Suppose that $Q_{e,i}$ is initialised for the final time at stage s^*. Then we may act for $Q_{e,i}$ at at most finitely stages after stage s^*.*

Proof. By Lemma 4.9 and Lemma 4.10, it suffices to show that for all $s > s^*$, if we do not act for any P- or T-requirement at any stage after stage s, then we act for $Q_{e,i}$ at at most finitely many stages after stage s. So let s be as above. If $Q_{e,i}$ is satisfied at stage s, then we cannot act for $Q_{e,i}$ at any stage after stage s. So suppose that $Q_{e,i}$ is not satisfied at stage s. Then we may act at most once for $Q_{e,i}$ after stage s. □

Lemma 4.12. *We act for every P-requirement at at most finitely many stages, for every Q-subrequirement at at most finitely many stages, and for every T-requirement at at most finitely many stages.*

Proof. By induction, with Lemma 4.9, Lemma 4.10, and Lemma 4.11. □

Let $A = \liminf_{s\to\infty} A_s$. The following is a straightforward consequence of Lemma 4.12.

Lemma 4.13. *A is Δ^0_2.*

Lemma 4.14. *Γ is consistent and $\Gamma(A)$ is total.*

Proof. The instructions we follow at the end of every stage ensure that $\Gamma(A)$ is total. These instructions also ensure that for all $n \in \omega$, then if n is not the follower of any P-requirement at some stage, then Γ is consistent for n.

So suppose that we choose the follower n for P_i at stage s. We define an axiom for n at some stage t only if n is the follower of P_i at stage t, so we may assume that we do not initialise P_i at any stage after stage s.

Claim 4.15. *For all $t \geqslant s$, $T_{i,t} = T_{i,s}$.*

Proof. This follows from the fact that we cannot act for T_i in case 2a at any stage after stage s. □

By the instructions in case 1c of the construction, it suffices to establish the following.

Claim 4.16. *For all $t \geqslant s$ and all strings δ which are maximal in $T_{i,t-1}$, there is no string in $\operatorname{dom}\Gamma_{t-1}(n)$ which is an initial segment of δ.*

Proof. By induction on t. First suppose that $t = s$. Since n is large at stage s, $\operatorname{dom}\Gamma_{s-1}(n) = \varnothing$. The inductive step follows from the previous claim, together with the fact a string which enters the domain of Γ at stage t must be of a large length. □

□

Lemma 4.17. *Every P-requirement is met.*

Proof. By Lemma 4.12, suppose that P_i is initialised for the final time at stage s^*. Then by Lemma 4.7 and Lemma 4.14, it can easily be seen that P_i is met. $\square$

We now turn to showing that the Q-requirements are met.

Lemma 4.18. *Suppose that X is set, $\delta \prec X$, Φ is a Turing functional with a length increasing computable enumeration, and that $\Phi(X)$ is total and noncomputable. Then for all $l, m \in \omega$, there is some set S of Φ-splits of cardinality m such that for all $\sigma \in S$, $\delta \prec \sigma$, and $|\Phi(\sigma)| > l$. Moreover, S can be chosen such that $\Phi(S)$ is sparse.*

Proof. The first part is standard. The fact that there are arbitrarily many splits means that S can always be chosen such that $\Phi(S)$ is sparse. $\square$

Lemma 4.19. *Suppose that d is such that $\Phi_d(A)$ is total and noncomputable. Then for all $i \geqslant d$, $Q_{d,i}$ is satisfied at all but finitely many stages.*

Proof. Suppose for contradiction that $\Phi_d(A)$ is total and noncomputable, and that there is some $i \geqslant d$ such that $Q_{d,i}$ is not satisfied at all but finitely many stages. Let i be the least such. By Lemma 4.12, suppose that we do not act for any $R \in \mathcal{R}$ with $R \leqslant_{\mathcal{R}} T_i$ at any stage after stage s. Let $l = |\lambda_{i,s}|$. Suppose that δ is the string in $T_{i,s}$ of level l which A_s extends. We have that $\delta \prec A_t$ for all $t \geqslant s$.

By the choice of s and the definition of requires attention, we cannot act for any T-requirement of weaker priority than T_i with e at any stage after stage s. Therefore, $Z_{e,t} = Z_{e,s}$ for all $t \geqslant s$.

First suppose that $d \in E_{i,s}$. Then by the choice of s, we could not remove d from E_i at any stage after stage s, and so $d \in E_{i,t}$ for all $t \geqslant s$. Again by the choice of s, and Lemma 4.17, all P-requirements P with $P \leqslant_{\mathcal{R}} Q_{d,i}$ are satisfied at stage s, and at all later stages. Yet again by the choice of s, if $\varphi_{d,i,s}$ were defined, then $\varphi_{d,i,t}$ would be defined for all $t \geqslant s$, and $Q_{d,i}$ would be satisfied at all but finitely many stages. So $\varphi_{d,i,s}$ is not defined. Therefore, $Q_{d,i}$ does not require attention at any stage after stage s, which contradicts the fact that $\Phi_d(A)$ is total.

Now suppose that $d \notin E_{i,s}$. Then by the choice of s, $d \notin E_{i,t}$ for all $t \geqslant s$. By the choice of i, every Q_d-subrequirement Q with $Q <_{\mathcal{R}} Q_{d,i}$ is satisfied at stage s, and by the choice of s, is satisfied at all later stages. Then T_i does not require attention via d at any stage after stage s, which contradicts Lemma 4.18. $\square$

Lemma 4.20. *If $e \in E_{i,s-1}$, then every Q_e-subrequirement of stronger priority than $Q_{e,i}$ is satisfied at stage s.*

Proof. Suppose that stage r is the last stage before stage s at which we enumerated e into E_i. Then we act for T_i with e at stage r, and by the definition of requires attention at stage r, every Q_e-subrequirement of stronger priority than $Q_{e,i}$ is satisfied at stage r. Let $Q_{e,j}$ be of stronger priority than $Q_{e,i}$. We show that $e \in E_{j,s-1}$ and that $\varphi_{e,j,s-1}$ is defined.

