THE REVERSE MATHEMATICS AND UNIFORM RELATIONSHIPS BETWEEN THEOREMS ON COLOURING OF PLANAR GRAPHS

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Abstract. Reverse mathematics is primarily interested in what set existence axioms are necessary and sufficient in a proof of a theorem. Much work has been done in classifying graph colouring theorems, studying k-regular graphs, k-chromatic graphs and forests. This paper takes inspiration from an old paper by Bean and studies graph colouring theorems restricted to planar graphs. We show that for any natural number $n > 3$, the n-colouring theorem for planar graphs is equivalent to $\mathtt{WKL}_0$. Further analysis of related principles, obtained by restricting the planar graphs in question to be connected, or with computable planar drawings also yield similar results. However, many of the proofs of equivalence are non-uniform; utilising tools from the study of Weihrauch reducibility, we show that in many instances such non-uniformity is necessary.

1. Introduction

Graph colourings have been widely studied classically. There are numerous results regarding existence of colourings for different classes of graphs. Some examples include Brooks theorem, "every graph with maximum degree n is $n+1$-colourable" [3], and the famous four colour theorem "every planar graph is 4-colourable" [14]. In the classical setting, most work has been focused on studying the colourings of finite connected graphs. To obtain the same results in the infinite case, one can simply apply the De-Bruijn Erdos theorem [5]. Nonetheless, colourings of infinite countable graphs might still be of interest in the sense that such colourings could be 'computationally hard' to find. For example, Bean showed that there exists a recursive planar graph with no recursive 4-colouring and that for each $n \geq 3$, there exists a n-colourable graph not recursively n-colourable [1]. Intuitively, this means that even if an (infinite) graph is n-colourable, such a colouring cannot be found algorithmically. In contrast, provided more colours are allowed, recursive colourings can be found [1, 13]. A natural question that arises is whether algorithmic colourings exists for certain classes of graphs, and if such colourings do not exist, how 'computationally difficult' it is to find them.

To investigate the computational content of graph colourings, one possible approach is to use reverse mathematics. Reverse mathematics is a systematic approach to classifying mathematical theorems by matching them up with different levels of set existence axioms (see [16] and [8] for reference texts). The idea is to search for the weakest set existence axiom

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necessary for the theorem of interest to hold. This then implies that the theorem cannot be proved by any weaker set existence axiom. Most mathematical theorems fall in one of five subsystems, listed in order of strength as follows, $\mathtt{RCA}_0, \mathtt{WKL}_0, \mathtt{ACA}_0, \mathtt{ATR}_0, \Pi^1_1\text{-}\mathtt{CA}_0$. Of particular interest to the present paper is the subsystem $\mathtt{WKL}_0$ which asserts that each infinite binary tree has a path. Under the framework of reverse mathematics, Hirst showed that the De-Bruijn Erdos theorem is equivalent to $\mathtt{WKL}_0$ over $\mathtt{RCA}_0$ and that every n-colourable graph has a low n-colouring [10], thus providing an upper bound for the algorithmic content of graph colourings. Various studies have been conducted on other graph colouring theorems and most have been shown to be equivalent to $\mathtt{WKL}_0$ over $\mathtt{RCA}_0$ [9, 12, 15].

Another possible approach is to utilise the tools of Weihrauch reducibility. Weihrauch reducibility provides a framework to study Π^1_2 statements (statements of the form $\forall X \exists Y \varphi$; X is generally referred to as an instance and Y a solution). Roughly speaking, if P, Q are Π^1_2 statements, and P is Weihrauch below Q, then it means that if for any Q-instance, we are able to find a Q-solution, then we can do the same for P; Q is algorithmically more complex than P. While in most cases, an implication in the setting of reverse mathematics corresponds to a Weihrauch reduction, there exists examples where an implication holds but no Weihrauch reductions can be found [7, 4, 11].

Following this pattern, we study theorems about planar graph colourings under the framework of reverse mathematics and Weihruach reductions. We show that for each $n \geq 4$, the existence of an n-colouring for planar graphs is equivalent to $\mathtt{WKL}_0$ over $\mathtt{RCA}_0$. However, for each $n > 4$, some non-uniform arguments were used in the reversal. Using the tools of Weihrauch reducibility, we show that for all $n \geq 7$, such non-uniformity is necessary by proving that no Weihrauch reduction exists. The cases for $n = 5, 6$ are left open but a possible approach is suggested using Lemma 3.5; which characterises the existence of Weihrauch reductions from $\mathtt{COL}(n)$ to $\mathtt{DNR}(k)$ (both to be defined later) as a finite graph theoretic property.

1.1. **The principles and formal definitions.** Since the main objective of this paper is to investigate the reverse mathematical strength of theorems on the colouring of planar graphs, we shall start with the following definitions:

Definition 1.1 ($\mathtt{RCA}_0$)**.**
- A graph $G = (V, E)$ is a countable set of vertices $V = \{v_0, v_1, \dots\}$ and a symmetric irreflexive binary relation $E \subseteq V^2$. We will only consider simple undirected graphs in this paper. Note that we *do not* require a graph to be locally finite or connected.
- A subgraph of $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. A subgraph is not necessarily induced.
- A finite graph $\widehat{G}$ is planar if neither $K_{3,3}$ nor K_5 is a minor of $\widehat{\mathrm{G}}$. An infinite countable graph G is planar iff every finite subgraph of G is planar.
- Given a graph $G = (V, E)$, we say that h is a c-colouring of G iff $h : V \to \{0, 1, \dots, c-1\}$ and for any two vertices v, w, if $h(v) = h(w)$, then $\{v, w\} \notin E$. A graph is said to be c-colourable if it has a c-colouring.

We use G to denote infinite countable planar graphs, and $\widehat{G}$ to refer to finite planar graphs, unless explicitly stated otherwise.

Definition 1.2 ($\mathtt{RCA}_0$)**.** Given a graph $G = (V, E)$, a *diagram* of G is a pair of injective functions $\psi : V \to \mathbb{Q}^2$ and $f : E \to \mathbb{N}$ satisfying the following.

- For each $\{u, v\} \in E$, $f(\{u, v\})$ encodes some j-tuple of pairs of rational numbers $\langle\langle p_0, q_0\rangle, \langle p_1, q_1\rangle, \ldots, \langle p_j, q_j\rangle\rangle$ where $\psi(u), \psi(v)$ are the first and last (or last and first) entry respectively.

In addition, we say that a *diagram is planar* (or simply a *plane diagram*) iff it is a diagram with the additional property that for each $\langle p_i, q_i\rangle \in f(\{u, v\})$ and $\langle p'_j, q'_j\rangle \in f(\{u', v'\})$ (where $\{u, v\} \neq \{u', v'\}$) the line segments from (p_i, q_i) to (p_{i+1}, q_{i+1}) do not intersect with those from (p'_j, q'_j) to (p'_{j+1}, q'_{j+1}), except possibly at the endpoints of $f(\{u, v\})$ and $f(\{u', v'\})$.[1]

The intention behind f in the definition above is to formalise embedding of edges by finitely many straight line segments. We note that in the classical setting, having an embedding of the graph into $\mathbb{R}^2$ is clearly equivalent to having a plane diagram in the sense of the definition above.

Definition 1.3 (Faces)**.** Given a plane diagram (ψ, f) of a finite connected planar graph $\widehat{G}$, we wish to define an encoding of the faces of the diagram. Classically, an inner face of the diagram can be defined as a bounded, non-empty, connected open region F of the plane that is enclosed by a closed walk C taken in the diagram with the property that F does not intersect the diagram. The outer face is the unbounded, connected open region F of the plane with a closed walk C forming the boundary of F with the property that F does not intersect the diagram.

Each face (inner and outer) of the plane diagram can be encoded by the closed walk forming the boundary of the face. (Recall that this is just a finite list of rational coordinates). A *face-list* of the plane diagram is a list of the codes of all faces of the diagram. Obviously, the face-list of a plane diagram of a finite connected planar graph can be found effectively from the plane diagram itself.

The common classical definition of a planar graph G is via the existence of a planar embedding of G into the plane (or the sphere). Wagner [17] showed that this definition is equivalent to the one given in Definition 1.2 for finite graphs. Erdős showed that the two definitions are equivalent for infinite countable graphs (see [6, Dirac and Schuster]). Note that in Definition 1.2 we do not put any other topological restrictions on the range of the embedding. For instance, Dirac and Schuster [6] pointed out that there is a certain countable planar graph G such that the range of every plane embedding of G into the plane contains a limit point.

[1] Given two line segments with rational endpoints, note that they intersect (not necessarily at a rational point) iff the following intersections of intervals $(p_i, p_{i+1}) \cap (p'_j, p'_{j+1})$ and $(q_i, q_{i+1}) \cap (q'_j, q'_{j+1})$ are both non-empty. This can be checked effectively by comparing the ordering amongst the rationals.

As usual, the fact that the two definitions (of planarity) are classically equivalent cannot be carried over to the effective setting. It is not hard to see that not every planar computable graph has a computable plane diagram. In fact, in Proposition 2.1, we will show that the statement "Every planar graph has a plane diagram" is equivalent to $\mathtt{WKL}_0$.

The main objective of this paper is to study the logical relationships between different statements surrounding the colouring of planar graphs. The most famous result in this area is the four colour theorem. This problem is widely believed to be first raised by Francis Guthrie in 1852 while trying to colour a map of the counties of England, and an incorrect proof of the four colour theorem was announced by Alfred Kempe in 1879. Kempe's proof was shown to be flawed by Percy Heawood eleven years later, who then proved the five colour theorem based on Kempe's incorrect proof. While these results are about finite planar graphs, the corresponding statements for infinite planar graphs still hold true classically, due to the well-known result of De Bruijn and Erdős [5].

This prompts us to consider the following principles, formalized in $\mathtt{RCA}_0$:

Definition 1.4 ($\mathtt{RCA}_0$)**.** Let $n \in \mathbb{N}$ where $n \geq 4$.

- $\mathtt{COL}(n)$ is the statement that every countable planar graph is n-colourable.
- $\mathtt{COL}^*(n)$ is the statement that every countable planar graph with a computable planar diagram is n-colourable.
- $\mathtt{ConnCOL}(n)$ is the statement that every countable connected planar graph is n-colourable.

Here, a connected graph is one where every two vertices is connected by a finite path. Trivially, $\mathtt{COL}(n)$ implies both $\mathtt{COL}^*(n)$ and $\mathtt{ConnCOL}(n)$ for each $n \geq 4$.

2. The Reverse Mathematics of Colouring Principles

We first begin by addressing the question of whether the two classical definitions of planarity:

- G is planar if no finite subgraph of G contains $K_{3,3}$ or K_5 as a minor, versus
- G is "planar" if there is a planar embedding of G into the plane,

are equivalent from the point of view of reverse mathematics. Firstly, notice that if G admits a plane diagram (refer to Definition 1.2), then G is planar: For this, note that Euler's formula and therefore the planarity of K_5 and $K_{3,3}$ can be verified in $\mathtt{RCA}_0$. Therefore, the principle of interest is "Every planar G admits a plane diagram". We prove that this is equivalent to $\mathtt{WKL}_0$:

Proposition 2.1. Over $\mathtt{RCA}_0$, the following are equivalent:

- $\mathtt{WKL}_0$.
- Every planar graph admits a plane diagram.

The "only if" direction is originally due to Erdős (again, see [6, Dirac and Schuster]), and we include a proof of it here for completeness.

