THE COMPLEXITY OF RECURSIVE SPLITTINGS OF RANDOM SETS

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Abstract. It is investigated how much information of a random set can be preserved if one splits the random set into two halves or, more generally, cuts out an infinite portion with an infinite recursive set. The two main results are the following ones: 1. Every high Turing degree contains a Schnorr random set Z such that $Z \equiv_T Z \cap R$ for every infinite recursive set R. 2. For each set X there is a Martin-Löf random set $Z \geq_T X$ such that for all recursive sets R, either $X \leq_T Z \cap R$ or $X \leq_T Z \cap \overline{R}$.

1. Introduction

We contribute to the ongoing investigation of the interplay between randomness and computational complexity. Our main question is: how is information distributed over a random set $Z \subseteq \mathbb{N}$? Our answer is that the higher the Turing degree the easier is it to find a random set in it whose parts contain useful information. Here we ask whether either the information is in every part or whether it is at least one one of the two parts of each given recursive splitting. Here our notion of a "part" of a set $Z \subseteq \mathbb{N}$ is intersections of the form $Z \cap R$, where R is an infinite and co-infinite recursive set. We refer to $Z \cap R$, $Z \cap \overline{R}$ as a (recursive) *splitting* of Z. We call the two components the *halves* of the splitting. We study the computational complexity of such parts $Z \cap R$. In particular, we ask:

(a) Is it possible that all parts of a random set Z compute Z?
(b) If not, can we still have complex information in at least one of the halves of every splitting of a random set Z?

Our first result, Theorem 3.1, provides a strong affirmative answer to (a) for Schnorr randomness. In every given high Turing degree we build a Schnorr random set Z where all parts compute Z. The mere existence of such a set Z was previously known by combining several results: if a degree is minimal, the parts of a random set in it, being non-recursive, automatically preserve the information. High minimal degrees exist and every high degree contains a Schnorr random set [18].

Let Z satisfy the stronger notion of being Martin-Löf random. We cannot expect an affirmative answer to (a). Any set A Turing below both halves of a splitting of Z must be a base for Martin-Löf randomness as each half

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is Martin-Löf random in the other. If A is a base for Martin-Löf randomness then A is K-trivial [10] and hence Δ^0_2. Since the K-trivials are closed downward under Turing reducibility [15], this shows that $Z \not\leq_{\mathrm{T}} A$. Every non-high Schnorr random set is Martin-Löf random [18]. Thus, we can conclude that Theorem 3.1 is optimal in that the high degrees are the largest class of Turing degrees where it works.

Our second result, Theorem 4.1, answers (b) in the affirmative for Martin-Löf randomness. We build a Martin-Löf random set Z such that for every splitting given by a recursive set R, at least one half computes Chaitin's Ω, which is Turing equivalent to the halting problem. Note that the two halves must be Turing incomparable (unless R is finite or co-finite), so we cannot expect the half to compute Z. However, it is in fact possible to replace Ω by an arbitrarily complex set at the cost of a more complicated argument.

While Theorem 3.1 dealt with Schnorr random sets where the information is distributed very evenly and every recursively cut out infinite part permits to reconstruct the whole set, Theorem 4.1 dealt with Martin-Löf random sets where the useful information is concentrated into the complement of an r-maximal set so that each splitting has a half containing almost all of this complement and therefore permitting the half to reconstruct Ω from this half. In our last result we show that for any random set Z of hyperimmune-free Turing degree (where the notions of Kurtz random, Schnorr random and Martin-Löf random coincide), the information is neither distributed evenly nor concentrated somewhere, instead for every non-recursive set A there is a recursive splitting such that neither of the halves computes A.

Please see the textbooks of Nies [16, Chapter 3] or Li and Vitányi [13] for definitions and context from algorithmic randomness and recursion theory.

2. Background

First we discuss the questions above in the purely recursion theoretic setting, without requiring that the set Z is random. Then we relate splitting to algorithmic randomness.