Suppose for contradiction that e were removed from E_j at some stage t with $t \in (r, s)$. Then we must either initialise T_j at stage t, or act for T_j in case 2a at stage t. In either case, we then initialise T_i and remove e from E_i at stage t, which is a contradiction.

Now suppose for contradiction that $\varphi_{e,j,s-1}$ is not defined. Then we must initialise $Q_{e,j}$ at some stage t with $t \in (r, s)$. Then we must initialise T_i and remove e from E_i at stage t, which is a contradiction. $\square$

Lemma 4.21. *If the Q_e-subrequirement Q is satisfied at stage s, then every Q_e-subrequirement of stronger priority than Q is satisfied at stage s.*

Proof. By the definition of satisfied and Lemma 4.20. $\square$

Lemma 4.22. *If we act for some Q-subrequirement at stage s, then $A_s = A_{s-1}$.*

Proof. This is immediate. □

Lemma 4.23. *Let $k, l < |\lambda_{i,s}|$. Then every string in $T_{i,s}$ of level l extends a string in $T_{i,s}$ of level k. Furthermore, distinct strings in $T_{i,s}$ of level l extend distinct strings in $T_{i,s}$ of level k.*

Proof. By induction on the stage number. □

Lemma 4.24. *Suppose that $a_{\nu,s-1}(\rho) \npreccurlyeq \Phi_{e,s}(A_{s-1})$. Then we cannot update the arrow for ρ in t_ν at stage s.*

Proof. In order to update the arrow for ρ in t_ν at stage s, we must act for some Q_e-subrequirement at stage s. Let φ be as at stage s. Then $\varphi \prec A_{s-1}$ and so $\Phi_e(\varphi) \preccurlyeq \Phi_{e,s}(A_{s-1})$. Now since $a_{\nu,s-1}(\rho) \npreccurlyeq \Phi_{e,s}(A_{s-1})$, we have that $a_{\nu,s-1}(\rho) \npreccurlyeq \Phi_e(\varphi)$, and we cannot update the arrow for ρ in t_ν at stage s. □

Lemma 4.25. *Suppose that $\rho \npreccurlyeq \Phi_{e,s}(A_{s-1})$. Then we cannot update the arrow for ρ in t_ν at stage s.*

Proof. By Lemma 4.24, noting that $a_{\nu,s-1}(\rho) \succcurlyeq \rho$. □

Lemma 4.26. *Suppose that $i < j$, and that $Q_{e,i}$ and $Q_{e,j}$ are satisfied at stage $t+1$. Then $\Phi_e(\varphi_{e,i,t}) \prec \Phi_e(\varphi_{e,j,t})$*

Proof. This follows immediately from the definition of requires attention. □

Fix some e such that $\Phi_e(A)$ is total and noncomputable. Let $\nu_e \in \{\mathtt{inf}, \mathtt{fin}\}^e$ be such that for all $d < e$, $\nu_e(d) = \mathtt{inf}$ if for all $i \geqslant d$, $Q_{d,i}$ is satisfied at all but finitely many stages, and $\nu_e(d) = \mathtt{fin}$ otherwise.

Lemma 4.27. *Suppose that d is such that $\nu_e(d) = \mathtt{fin}$. Then there are at most finitely many stages at which we act for some T_i with d.*

Proof. By the definition of ν_e, there is some $i \geqslant d$ such that $Q_{d,i}$ is not satisfied at infinitely many stages t. By Lemma 4.12, $Q_{d,i}$ must not be satisfied at all but finitely many stages. Let s be such that $Q_{d,i}$ is not satisfied at any stage t with $t \geqslant s$. Then by the definition of requires attention, we could not act for any T-requirement T with $T_i \leqslant_{\mathcal{R}} T$ with d at any stage after stage s. The lemma now follows by Lemma 4.12. □

Lemma 4.28. *t_{ν_e} is reset at at most finitely many stages.*

Proof. t_{ν_e} is reset at some stage s only if we either act for some $R \in \mathcal{R}$ with $R < Q_e$ at stage s, or for some $d < e$ such that $\nu_e(d) = \mathtt{fin}$, we act for some T_i with d at stage s. The lemma now follows from Lemma 4.12 and Lemma 4.27. □

Lemma 4.29. *t_{ν_e} is an α-change test.*

Proof. We write ν for ν_e. By Lemma 4.28, suppose that the last stage at which t_ν is reset is stage s_0. It is clear that for all strings ρ and all $s \in \omega$, $a_{\nu,s}(\rho) \succcurlyeq \rho$. Therefore, it suffices to show that for all strings ρ and all $s > s_0$, if we update the arrow for ρ in t_ν at stage s, then $b_{\nu,s}(\rho) < b_{\nu,s-1}(\rho)$.