Proof. First, we prove the "only if" direction. Let $G = (V, E)$ be a planar graph, and $\{v_i\}_{i\in\mathbb{N}}$ be an enumeration of V. To obtain a plane diagram D, we encode for each n, all

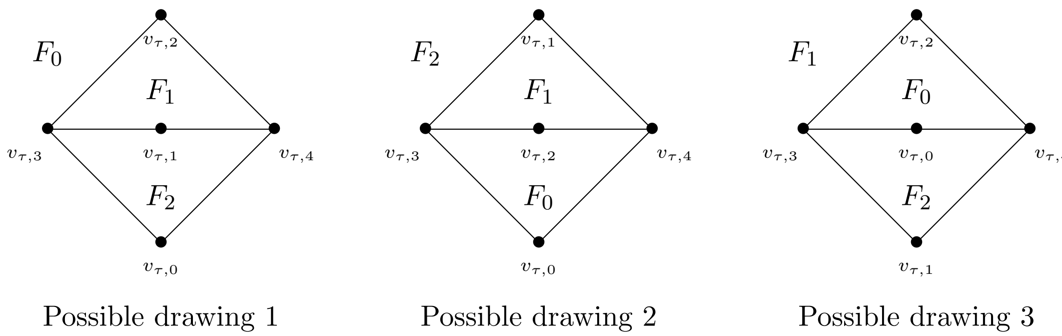

Figure 1. Reversal of Proposition 2.1

possible plane diagrams of the induced subgraph $\widehat{G}$ of G containing $v_0, v_1, \ldots, v_{n-1}$, as a node on a tree T. Clearly, the intuition should be that a node σ is extended by τ if the diagram encoded by σ is a sub-diagram of the one encoded by τ. We provide the details below.

Recall from Definition 1.3, that given any plane diagram of a finite graph $\widehat{G}$, we are able to list all its faces. Given $\sigma \in T$ encoding a plane diagram of the induced subgraph containing vertices $v_0, v_1, \ldots, v_{n-1}$, since there are only finitely many vertices, then there can be only finitely many faces. For a given face, embedding v_n anywhere within that face will result in an equivalent diagram; that is to say, there are only finitely many possible extensions of σ. Of course, this is not to say that every choice of face to embed v_n in results in a valid plane diagram extending σ. In particular, if v_n is adjacent to some vertex not on the boundary of the face it was embedded in, this cannot be extended to a valid plane diagram extending σ. Nevertheless, since there are only finitely many faces, and each face corresponds to at most one valid choice to embed v_n, then T is computably branching.

Suppose to the contrary that T as described is finite. Then, there is some n such that for all $t \in T$, $|t| < n + 1$. For such an n, consider the induced subgraph $\widehat{G}$ containing vertices $v_0, v_1, \ldots, v_n$. Since G is planar, then $\widehat{G}$ must also be planar; then $\widehat{G}$ must have a plane diagram D. For each $i \leq n$, considering the positions of the vertices $v_0, v_1, \ldots, v_i$ relative to the faces, must produce some plane diagram equivalent to one encoded by some node $\sigma \in T$ of length $i + 1$. Then there must be some node of length $n + 1$ in T, a contradiction to the assumption. Thus T is infinite. By bounded Konig's lemma, there is a path h through T. Furthermore, it is clear that a planar diagram of G can then be constructed by considering $h(n)$ at each n. Thus we have that $\mathtt{WKL}_0$ proves that every planar graph G has a plane diagram.

Now we reason in $\mathtt{RCA}_0$ and assume that every planar graph G has a computable plane diagram. Let T be an arbitrary infinite binary tree. Construct a countable planar graph, $G = \langle V, E \rangle$ as follows. G will consist of countable many components, each corresponding to some binary string τ. We thus refer to each component as the τ-th component. The τ-th component will originally consist of six vertices, $v_{\tau,i}$ for each $i < 6$. The vertices

$v_{\tau,j}$ for each $j < 5$ are connected as shown in Fig. 1 (note that the three drawings are all isomorphic as graphs), and $v_{\tau,5}$ is connected to $v_{\tau,3}$. The idea is that in the τ-th component, the position of $v_{\tau,5}$ in the diagram encodes which subtree extending σ is infinite. Since there are only three distinct drawings (up to equivalence) of the first five vertices in a component, the choice of embedding of $v_{\tau,5}$ will indicate the subtree extending τ which is infinite.

(1) If $v_{\tau,5}$ is contained in F_i for some $i < 2$, then the subtree above $\tau^\frown i$ is infinite.
(2) If $v_{\tau,5}$ is contained in F_2, then both subtrees above $\tau^\frown 0$ and $\tau^\frown 1$ are infinite.

We provide the details below.

For convenience, we assume that at each stage, at most one node is enumerated into T or its complement. We compute membership in T in increasing order of the length and lexicographic order; $\epsilon, 0, 1, \dots$. At stage s, if no new nodes are enumerated into the complement of T, do nothing and proceed to the next stage. If some node is enumerated into the complement, we may assume that it is of the form $\tau^\frown x$ for some $\tau \in T$ (otherwise $\epsilon \notin T$), then search for the maximal σ such that $\sigma \subseteq \tau$ and the σ-th component has yet to *act*. Also let $y \in \{0, 1\}$ be such that $\sigma^\frown y \subseteq \tau^\frown x$. Then act for the σ-th component as follows. Enumerate a new vertex $v_{\sigma,6}$ into the σ-th component and connect $v_{\sigma,6}$ to $v_{\sigma,5}, v_{\sigma,2}$ and $v_{\sigma,1-y}$. We will then say that the σ-th component has *chosen* $\sigma^\frown(1-y)$.

Proposition 2.2. For each $\tau^\frown x$ enumerated into the complement of T and $\tau \in T$, there always exists a maximal $\sigma \subseteq \tau$ such that the σ-th component has not acted. Furthermore, if the σ-th component has acted and chosen $\sigma^\frown(1-y)$, then the subtree extending $\sigma^\frown y$ is finite.

Proof of Proposition 2.2. We proceed by induction on the order of nodes $\tau^\frown x$ enumerated into the complement of T for some $\tau \in T$. Let $\tau^\frown x$ be the first node discovered to be in the complement of T. According to the construction, we must have picked the maximal $\sigma \subseteq \tau$ that has yet to act. Since $\tau^\frown x$ is the first node to be enumerated into the complement, then no component could have acted yet and thus $\sigma = \tau$. Furthermore, y would have been chosen such that $\sigma^\frown y \subseteq \tau^\frown x$, and thus $y = x$. It is also evident that the subtree extending $\sigma^\frown y = \tau^\frown x$ is finite (empty).

Suppose inductively that the proposition holds for the first s many nodes discovered to be in the complement of T. Let $\tau^\frown x$ be the $s+1$-th node discovered to be in the complement of T, and suppose for a contradiction that for each $\xi \subseteq \tau$, the ξ-th component of G has already acted at some earlier stage. Since $\tau^\frown x$ only just entered the complement of T, when each of the ξ-th component acted, it must have been for the sake of some subtree extending $\xi^\frown z \nsubseteq \tau$ discovered to be finite. But this cannot be, as this would imply that T is finite. Thus, there must be some prefix σ of τ that has not acted.

By the inductive hypothesis, for each ξ where $\sigma \subsetneq \xi \subseteq \tau$, the subtree extending $\xi^\frown z \nsubseteq \tau^\frown x$ must have been discovered to be finite. That is to say, any infinite path through $\sigma^\frown y \subseteq \tau^\frown x$ must also be a path through $\tau^\frown x$. But since $\tau^\frown x$ is found to be in the complement of T, then the subtree extending $\sigma^\frown y$ must be finite. $\square$

It should be evident that G is planar in the sense of Definition 1.1 as each τ-th component is finite and does not contain K_5 or $K_{3,3}$ as a minor. Then by the principle that every planar graph has a plane diagram, there is some diagram D of G as described above. We now extract a path h through T using G as follows. For each n, consider the $h \upharpoonright n$-th component.

- If $v_{h\upharpoonright n,5}$ is in F_0 (as illustrated in Fig. 1), then let $h(n) = 0$.
- Otherwise, let $h(n) = 1$.

We claim that for each n, the subtree extending $h \upharpoonright n$ is infinite. The base case is trivial. Suppose inductively that for some n, the subtree extending $\tau = h \upharpoonright n$ is infinite. Consider the following cases.

Case 1: $h(n) = 0$. Since $h(n) = 0$, then $v_{\tau,5}$ is contained in F_0 of the τ-th component. If the τ-th component never acts, then by applying Proposition 2.2, we obtain that both subtrees extending $\tau^\frown 0$ and $\tau^\frown 1$ are infinite.

We may thus assume that the τ-th component acts at some finite stage. By inductive hypothesis, we know that at least one of the subtrees extending $\tau^\frown 0$ or $\tau^\frown 1$ should be infinite. Suppose for a contradiction that the one extending $\tau^\frown 0$ is finite. Then the τ-th component must have acted at the stage where this was discovered and connected $v_{\tau,6}$ to $v_{\tau,2}, v_{\tau,5}$ and $v_{\tau,1}$. However, if $v_{\tau,5}$ was indeed contained in F_0 (see Fig. 1), D cannot possibly be a planar diagram. Thus the subtree above $\tau^\frown 0$ must be infinite.

Case 2: $h(n+1) = 1$. There are two further possibilities here. First, if $v_{\tau,5}$ is contained in F_2 in the τ-th component, observe that the τ-th component never acts throughout the construction. If the τ-th component ever acts, it enumerates a new vertex $v_{\tau,6}$ connected to both $v_{\tau,5}$ and $v_{\tau,2}$; this is impossible to do while keeping the edges non-intersecting. Thus, the τ-th component never acts, and by Proposition 2.2, both subtrees extending $\tau^\frown 0$ and $\tau^\frown 1$ are infinite.

Next, consider the possibility that $v_{\tau,5}$ is contained in F_1. The argument here is similar to Case 1 above and we conclude that the subtree extending $\tau^\frown 1$ must also be infinite.

Therefore, h is a path through T and h clearly exists by Δ^0_1-comprehension. ☐

Observe that we can also arrange the components based on the string associated with them as a full binary tree. By connecting the components via $v_{\tau,4}$, the graph constructed in the proof above can be made connected without losing its planarity.

Corollary 2.3.

- There is a planar computable graph with no computable plane diagram.
- There is a planar connected computable graph with no computable plane diagram.
- For each $k \geq 4$, the principles $\mathtt{COL}(k)$ and $\mathtt{COL}^*(k)$ are equivalent over $\mathtt{WKL}_0$.

Proposition 2.1 justifies the different formalizations of the k Colour Theorem in Definition 1.4, since $\mathtt{COL}(k)$ and $\mathtt{COL}^*(k)$ are not obviously computably equivalent: Given an arbitrary planar computable graph G, we cannot simply extract a computable plane diagram for G. This however does not rule out the possibility that $\mathtt{COL}(k)$ and $\mathtt{COL}^*(k)$ are still computably equivalent, for instance, in some non-uniform way. Interestingly, it actually turns out that they *are* equivalent over $\mathtt{RCA}_0$, and this shall be proved later on in this section.

2.1. **Recursive Comprehension Axiom.** We first work in $\mathtt{RCA}_0$ to show that for each n, $\mathtt{RCA}_0$ does not prove $\mathtt{COL}(n)$. This is clearly a weaker result than $\mathtt{RCA}_0 + \mathtt{COL}(n) \vdash \mathtt{WKL}_0$ proved in Section 2.4, but some ideas and objects introduced here will become relevant in Section 3.

As seen in the proof of Proposition 2.1, the gadgets that will be used in the proofs will include enumerating some initial part of a graph before 'extending' it to a larger graph. Since we are dealing with colourings, we would like to forbid 'extensions' of graphs adds edges between two previously enumerated vertices. Since the graphs in this paper will all be computable, given any finite subset of vertices, we should be able to compute the edge relation between any two vertices. In this sense, we may assume that a new vertex is only enumerated whenever its relation with all currently enumerated vertices have been decided.