Recursion theoretic facts. Dekker and Myhill [2] showed that every Turing degree contains an introreducible set, that is, a set A which is Turing reducible to each of its infinite subsets. Hence for every recursive set R, either $A \equiv_T A \cap R$ or $A \cap R$ is finite.

Let us call a set A *self-reducing* if $A \equiv_T A \cap R$ for every infinite recursive set R. Equivalently, whenever $B \leq_m A$ via a function with infinite range then $B \equiv_T A$.

Recall that an infinite co-infinite set A is called *bi-immune* if neither A nor $\overline{A}$ contain an infinite recursive set R. Clearly every non-recursive self-reducing set is bi-immune. Conversely, if A is bi-immune and of minimal Turing degree, then A is self-reducing. Jockusch [11] constructed a non-recursive degree $\mathbf{a} \leq \mathbf{0}''$ such that no degree $\mathbf{b} \leq \mathbf{a}$ contains any bi-immune sets. This gives examples of degrees without self-reducing sets. He also showed that every hyperimmune degree contains a bi-immune set. We now show that every Δ^0_2 degree contains a self-reducing set.

Proposition 2.1. *Every Δ^0_2 Turing degree contains a set A such that for every infinite r.e. set R we have $A \leq_T A \cap R$.*

Proof. We fix a given non-recursive set $B \in \Delta^0_2$ with a recursive approximation B_s, that is, for all x and almost all s, $B_s(x) = B(x)$. The convergence module

$$c_B(x) = \min\{s > x : \forall y \leq x\,[B_s(y) = B(x)]\}$$

satisfies that $B \leq_T f$ for every function f majorising c_B – see for instance [16, Proposition 1.5.12]. Now one B-recursively splits the integers into intervals $I_0, I_1, \ldots$ with I_n consisting of $c_B(n)$ elements. Note that there is no recursive function f with $\forall n\,[f(\min(I_n)) \geq \max(I_n)]$; if such an f would exist, one could directly compute a function g majorising c_B by letting $g(0) = f(0)$ and $g(n+1) = \max\{f(m) : m \leq g(n) + 1\}$.

One constructs a set A which is constant on each interval I_n, that is, satisfies $A(x) = A(y)$ for all $x, y \in I_n$. This is done via a priority construction similar to the one of Jockusch [11] which shows that every hyperimmune degree contains a bi-immune set.

In stage n one defines $A(x)$ for all $x \in I_n$. The requirement $R_{\langle i,j\rangle}$ needs attention if the requirement has not acted in stages $0, 1, \ldots, n-1$ and W_i enumerates within $c_B(n)$ steps an element into I_n. If $R_{\langle i,j\rangle}$ receives attention then it acts as follows: If j is even then let $A(x) = 0$ for all $x \in I_n$ else let $A(x) = 1$ for all $x \in I_n$.

Assuming that W_0 enumerates all elements of $\mathbb{N}$ within 0 steps, it is clear that in every stage some requirement needs attention and therefore A will be defined on all intervals I_n. Assume that there is a pair $\langle i,j\rangle$ such that W_i is infinite and the requirement $R_{\langle i,j\rangle}$ never acts; without loss of generality let $\langle i,j\rangle$ be the least such pair. If n is sufficiently large then all requirements of higher priority do require attention only before stage n – either they require attention only finitely often (without being successful) or they act eventually and will never require attention again. Let $f(m) = x + t$ for the first pair $\langle x,t\rangle$ found such that $x \geq m$ and $x \in W_{i,t}$. The function f is total as W_i is infinite. If now $f(\min(I_n)) > c_B(n)$ for almost all n, then c_B would be majorised by a recursive function; hence there are infinitely many n such that $f(\min(I_n)) < c_B(n)$ and the requirement $R_{\langle i,j\rangle}$ would require attention in these stages n and therefore act eventually in contradiction to the assumption.