Fix some string ρ. If we do not update the arrow for ρ in t_ν at any stage after stage s_0, then we are done. So suppose we do. Let $s_1, s_2, \ldots$ be the nonempty and possibly infinite sequence of stages after stage s_0 at which we update the arrow for ρ in t_ν. We have $b_{\nu,s_1-1}(\rho) = \infty$ and $b_{\nu,s_1}(\rho) < \infty$. So suppose by induction that for some $k > 1$, s_k is defined, and that for all $j \in [1,k]$, $b_{\nu,s_j}(\rho) < b_{\nu,s_j-1}(\rho)$. If s_{k+1} is not defined, then we are done. So suppose that s_{k+1} is defined. We show that $b_{\nu,s_{k+1}}(\rho) < b_{\nu,s_k}(\rho)$.

To slightly ease the notational burden, we write s for s_k, w for s_{k+1}, and I_s for $I_{\nu,\rho,s}$.

Claim 4.30. *Suppose that $a_{\nu,s}(\rho)$ is incomparable with $\Phi_e(\theta)$. Then there is some $t \in (s, w)$ such that $\theta \nprec A_t$.*

Proof. Suppose not. Then $\theta \prec A_{w-1}$, and so $\Phi_e(\theta) \preccurlyeq \Phi_{e,w}(A_{w-1})$. By the choice of w, $a_{\nu,w-1}(\rho) = a_{\nu,s}(\rho)$. Since $a_{\nu,s}(\rho)$ is incomparable with $\Phi_e(\theta)$, $a_{\nu,w-1}(\rho)$ is incomparable with $\Phi_{e,w}(A_{w-1})$, so by Lemma 4.24, we cannot update the arrow for ρ in t_ν at stage w, which is a contradiction. □

Claim 4.31. *We cannot act for any $R \in \mathcal{R}$ with $R < Q_e$ at any stage after stage s_0.*

Proof. Doing so would reset t_ν. □

We must act for some Q_e-subrequirement at stage s. Suppose that we act for $Q_{e,i}$ at stage s. Let M be as in the construction at stage s.

Claim 4.32. *Suppose that for some j, we act for P_j or T_j at some stage u. Further suppose that T_j was initialised at some stage t with $t < u$, and that t is the greatest such. Then if $\alpha \prec A_{t-1}$ is of length at most t, $\alpha \prec A_u$.*

Proof. At stage t, we initialise all P- and T-requirements R with $T_j \leqslant_{\mathcal{R}} R$.

Since we initialise T_j at stage t, $\varepsilon_{j,t}$ is undefined. In order to act for P_j or T_j at stage u, $\varepsilon_{j,u-1}$ must be defined. Therefore, at some stage v with $v \in (t, u)$, we must define $\varepsilon_{j,v}$. By the choice of t, there is in fact some unique such stage v. Let $\alpha \prec A_{t-1}$ be of length at most t.

Claim 4.33. $\alpha \prec A_{v-1}$.

Proof. We cannot act for any $R \in \mathcal{R}$ with $R <_{\mathcal{R}} T_j$ at any stage after stage t and before stage v, since this would initialise T_j. Now by the definitions of requires attention for the different requirements and subrequirements, we cannot act for any $R \in \mathcal{R}$ with $T_j \leqslant_{\mathcal{R}} R$ at any stage after stage t and before stage v. □

We define $\varepsilon_{j,v}$ to be some initial segment of A_{v-1} of a large length, and so in particular, $|\varepsilon_{j,v}| > t$, and $\alpha \prec \varepsilon_{j,v}$. A straightforward induction shows that $\varepsilon_{j,v} \prec A_w$ for all $w \in (v, u]$. □

Claim 4.34. *Suppose that $u \geqslant s$, and that we do not initialise T_i at any stage t with $t \in (s, u]$. Then A_u extends some string in M.*

Proof. By induction on u, using Lemma 4.23. □

Claim 4.35. *There is some $j \in I_s$ and some $u \in (s, w)$ such that we act either for P_j or for T_j at stage u.*

Proof. First suppose that we update the arrow for ρ in step 1 or in case 3b at stage s. Then $I_s = [e, i]$. Let φ be as in the construction at stage s. We define $a_{\nu,s}(\rho)$ to be incomparable with $\Phi_e(\varphi)$. By Claim 4.30, there must be some stage u with $u \in (s, w)$ such that $\varphi \nprec A_u$. Let u be the least such. By Lemma 4.22, there is some $j \in \omega$ such that we act for P_j or for T_j at stage u. By Claim 4.31, we cannot act for any $R \in \mathcal{R}$ with $R < Q_e$ at stage u. We initialise all $R \in \mathcal{R}$ with $Q_{e,i} < R$ at stage s. Note that by the definition of requires attention, $|\varphi| < s$. In addition, we have $\varphi \prec A_{s-1}$. Then by Claim 4.32, we must have $j \in [e, i]$.

Now suppose that we act in case 3a at stage s. Then $I_s = [e, i)$. We define $a_{\nu,s}(\rho)$ to be incomparable with $\Phi_e(\mu)$ for all $\mu \in M$. By Claim 4.34, A_s extends some string in M. Now by Claim 4.30, there must be some stage u with $u \in (s, w)$ such that $\mu \nprec A_u$ for all $\mu \in M$. Let u be the least such. By Lemma 4.22, there is some $j \in \omega$ such that we act for P_j or for T_j at stage u. By Claim 4.31, we cannot act for any $R \in \mathcal{R}$ with $R < Q_e$ at stage u. We initialise all $R \in \mathcal{R}$ with $Q_{e,i} < R$ at stage s. Note that for all $\mu \in M$, by the definition of requires attention, $|\mu| < s$. Then by Claim 4.32, we have $j \in [e, i]$. Suppose for contradiction that $j = i$. Again by Claim 4.32, we cannot initialise T_i at any stage t with $t \in (s, u)$. Then by Claim 4.34, $A_u > \mu$ for some $\mu \in M$, which is a contradiction. Therefore, $j \neq i$, and $j \in I_s$. □

By Claim 4.35, let $k \in I_s$ be least such that we act for P_k or for T_k at some stage u with $u \in (s, w)$.