Definition 2.4. Let G be a subgraph of G'. G' is said to be an extension of G if G is an induced subgraph of G'. Furthermore, we will denote this with $G \subseteq G'$.

Theorem 2.5. $\mathtt{RCA}_0 \nvdash \mathtt{COL}(4)$.

Proof. We construct a planar computable graph that does not have a computable 4-colouring. Let $\{\varphi_e\}_{e\in\mathbb{N}}$ be an enumeration of the partial computable functions from $\mathbb{N}$ to $\{0,1,2,3\}$. We construct a graph $G = \langle V, E\rangle$ satisfying the following requirement for each $e \in \mathbb{N}$.

$$R_e : \text{If } \varphi_e \text{ is total, then } \varphi_e \text{ is not a colouring of } G$$

We build G in stages, and say that a requirement R_e is met when it has found some *witness* v_e^* such that there is some v adjacent to v_e^* and $\varphi_e(v) = \varphi_e(v_e^*)\downarrow$. In other words, φ_e fails to be a 4-colouring of G.

Stage 0**:** Enumerate a K_4 graph into G and let the vertices of this K_4 be denoted as $v_{0,i}$ for each $i < 4$. In addition to these vertices, also enumerate a single (temporarily) isolated vertex $v_{0,4}$ into G. The vertices $v_{0,j}$ for various j will be referred to as the 0^{th} component of G.

Stage $s > 0$**:** Begin the s^{th} component of G by enumerating 5 new vertices $v_{s,j}$ for each $j < 5$ and connect $v_{s,i}$ for each $i < 4$ as a K_4 graph.

For each $e < s$, if $\varphi_e(v_{e,j})\downarrow$ for all $j < 5$ and R_e has yet to act, then enumerate $v_{e,5}$ into the e^{th} component and connect $v_{e,5}$ to $v_{e,4}$ and $v_{e,i}$ for each $i < 4$ where $\varphi_e(v_{e,i}) \neq \varphi_e(v_{e,4})$. Then declare $v_{e,5}$ as the witness for R_e and proceed with the construction.

Observe that if some requirement R_e never acts, then there must be some $i < 5$ for which $\varphi_e(v_{e,i})[s]\uparrow$ at every stage s. Then φ_e cannot possibly be total and thus R_e is satisfied. Suppose then that there is some stage at which R_e is discovered to be ready. Since there is no injury in the construction, (each requirement acts only on the component reserved for it), then R_e must act at some stage s. When it does, recall that $v_{e,5}$ is then

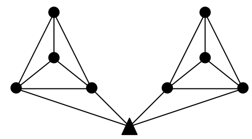

FIGURE 2. W_1

enumerated into the e^{th} component and connected with vertices $v_{e,4}$ and $v_{e,i}$ for each $i < 4$ where $\varphi_e(v_{e,i}) \neq \varphi_e(v_{e,4})$. Then φ_e must fail as a 4-colouring as the neighbours of $v_{e,5}$ are coloured 4 different colours. □

The main idea of the proof above was simply to wait until a potential colouring has used up all of its allowable colours in a certain configuration and then enumerating a new vertex connected to all the vertices coloured with the allowable colours. Since K_4 is planar and must be coloured with 4 colours, we can ensure that any potential 4-colouring uses all 4 colours available to it. Evidently, we cannot simply enumerate a new vertex and connect it to all vertices currently in the K_4 otherwise the resulting graph is no longer planar. Thus we have an additional isolated vertex in each component to occupy the final colour. It is not difficult to see that when the number of colours increases, the diagonalisation strategy will become more complex as it has to be able to get any potential colouring to commit to more colours. However, since every planar graph is 4-colourable, it is evident that each component should become 'layered' in some way.

The general gadget will be as follows. Start by enumerating sufficiently many K_4 graphs. By pidgeonhole principle, there will be at least two, say K_4^0, K_4^1, that are coloured the same 4 colours. Now connect a new vertex ▲ to two vertices each from K_4^0 and K_4^1 all of different colours. As shown in Fig. 2, this is still a planar graph. Furthermore, ▲ cannot be coloured any of the 4 colours used to colour K_4^0, K_4^1. We shall refer to the configuration in Fig. 2 as W_1 and ▲ as the *special* vertex of W_1.

We refer the reader to Fig. 3 for the illustration of the general gadget. To construct W_{n+1}, start by enumerating a large number of K_4 graphs and wait for the colouring c, to converge on these graphs. Once it does, there must be sufficiently many K_4 graphs all coloured the same 4 colours, which we may assume to be $0, 1, 2, 3$. Then extend the K_4 graphs (in the sense of Definition 2.4) into W_1 graphs by pairing them up and enumerating the special vertices.

Notation 2.6. Let $l \leq 4$ and $i \in \mathbb{N}$ be given. We write $K_l(W_i)$ to denote l many W_i graphs, where the special vertices of the W_i graphs are connected as a K_l graph. For convenience, we also use W_0 to denote a single vertex; $K_4(W_0)$ is isomorphic to K_4.

In addition, extend the resulting W_1 graphs into $K_4(W_1)$ graphs by connecting the special vertices of four W_1 graphs as a K_4. The idea here is that since the special vertex of each W_1 cannot be coloured any of $0, 1, 2, 3$, then the new $K_4(W_1)$ configuration forces the colouring to use the next four colours.

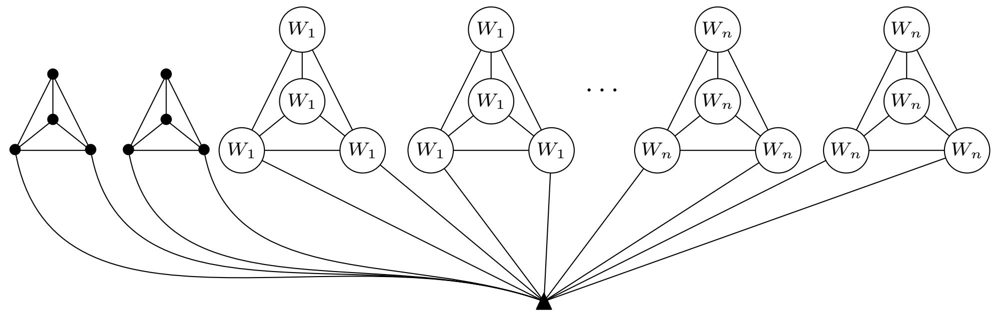

FIGURE 3. W_{n+1}

In general, once the colouring has converged on the vertices of all the $K_4(W_n)$ graphs, we extend it to W_{n+1} by enumerating a special vertex ▲ and connecting it to two copies each of $K_4, K_4(W_1), \dots, K_4(W_n)$ as shown in Fig. 3. We may assume that each pair of $K_4(W_i)$ are coloured with $4i, 4i+1, 4i+2$, and $4i+3$ (by pidgeonhole principle). Then by connecting ▲ the special vertices of these various $K_4(W_i)$, each coloured $4i, 4i+1, 4i+2, 4i+3$, the colouring cannot use any of the first $4n$ many colours to colour ▲. Now take four copies of W_{n+1} which are coloured the same $4n$ colours (except the special vertex) and connect the special vertices as a K_4 graph to form $K_4(W_{n+1})$. Evidently, for a fixed computable k-colouring φ, W_n cannot be coloured by φ for any $n \geq \lceil k/4 \rceil$.

Notation 2.7. Since W_n will generally be defined based on some colouring, to emphasise this, we will write the function used to define W_n in the superscript. In particular, if φ is a k-colouring, then for any $n \geq \lceil k/4 \rceil$, W_n^{φ} cannot be coloured by φ.

Theorem 2.8. For each $n \geq 4$, $\mathtt{RCA}_0 \nvdash \mathtt{COL}(n)$.

Sketch of proof. Let some $n \geq 4$ be given and let $\{\varphi_e\}_{e \in \mathbb{N}}$ be a listing of the computable functions $\varphi_e : \mathbb{N} \to \{0, 1, \dots, n-1\}$. Once again we aim to construct a graph G satisfying the conditions below.

$$R_e : \text{If } \varphi_e \text{ is total, then } \varphi_e \text{ is not a colouring of } G.$$

The graph will be constructed in components where each component aims to satisfy a single R_e. The action on the e^{th} component will depend solely on the behaviour of φ_e.

Fix some $e \in \mathbb{N}$. In the e^{th} component, we aim to construct a $W_m^{\varphi_e}$ where m is the least such that $m \geq \lceil n/4 \rceil$. Since we know exactly how many $K_4, K_4(W_1), \dots, K_4(W_{m-1})$ are needed, the required starting amount of K_4 graphs can be found recursively. Begin by enumerating a sufficiently large number of K_4 graphs into the e^{th} component. Proceed to construct $W_m^{\varphi_e}$ as described previously; wait for φ_e to converge on the vertices in $K_4(W_i^{\varphi_e})$ before extending them to $W_{i+1}^{\varphi_e}$.

If φ_e is total, it must converge on every vertex within the e^{th} component. Then we must succeed in constructing $W_m^{\varphi_e}$. However, in order for φ_e to successfully colour $W_m^{\varphi_e}$, $4m+1$

colours are needed. By choice of m, we obtain that $4m+1 \geq k+1 > k$; φ_e cannot be a computable k-colouring. □

Remark 2.9. By changing the requirements to

$$R_{\langle e,n\rangle} : \text{ If } \varphi_e \text{ is total, then } \varphi_e \text{ is not an } n\text{-colouring of } G,$$

essentially the same ideas can show that $\mathtt{RCA}_0$ does not prove the principle "every planar graph G has some n-colouring". However, the techniques used in this paper generally depend on the number of colours allowed, and might not extend easily to a proof of the same results for this principle. We leave the question of the axiomatic strength of this principle open.

2.2. **Reversing the 4 colour theorem.** To show that $\mathtt{COL}(4)$ and $\mathtt{WKL}_0$ are equivalent over $\mathtt{RCA}_0$, we first that show $\mathtt{WKL}_0 \vdash \mathtt{COL}(4)$. Going through the proof of the four colour theorem in [14], one can verify that the proof can be done with at most Σ^0_1-induction and thus $\mathtt{RCA}_0$ proves that every finite planar graph is 4-colourable. Thus, the remaining ingredient to complete the proof of $\mathtt{COL}(4)$ would be the ability to extend colourings of finite planar graphs to obtain $\mathtt{COL}(4)$. This can be done using the following theorem.

Theorem 2.10. (De Bruijn-Erdös theorem) Given a graph G, if all finite subgraphs of G are k-colourable, then G is k-colourable.

Remark 2.11. For our purposes, restricting G to be countable and planar is sufficient.

The first ingredient required in the proof of $\mathtt{WKL}_0 \vdash \mathtt{COL}(4)$ (and in fact for $\mathtt{COL}(n)$), would thus be that Theorem 2.10 holds in $\mathtt{WKL}_0$.

Theorem 2.12 (Hirst, [10])**.** Over $\mathtt{RCA}_0$, the following are equivalent:

- Weak König's lemma,
- De Bruijn-Erdõs theorem.

Theorem 2.13. Over $\mathtt{RCA}_0$, the following are equivalent:

- $\mathtt{WKL}_0$,
- $\mathtt{COL}(4)$.

Proof. It is known that $\mathtt{WKL}_0$ proves De Brujin-Erdös theorem. Putting this together with the fact that $\mathtt{RCA}_0$ produces a 4-colouring for every finite planar graph, we obtain that $\mathtt{WKL}_0$ produces a 4-colouring for every countable planar graph.

All that remains is the reversal. We reason in $\mathtt{RCA}_0$ and assume that every countable planar graph has a 4-colouring. Let $T \subseteq 2^{<\mathbb{N}}$ be an infinite tree. We construct a countable planar graph, $G = \langle V, E\rangle$ in stages as follows. An illustration of G can be found in Fig. 4. The vertex set V will contain the following type of vertices.