It follows from the construction that every infinite r.e. set R intersects infinitely many $I_n \subseteq \overline{A}$ and infinitely many $I_n \subseteq A$. Thus one can construct relative to $R \cap A$ a function h with $h(n) \in R$ for all n, $h(n) < h(n+1)$ for all n and $h(n) \in A$ iff n is even. As $R_{\langle 0,0\rangle}$ requires attention at stage 0, $h(0) \notin I_0$. Furthermore, one can see by induction that the facts that $h(n) \notin \cup_{m \leq n} I_m$, $h(n), h(n+1)$ being in different intervals and h being strictly monotonically increasing imply that $h(n+1) \notin \cup_{m \leq n+1} I_m$. Thus $h(n) \geq \max(I_n) \geq c_B(n)$ and so $B \leq_T h$. As $h \leq_T A \cap R$, it follows that $B \leq_T A \cap R$. □

Facts related to randomness. One can split every complete Turing degree into Turing incomplete Martin-Löf random degrees. We would like to thank Yu Liang for a simplification of our original proof of this fact.

Proposition 2.2. *Every Turing degree above that of Ω contains a Martin-Löf random set of the form $X \oplus Y$ such that X is low and Y is Turing incomplete.*

Proof. Let $Z \geq_T \Omega$ be in the given Turing degree and let X be a low Martin-Löf random set, for example a half of Ω. Relativising the Theorem of Kučera and Gács to X gives that there is a set Y which is Martin-Löf random relative to X and which satisfies $Z \equiv_T X \oplus Y$. Now X is Martin-Löf random relative to Y by the Theorem of van Lambalgen [12] and therefore $X \not\leq_T Y$; hence Y is Turing incomplete. □

In the first section, we discussed the fact that every set A below both halves of a splitting of a Martin-Löf random set is K-trivial. In recent work, Bienvenu, Kučera, Greenberg, Nies and Turetsky [1] have built a K-trivial set A that is not below both halves of any such splitting. On the other hand, every strongly jump traceable set is below every ω-r.e. Martin-Löf random set [9] and hence below, say, $\Omega \cap R$ for every infinite recursive set R.

For some Demuth random sets Z, there is still such a set A: Nies [17] builds a Demuth random ω^2-r.e. set Z; he also observes that there is a non-recursive r.e. set Turing below all ω^2-r.e. Martin-Löf random sets. Note that this stands in contrast to Martin-Löf random sets Z of hyperimmune-free degree where it is impossible to find any non-recursive set A which is retrievable from at least one half of every recursive splitting, see Theorem 4.3 below.

3. Schnorr random sets

Theorem 3.1. *Every high Turing degree contains a Schnorr random set Z such that $Z \leq_T R \cap Z$ for every infinite r.e. set R. Thus, if R is an infinite recursive set then $Z \equiv_T R \cap Z$.*

Proof. Let g be a function in the given Turing degree which dominates all recursive functions. Define recursively in g a function f such that $f(n)$ is the sum of all $\varphi_k(n)$ where $k \leq n$ and $\varphi_k(n)$ is computed within $g(n)$ computation steps. Note that the function f also dominates all recursive functions.

Fix a set X. We will define a Schnorr random set $Z \leq_T g \oplus X$ such that $X \leq_T R \cap Z$ for every infinite r.e. set R. The theorem is then obtained by letting X be any set in the given high degree.

Inductively define for each n and each $k = 0, 1, \ldots, n$, the set $E_{n,k}$ to be the first $2n + 5$ elements enumerated into

$$W_{k,g(f(n))} - \{0, 1, \ldots, f(n)\} - \bigcup_{(n',k'):(n',k')<_{lex}(n,k) \wedge k' \leq n'} E_{n',k'};$$

whenever they exist; if they do not exist then $E_{n,k} = \emptyset$. Note that the double sequence $(E_{n,k})_{n,k \in \mathbb{N}}$, when viewed as a double sequences of strong indices for finite sets, is g-recursive.