Claim 4.36. *There is some $\zeta \in Z_{e,s-1}$ such that $\Phi_e(\zeta) = \rho$.*

Proof. As noted above, $b_{\tau,s_1-1}(\rho) = \infty$ and $b_{\tau,s_1}(\rho) < \infty$. So we must act in step 1 at stage s_1. Let δ_{s_1} be the string δ as at stage s_1 of the construction. Then $\Phi_e(\delta_{s_1}) = \rho$. We have that $\delta_{s_1} \in T_{i,s_1-1}$ and in addition, $\delta_{s_1} \in Z_{e,s_1-1}$. Finally, note that we never remove any elements from Z_e at any stage, and $s_1 < s_k = s$. □

Claim 4.37. *All Q_e-subrequirements Q with $Q \leqslant_{\mathcal{R}} Q_{e,i}$ are satisfied at stage s.*

Proof. We act for $Q_{e,i}$ at stage s, so it is clear that $Q_{e,i}$ is satisfied at stage s. Now use Lemma 4.20. □

Claim 4.38. *Suppose that $Q_{e,j} \leqslant_{\mathcal{R}} Q_{e,i}$, and that $Q_{e,j}$ is the Q_e-subrequirement of strongest priority which is not satisfied at some stage t with $t \in (s, w]$. Then we must act for $Q_{e,j}$ at stage w.*

Proof. Suppose not. We must act for some Q_e-subrequirement at stage w, so suppose that we act for $Q_{e,l}$ at stage w.

First suppose that $Q_{e,l}$ is of stronger priority than $Q_{e,j}$. By Claim 4.37, $Q_{e,l}$ is satisfied at stage s, and by assumption, $Q_{e,l}$ is satisfied at stage w. So we cannot act for $Q_{e,l}$ at stage w.

Now suppose that $Q_{e,l}$ is of weaker priority than $Q_{e,j}$. Since we act for $Q_{e,l}$ at stage w, $e \in E_{l,w-1}$, so by Lemma 4.20, all Q_e-subrequirements of stronger priority than $Q_{e,l}$ are satisfied at stage w, and in particular, $Q_{e,j}$ is satisfied at stage w. Since $Q_{e,j}$ is not satisfied at some stage t with $t \in (s, w)$, we must act for $Q_{e,j}$ at some stage v with $v \in (t, w)$. Let v be the greatest such. Then $\varphi_{e,j,w} = \varphi_{e,j,v}$. By Claim 4.36, let ζ be such that $\zeta \in Z_{e,s-1}$ and $\Phi_e(\zeta) = \rho$. Since $s < v$, we have that $\zeta \in Z_{e,v-1}$. Now by the definition of requires attention at stage v, we have that $|\Phi_e(\varphi_{e,j,v})| > |\rho|$. Then by Lemma 4.26, and the fact that we act for $Q_{e,l}$ at stage w and update the arrow for ρ in t_ν, we have that

$$\Phi_e(\varphi_{e,j,v}) = \Phi_e(\varphi_{e,j,w}) \preccurlyeq \Phi_e(\varphi_{e,l-1,w}) \prec \rho \preccurlyeq \Phi_e(\varphi_{e,l,w})$$

which is a contradiction. □

Claim 4.39. *Suppose that we act for $Q_{e,j}$ at stage w, and that e was removed from E_j at some stage u with $u \in (s, w)$. Then we act in case 3a at stage w.*

Proof. Let $u \in (s, w)$ be greatest such that we remove e from E_j at stage u. Since we act for $Q_{e,j}$ at stage w, we must have that $e \in E_{j,w-1}$. Therefore, we must enumerate e into E_j at some stage v with $v \in (u, w)$. Let v be the greatest such. Let T be as at stage v. Then $T \subset Y_{j,e,v}$. We empty $Y_{j,e}$ at stage u, so every element of T was enumerated into $Y_{j,e}$ at some stage after stage u. Let $\sigma \in T$. By Claim 4.36, let ζ be such that $\zeta \in Z_{e,s-1}$ and $\Phi_e(\zeta) = \rho$. Then since $s < u$, we have that $\zeta \in Z_{e,u-1}$. Now by the definition of requires attention, we have that $|\Phi_e(\sigma)| > |\Phi_e(\zeta)| = |\rho|$.

Let δ be as at stage w. Then by the choice of v, $\delta \in T$. Since ρ and $\Phi_e(\delta)$ are comparable, by the above, we must have that $\rho \prec \Phi_e(\delta)$. □

Claim 4.40. *Suppose that $I \subseteq [e, i]$, and that for all $j \in I$, we do not initialise P_j or act for P_j in case 1b at any stage t with $t \in (s, w)$. Then for all $j \in I$ and all $t \in [s, w]$,*

- *P_j has some follower $n_{j,s}$ at the beginning of stage s, and P_j has follower $n_{j,s}$ at the beginning of stage t, and*

- *P_j has been realised with its follower by the beginning of stage t if and only if P_j has ben realised with its follower by the beginning of stage s.*

Proof. We act for $Q_{e,i}$ at stage s, and so every P-requirement P with $P \leqslant_{\mathcal{R}} Q_{e,i}$ is satisfied at stage s, and so in particular, P must have a follower at the beginning of stage s. The claim now follows directly from the assumptions. □

First suppose that we initialise T_k at some stage t with $t \in (s,w)$. Then by the choice of k, we must act for some Q-subrequirement of stronger priority than T_k at some stage t with $t \in (s,w)$. Suppose that $Q_{e,l}$ is of equally strong priority as the strongest priority Q-subrequirement for which we act at some stage t with $t \in (s,w)$.