- $W = \{w_\tau, v_\tau \mid \tau \in 2^{<\mathbb{N}}\}$
- $F = \{f_\tau \mid \tau \in 2^{<\mathbb{N}}\}$.
- $N = \{n_\tau \mid \tau \in 2^{<\mathbb{N}}\}$.
- A, some infinite set of vertices to be used in the construction.

The τ-th, $\tau0$-th and $\tau1$-th component. Acting in the σ^{th} component.

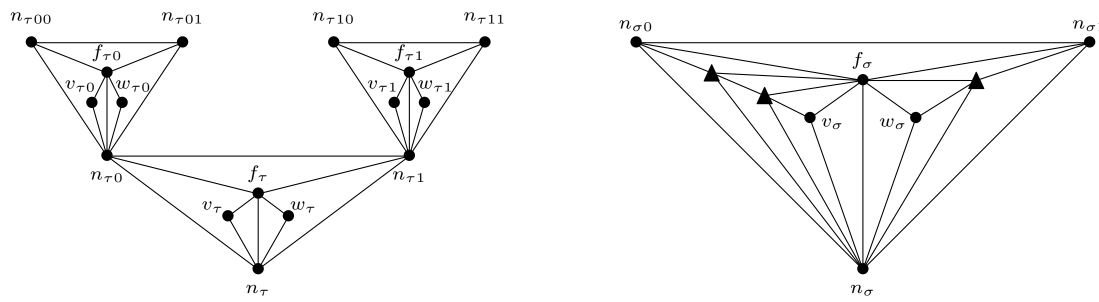

FIGURE 4. Reversal of COL(4).

Stage 0: Enumerate $f_\epsilon, w_\epsilon, v_\epsilon, n_\epsilon, n_0$ and n_1. Connect them as shown in Fig. 4 (take $\tau = \epsilon$).

Stage $s > 0$: For each $\tau \in 2^{<\mathbb{N}}$ of length s, enumerate the vertices $n_{\tau0}, n_{\tau1}, f_\tau, v_\tau$, and w_τ, and connect them as shown in Fig. 4. We shall refer to this as the τ-th component of the graph.

We say that the σ-th component has *acted* when a vertex from A has been enumerated and connected to the vertices in the σ-th component. We once again refer the reader to Fig. 4 for an illustration, where the vertices from A are denoted by the triangular vertices. Furthermore, once the σ-th component has acted, we also say that it has *chosen* σx if there is only one vertex from A between w_σ or v_σ and $n_{\sigma x}$. As an example, the σ-th component in Fig. 4 has chosen $\sigma 1$.

For convenience, we assume that at each stage, we compute the value of $T(\tau)$ for the next τ. If at stage s, τx is found to be in the complement, then pick the maximal prefix $\sigma \subseteq \tau$ such that the σ-th component has yet to act. Let y be such that $\sigma y \subseteq \tau x$ and act for the σ-th component, choosing $\sigma^\frown(1-y)$.

Since each (diagonalisation) component is clearly planar and they are connected as shown in Fig. 4, the graph as defined above is connected and planar. Applying Proposition 2.2 allows us to conclude that whenever some τx is discovered to be in the complement of T, the maximal prefix $\sigma \subseteq \tau$ that has yet to act can always be found. Furthermore, if the σ-th component acts and kills σy, then the subtree above σy is finite.

By COL(4), there is a colouring $c : V \to \{0, 1, 2, 3\}$ of G. We extract a path h through T using c as follows. Let $\tau_i = h \restriction i$.

- If $c(v_{\tau_i}) = c(n_{\tau_i}0)$, then let $h(i) = 0$.
- Otherwise, let $h(i) = 1$.

We proceed by induction on i to verify that $\tau_i \in T$ for every i. Base case is trivial since $\tau_0 = \epsilon$. Suppose inductively that $\tau_i \in T$ and consider the following cases.

Case 1: $h(i) = 0$. If the τ_i-th component never acts, since $\tau_i \in T$, then both subtrees extending $\tau_i 0$ and $\tau_i 1$ must be infinite (by Proposition 2.2). We may thus assume for a contradiction that the τ_i-th component has acted and kills $\tau_i 0$; there are two vertices from A between v_{τ_i} and $n_{\tau_i 0}$. However, by definition of h, we also have that $c(v_{\tau_i}) = c(n_{\tau_i 0})$. This leads to a contradiction since c has only one colour to colour the two adjacent vertices from A between v_{τ_i} and n_{τ_i}.

Case 2: $h(i) = 1$. Once again, we may assume for a contradiction that the τ_i-th component acts and kills $\tau_i 1$. In other words, there is exactly one vertex from A between v_{τ_i} and $n_{\tau_i 0}$ (the τ-th component chooses $\tau_i 0$). Since $h(i) = 1$, then $c(v_{\tau_i}) \neq c(n_{\tau_i 0})$; the vertex from A between v_{τ_i} and $n_{\tau_i 0}$ is connected to four vertices all coloured different colours by c. Therefore, c cannot be a valid colouring of G.

Thus, h must be a path through T. Since h depends only on c, then h exists by Δ^0_1-comprehension as desired. ☐

Remark 2.14. Notice that the graph G constructed in the proof of the theorem above has a computable plane diagram and is also connected. Therefore, the principles $\mathtt{ConnCOL}(4)$ and $\mathtt{COL}^*(4)$ are also able to produce a 4-colouring of G.

Corollary 2.15. Over $\mathtt{RCA}_0$, the following are equivalent:

- $\mathtt{WKL}_0$,
- $\mathtt{COL}(4)$,
- $\mathtt{ConnCOL}(4)$,
- $\mathtt{COL}^*(4)$.

2.3. **Reversing to $\mathtt{DNR}(k)$.** As discussed in Section 2.1, allowing the colouring access to more than 4 colours adds some level of complexity to the required proof. In particular, the proof as presented for Theorem 2.13 will not work if the principle is $\mathtt{COL}(5)$. The gadget used to obtain the reversal for $\mathtt{COL}(4)$ mainly uses the property that 4 colours are required to colour K_4. However, a graph that requires 5 colours to colour is obviously not going to be planar. In order to obtain reversals for the weaker colouring principles ($n > 4$), we consider the following principles.

Definition 2.16. The following definitions can be made in $\mathtt{RCA}_0$. Let $\{\varphi_x\}_{x\in\mathbb{N}}$ be a listing of the partial computable functions.

- $\mathtt{DNR}$ is the principle:
$$\forall f,\, \exists g : \mathbb{N} \to \mathbb{N} \text{ such that } \forall x,\, g(x) \neq \varphi^f_x(x).$$
- $\mathtt{DNR}(k)$ is the principle:
$$\forall f,\, \exists g : \mathbb{N} \to \{0, 1, 2, \dots, k-1\} \text{ such that } \forall x, g(x) \neq \varphi^f_x(x).$$

It is known that $\mathtt{WKL}_0$ and $\mathtt{DNR}(2)$ are equivalent (uniformly) over $\mathtt{RCA}_0$ [11]. As noted earlier, allowing more colours gives the 'opponent' more options to colour the gadgets that we use. In order to obtain the reversals for the colouring principles in general, we instead reverse to $\mathtt{DNR}(k)$ which are known to be equivalent to $\mathtt{WKL}_0$ (non-uniformly). Before we prove the general case for $n \geq 4$, we attempt to find the strongest principle $\mathtt{DNR}(k)$ that we are able to obtain a uniform reversal for in the cases $n = 5, 6, 7$. The results of this section can be found in Fig. 5; the arrows $P \to Q$ represent implication over $\mathtt{RCA}_0$ uniformly

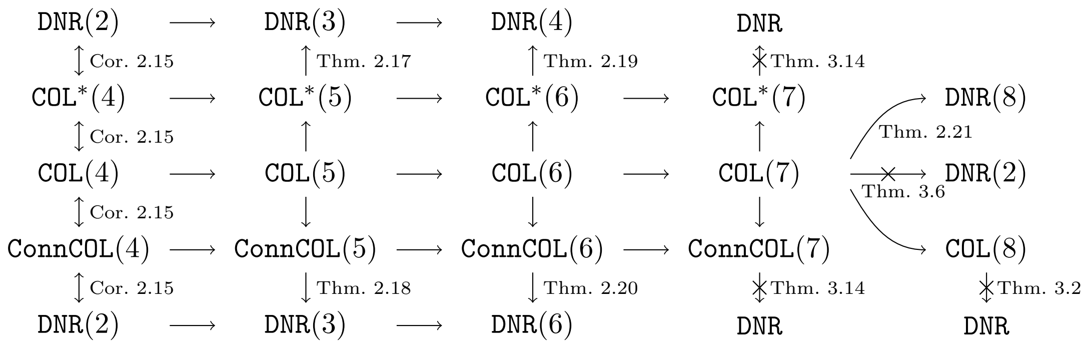

Figure 5. Hierarchy of Principles

(in fact $Q \leq_{sW} P$, to be defined later), while a crossed out arrow represents that a non-uniform proof is necessary (see Section 3). Since $\mathtt{DNR}(2), \mathtt{DNR}(3), \mathtt{DNR}(6), \mathtt{DNR}(8)$ are known to be equivalent to $\mathtt{WKL}_0$, this allows us to obtain reversals for $\mathtt{COL}(7)$ and all the colouring principles for $n \leq 6$.

Theorem 2.17. $\mathtt{RCA}_0 \vdash \mathtt{COL}^*(5) \to \mathtt{DNR}(3)$.

Proof. We reason in $\mathtt{RCA}_0$. Fix some f, and let $\left\{\varphi_e^f\right\}_{e\in\mathbb{N}}$ be an enumeration of the f-computable functions. G will consist of *components*, each consisting of six K_3 graphs originally, labelled K_3^j for each $j < 6$. At each stage s, check if $\varphi_e^f(e)[s]\downarrow$. If $\varphi_e^f(e)[s]\uparrow$, then do nothing. Otherwise, enumerate vertices into the e^{th} component and connect them as illustrated in Fig. 6, where $\circ$ represents the newly enumerated vertices, and the numbered K_3 graphs correspond to K_3^j for $j < 6$.

G as constructed above is clearly planar, and has a computable planar drawing, and thus by $\mathtt{COL}^*(5)$ has a 5-colouring c. For convenience, in the e^{th} component, define $A_e = \{c(v) \mid v \in K_3^j \text{ for } j < 3\}$ and $B_e = \{c(v) \mid v \in K_3^j \text{ for } 3 \leq j < 6\}$. Then define g as follows.

- Define $g(e) = 0$ if $|A_e| = 3$ or $|B_e| = 3$.

  We now claim that if this is the case, $g(e) \neq \varphi_e^f(e)$. Suppose otherwise, then $\varphi_e^f(e) = g(e) = 0$, and hence it must be that one of the sets A_e or B_e has size 3. Referring to Fig. 6, observe that in either case, c cannot possibly extend to a 5-colouring of the e^{th} component.
- Define $g(e) = 1$, if $A_e = B_e$ and $|A_e| = |B_e| = 4$.

  Once again, we check that $g(e) \neq \varphi_e^f(e)$. Suppose to the contrary that $\varphi_e^f(e)\downarrow = g(e) = 1$. Then the two new vertices v, w enumerated in this case cannot be coloured any of the 4 colours in A_e, B_e. However, as v is also adjacent to w, c cannot colour both v, w with the single remaining colour.
- Define $g(e) = 2$ if both of the above cases do not hold.

