MARTIN-LÖF RANDOM POINTS SATISFY BIRKHOFF'S ERGODIC THEOREM FOR EFFECTIVELY CLOSED SETS

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ABSTRACT. We show that if a point in a computable probability space X satisfies the ergodic recurrence property for a computable measure-preserving $T\colon X \to X$ with respect to effectively closed sets, then it also satisfies Birkhoff's ergodic theorem for T with respect to effectively closed sets. As a corollary, every Martin-Löf random sequence in the Cantor space satisfies Birkhoff's ergodic theorem for the shift operator with respect to Π_1^0 classes. This answers a question of Hoyrup and Rojas.

Several theorems in ergodic theory state that almost all points in a probability space behave in a regular fashion with respect to an ergodic transformation of the space. For example, if $T\colon X \to X$ is ergodic,[1] then almost all points in X recur in a set of positive measure:

Theorem 1 (See [4]). *Let (X, μ) be a probability space, and let $T\colon X \to X$ be ergodic. For all $E \subseteq X$ of positive measure, for almost all $x \in X$, $T^n(x) \in E$ for infinitely many n.*

Recent investigations in the area of algorithmic randomness relate the hierarchy of notions of randomness to satisfaction of computable instances of ergodic theorems. This has been inspired by Kučera's classic result characterising Martin-Löf randomness in the Cantor space. We reformulate Kučera's result using the general terminology of [3].

Definition 2. Let (X, μ) be a probability space, and let $T\colon X \to X$ be a function. Let $\mathcal{C}$ be a collection of measurable subsets of X. A point $x \in X$ is a *Poincaré point for T with respect to $\mathcal{C}$* if for all $E \in \mathcal{C}$ of positive measure, for infinitely many n, $T^n(x) \in E$.

The Cantor space 2^ω is equipped with the fair-coin product measure λ. The *shift operator* σ on the Cantor space is the function $\sigma(a_0 a_1 a_2 \dots) = a_1 a_2 \dots$. The shift operator is ergodic on $(2^\omega, \lambda)$.

Theorem 3 (Kučera [6]). *A sequence $R \in 2^\omega$ is Martin-Löf random if and only if it is a Poincaré point for the shift operator with respect to the collection of effectively closed (i.e., Π_1^0) subsets of 2^ω.*

[1]Recall that if (X, μ) is a probability space, then a measurable map $T\colon X \to X$ is *measure-preserving* if for all measurable $A \subseteq X$, $\mu\left(T^{-1}A\right) = \mu(A)$. We say that a measurable set $A \subseteq X$ is *invariant* under a map $T\colon X \to X$ if $T^{-1}A = A$ (up to a null set). A measure-preserving map $T\colon X \to X$ is *ergodic* if every T-invariant measurable subset of X is either null or co-null

Building on work of Bienvenu, Day, Mezhirov and Shen [1], Bienvenu, Hoyrup and Shen generalised Kučera's result to arbitrary computable ergodic transformations of computable probability spaces.

Theorem 4 (Bienvenu, Hoyrup and Shen [2]). *Let (X, μ) be a computable probability space, and let $T\colon X \to X$ be a computable ergodic transformation. A point $x \in X$ is Martin-Löf random if and only if it is a Poincaré point for T with respect to the collection of effectively closed subsets of X.*[2]

The seminal regularity theorem is due to Birkhoff (see [4]).

Birkhoff's Ergodic Theorem. *Let (X, μ) be a probability space, and let $T\colon X \to X$ be ergodic. Let $f \in L^1(X)$. Then for almost all $x \in X$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i < n} f\left(T^i(x)\right) = \int f \, d\mu.$$

Of particular interest is the case when f is the characteristic function of a measurable subset of a space. Let (X, μ) be a probability space and let $T\colon X \to X$ be a measurable function. For any $f \in L^1(X)$ and $n < \omega$ we let

$$f^{(n)} = \frac{1}{n} \sum_{i < n} f\left(T^i(x)\right).$$

If $A \subseteq X$ is measurable, we let 1_A denote A's characteristic function; so for all $n < \omega$ and $x \in X$,

$$1_A^{(n)}(x) = \frac{\#\left\{i < n : T^i(x) \in A\right\}}{n}.$$

Birkhoff's ergodic theorem implies that if T is ergodic, then for almost all $x \in X$,

$$\lim_{n \to \infty} 1_A^{(n)}(x) = \mu(A).$$

We can therefore make an analogue of Definition 2:

Definition 5. Let (X, μ) be a probability space, and let $T\colon X \to X$ be a function. Let $\mathcal{C}$ be a collection of measurable subsets of X. A point $x \in X$ is a *Birkhoff point for T with respect to $\mathcal{C}$* if for all $E \in \mathcal{C}$,

$$\lim_{n \to \infty} 1_E^{(n)}(x) = \mu(E).$$

Gács, Hoyrup and Rojas [3] characterised the Schnorr random points as the Birkhoff points for computable ergodic transformations with respect to effectively closed sets whose measure is computable. They asked [7] what happens if we omit the requirement that the measure of the sets be computable.

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[2]Here we use the notion of a computable probability space of Gács, Hoyrup and Rojas [3], although all reasonable definitions of this concept are equivalent. For completeness, we recall the definition here. A *computable metric space* is a complete metric space (X, d) that is equipped with an enumeration $\langle s_i \rangle$ of a countable dense subset of X such that $(i, j) \mapsto d(s_i, s_j)$ is a computable function. This enumeration gives rise to an enumeration $\langle B_i \rangle$ of basic open balls of X that forms a basis for its topology. A *c.e. open* subset of X is a set of the form $\bigcup_{i \in S} B_i$ where $S \subseteq \omega$ is c.e. A function $f\colon X \to Y$ between two computable metric spaces is *computable* if uniformly, the inverse of a c.e. open set is c.e. open. An *effectively closed* set is the complement of a c.e. open set. A Borel probability measure μ on X is *computable* if uniformly, the measure of a finite union of basic open balls is a left-c.e. real; equivalently, if it is a computable point in the space of Borel probability measures on X equipped with the topology of weak convergence. A *computable probability space* is a computable metric space equipped with a computable measure.

Theorem 6. *Let X be a computable probability space, and let $T\colon X \to X$ be a computable ergodic map. Then a point $x \in X$ is Martin-Löf random if and only if it is a Birkhoff point for T with respect to the collection of effectively closed subsets of X.*

Of course if $x \in X$ is a Birkhoff point for T with respect to $\mathcal{C}$ then it is a Poincaré point for T with respect to $\mathcal{C}$. Our main result is an instance of the converse.

Theorem 7. *Let (X, μ) be a computable probability space, and let $T\colon X \to X$ be a computable, measure-preserving transformation. Let $x \in X$ be a Poincaré point for T with respect to the collection of effectively closed subsets of X. Then x is also a Birkhoff point for T with respect to the collection of effectively closed subsets of X.*

Theorem 6 follows immediately from the combination of Theorems 7 and 4, together with the fact that Birkhoff-ness implies Poincaré-ness. We should note that as the present paper was in preparation, Hoyrup independently announced a proof of Theorem 6.

We set about to prove Theorem 7. To prove a limit exists and has the required value, we discuss the partial limits, the inferior and superior, separately.

Proposition 8. *Let (X, μ) be a computable probability space, and let $T\colon X \to X$ be a computable, measure-preserving transformation. Let $x \in X$ be a Poincaré point for T with respect to the collection of effectively closed subsets of X. Then for any effectively closed subset P of X,*

$$\liminf_n 1_P^{(n)}(x) \geqslant \mu(P).$$

We first prove Proposition 8 and then show that it implies the full Theorem 7. We need the concept of left-c.e. functions on a computable metric space. These are the effectively lower semi-continuous functions.

Definition 9. Let X be a computable metric space. A function $f\colon X \to \mathbb{R}$ is *left-c.e.* if uniformly in $q \in \mathbb{Q}$, $f^{-1}(-\infty, q)$ is c.e. open in X.