Claim 4.41. *We act for $Q_{e,l+1}$ in case 3a at stage w.*

Proof. By the choice of l, $l < k$. Let $t \in (s,w)$ be the greatest such that we act for some Q-subrequirement of equally strong priority as $Q_{e,l}$ at stage t. We initialise $Q_{e,l+1}$ at stage t, and again by the choice of l, $Q_{e,l+1}$ is the Q_e-subrequirement of strongest priority which is not satisfied at some stage after stage s and before stage w. Therefore, by Claim 4.38, we must act for $Q_{e,l+1}$ at stage w. At stage t, we reset E_{l+1}, and so we remove e from E_{l+1}. Now by Claim 4.39, we must act in case 3a at stage w. □

Claim 4.42. $b_{\nu,w}(\rho) < b_{\nu,s}(\rho)$.

Proof. By Claim 4.41, $I_w = [e, l+1)$. By the choice of l, for all $j \in I_w$, we do not initialise P_j or act for P_j in case 1b at any stage after stage s and before stage w. By Claim 4.40 and the definition of $b_{\nu,w}(\rho)$, it suffices to show that $|I_w| < |I_s|$. First suppose that we act for $Q_{e,i}$ via step 1 or in case 3b at stage s. Then $I_s = [e,i]$. Since $l < k \leqslant i$ and $I_w = [i,l]$, we have $|I_w| < |I_s|$. Now suppose that we act for $Q_{e,i}$ in case 3a at stage s. Then $I_s = [e,i)$. Then $l < k < i$, which suffices as above. □

Now suppose that we do not initialise T_k at any stage t with $t \in (s,w)$. By Claim 4.37, $e \in E_{k,s-1}$. Suppose that stage r is the last stage before stage s at which e was enumerated into E_k. Let $l_{e,k} = |\lambda_{k,r}|$, and let $\delta_{k,s}$ be the string in $T_{k,s}$ of level $l_{e,k}$ which A_s extends.

Claim 4.43. $\Phi_e(\delta_{k,s}) \preccurlyeq \rho$.

Proof. First suppose that $k < i$. $\Phi_e(\varphi_{e,i-1,s}) \prec \rho \preccurlyeq \Phi_e(\varphi_{e,i,s})$. By Claim 4.37, and Lemma 4.26, $\Phi_e(\varphi_{e,k,s}) \prec \Phi_e(\varphi_{e,i,s})$ and so $\Phi_e(\varphi_{e,k,s}) \preccurlyeq \Phi_e(\varphi_{e,i-1,s})$ Let stage q be the last stage before stage s at which we act for $Q_{e,k}$. By the choice of q, we cannot initialise $Q_{e,k}$ at any stage t with $t \in (q,s)$. Therefore, we cannot initialise T_k at any stage t with $t \in (q,s)$, and so cannot remove any string from T_k at any stage t with $t \in (q,s)$. So $\delta_{k,s}$ is in fact the string in $T_{i,q}$ which A_q extends, and $\delta_{k,s} \prec \varphi_{e,k,q}$. Then, using the fact that we act for $Q_{e,i}$ and update the arrow for ρ in t_ν at stage s,

$$\Phi_e(\delta_{k,s}) \preccurlyeq \Phi_e(\varphi_{e,k,q}) = \Phi_e(\varphi_{e,k,s}) \preccurlyeq \Phi_e(\varphi_{e,i-1,s}) \prec \rho.$$

Now suppose that $k = i$. Since $k \in I_s$, $i \in I_s$, so we must have that $I_s = [e,i]$. Therefore, we must update the arrow for ρ via step 1 or case 3b at stage s. First suppose that we update the arrow for ρ via step 1 at stage s. Then by the choice of $\delta_{k,s}$ and our assumptions, $\rho = \Phi_e(\delta_{k,s})$. So now suppose that we update the arrow for ρ via case 3b at stage s. Then $\Phi_e(\delta_{k,s}) \preccurlyeq \rho$. □

There are two further cases that we consider. First consider the case where we act for T_k or for P_k in case 1b at some stage u with $u \in (s,w)$.

Claim 4.44. *Suppose that we act for T_k in at some stage u with $u \in (s,w)$ and that u is the least such. Then we must act with some $f > e$ at stage u.*

Proof. Suppose for contradiction that we act with some $d \leqslant e$ at stage u.

First suppose that $d = e$. We act for $Q_{e,i}$ at stage s, so $e \in E_{i,s-1}$. By Claim 4.37, we have that $e \in E_{k,s-1}$. We do not initialise T_k at any stage t with $t \in (s, w)$, and by the choice of u, we cannot remove any elements from E_i at any stage after stage s and before stage u. Therefore, $e \in E_{k,u-1}$, so by the definition of requires attention, we cannot act with e at stage u, which is a contradiction.

So suppose that $d < e$. Since we update the arrow for ρ in t_ν at stage s, we have that $\nu \in N_{e,i,s}$. There are two cases to consider. First consider the case where $\nu(d) = \texttt{inf}$. By the definition of $N_{e,i,s}$, we have that $d \in E_{i,s-1}$. Then by the same argument as above, we cannot act with d at stage u, which is a contradiction. Now consider the case where $\nu(d) = \texttt{fin}$. Then we reset ν at stage v, which is a contradiction, since $v > s_0$. □

Claim 4.45. *Suppose that we act for T_k or for P_k in case 1b at some stage u with $u \in (s, w)$. Then we must act for T_k in case 2a at some stage v with $v \in [u, w)$.*

Proof. Let $u \in (s, w)$ be least such that we act for T_k or for P_k in case 1b at stage u. If we act for T_k in case 2a at stage u, then we are done. So suppose that we act for T_k in case 2b or for P_k in case 1b at stage u.