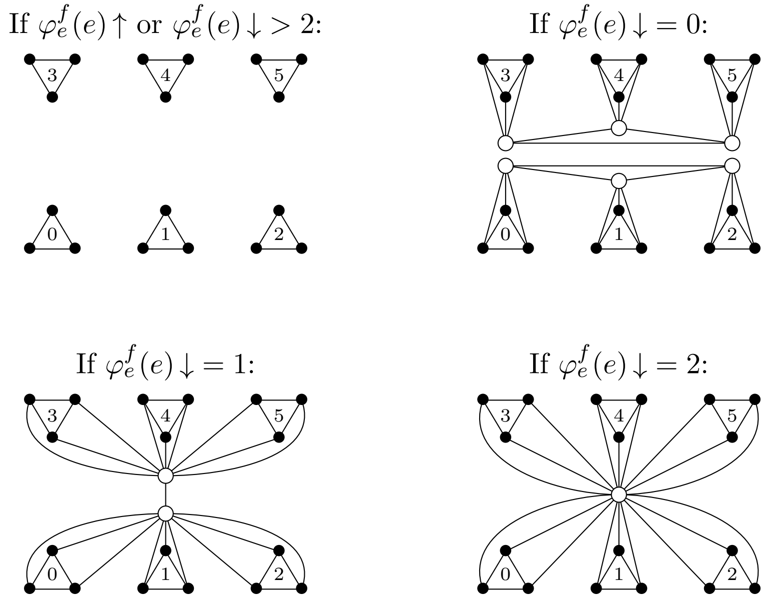

Figure 6. The gadget for Theorem 2.17.

Assume for a contradiction that $\varphi_e^f(e) = g(e) = 2$, then a single vertex v must have been enumerated into the e^{th} component. Furthermore, this vertex is connected to all other vertices in the e^{th} component. But c is assumed to have already used up all 5 colours to colour the original vertices in the e^{th} component, leaving it no colour for v.

Thus g as defined above satisfies $\mathtt{DNR}(3)$ and must exist by Δ^0_1-comprehension; we are able to compute the sets A_e, B_e using c as an oracle. □

To obtain the reversal for $\mathtt{ConnCOL}(5)$, we simply need to modify the gadget in a way that makes it connected. For each K_3 graph enumerated as the initial part of a gadget, we enumerate a single vertex and connect it to a vertex of the K_3. Now connect all of these new vertices as an infinite path. Observe that this results in a planar connected graph. As a result, $\mathtt{ConnCOL}(5)$ can therefore produce a 5-colouring of this new graph and g can be defined in the same way as before. We thus obtain the following as a corollary.

Theorem 2.18. $\mathtt{RCA}_0 \vdash \mathtt{ConnCOL}(5) \rightarrow \mathtt{DNR}(3)$.

We now address the case for $n = 6$. The key ingredients in the proof for the case $n = 5$ was to have some base configuration that can be extended (recall Definition 2.4) into three different graphs with the following property. Any colouring of the extended graphs, when restricted back to the original configuration, results in different colourings. This allows us to successfully avoid the value of $\varphi_e^f(e)$ by considering the colouring of the base configuration. We adopt a similar idea for the remaining cases. For $\mathtt{COL}^*(6)$, we use a

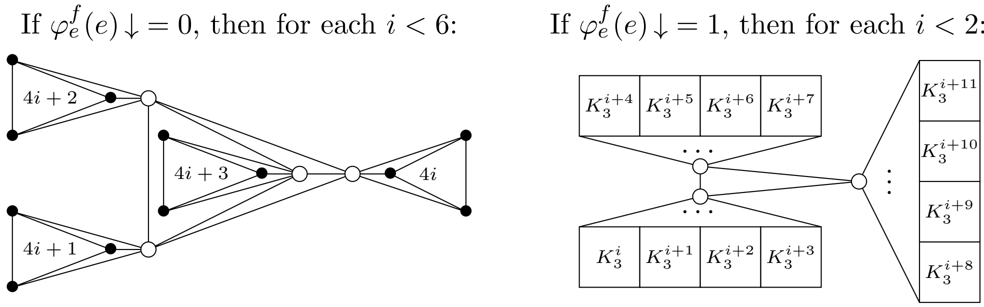

FIGURE 7. The gadget for Theorem 2.19. We use '$\cdots$' to replace the edges to avoid cluttering the diagram.

base configuration that can be extended into four different graphs such that any colouring of the base configuration fails to extend to at least one of the four graph extensions. We specify the details below.

Theorem 2.19. $\mathtt{RCA}_0 \vdash \mathtt{COL}^*(6) \to \mathtt{DNR}(4)$.

Proof. We reason in $\mathtt{RCA}_0$. Fix some f, and let $\left\{\varphi_e^f(e)\right\}_{e\in\mathbb{N}}$ be an enumeration of the f-computable functions. Once again, G will be made up of *components*, each of which now contains 24 K_3 graphs, denoted as K_3^j for each $j < 24$.

At each stage s, do the following.

Stage s**:** Enumerate the new s^{th} component by enumerating 24 new K_3 graphs into G. For each $e < s$, which is not yet declared satisfied, do the following.

- If $\varphi_e^f(e)[s]\downarrow = 0$, then enumerate 24 new vertices, v_j, for each $j < 24$, into the e^{th} component. For each j, connect v_j to all vertices in K_3^j, and then for each $i < 6$, connect the vertices v_j where $4i \leq j < 4(i+1)$ in a K_4 graph. (See Fig. 7 for an illustration; the labels of the K_3 graphs correspond to j and the newly enumerated vertices are represented by $\circ$.)
- If $\varphi_e^f(e)[s]\downarrow = 1$, then enumerate 6 new vertices, v_j for each $j < 6$, into the e^{th} component and connect v_j to all vertices of K_3^i for each $4j \leq i < 4(j+1)$. Then connect v_0, v_1, v_2 and v_3, v_4, v_5 as two K_3 graphs respectively. (See Fig. 7 for an illustration.)
- If $\varphi_e^f(e)[s]\downarrow = 2$, then enumerate 2 new vertices, v_0, v_1 into the e^{th} component. Connect v_0 to v_1 and all vertices in K_3^j for each $j < 12$, and connect v_1 to all vertices in K_3^j for each $j \geq 12$.
- If $\varphi_e^f(e)[s]\downarrow = 3$, then enumerate 1 new vertex v_0 into the e^{th} component and connect v_0 to all vertices in each K_3^j for every $j < 24$.

In any of the cases, once the new vertices are enumerated, we declare e to be satisfied.

The graph G as described above is easily seen to be planar and having a computable plane diagram. Thus, a 6-colouring c may be obtained by applying $\texttt{COL}^*(6)$. Using c, define define the following sets (via Δ^0_1-comprehension using c). Fix some $e \in \mathbb{N}$,

- Let $A_i = \{c(v) \mid v \in K_3^j \text{ for some } 4i \leq j < 4(i+1)\}$ for each $i < 6$.
- Let $B_i = \{c(v) \mid v \in K_3^j \text{ for some } 12i \leq j < 12(i+1)\}$ for each $i < 2$.

We are now ready to define $g(e)$ satisfying $\texttt{DNR}(4)$ as follows.

- If there is some i for which $|A_i| = 3$, then define $g(e) = 0$.

  Suppose now for a contradiction that $\varphi_e^f(e)\downarrow = g(e) = 0$. By the construction, vertices v_j for each $4i \leq j < 4(i+1)$ must have been enumerated into the e^{th} and component and each v_j is connected to all the vertices in K_3^j. Since $|A_i| = 3$ and c is a colouring, we have that $c(v_j) \notin A_i$ for each $4i \leq j < 4(i+1)$. But the four vertices $v_{4i}, v_{4i+1}, v_{4i+2}, v_{4i+3}$ are connected as a K_4 graph. Then c cannot possibly be a valid 6-colouring.
- If the previous case does not hold and there is some i such that $|B_i| = 4$, then define $g(e) = 1$.

  Once again, suppose that $\varphi_e^f(e)\downarrow = g(e) = 1$. Since the previous case does not hold, then it must be that for each $j < 6$, we have that $|A_j| \geq 4$. An analysis of the second case of the construction would allow us to conclude that the 6 new vertices enumerated, v_j for each $j < 6$ is such that $c(v_j) \notin A_j$. In particular, since $A_j \subseteq B_i$ for each $3i \leq j < 3(i+1)$, and $|B_i| = 4$, then $c(v_j) \notin B_i$ for each $3i \leq j < 3(i+1)$. Then c must fail as a 6-colouring as the vertices $v_{3i}, v_{3i+1}, v_{3i+2}$ are connected as a K_3.
- If neither of the previous cases hold and $B_0 = B_1$, then define $g(e) = 2$.

  Assume that $\varphi_e^f(e)\downarrow = 2$. Then the 2 new vertices v_0, v_1 enumerated into the e^{th} component are respectively connected to all vertices in K_3^j for each $j < 12$ and K_3^l for each $12 \leq l < 24$. In particular, $c(v_0) \notin B_0$ and $c(v_1) \notin B_1$. Since the previous cases all do not hold and $B_0 = B_1$, then $|B_0| = |B_1| \geq 5$. c cannot possibly be a 6-colouring.
- If none of the previous cases hold, then define $g(e) = 3$.

  If none of the previous cases hold, then $|B_0|, |B_1| \geq 5$ and $B_0 \neq B_1$. This means that $|B_0 \cup B_1| = 6$. If $\varphi_e^f(e)\downarrow = 3$, then in the e^{th} component, a vertex v_0 must have been enumerated into the e^{th} component and connected to all other vertices within the component. In particular, $c(v_0) \notin B_0 \cup B_1$, and hence c fails to be a colouring.

Since the sets A_i and B_j can be computed using c, then by Δ^0_1-comprehension, g exists. Thus, $\texttt{RCA}_0 \vdash \texttt{COL}^*(6) \to \texttt{DNR}(4)$. □

Unfortunately, it is not clear how the proof above can be easily modified to make the resulting graph connected while retaining a computable plane diagram. The main issue lies in using K_4 as a subgraph of the components (which was not required in the proof of Theorem 2.17). We thus present the reversal for $\texttt{ConnCOL}(6)$ separately below.

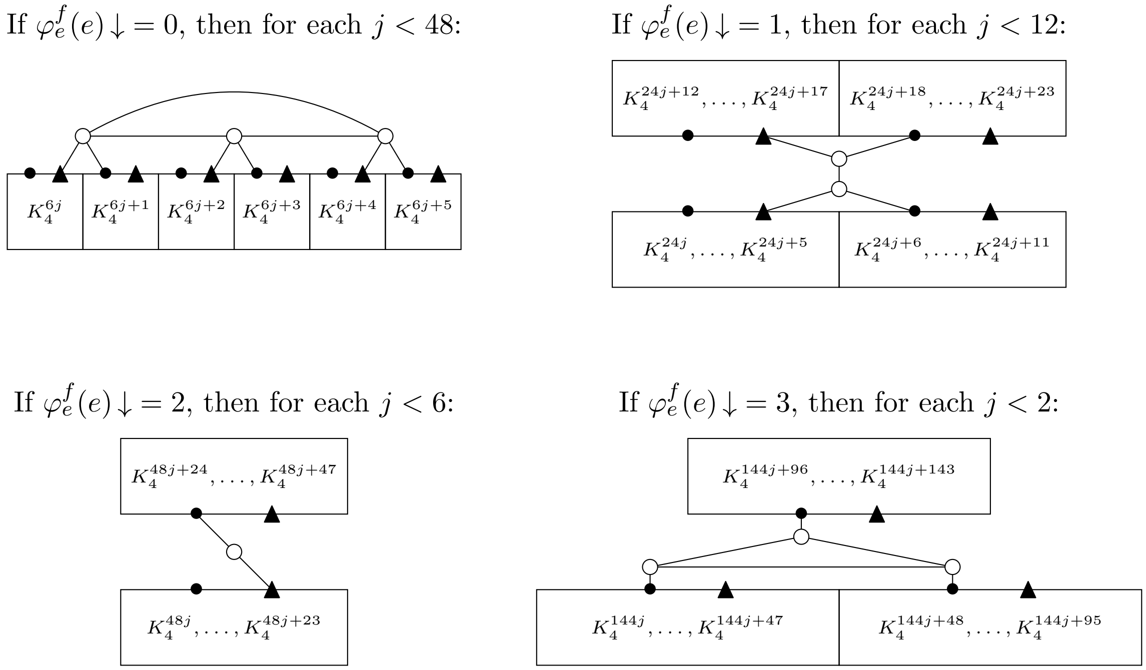

FIGURE 8. The gadget for Theorem 2.20. To reduce clutter, we use • and ▲ to represent all the non-special and special vertices of the K_4^m respectively.