We define the set Z in two parts. First, we specify the digits of Z at the coding locations in $E_{n,k} \neq \emptyset$ by the following. If $i = \min E_{n,k}$ we define $Z(i) = X(n)$. If i is the $(j+1)^{th}$ element in $E_{n,k}$ we put $Z(i) = 1$ if and only if $\varphi_j(n+1)$ contributes to $f(n+1)$, where $j = 0, 1, \ldots, n+1$. If i is the $(n+3+j)^{th}$ element in $E_{n,k}$ we put $Z(i) = 1$ if and only if $E_{n+1,j}$ is non-empty, where $j = 0, 1, \ldots, n+1$. So in total $2n+5$ coding bits are used.

The intuition behind this coding is the following. $E_{n,k}$ is a set of coding locations to allow $W_k \cap Z$ to decipher $X(n)$. Since $W_k \cap Z$ may be empty outside of the W_k-reserved blocks $E_{-,k}$, the location of the next block $E_{n+1,k}$ must therefore be coded into $W_k \cap Z$ within the block $E_{n,k}$. This is coded in the $(n+3+j)^{th}$ locations. Furthermore, to make Z Schnorr random, we need to ensure that the coding blocks (where we have no control over the value Z takes) are spaced out in a sparse way This will force us to place $E_{n,k}$ above the value $f(n)$. We must then code the value of $f(n+1)$ (indirectly) into $E_{n,k}$ via the $(k+1)^{th}$ element in order to allow $W_k \cap Z$ when reading the block $E_{n,k}$ to find the next block.

We now describe how to define Z on the bits which are not in $E_{n,k}$ for any n, k. By a result of Schnorr – see [16, Lemma 7.5.1] – there is a fixed g-recursive martingale L with rational values which dominates all recursive martingales M up to a multiplicative constant v, namely, $M(\sigma) \leq vL(\sigma)$ for each σ. For every k, $W_k \cap Z$ does not look at these digits when recovering X. Hence we define Z on these digits so that L does not increase. Namely if we have already defined $Z(0)Z(1)\ldots Z(x-1)$ and x is not in a coding block then we define $Z(x) = 0$ iff $L(Z(0)Z(1)\ldots Z(x-1)0) \leq L(Z(0)Z(1)\ldots Z(x-1)1)$.

We now verify that Z is Schnorr random. For each n, there are at most n^2 many different coding blocks below $f(n)$, hence $L(Z(0)Z(1)\ldots Z(f(n)-1)) \leq 2^{(2n+5)^2}$. Now if Z is not Schnorr random there is a recursive martingale M and a recursive function h such that $M(Z(0)Z(1)\ldots Z(h(n)-1)) > n$ for infinitely many n – see [16, Theorem 7.3.3]. It is not hard to see that for each constant v there is a recursive function p such that for infinitely many k, we have $M(Z(0)Z(1)\ldots Z(\ell-1)) > v2^{(2k+5)^2}$ for some $\ell \leq p(k)$. Since f dominates every recursive function, we have a contradiction to the dominating property of L.

Clearly Z is recursive in $X \oplus g$. Now let W_e be an infinite r.e. set. As g dominates all recursive functions, one can find for almost all n more than $(2n+5)^3$ elements in W_e above $f(n)$ in time $g(f(n))$. Therefore, $E_{n,e}$ is non-empty for almost all n. So when starting with a sufficiently large n, using Z and the enumeration of W_e one can find all the entries for $E_{m,e}$ with $m \geq n$, and therefore compute X using only $Z \cap W_e$. Note that $Z \cap W_e$ allows us to compute the elements of $E_{n',e'}$ for every n', e', but does not tell us whether such an element is in Z (we do not need to know this), unless $e' = e$. □

The following corollary is now easily obtained.