Suppose for contradiction that we do not act for T_k in case 2a at any stage v with $v \in (u, w)$. Then by the choice of u, $\delta_{k,s}$ is the string in $T_{k,u-1}$ of level $l_{e,k}$ which A_{u-1} extends.

First suppose that we act for T_k in case 2b at stage u. Then by Claim 4.44, for some $f > e$, we say that we have found Φ_f-splits above $\delta_{k,s}$. Using the fact that the construction can be carried out as described, an induction using Lemma 4.23 shows that for all $v \in (u, w)$, A_{v-1} extends some string in $T_{k,v-1}$ of level $l_{e,k}$ not equal to $\delta_{k,s}$. Now since the set of strings in $T_{k,v-1}$ of level $l_{e,k}$ is a set of Φ_e-splits, and by Claim 4.43, for all $v \in (u, w)$, $\rho \not\preccurlyeq \Phi_{e,v}(A_{v-1})$. By Lemma 4.25, this is a contradiction.

Now suppose that we act for P_k in case 1b at stage u. Then we enumerate $\delta_{k,s}$ into F_k at stage u. By part (2) of Lemma 4.2, for all $v \in (u, w)$, A_{v-1} does not extend $\delta_{k,s}$. Then as above, for all $v \in (u, w)$, $\rho \not\preccurlyeq \Phi_{e,v}(A_{v-1})$, which is a contradiction. □

Claim 4.46. *We act for $Q_{e,k}$ in case 3a at stage w.*

Proof. Let $u \in (s, w)$ be least such that we act for T_k or for P_k in case 1b at stage u. Then at stage u, we initialise $Q_{e,k}$, so $Q_{e,k}$ is not satisfied at stage u. By the choice of k, $Q_{e,k}$ is the Q_e-subrequirement of strongest priority which is not satisfied at some stage t with $t \in (s, w)$. So by Claim 4.38, we must act for $Q_{e,k}$ at stage w. By Claim 4.45, we must act for T_k in case 2a at some stage v with $v \in [u, w)$, and by Claim 4.44, we must act with some $f > e$ at stage v. Then we remove e from E_k at stage v. Now by Claim 4.39, we must act in case 3a at stage w. □

Claim 4.47. $b_{\nu,w}(\rho) < b_{\nu,s}(\rho)$.

Proof. By Claim 4.46, we have that $I_w = [e, k)$. Again by the choice of k, for all $j \in I_w$, we do not initialise P_j or act for P_j in case 1b at any stage after stage s and before stage w. By Claim 4.40 and the definition of $b_{\nu,w}(\rho)$, it suffices to show that $|I_w| < |I_s|$. First suppose that we act for $Q_{e,i}$ via step 1 or in case 3b at stage s. Then $I_s = [e, i]$. Since $k \leqslant i$ and $I_w = [i, k)$, we have $|I_w| < |I_s|$. Now suppose that we act for $Q_{e,i}$ in case 3a at stage s. Then $I_s = [e, i)$. Then $k < i$, which suffices as above. □

Now suppose that we do not act for T_k or for P_k in case 1b at any stage u with $u \in (s, w)$.

Claim 4.48. *For all $j \in [e, i]$, P_j is satisfied at stage s, and in particular, P_j has a follower at the beginning of stage s.*

Proof. We act for $Q_{e,i}$ at stage s, so every P-requirement P with $P \leqslant_{\mathcal{R}} Q_{e,i}$ is satisfied at stage s. $\square$

Claim 4.49. *We must act for P_k in case 1d at some stage v with $v \in (s, w)$.*

Proof. By the choice of k, we must act for T_k or for P_k at some stage v with $v \in (s, w)$. Let v be the least such. From our assumption, we must act for P_k in either case 1a, case 1c, or case 1d at stage v. By Claim 4.48, P_k is satisfied at stage s, and by the choice of v, P_k must also be satisfied at stage v. Then we cannot act in case 1a or in case 1c at stage v, so we must act in case 1d at stage v. $\square$

Claim 4.50. *We act for $Q_{e,k}$ in case 3b at stage w.*

Proof. By Claim 4.49, suppose that stage v is the first stage after stage s at which we act for P_k in case 1d. Then at stage v, we initialise $Q_{e,k}$. By the choice of k, $Q_{e,k}$ is the Q_e-subrequirement of strongest priority which is not satisfied at some stage t with $t \in (s, w]$. By Claim 4.38, we must act for $Q_{e,k}$ at stage w. Let δ_w be the string δ as at stage w of the construction. Then by our assumptions and the choice of v, we have that $\delta_w = \delta_{k,s}$. To show that we act in case 3b at stage w, we must show that $\Phi_e(\delta_w) \preccurlyeq \rho$. However, this now follows from Claim 4.43. $\square$

Claim 4.51. *$I_w \subseteq I_s$.*

Proof. By Claim 4.50, $I_w = [e, k]$. First suppose that we update the arrow for ρ via step 1 or case 3b at stage s. Then $I_s = [e, i]$. We have that $k \in I_s$, so $k \leqslant i$, and $I_w \subseteq I_s$. Now suppose that we update the arrow for ρ via case 3a at stage s. Then $I_s = [e, i)$. Then $k \in I_s$, so $k < i$, and $I_w \subseteq I_s$. $\square$