Theorem 2.20. $\mathtt{RCA}_0 \vdash \mathtt{ConnCOL}(6) \rightarrow \mathtt{DNR}(6)$.

Proof. We reason in $\mathtt{RCA}_0$. Fix some f and let $\left\{\varphi_e^f(e)\right\}_{e\in\mathbb{N}}$ be an enumeration of the f-computable functions. Once again we construct the graph via components. Each component will consist of 288 K_4 graphs indexed with $j < 288$. For each K_4 graph, we pick an arbitrary vertex and label it the *special* vertex.

We proceed with the construction as follows.

Stage s**:** For each $e < s$ not yet declared satisfied, consider the following cases. (Refer to Fig. 8 for an illustration. The nodes $\circ$ represent the newly enumerated vertices whilst ▲ and • represent the special and non-special vertices respectively.)

- If $\varphi_e^f(e)[s]\downarrow = 0$, then for each $j < 48$, enumerate vertices $v_{3j}, v_{3j+1}, v_{3j+2}$ connected as a K_3 graph. In addition, for each $i < 3$, connect each v_{3j+i} to the special vertex of K_4^{6j+2i} and the non-special vertices of $K_4^{6j+2i+1}$.
- If $\varphi_e^f(e)[s]\downarrow = 1$, then for each $j < 12$, enumerate vertices v_{2j}, v_{2j+1} and connect v_{2j} to v_{2j+1}. We also connect v_{2j} to all the special vertices of K_4^{24j+i} and all the non-special vertices of $K_4^{24j+i+6}$ for each $i < 6$. Similarly, connect v_{2j+1} to all the special vertices of $K_4^{24j+i+12}$ and all the non-special vertices of $K_4^{24j+i+18}$ for each $i < 6$.

- If $\varphi_e^f(e)[s]\downarrow = 2$, then for each $j < 6$, enumerate a vertex v_j and for each $i < 24$, connect v_j to all the special vertices of K_4^{48j+i} and to all the non-special vertices of $K_4^{48j+i+24}$.
- If $\varphi_e^f(e)[s]\downarrow = 3$, then for each $j < 2$, enumerate $v_{3j}, v_{3j+1}, v_{3j+2}$ connected as a K_3 graph. Also connect v_{3j+i} to the non-special vertices of $K_4^{144j+48i+l}$ for each $l < 48$.
- If $\varphi_e^f(e)[s]\downarrow = 4$, then enumerate v_0, v_1 and connect them. Connect v_0 to all the non-special vertices of K_4^i for each $i < 144$ and connect v_1 to all the non-special vertices of K_4^j for each $144 \leq j < 288$.
- If $\varphi_e^f(e)[s]\downarrow = 5$, then enumerate a single vertex v_0 and connect it to all the non-special vertices of K_4^i for each $i < 288$.

If one of the above cases hold, then declare e satisfied.

To maintain connectedness of the graph described above, we can arrange the various K_4 graphs in a 'row', and for each K_4, enumerate some vertex and connect it to one of the outer vertices. Then connect these newly enumerated vertices as a path. It is clear that planarity is preserved and the resultant graph is now connected. Hence by `ConnCOL`(6), there is a 6-colouring c of G. It remains to prove that there exists a g satisfying `DNR`(6).

Using Δ_1^0-comprehension, define the following sets. Fix some e, and within the e^{th} component, for each $i < 288$, let

- $N_i = \{c(v) \mid v \in K_4^i \text{ and } v \text{ non-special}\}$,
- $S_i = \{c(v) \mid v \in K_4^i \text{ and } v \text{ special}\}$.

Now consider the following cases.

(1) If there exists some j such that $|\bigcup_{48j\leq i<48(j+1)} N_i| = 3$, then we consider the following subcases.
  (a) If there exists some i such that $8j \leq i < 8(j+1)$ and $|\bigcup_{l<6} S_{6i+l}| = 1$, then define $g(e) = 0$. In other words, for each $l < 6$, $N_{6i} = N_{6i+l}$ and $S_{6i} = S_{6i+l}$.
  (b) If the previous subcase does not hold and there exists some i such that $2j \leq i < 2(j+1)$ and $|\bigcup_{l<24} S_{24i+l}| = 2$, then define $g(e) = 1$. Since the previous subcase does not hold, then $|\bigcup_{l<6} S_{24i+6m+l}| > 1$ for each $m < 4$. In particular, it must be exactly 2. Therefore, we obtain that for each $m < 4$,

$$\bigcup_{l<6} S_{24i+l} = \bigcup_{l<6} S_{24i+6m+l} \text{ and } \bigcup_{l<6} N_{24i+l} = \bigcup_{l<6} N_{24i+6m+l}.$$

  (c) If neither of the previous subcases hold, then define $g(e) = 2$. In particular, it must be the case that for each i such that $2j \leq i < 2(j+1)$, $|\bigcup_{l<24} S_{24i+l}| = 3$ since c is a 6-colouring. Therefore,

$$\bigcup_{l<24} S_{48j+l} = \bigcup_{l<24} S_{48j+24+l} \text{ and } \bigcup_{l<24} N_{48j+l} = \bigcup_{l<24} N_{48j+24+l}.$$

Combining the conclusions above with the corresponding actions taken during the construction if $\varphi_e^f(e) = k \leq 2$ allows us to conclude that c cannot possibly extend to a 6-colouring if $g(e) = \varphi_e^f(e) = k \leq 2$.

(2) Suppose that the previous case does not hold; for every j, $|\bigcup_{48j \leq i < 48(j+1)} N_i| \geq 4$. Now consider the following subcases.

 (a) If in addition, there is some j for which $|\bigcup_{144j \leq i < 144(j+1)} N_i| = 4$, then define $g(e) = 3$. This implies that for each $m < 3$,

$$\bigcup_{l<48} N_{144j+l} = \bigcup_{l<48} N_{144j+48m+l}.$$

 (b) If the previous subcase does not hold and $|\bigcup_{j<288} N_j| = 5$, then define $g(e) = 4$. Note that we have

$$\bigcup_{l<144} N_l = \bigcup_{l<144} N_{144+l}.$$

 (c) Otherwise, define $g(e) = 5$. If neither of the previous subcases hold, we must have that $|\bigcup_{l<288} N_l| = 6$.

A simple analysis of the actions taken during the construction if $\varphi_e^f(e) = k \geq 3$ allows one to conclude that it cannot be that $g(e) = \varphi_e^f(e) = k \geq 3$.

Thus, g satisfying $\mathtt{DNR}(6)$ exists by Δ_1^0-comprehension. □

Observe that in the proof above, the graph constructed need not have a computable plane diagram. In particular, we cannot compute whether or not the special vertex in each K_4 should be embedded as the 'inner' or one of the 'outer' vertices in a standard drawing of K_4.

When $n = 7$, the principles separate even more in the sense that we are only able to obtain a uniform proof of $\mathtt{RCA}_0 \vdash \mathtt{COL}(7) \to \mathtt{DNR}(k)$ for some k. For the other principles, in particular $\mathtt{RCA}_0 \vdash \mathtt{ConnCOL}(7) \to \mathtt{DNR}(k)$ and $\mathtt{RCA}_0 \vdash \mathtt{COL}^*(7) \to \mathtt{DNR}(k)$, a non-uniform proof is necessary (see Theorem 3.14).

Theorem 2.21. $\mathtt{RCA}_0 \vdash \mathtt{COL}(7) \to \mathtt{DNR}(8)$.

Proof. Reasoning in $\mathtt{RCA}_0$, fix some f and let $\left\{\varphi_e^f\right\}$ be an enumeration of the f-computable functions. We adopt a similar gadget as before and construct the graph G in diagonalisation components. Each component will contain 9216 K_4 graphs (indexed in the superscript).

Now construct G as follows.

Stage s: For each $e < s$ not yet declared satisfied, do the following.

- If $\varphi_e^f(e)[s]\downarrow = 0$, then for each $j < 1152$, enumerate vertices $4j + i$ for each $i < 4$ connected as a K_4 into the e^{th} component. For each $i < 4$, connect each v_{4j+i} to the special vertex of K_4^{8j+2i} and the non-special vertices of $K_4^{8j+2i+1}$.

- If $\varphi_e^f(e)[s]\downarrow = 1$, then for each $j < 192$, enumerate vertices $3j+i$ for each $i < 3$ connected as a K_3. For each $i < 3$, and $l < 8$, connect each v_{3j+i} to all the special vertices of $K_4^{48j+16i+l}$ and the non-special vertices of $K_4^{48j+16i+8+l}$.
- If $\varphi_e^f(e)[s]\downarrow = 2$, then for each $j < 48$, enumerate vertices v_{2j}, v_{2j+1} and connect v_{2j} to v_{2j+1}. For each $i < 2$, and $l < 48$, connect v_{2j+i} to all the special vertices of $K_4^{192j+96i+l}$ and all the non-special vertices of $K_4^{192j+96i+48+l}$.
- If $\varphi_e^f(e)[s]\downarrow = 3$, then for each $j < 24$, enumerate a vertex v_j into the e^{th} component. In addition, for each $l < 192$, connect v_j to all the special vertices of K_4^{384j+l} and all the non-special vertices of $K_4^{384j+192+l}$.
- If $\varphi_e^f(e)[s]\downarrow = 4$, then for each $j < 6$, enumerate vertices v_{4j+i} for each $i < 4$ connected as a K_4. In addition, for each $i < 4$ and $l < 384$, connect v_{4j+i} to all the non-special vertices of $K_4^{1536j+384i+l}$.
- If $\varphi_e^f(e)[s]\downarrow = 5$, then for each $j < 2$, enumerate vertices v_{3j+i} for each $i < 3$ connected as a K_3. Also connect for each $i < 3$ and $l < 1536$, v_{3j+i} to all the non-special vertices of $K_4^{4608j+1536i+l}$.
- If $\varphi_e^f(e)[s]\downarrow = 6$, then enumerate vertices v_0 and v_1 adjacent to each other. For each $l < 4608$, connect v_0, v_1 to all the non-special vertices of K_4^l and K_4^{4608+l} respectively.
- If $\varphi_e^f(e)[s]\downarrow = 7$, then enumerate a single vertex v_0 and connect it to all the non-special vertices of K_4^l for each $l < 9216$.

We note here that if we were to make a similar modification to the gadgets above by attempting to connect them as before, the graph will no longer be planar. In fact there should be no easy modification of the gadget which would make the proof work for $\mathtt{RCA}_0 \vdash \mathtt{ConnCOL}(7) \to \mathtt{DNR}(k)$ for some k (as proved formally in Theorem 3.14). Similarly, this graph G also might not have a computable plane diagram since we are unable to compute if a special vertex is the 'inner' or 'outer' vertex in a standard drawing of K_4.

It follows from the construction above that the graph is computable and planar. Thus, by $\mathtt{COL}(7)$, there exists a 7-colouring c of the graph. In order to define $g(e)$, we define the following sets using the behaviour of c in the e^{th} component.