Thus every computable function $f\colon X \to \mathbb{R}$ is left-c.e., but not every left-c.e. function is computable:

Example 10. Let $P \subseteq X$ be effectively closed. Then 1_P is left-c.e.: for $q \leqslant 0$, $1_P^{-1}(-\infty, q) = \emptyset$; for $q \in (0, 1]$, $1_P^{-1}(-\infty, q) = X \backslash P$; and for $q > 1$, $1_P^{-1}(-\infty, q) = X$. If P is not c.e. open, then 1_P is not computable, since $1_P^{-1}(-1/2, \infty) = P$. Indeed, if P is not clopen, then 1_P is not continuous, whereas every computable function is continuous.

Lemma 11. *Let X be a computable metric space.*
  (1) *A finite sum of left-c.e. functions on X is left-c.e.*
  (2) *If $f\colon X \to \mathbb{R}$ is left-c.e. and $q \in \mathbb{Q}$, then qf is left-c.e.*
  (3) *If $T\colon X \to X$ is computable and $f\colon X \to \mathbb{R}$ is left c.e., then $f \circ T$ is left-c.e.*
Furthermore, all these closure operations are uniform: a left-c.e. index for $f+g$ can be effectively obtained from left-c.e. indices for f and g; for $f \circ T$, from a left-c.e. index for f and a computable index for T; etc.

Proof. For (1), $(f+g)(x) < q$ if and only if there are rational numbers s and r such that $f(x) < s$, $g(x) < r$ and $r+s \leqslant q$. (2) is immediate. For (3), $(f \circ T)^{-1}(-\infty, q) = T^{-1}[f^{-1}(-\infty, q)]$, and since T is computable, T^{-1} preserves c.e. open sets. $\square$

Corollary 12. *Let X be a computable metric space, let $T \colon X \to X$ be computable, and let $f \colon X \to \mathbb{R}$ be left-c.e. Then the sequence $\langle f^{(n)} \rangle$ is uniformly left-c.e.*

We are ready to prove Proposition 8. Let (X, μ) be a computable probability space. Let P be an effectively closed subset of X, and let $x \in X$ be a Poincaré point for T with respect to the collection of effectively closed subsets of X. We need to show that $\liminf_n 1_P^{(n)}(x) \geqslant \mu(P)$.

In our proof we make use of another classical result (see [4]):

Maximal Ergodic Theorem. *Let (X, μ) be a probability space, let $T \colon X \to X$ be measure preserving, and let $f \in L^1(X)$. Let*

$$E = \left\{ y \in X \ : \ f^{(n)}(y) \geqslant 0 \text{ for some } n < \omega \right\}.$$

Then

$$\int_E f \, d\mu \geqslant 0.$$

Let $q < \mu(P)$ be a rational number; we show that $\liminf_n 1_P^{(n)}(x) \geqslant q$. Define $g \colon X \to \mathbb{R}$ by letting $g(y) = q - 1_P(y)$. Note that for all $n < \omega$ and $y \in X$, $g^{(n)}(y) = q - 1_P^{(n)}(y)$. Let

$$E = \left\{ y \in X \ : \ 1_P^{(n)}(y) \leqslant q \text{ for some } n < \omega \right\}.$$

By the maximal ergodic theorem,

$$\int_E g \, d\mu \geqslant 0.$$

Lemma 13. $\mu(E) < 1$.

Proof. Suppose, for a contradiction, that $\mu(E) = 1$. Then

$$\int_E 1_P \, d\mu = \mu(P),$$

and

$$\int_E q \, d\mu = q.$$

Then

$$0 \leqslant \int_E g \, d\mu = q - \mu(P) < 0$$

by the choice of q, for a contradiction. $\square$

In fact, calculations show that $\mu(E) \leqslant (1 - \mu(P))/(1 - q)$, assuming $q > 0$. Now E may not be c.e. open, but a close associate of E is. Let

$$F = \left\{ y \in X \ : \ 1_P^{(n)}(y) < q \text{ for some } n < \omega \right\}.$$

Lemma 14. *F is a c.e. open subset of X.*

Proof. Since P is effectively closed, by Example 10, 1_P is a left-c.e. function. By Corollary 12, the sequence $\langle 1_P^{(n)} \rangle$ is uniformly left-c.e.; the result follows. $\square$