Claim 4.52. *$I^+_w = I^+_s \cap [e, k]$.*

Proof. By Claim 4.50 and Claim 4.40. $\square$

Claim 4.53. *Suppose that P_k has follower n at the beginning of stage s. Then $o_{e,w}(n) < o_{e,s}(n)$.*

Proof. As above, P_k is satisfied at the beginning of stage s. By Claim 4.49 and Claim 4.40, P_k has been realised with n by stage s. Therefore, $\Gamma_{s-1}(A_{s-1}, n)\downarrow$, and $\Gamma_{s-1}(A_{s-1}, n) > f_{i,s}(n)$. Let $\gamma \prec A_{s-1}$ be least such that $\Gamma_{s-1}(\gamma, n)\downarrow$. Let v be as in Claim 4.49. Then P_k requires attention via (4) at stage v, so $\Gamma_{v-1}(A_{v-1}, n) \leqslant f_{i,v}(n)$. By the choice of v, $\gamma \prec A_{v-1}$, and so $\Gamma_{s-1}(A_{s-1}, n) = \Gamma_{v-1}(A_{v-1}, n)$. Therefore, $f_{i,v}(n) \neq f_{i,s}(n)$, and so $o_{i,v}(n) < o_{i,s}(n)$, and $o_{e,w}(n) < o_{e,s}(n)$. $\square$

Claim 4.54. *$b_{\nu,w}(\rho) < b_{\nu,s}(\rho)$.*

Proof. By Claim 4.50, $I_w = [e, k]$, and by Claim 4.49, $k \in I^+_w$. Now by Claim 4.51, Claim 4.52, Claim 4.53, and the definition of $b_{\nu,w}(\rho)$, the lemma follows. $\square$

$\square$

Lemma 4.55. *$\Phi_e(A)$ does not avoid* $\operatorname{range} t_{\nu_e}$.

Proof. We write ν for ν_e. By Lemma 4.27, let i^* be such that for all $i \geqslant i^*$ and for all $d < e$ with $\nu(d) = \texttt{fin}$, we do not act for T_i with d at any stage.

Let $i \geqslant i^*$. By Lemma 4.12, let $s_{i,0}$ be the last stage at which T_i is initialised. By Lemma 4.19, $Q_{e,i}$ is satisfied at all but finitely many stages. Therefore, e is an element of E_i at all but finitely many stages. By Lemma 4.12, let r_i be the last stage at which we enumerate e into E_i, and let $l_{e,i} = |\lambda_{i,r_i}|$. Again by Lemma 4.12, suppose that stage $s_{i,1}$ is the last stage at which we act for T_i or for P_i in case 1b.

Claim 4.56. *$\nu \in N_{e,i,s}$ for all $s \geqslant s_{i,1}$.*

Proof. Suppose that $s \geqslant s_{i,1}$, $d < e$, and $\nu(d) = \mathtt{inf}$. It suffices to show that $d \in E_{i,s}$. By the definition of ν, d is in E_i at all but finitely many stages. Now by the choice of $s_{i,1}$, $d \in E_{i,s}$. □

We have that $s_{i,1} \geqslant r_i$. Let δ_i be the string in $T_{i,s_{i,1}}$ of level $l_{e,i}$ which $A_{s_{i,1}}$ extends. Then by the choice of $s_{i,1}$, $\delta_i \prec A_s$ for all $s \geqslant s_{i,1}$. By Lemma 4.17, P_i is satisfied at all but finitely many stages, and so we must act in case 1c at some stage after stage $s_{i,1}$. Let $s_{i,2}$ be the first such stage. We initialise $Q_{e,i}$ at stage $s_{i,2}$. Since $\Phi_e(A)$ is total and noncomputable, we must act for $Q_{e,i}$ in case 3 at some later stage. Let $s_{i,3}$ be the first such stage.

Let $\rho_i = \Phi_e(\delta_i)$. By the choice of $s_{i,3}$ and since $\nu \in \nu_{e,i,s_{i,3}}$, we must act for $Q_{e,i}$ via step 1 at stage $s_{i,3}$ and define $b_{\nu,s_{i,3}}(\rho_i) < \infty$. Since $\delta_i \prec A_s$ for all $s \geqslant s_{i,1}$, $\rho_i \prec \Phi_e(A)$. By the choice of $s_{i,0}$, and since $i \geqslant i^*$, we cannot reset ν at any stage after stage $s_{i,1}$. Therefore, $\rho_i \in \operatorname{range} \nu$.

It now suffices to establish the following.

Claim 4.57. *$\{\rho_i : i \geqslant i^*\}$ is cofinal along $\Phi_e(A)$*

Proof. Let $i < j$. For $k \in \{i, j\}$, since δ_k is a Φ_e-split in $T_{k,s_{k,1}}$, by part (10) of Lemma 4.2, δ_k was enumerated into Z_e at some stage q_k with $q_k \leqslant s_{k,1}$. Then $q_i < q_j$, and $\delta_i \in Z_{e,q_j-1}$. Now by the definition of requires attention at stage q_j, we have that $|\Phi_e(\delta_j)| > |\Phi_e(\delta_i)|$. □

□

Lemma 4.58. *$\Phi_e(A)$ does not meet* $\operatorname{range} t_{\nu_e}$.