- $N_i = \{c(v) \mid v \in K_4^i \text{ and } v \text{ non-special}\}$,
- $S_i = \{c(v) \mid v \in K_4^i \text{ and } v \text{ special}\}$.

Consider the following possibilities.

(1) First suppose that there exists j such that $|\bigcup_{384j \le i < 384(j+1)} N_i| = 3$. We have the following subcases.
  (a) If there also exists i such that $48j \le i < 48(j+1)$ and $|\bigcup_{l<8} S_{8i+l}| = 1$, then define $g(e) = 0$. In this subcase, it is easy to see that for each $l < 8$,
$$S_{8i} = S_{8i+l} \text{ and } N_{8i} = N_{8i+l}.$$
  (b) If the previous subcase does not hold and there exists i such that $8j \le i < 8(j+1)$ and $|\bigcup_{l<48} S_{48i+l}| = 2$, then define $g(e) = 1$. Since the previous

subcase does not hold, for each $m < 6$, $|\bigcup_{l<8} S_{48i+8m+l}| > 1$ and thus must be exactly 2. Therefore, for each $m < 6$,

$$\bigcup_{l<8} S_{48i+l} = \bigcup_{l<8} S_{48i+8m+l} \text{ and } \bigcup_{l<8} N_{48i+l} = \bigcup_{l<8} N_{48i+8m+l}.$$

(c) If both the previous subcases does not hold and there exists i such that $2j \leq i < 2(j+1)$ and $|\bigcup_{l<192} S_{192i+l}| = 3$, then define $g(e) = 2$. Since Case (1b) does not hold, then for each $m < 4$, $|\bigcup_{l<48} S_{192i+48m+l}| > 2$. That is, it must be exactly 3. We thus obtain that for each $m < 4$,

$$\bigcup_{l<48} S_{192i+l} = \bigcup_{l<48} S_{192i+48m+l} \text{ and } \bigcup_{l<48} N_{192i+l} = \bigcup_{l<48} N_{192i+48m+l}.$$

(d) If none of the previous subcases hold, then define $g(e) = 3$. Since we also have that c is a 7-colouring, then for each i such that $2j \leq i < 2(j+1)$, $|\bigcup_{l<192} S_{192i+l}| = 4$. Which is to say that

$$\bigcup_{l<192} S_{384j+l} = \bigcup_{l<192} S_{384j+192+l} \text{ and } \bigcup_{l<192} N_{384j+l} = \bigcup_{l<192} N_{384j+192+l}.$$

Once again, using the conclusions obtained above and some careful analysis of the cases in the construction, we can conclude that if $g(e) = \varphi_e^f(e) = k \leq 3$, c fails to 7-colour the e^{th} component.

(2) Now suppose that for each j, $|\bigcup_{384j \leq i < 384(j+1)} N_i| \geq 4$. We consider the following subcases.

(a) If in addition, there exists j such that $|\bigcup_{l<1536} N_{1536j+l}| = 4$, then define $g(e) = 4$. With the assumption made in Case (2), we obtain the following. For each $m < 4$

$$\bigcup_{l<384} N_{1536j+l} = \bigcup_{l<384} N_{1536j+384m+l}.$$

(b) If there exists j such that $|\bigcup_{l<4608} N_{4608j+l}| = 5$ and the previous subcase does not hold, then define $g(e) = 5$. Using a similar analysis as before, for each $m < 3$,

$$\bigcup_{l<1536} N_{4608j+l} = \bigcup_{l<1536} N_{4608j+1536m+l}.$$

(c) If $|\bigcup_{l<9216} N_l| = 6$ and neither of the previous subcases hold, then define $g(e) = 6$. Since the assumption in Case (2b) fails, then

$$\bigcup_{l<4608} N_l = \bigcup_{4608 \leq l < 9216} N_l.$$

(d) Otherwise, define $g(e) = 7$. It is evident that in this subcase, $|\bigcup_{l<9216} N_l| = 7$.

A similar analysis as before can be done to conclude that if $g(e) = \varphi_e^f(e) = k \geq 4$, c must fail to be a 7-colouring of the e^{th} component.

Therefore, g satisfying $\texttt{DNR}(8)$ exists by Δ^0_1-comprehension. □

2.4. **Reversing the n colour theorem.** As suggested in Fig. 5, when $n = 7, 8$, the colouring principles seem to weaken considerably as compared to $n < 7$. In fact, $\texttt{COL}(8)$ does not even reverse uniformly to any $\texttt{DNR}(k)$ principle (Theorem 3.2). In order to obtain the reversal to $\texttt{WKL}_0$ over $\texttt{RCA}_0$, we use the following.

Definition 2.22. The following definitions can be made in $\texttt{RCA}_0$.

- A trace $\{T_n\}_{n\in\mathbb{N}}$ is a sequence of sets. In $\texttt{RCA}_0$, we can interpret such an object as an effective listing of the indexes of the c.e. sets required.
- Given a function h, a trace is h-bounded if for each n, $|T_n| \leq h(n)$.
- A trace $\{T_n\}_{n\in\mathbb{N}}$ is a c.e. trace if there exists some computable g such that $T_n = W_{g(n)} = \{x \mid \varphi_{g(n)}(e)\downarrow\}$ for all n. A universal c.e. trace is a c.e. trace where every c.e. set A is equal to T_n for some n.

As usual, the definition above can be relativised to any oracle f.

Definition 2.23. The following definitions can be made in $\texttt{RCA}_0$.

- A function g is approximated by $\tilde{g}$ iff for every x, $\lim_{s\to\infty} \tilde{g}(x, s) = g(x)$.
- We say that $\tilde{g}$ is a l-approximation of g if $\tilde{g}$ approximates g and

$$|\{s \mid \tilde{g}(x, s) \neq \tilde{g}(x, s + 1)\}| < l + 1.$$

Definition 2.24. Let $\{T_x\}_{x\in\mathbb{N}}$ be the universal $l + 1$-bounded c.e. trace, then $\texttt{DNR}(k, l)$ is the principle:

$$\forall f,\ \exists \tilde{g} : \mathbb{N}^2 \to \{0, 1, 2, \dots, k - 1\} \text{ an } l\text{-approximation such that } \forall x,\ \lim_{s\to\infty} \tilde{g}(x, s) \notin T^f_x.$$

Recall that in Section 2.1, in order to diagonalise against the n-colourings when n is large, we use a 'layered' gadget. More specifically, we wait for the colouring to first commit on some initial part of the gadget which has been revealed before extending the gadget. Each extension forces the colouring to use up more of its available colours before we finally obtain a diagonalisation. This lends itself nicely into the idea behind using l-approximable functions; roughly speaking, each 'layer' in the gadget will correspond to a stage at which the approximation changes. In order to obtain the reversal of $\texttt{COL}(n)$ over $\texttt{RCA}_0$ to $\texttt{WKL}_0$, we will first reverse $\texttt{COL}(n)$ to some $\texttt{DNR}(k, l)$ and reverse $\texttt{DNR}(k, l)$ to $\texttt{WKL}_0$. As such, we first prove that for each k, l, $\texttt{RCA}_0 \vdash \texttt{DNR}(k, l) \to \texttt{WKL}_0$.

Theorem 2.25. $\texttt{RCA}_0 \vdash \texttt{DNR}(k, 1) \to \texttt{DNR}(k)$.

Proof. Fix some function f and an enumeration of the computable functions $\{\varphi_e\}_{e\in\mathbb{N}}$. Let $\left\{T^f_x\right\}_{x\in\mathbb{N}}$ be a universal 2-bounded c.e. trace. Define p a computable function such that

$$T^f_{p(e,x)} = \begin{cases} \emptyset, & \text{if } \varphi^f_x(x)\uparrow \\ \left\{\varphi^f_x(x)\right\}, & \text{if } \varphi^f_x(x)\downarrow \wedge \varphi^f_e(e)\uparrow \\ \left\{\varphi^f_x(x), \varphi^f_e(e)\right\}, & \text{if } \varphi^f_x(x)\downarrow \wedge \varphi^f_e(e)\downarrow \end{cases}$$

By $\mathtt{DNR}(k,1)$, there exists some $g:\mathbb{N}\to\{0,1,\ldots,k-1\}$ such that for all e,x, $g(p(e,x))\notin T^f_{p(e,x)}$, and g has a computable 1-approximation $\tilde{g}$. Define g^* (non-uniformly) satisfying $\mathtt{DNR}(k)$ as follows.

Case 1: If there is some e such that for every x, $\tilde{g}(p(e,x),0)\neq\varphi^f_x(x)$, then define $g^*(x)=\tilde{g}(p(e,x),0)$. g^* clearly exists via Δ^0_1-comprehension and it is evident that $g^*(x)\downarrow\neq\varphi^f_x(x)$ for every x.

Case 2: If for every e, there is some x such that $\tilde{g}(p(e,x),0)=\varphi^f_x(x)$, then define $g^*(e)=\tilde{g}(p(e,x),s_e)$ where s_e is the first stage at which $\tilde{g}(p(e,x),s_e)\neq\tilde{g}(p(e,x),0)$.

Since $\tilde{g}$ is a 1-approximation to g (satisfying $\mathtt{DNR}(k,1)$), then there must be some stage at which $\tilde{g}(p(e,x),s)\neq\tilde{g}(p(e,x),0)$. Otherwise $\tilde{g}(p(e,x),s)=\tilde{g}(p(e,x),0)=\varphi^f_x(x)\downarrow$ at every stage s which contradicts the assumption that $g(p(e,x))\notin T^f_{p(e,x)}$. Since s must exist, then there is always a least one and thus g^* exists by Δ^0_1-comprehension.

Suppose now for a contradiction that $\varphi^f_e(e)\downarrow=g^*(e)=\tilde{g}(p(e,x),s_e)$. Since $\tilde{g}$ is a 1-approximation to g, that means that $g(p(e,x))=\tilde{g}(p(e,x),s_e)$. Then we have that $g(p(e,x))=\varphi^f_e(e)\downarrow$ a contradiction. Thus g^* satisfies $\mathtt{DNR}(k)$.

In any case, we have that g^* exists and satisfies $\mathtt{DNR}(k)$. □

Theorem 2.26. For any $l>1$, $\mathtt{RCA}_0\vdash\mathtt{DNR}(k,l+1)\to\mathtt{DNR}(k)\vee\mathtt{DNR}(k,l)$.

Proof. Fix some function f and an enumeration of the computable functions $\{\varphi_e\}_{e\in\mathbb{N}}$. Let $\left\{T^f_x\right\}_{x\in\mathbb{N}}$ and $\{U^f_x\}_{x\in\mathbb{N}}$ be a $l+1$-bounded and $l+2$-bounded universal c.e. trace respectively. Define p a computable functions such that

$$U^f_{p(e,x)}=\begin{cases}\emptyset, & \text{if } \varphi^f_x(x)\uparrow\\ \left\{\varphi^f_x(x)\right\}\cup T^f_e, & \text{if } \varphi^f_x(x)\downarrow\end{cases}$$

By $\mathtt{DNR}(k,l+1)$, there must exists g such that for all e,x, $g(p(e,x))\notin U^f_{p(e,x)}$ with a computable $l+1$-approximation $\tilde{g}$. We proceed non-uniformly and aim to define a function h which satisfies $\mathtt{DNR}(k)\vee\mathtt{DNR}(k,l)$.

Case 1: If there is some e such that for every x, $\tilde{g}(p(e,x),0)\neq\varphi^f_x(x)$, then define $h(x)=\tilde{g}(p(e,x),0)$. h clearly exists via Δ^0_1-comprehension and satisfies $\mathtt{DNR}(k)$.