Corollary 3.2. *A degree is high if and only if it contains a Schnorr random set Z such that we have $Z \equiv_T R \cap Z$ for every infinite recursive set R.*

Proof. The implication from left to right is immediate from the theorem. For the converse implication, note that Z is not Martin-Löf random by van Lambalgen's theorem. Nies, Stephan and Terwijn [18] showed that the degree of each Schnorr but not Martin-Löf random real is high, therefore it follows that the degree of Z is high. □

4. Martin-Löf random sets

Theorem 4.1. *There is a Martin-Löf random set Z such that $\Omega \leq_T Z \cap R$ or $\Omega \leq_T Z \cap \overline{R}$ for every recursive set R.*

Proof. The proof is in two steps:

- First we construct an r-maximal set S with complement $E_0 \cup E_1 \cup E_2 \cup \ldots$ where the parts $E_0, E_1, E_2, \ldots$ are finite sets with $\max E_n < \min E_{n+1}$ which each maximise their e-state (recall that an r-maximal set S is one where there is no recursive set R for which both $R \cap \overline{S}$ and $\overline{R} \cap \overline{S}$ are infinite). This construction is different from the usual e-state construction of a maximal set – see Friedberg [8] – in the following way. In the usual construction each current n^{th} element of the complement of S is given an e-state to maximise. Here we collect elements in the complement of S into groups E_n. To each group E_n we assign a single e-state to maximise.
- Once the r.e. set S is constructed, we define Z in two parts. We code Ω into Z on the digits specified by the complement of S. The other digits of Z are filled by a sufficiently random sequence, which is chosen to ensure that Z is Martin-Löf random.

We first construct S. Thereafter we give the detailed definition of Z. Lastly we will show that Z is Martin-Löf random, and that for every recursive splitting of Z, one half is Turing above Ω.

Construction of the r-maximal set S. We modify the usual Friedberg construction of an r.e. maximal set. Given a finite set D, we define the n-th e-state of D as the sum $3^n a_0 + 3^{n-1} a_1 + \ldots + 3a_{n-1} + a_n$ where for each $k \leq n$, we have

$$a_k = \begin{cases} 2 & \text{if } \varphi_k(x) \text{ is defined and positive for every } x \in D, \\ 1 & \text{if } \varphi_k(x) \text{ is defined and equal 0 for every } x \in D, \\ 0 & \text{otherwise.} \end{cases}$$

For each D and each n there is a natural uniform approximation to the n-th e-state at any given time. We now build the r.e. set S by initially setting each $E_{n,0}$ to be an interval of length $2^{3^{n+1}} \cdot n^2$. Its initial n-th e-state is 0. The construction ensures that for every n, s, $\max E_{n,s} < \min E_{n+1,s}$ and $\cup_n E_{n,s} = \overline{S}_s$ at stage s. Furthermore, when the n-th e-state of E_n is p then we ensure that E_n has $n^2 \cdot 2^{3^{n+1}-p}$ many elements.

Now at stage s search for the least $n < s$ such that $E_{n,s}$ can be redefined to improve its n-th e-state. This means that there exists a finite set $D \subseteq \overline{S}_s$ with $\max E_{n,s} < \min D$ such that D has n-th e-state p larger than the n-th e-state of $E_{n,s}$, where D has $n^2 \cdot 2^{3^{n+1}-p}$ many elements. If such least $n < s$ is found at stage s we redefine $E_{n,s+1} = D$ and for every $m > n$ we redefine $E_{m,s+1}$ larger than s. All elements below s which are not on an interval E_n at stage s are enumerated into S.

Verification of the construction. It is easy to argue, as in the usual construction of a maximal set, that each E_n moves only to improve its n-th e-state and hence is moved only finitely often. For each n let $a_0^n, a_1^n, \ldots, a_n^n$ be the final limiting values of the parameters $a_0, a_1, \ldots, a_n$ in the definition of the n-th e-state. It follows easily by induction that for every $k \in \omega$, $\lim_{n\to\infty} a^n_k$ exists.