Proof. We write ν for ν_e. Let $i \geqslant e$. By Lemma 4.12, let $s_{i,0}$ be the last stage at which T_i is initialised. By Lemma 4.19, $Q_{e,i}$ is satisfied at all but finitely many stages. Therefore, e is an element of E_i at all but finitely many stages, and we must act for T_i at some stage after stage $s_{i,0}$. Again by Lemma 4.12, suppose that stage $s_{i,1}$ is the last stage at which we act for T_i or for P_i in case 1b.

Claim 4.59. *For all $i \geqslant e$ and all $s \geqslant s_{i,1}$, $\nu \in N_{e,i,s}$.*

Proof. As for Claim 4.56 □

By Lemma 4.17, P_i is satisfied at all but finitely many stages. Let $s_{i,2}$ be least such that P_i is satisfied at all stages s with $s \geqslant s_{i,2}$. Note that $s_{i,2} > s_{i,1}$. We must act for P_i in case 1c at stage $s_{i,2}$. We initialise $Q_{e,i}$ at stage $s_{i,2}$. Since $\Phi_e(A)$ is total and noncomputable, we must act for $Q_{e,i}$ in case 3 at some later stage. Let stage $s_{i,3}$ be the first such stage.

For all $i \geqslant e$, by the choice of $s_{i,2}$ and $s_{i,3}$, we do not act for $Q_{e,i}$ at any stage after stage $s_{i,3}$, which gives the following.

Claim 4.60. *$\varphi_{e,i,s_{i,3}} \prec A_s$ for all $s \geqslant s_{i,3}$, and so $\varphi_{e,i,s_{i,3}} \prec A$.*

We therefore let $\varphi_{e,i} = \varphi_{e,i,s_{i,3}}$.

Claim 4.61. *$\{\Phi_e(\varphi_{e,i}) : i \geqslant e\}$ is cofinal along $\Phi_e(A)$.*

Proof. By Claim 4.60, $\varphi_{e,i} \prec A$, and so $\Phi_e(\varphi_{e,i}) \prec \Phi_e(A)$. For all $i, j \geqslant e$, if $i < j$, then by Lemma 4.26, $\Phi_e(\varphi_{e,i}) \prec \Phi_e(\varphi_{e,j})$. □

Suppose that $\rho \prec \Phi_e(A)$ and that $\lim_s b_{\nu,s}(\rho) < \infty$. By Claim 4.61, let $i \geqslant e$ be least such that $\rho \preccurlyeq \Phi_e(\varphi_{e,i})$.

Claim 4.62. *$b_{\nu,s_{i,3}}(\rho) < \infty$.*

Proof. Suppose for contradiction that $b_{\nu,s_{i,3}}(\rho) = \infty$. Then since $\lim_s b_{\nu,s}(\rho) < \infty$, there is some $t > s_{i,3}$ such that $b_{\nu,t-1}(\rho) = \infty$ and $b_{\nu,t}(\rho) < \infty$. At stage t, we must act for some Q_e-subrequirement via step 1 and update the bound for ρ in t_ν. Since $t > s_{i,3}$, we must act for some Q_e-subrequirement of weaker priority than $Q_{e,i}$.

Suppose that we act for $Q_{e,j}$ at stage t. Then $e \in E_{j,t-1}$. Let stage r be the last stage before stage t at which e was enumerated into E_j. Then A_t extends some string θ in T_j of level $|\lambda_{j,r}|$. Since we update the bound for ρ at stage t, we have that $\Phi_e(\theta) = \rho$. Suppose that θ was enumerated into $Y_{j,e}$ at stage q. Then by part (10) of Lemma 4.2, $q > s_{i,3}$. By the properties listed in the definition of requires attention at stage q, we have that $|\Phi_e(\theta)| > |\Phi_e(\varphi_{e,i,q-1})|$. By the choice of $s_{i,3}$, $\varphi_{e,i,q-1} = \varphi_{e,i,s_{i,3}} = \varphi_{e,i}$. This contradicts the fact that $\rho \preccurlyeq \Phi_e(\varphi_{e,i})$. □

The instructions in step 2 at stage $s_{i,3}$ of the construction ensure that $a_{\nu,s_{i,3}}(\rho)$ is incomparable with $\Phi_e(\varphi_{e,i})$. Again by the choice of $s_{i,2}$ and $s_{i,3}$, $a_{\nu,s}(\rho) = a_{\nu,s_{i,3}}(\rho)$ for all $s \geqslant s_{i,3}$. Now by Claim 4.60, $\varphi_{e,i} \prec A_s$ for all $s \geqslant s_{i,3}$, so $\lim_s a_{\nu,s}(\rho)$ is incomparable with $\Phi_e(A)$. □

Lemma 4.63. *Q_e is met.*

Proof. By Lemma 4.29, t_{ν_e} is an α-change test, and by Lemma 4.55 and Lemma 4.58, $\Phi_e(A)$ neither meets nor avoids range t_{ν_e}. □

□

The following theorem, together with Theorem 4.16 and Theorem 5.4 of [13], gives a Δ^0_2 Turing degree separation of each level in the hierarchy of multiple genericity notions.

Theorem 4.64. *Let $\alpha \leqslant \varepsilon_0$ be a power of ω. Then there is a Δ^0_2 Turing degree which is weakly α-change generic but not α-change generic.*

Proof. By Theorem 4.1, let $\mathbf{a}$ be a Δ^0_2 degree which is not α-c.a. dominated, and which does not compute any α-change generic degree. Then by Theorem 4.13 of [13], $\mathbf{a}$ computes a weakly α-change generic degree $\mathbf{b}$. As $\mathbf{a}$ does not compute any α-change generic degree, neither does $\mathbf{b}$. In particular, $\mathbf{b}$ itself is not α-change generic. □

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