It follows that $X - F$ is an effectively closed subset of X. Since $F \subseteq E$, by Lemma 13, $\mu(F) < 1$, so $X - F$ has positive measure. Since x is a Poincaré point for T with respect to all effectively closed subsets of X, there is an $n < \omega$ such that $T^n(x) \notin F$. That is, $1_P^{(m)}(T^n(x)) \geqslant q$ for all $m < \omega$. Now for all $m < \omega$,

$$1_P^{(n+m)}(x) \geqslant 1_P^{(m)}(T^n(x)) \frac{m}{m+n}$$

and so for all $m < \omega$,

$$1_P^{(n+m)}(x) \geqslant q \frac{m}{m+n}.$$

As $m/(m+n) \to 1$, we see that $\liminf 1_P^{(m)}(x) \geqslant q$ as required. This concludes the proof of Proposition 8.

Now we prove Theorem 7. We use the fact that the measure μ is effectively outer regular.

Lemma 15. *Let (X, μ) be a computable probability space. Then for all $\varepsilon > 0$, for any effectively closed $P \subseteq X$ there is a c.e. open $A \supseteq P$ such that $\mu(A - P) < \varepsilon$.*

Proof. This follows from the fact that the measure μ on X is σ-additive and the fact that the topology on X originates from a metric.

Let $\langle s_i \rangle$ be the sequence of "ideal" (or "rational") points of X. If $B = B(s_i, q)$ is a basic open ball, then its closure

$$\bar{B} = \bar{B}(s_i, q) = \{z \in X \,:\, d(z, s_i) \leqslant q\}$$

is, uniformly in i and q, effectively closed, as its complement is the union of all basic open balls $B(s_j, r)$ where $r < d(s_i, s_j) - q$; the collection of such s_j and rational r is c.e. because r and q are rational numbers and $d(s_i, s_j)$ is computable.

Let $P \subseteq X$ be effectively closed; there is a c.e. set $S \subseteq \omega$ such that

$$X - P = \bigcup_{(j,q) \in S} B(s_j, q).$$

Let $\langle S_s \rangle$ be an effective enumeration of S. We let

$$F_s = \bigcup_{(j,q) \in S_s} \bar{B}(s_j, q - 2^{-s}).$$

By [5], the intersection of finitely many c.e. open subsets of X is a c.e. open set, so the union of finitely many effectively closed sets is effectively closed. It follows that every F_s is effectively closed. We have $X - P = \bigcup_s F_s$. Since μ is σ-additive, for all $\varepsilon > 0$ there is an s such that $\mu(F_s) > \mu(X - P) - \varepsilon$. Then $X - F_s$ is c.e. open, contains P, and $\mu(X - F_s) - \mu(P) < \varepsilon$. $\square$

For the proof of Theorem 7, let (X, μ) be a computable probability space, let $T\colon X \to X$ be computable and measure-preserving, and let $P \subseteq X$ be effectively closed. Let $x \in X$ be a Poincaré point for T with respect to effectively closed sets. We want to show that $\lim_n 1_P^{(n)}(x) = \mu(P)$. By Proposition 8, $\liminf 1_P^{(n)}(x) \geqslant \mu(P)$, so it only remains to be shown that $\limsup_n 1_P^{(n)}(x) \leqslant \mu(P)$.

Let $\varepsilon > 0$; by Lemma 15, let $A \supseteq P$ be c.e. open such that $\mu(A - P) < \varepsilon$. By Proposition 8,

$$\liminf_n 1_{X-A}^{(n)}(x) \geqslant \mu(X - A) = 1 - \mu(A).$$

Since for all n, $1_A^{(n)}(x) + 1_{X-A}^{(n)}(x) = 1$, we get that

$$\limsup_n 1_A^{(n)}(x) = 1 - \liminf_n 1_{X-A}^{(n)}(x) \leqslant \mu(A).$$

Since $P \subseteq A$, for all n, $1_P^{(n)}(x) \leqslant 1_A^{(n)}(x)$. It follows that

$$\limsup_n 1_P^{(n)}(x) \leqslant \limsup_n 1_A^{(n)}(x) \leqslant \mu(A) \leqslant \mu(P) + \varepsilon.$$

Since this inequality holds for all $\varepsilon > 0$, we are done. This completes the proof of Theorem 7.

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