We argue that S is r-maximal. This is the same as showing that for each k, if φ_k is total then $\lim_{n\to\infty} a^n_k \neq 0$. Suppose not, and fix a least counterexample k. Fix a large enough n so that $a^n_0, \ldots, a^n_k$ are all stable. Since φ_k is total it has to converge to the same value, say 0, on at least half of the elements in E_{n+1}. Since at the end we only care about the functions which are total, we may adopt the convention that if $\varphi_i(j)$ converges at a stage s then for every $j' \leq j$, $\varphi_i(j')$ has also converged by stage s. Then, for every subset X of E_{n+1}, the $k-1$-th e-state of X equals $3^n a^n_0 + \ldots + 3^{n-k+1} a^n_{k-1}$. Now pick X to be the subset of E_{n+1} where φ_k converges to 0. The size of X is at least $\frac{1}{2}(n+1)^2 2^{3^{n+2}-q}$ where q is the $(n+1)$-th e-state of E_{n+1}. If p is the n-th e-state of X then clearly p is larger than the n-th e-state of E_n and furthermore $3p > q$. It is easy to see that X has at least $n^2 \cdot 2^{3^{n+1}-p}$ many elements, which means the construction would have ensured that E_n is moved, a contradiction.

Construction of Z. Let $E = \overline{S} = \cup_{k\in\omega} E_k$ and let e_k be the k-th element of E (in ascending order). Now let $Z(e_k) = \Omega(k)$. Note that whenever $e_k \in E_n$ then $e_{k+1} \in E_n \cup E_{n+1}$. By the low for Ω basis theorem [5, 14], we fix a set P which is low for Ω and PA-complete. Let V be a set which is Martin-Löf random relative to $P \oplus \Omega$ and for $x \in S$, let $Z(x) = V(x)$. This completes the construction of Z.

Ω is computable from $Z \cap R$ or $Z \cap \overline{R}$. Note that from the final e-states of $E_0, E_1, \ldots, E_n$ we can compute the sets $E_0, E_1, \ldots, E_n$ explicitly by simply simulating the construction until for all $k \leq n$ the e-states of $E_{k,t}$ have reached the corresponding values.

The size of $E_0 \cup E_1 \cup \ldots \cup E_n$ is at least $O(n^3)$ while the e-states of $E_0, E_1, \ldots, E_n, E_{n+1}$ together can be described with $\log(3) \cdot \frac{n(n+1)}{2} \in O(n^2)$ bits. Therefore, the positions of $E_0, E_1, \ldots, E_n, E_{n+1}$ are reached at some stage t before the left-r.e. approximation of Ω stabilizes on the first $|E_0 \cup E_1 \cup \ldots \cup E_n|$ many bits. To see this, assume not, and let $F(n)$ be the first stage t such that $E_{0,t}, E_{1,t}, \ldots, E_{n,t}, E_{n+1,t}$ have all reached their final positions. Then for infinitely many n, $F(n+1)$ and hence $\Omega(0)\Omega(1)\ldots\Omega(n^3)$ can be described using $O((n+1)^2)$ many bits of information, which is a contradiction.

This fact can then be used to compute Ω from Z in an iterative manner: Knowing the bits of Ω coded on $E_0 \cup E_1 \cup \ldots \cup E_n$ allows us to compute the position of E_{n+1}, which then again enables us to look up in Z the bits of Ω coded on E_{n+1}. So $\Omega \leq_T Z$ by only taking into consideration the positions in E. As S is r-maximal, for any recursive set R, either almost all elements of E are in R, or almost all elements are out of R; depending on which case holds, one can compute Ω from either $Z \cap R$ or $Z \cap \overline{R}$.

Z is Martin-Löf random. Assume that this is not the case. Then, since P is PA-complete, there is a P-recursive martingale M succeeding on Z. There is a partial recursive function γ from finite strings to finite strings

that while processing the input from Z, computes the set E_{n+1}: whenever it has processed all the members of $E_0 \cup E_1 \cup \ldots \cup E_n$ and found the corresponding values of Ω, it uses the time the left-approximation takes to reach these values to get the position of E_{n+1}. Formally we have for every n, $\gamma(Z(0)Z(1)\ldots Z(n)) = S(n+1)$. Of course γ might be wrong or undefined if the input is not a prefix of Z. Now let

$$\begin{aligned} q_n &= \frac{M(Z(0)Z(1)\ldots Z(n)Z(n+1))}{M(Z(0)Z(1)\ldots Z(n))}; \\ s_n &= \prod_{m \in \{0,1,\ldots,n\} \cap S} q_m; \\ t_n &= \prod_{m \in \{0,1,\ldots,n\} \cap E} q_m. \end{aligned}$$

As the limit superior of $s_n \cdot t_n = \frac{M(Z(0)Z(1)\ldots Z(n)Z(n+1))}{M(Z(0))}$ is ∞, it follows that (a) the limit superior of the s_n is ∞ or (b) the limit superior of the t_n is ∞. We now show that in both cases (a) and (b) a contradiction can be derived.

Recall that by hypothesis, Ω is Martin-Löf random relative to P, V is Martin-Löf random relative to $\Omega \oplus P$. Then, by the Theorem of van Lambalgen relativized to P, Ω is Martin-Löf random relative to $V \oplus P$.

Case (a): the limit superior of the s_n is ∞. In this case, one can define a modification $\widetilde{M}$ of M which bets using information from γ. The modified martingale $\widetilde{M}$ will be recursive in $P \oplus \Omega$ and succeeds on V. The martingale $\widetilde{M}$ will use P to consult M and use Ω to obtain $Z(0)Z(1)\ldots Z(n-1)$ from $V(0)V(1)\ldots V(n-1)$. More specifically when given a string $\sigma \subset V$ and the members of E below $|\sigma|$, we check $\gamma(\sigma')$ to see if $|\sigma| \in E$, where σ' is modified from σ by filling in all the positions of E on σ using the bits of Ω (hence $\sigma' \subset Z$). If γ tells us that $|\sigma| \in E$ then $\widetilde{M}$ refrains from betting else $|\sigma| \in S$ and $\widetilde{M}$ bets proportionally using the ratio from M. This allows $\widetilde{M}$ to compute $\widetilde{M}(\sigma 0)$ and $\widetilde{M}(\sigma 1)$ given $\widetilde{M}(\sigma)$ and also to compute a guess at whether $|\sigma| \in E$.

This procedure clearly applies (inductively) for all strings σ. While $\widetilde{M}$ is not a total martingale, it is defined on all initial segments of V. Thus, we can alternately use the bits of Ω and the function γ to correctly predict E along V, and bet according to M along Z. Hence the partial martingale $\widetilde{M}$ succeeds on V and V is not Martin-Löf random relative to $P \oplus \Omega$, a contradiction.

Case (b): the limit superior of the t_n is ∞. In this case, we make another modification $\widehat{M}$ of M such that the resulting partial $P \oplus V$-recursive martingale $\widehat{M}$ bets the value $q_{e_{n+1}} = t_{e_{n+1}}/t_{e_n}$ on $\Omega(n+1)$. Given a string $\sigma \subset \Omega$ and the first $|\sigma|$ elements of E, $\widehat{M}$ uses V to fill in the bits on S and σ to fill in the bits on E. From this resulting string $\eta \subset Z$ it asks γ for the next element of E, padding η with the bits of V until γ returns the next element of E. We can then obtain the ratio $q_{e_{n+1}}$ from M.

Thus along Ω it is easy to see that each η is an initial segment of Z. So γ is always returns the correct answers, whence $\widehat{M}$ is defined along Ω using the correct ratios. This shows that Ω is not Martin-Löf random relative to $P \oplus V$, a contradiction. □

Remark 4.2. This argument can be extended in order to show that for any given set Y, there is a Martin-Löf random Z such that either $Y \leq_T Z \cap R$ or $Y \leq_T Z \cap \overline{R}$ for every recursive set R.

The next result shows that for Z of low computational power, one cannot code any non-recursive set into Z such that it can always be retrieved from some half of a recursive splitting.

Theorem 4.3. *Assume that Z is Martin-Löf random and has hyperimmune-free Turing degree. Then there is no non-recursive set A such that for every recursive set R, either $A \leq_T Z \cap R$ or $A \leq_T Z \cap \overline{R}$.*

Proof. Let Z be as given and assume by way of contradiction that such a non-recursive $A \leq_T Z$ exists. Then $A \leq_{tt} Z$ as Z has hyperimmune-free Turing degree. Demuth [3] showed that there is a Martin-Löf random set $B \equiv_T A$; as the Turing degree of A is hyperimmune-free as well, $B \leq_{tt} Z$. Let f be a recursive use function for the reduction witnessing $B \leq_{tt} Z$ such that it computes $B(0)B(1)\ldots B(n)$ from $Z(0)Z(1)\ldots Z(f(n))$. Now one can choose a recursive set $R = \bigcup_n \{a_{2n}, a_{2n}+1, \ldots, a_{2n+1}-1\}$ where $a_0 = 0$ and $a_{m+1} = a_m + 1 + f(a_m + 2m)$ for all m. Let X and Y be the splitting of Z along R, that is $X(n) = Z(p_R(n))$ and $Y(n) = Z(p_{\overline{R}}(n))$ where $p_S(n)$ is the n-th member of a set S. By assumption $B \leq_{tt} X$ or $B \leq_{tt} Y$, say the second and let Φ be the corresponding truth-table reduction. Furthermore, $\Phi^{-1}(\sigma)$ denotes the class $\{\tilde{Y} : \Phi(\tilde{Y}) \succeq \sigma\}$. There are now two cases.

First, assume that for infinitely many even n the set $\Phi^{-1}(B(0)B(1)\ldots B(a_n + 2n - 1))$ has at least the measure $2^{-a_n - n}$. There are at most 2^{a_n+n} strings $\sigma \in \{0,1\}^{a_n+2n}$ for which the class $\Phi^{-1}(\sigma)$ has at least measure 2^{-a_n-n}. Now let $U_n = \{\tilde{B} : \Phi^{-1}(\tilde{B}(0)\tilde{B}(1)\ldots\tilde{B}(a_n + 2n - 1))$ has at least measure $2^{-a_n-n}\}$; each U_n has at most measure 2^{-n}. As B is in infinitely many of the U_n, B cannot be Martin-Löf random contradicting the above choice of B, so this case does not occur.

Second, assume that for infinitely many even n the set $\Phi^{-1}(B(0)B(1)\ldots B(a_n + 2n - 1))$ has at most the measure 2^{-a_n-n}. For even n, one can compute $\sigma = B(0)B(1)\ldots B(a_n + 2n - 1)$ relative X using as an additional input τ the first a_n bits of Z via the following truth-table reduction with oracle X: let $\tau \mapsto \sigma(\tau)$ be the X-recursive function which is defined for all $\tau \in \{0,1\}^{a_n}$ for even n and which evaluates the truth-table reduction from B to Z by taking $Z(k) = \tau(k)$ for $k < a_n$ and by retrieving $Z(k)$ from X for $k \in \{a_n, a_n{+}1, \ldots, f(a_n{+}2n{-}1)\}$. Note that $\sigma(Z(0)Z(1)\ldots Z(a_n{-}1))$ is the correct value while other strings of length a_n can result in incorrect values. For even n, let V_n be the union of all those $\Phi^{-1}(\sigma(\tau))$ where $\tau \in \{0,1\}^{a_n}$ and the measure of $\Phi^{-1}(\sigma(\tau))$ is at most 2^{-a_n-n}. As this is the union over at most 2^{a_n} components, the overall class V_n has at most measure 2^{-n}. Again, there are infinitely many even n such that V_n contains Y, a contradiction to the Theorem of van Lambalgen which implies that Y is Martin-Löf random relative to X. So this case also cannot occur.

Hence, in both cases, there is a contradiction. Since at least one of these two cases has to apply, it follows that A and B cannot exist in contrary to the assumption. This proves the theorem. $\square$

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