Case 2: If for every e, there is some x such that $\tilde{g}(p(e,x),0)=\varphi^f_x(x)$, then define $h(e)=\lim_{s\to\infty}\tilde{g}(p(e,x),s+s_e)$ where s_e is the first stage at which $\tilde{g}(p(e,x),s_e)\neq\tilde{g}(p(e,x),0)$. By assumption that g satisfies $\mathtt{DNR}(k,l+1)$, then $g(p(e,x))\notin U^f_{p(e,x)}$. That is, $h(e)=g(p(e,x))\notin T^f_e$. It remains thus to show that there is some computable $\tilde{h}$ which is an l-approximation of h.

For similar reasons as before, s_e exists and thus by defining $\tilde{h}(e,s)=\tilde{g}(p(e,x),s+s_e)$, $\tilde{h}$ exists via Δ^0_1-comprehension. Since we have that $\tilde{g}$ is a $l+1$-approximation and $\tilde{g}(p(e,x),s_e-1)\neq\tilde{g}(p(e,x),s_e)$, then it is clear that $\tilde{h}$ is a l-approximation.

Thus we have that the h defined satisfies either $\mathtt{DNR}(k)$ or $\mathtt{DNR}(k,l)$. □

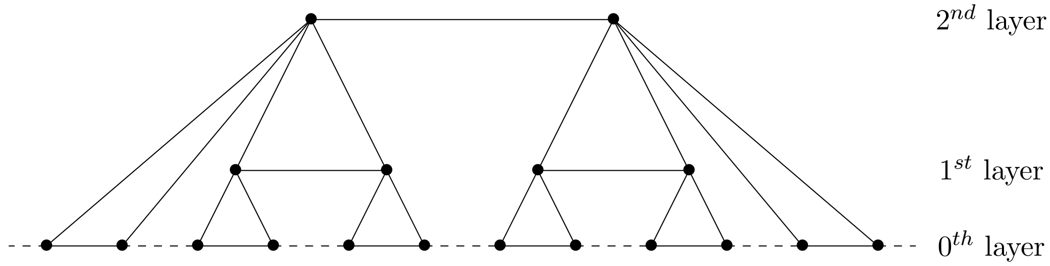

FIGURE 9. Reversal of $\texttt{COL}(n)$. The dashed lines represent some number of vertices and edges; the 0^{th} layer is a path.

Corollary 2.27. For any k, l, $\texttt{RCA}_0 \vdash \texttt{DNR}(k, l) \rightarrow \texttt{DNR}(k)$

We are now ready to prove the main result of this section.

Theorem 2.28. Over $\texttt{RCA}_0$, the following are equivalent

- Weak König's lemma,
- $\texttt{COL}(n)$, $\texttt{ConnCOL}(n)$, $\texttt{COL}^*(n)$.

Proof. We already have that $\texttt{WKL}_0 \vdash \texttt{COL}(n)$. Since $\texttt{COL}(n)$ trivially implies both $\texttt{ConnCOL}(n)$ and $\texttt{COL}^*(n)$, it remains to show the reversal. In order to do so, we show that for each n, $\texttt{ConnCOL}(n)$ proves $\texttt{DNR}(k_n, l_n)$ where k_n and l_n are constants dependent on n.

We define the following:

- $l_n = \left\lfloor \frac{n}{2} \right\rfloor - 1$.
- $q_{n,l_n+1} = 2$, the number of vertices in the $l_n + 1$-th *layer* of a *component* (both properly defined in the construction).
- Proceed recursively for each $i < l_n + 1$ and define $q_{n,i-1}$ as follows. Let $p_{l_n+1} = 2$, and let $p_{i-1} = \sum_{j=i}^{l_n+1} 2p_j$. Then define $q_{n,i-1} = (p_{i-1} - 1)C(n, 2) + 1$, where $C(x, y)$ is the binomial coefficient. Observe that by pidgeonhole principle, $q_{n,i-1}$ is the number of pairs of vertices that when coloured with some valid n-colouring, at least p_{i-1} many pairs of vertices have the same two colours.
- Define $k_n = \max\{\langle i, p_i^* \rangle \mid i < l_n + 1\} + 1$, where p_i^* is an encoding of a possible p_i-tuple of pairs of vertices in the i^{th} layer (to be defined later).

Fix some function f and the l_n+1-bounded universal trace $\{T_e^f\}$. The rough idea is that in order to obtain the $\texttt{DNR}(k_n, l_n)$ function, we build the gadget in layers. Each time some new element is enumerated into the universal trace, we extend (in the sense of Definition 2.4) the gadget to the next layer, ensuring that any valid colouring encodes information regarding the element which just entered the universal trace. We now construct the graph in (diagonalisation) components as before.

Stage s: Begin the s^{th} component by enumerating $2q_{n,0}$ many vertices and connecting them as a path. These vertices shall be referred to as the 0^{th} layer of the s^{th} component.

For each $e < s$, let i be the least such that the i^{th} layer has been defined in the e^{th} component and do the following.

For each $t \in T_e^f[s]$, check if $t = \langle i, p_i^* \rangle$ for some p_i^*, an encoding of a p_i-tuple consisting of entries $< q_{n,i}$. If there are no $t \in T_e^f[s]$ which satisfy the desired condition, then do nothing and attend to the next $e < s$ or proceed to the next stage if there are no more $e < s$ to attend to.

If there is some $t_i \in T_e^f[s]$ such that $t_i = \langle i, p_i^* \rangle$ for some p_i^*, then enumerate new vertices v_j for each $j < 2q_{n,i+1}$ into the e^{th} component. We refer to these vertices as the $i+1$-th layer of the e^{th} component. For each $j < 2q_{n,i+1}$, and for each $r < i+1$, connect v_j to some pair of vertices encoded by p_r^* in the r^{th} layer, where $\langle r, p_r^* \rangle \in T_e^f[s]$ (see Fig. 9 for an illustration[2]). Finally, within the $i+1$-th layer, connect v_{2j} to v_{2j+1} for each $j < q_{n,i+1}$. Note that this procedure is well-defined as the $i+1$-th layer can only exist at the s^{th} stage if for each $r \leq i$, some element in $T_e^f[s]$ is of the form $\langle r, p_r^* \rangle$. By some careful arrangement of the connections as illustrated in Fig. 9, each component can be made planar.

Once we are done acting for each $e < s$, proceed to stage $s+1$.

It is easy to see that the graph can be made connected with a computable plane diagram. By the choice of $q_{n,i}$, there is always 'sufficient space' to connect each new layer to the previous ones.

Applying $\texttt{ConnCOL}(n)$, there is a n-colouring c of the graph G as described in the construction (we could also use $\texttt{COL}(n)$ or $\texttt{COL}^*(n)$ in place of $\texttt{ConnCOL}(n)$). Now we attempt to define a l_n-approximation $\tilde{g}$ satisfying $\texttt{DNR}(k_n, l_n)$.

To define an l_n-approximation, the intuitive idea is that each layer aims to 'block' off two colours and corresponds roughly to each change in the approximation. In the case that all the colours are used up, we then show that the final value must avoid the trace. We provide the details as follows.

Fix some $e \in \mathbb{N}$ and consider the colouring c within the e^{th} component. Define $\tilde{g}(e,0) = \langle 0, p_0^* \rangle$, where p_0^* is the encoding of the first $q_{n,0}$ many pairs in the 0^{th} layer of the e^{th} component that is coloured the same two colours by c. Without loss of generality, we may assume that these two colours are 0 and 1. Now suppose inductively that $\tilde{g}(e,s)$ is some pair $\langle i, p_i^* \rangle$ where p_i^* is an encoding of a p_i-tuple with entries $< q_{n,i}$. Furthermore, this p_i-tuple encoded by $\tilde{g}(e,s)$ is coloured $2i, 2i+1$ by c and the i^{th} layer exists in the e^{th} gadget. We now aim to define $\tilde{g}(e, s+1)$.

If $\tilde{g}(e,s) \notin T_e^f[s+1]$, then let $\tilde{g}(e,s+1) = \tilde{g}(e,s)$. In particular, observe that the desired property is satisfied. If we instead have that $\tilde{g}(e,s) \in T_e^f[s+1]$, then some element t must have entered T_e^f at stage $s+1$ and must be equal to $\langle i, p_i^* \rangle$ (the value of $\tilde{g}(e,s)$). When this happens in the construction, the $i+1$-th layer consisting of $q_{n,i+1}$ many pairs of

[2]The dashed lines in the figure represent some unknown number of nodes and edges. The nodes chosen to be connected to higher layers are based solely on the action of the c.e. trace as described.

vertices must have been enumerated into the e^{th} component. Furthermore, each vertex in the $i+1$-th layer is connected to some pair of vertices in each layer $\leq i$, which is encoded by some $t_r = \langle r, p_r^* \rangle \in T_e^f[s+1]$ for each $r \leq i$ as described in the construction. By the inductive hypothesis, this means that each vertex in the $i+1$-th layer is adjacent to vertices coloured $0, 1, \ldots, 2i, 2i+1$. In order for c to be a valid colouring, none of the vertices in the $i+1$-th layer can be coloured with the first $2i+2$ colours. In addition, the vertices in the $i+1$-th layer are also connected as pairs, and by pidgeonhole principle, there are at least p_{i+1} many pairs coloured the same two colours. Without loss of generality, we may assume that these colours are $2i+2, 2i+3$ as desired.

Since we are defining $\tilde{g}$, in order to make it a l_n-approximation, we simply stop changing its value once it has used up all l_n many changes. Now suppose for a contradiction that $\lim_{s\to\infty} \tilde{g}(e,s) \in T_e^f$. Let $s_0 = 0$ and s_i be the least stage such that $\tilde{g}(e, s_{i-1}) \neq \tilde{g}(e, s_i)$. Since $\tilde{g}$ is an l_n-approximation, we obtain that

$$\{s \mid \tilde{g}(e, s-1) \neq \tilde{g}(e,s)\} = \{s_i \mid 1 \leq i \leq l_n\}.$$

By choice of s_i, we also have that $\tilde{g}(e, s_i) = \langle i, p_i^* \rangle$ for each i. This implies that

$$T_e^f = \{s_i \mid 0 \leq i \leq n\},$$

because $\tilde{g}(e,s) \neq \tilde{g}(e, s-1)$ only if $\tilde{g}(e, s-1) \in T_e^f[s]$. We also have that $\tilde{g}(e, s_i) = \langle i, p_i^* \rangle$ encodes p_i many pairs of vertices in the i^{th} layer coloured with $2i, 2i+1$. Consider the l_n+1-th layer (consisting of a pair of vertices) in the e^{th} component. Each vertex in this layer is connected to a pair of vertices from each previous layer; in order for c to be a valid colouring, the vertices in the l_n+1-th layer cannot be coloured with colours $0, 1, \ldots, 2l_n+1$. However, by choice of l_n, we obtain that $2l_n + 1 \geq n-2$. Then c has only one remaining colour left to colour the two adjacent vertices in the l_n+1-th layer. Thus, it must be that $\lim_{s\to\infty} \tilde{g}(e,s) \notin T_e^f$. Furthermore, by choice of k_n, it is also evident that for each s, $\tilde{g}(e,s) < k_n$. Therefore, $\tilde{g}$ satisfies $\mathtt{DNR}(k_n, l_n)$. □

Under the framework of reverse mathematics, Theorem 2.28 implies the theorems in Section 2.3. However, in this more general reversal of the colouring principles, we note that the proof is rather inefficient with the parameters. In particular, for the cases of $n = 5, 6, 7$, one can easily compute that we reverse the principles $\mathtt{COL}(n)$, $\mathtt{ConnCOL}(n)$, and $\mathtt{COL}^*(n)$ to some $\mathtt{DNR}(k_n, l_n)$ for $l_n \geq 1$, and a rather large k_n. In order to reverse these principles to $\mathtt{WKL}_0$ over $\mathtt{RCA}_0$, we utilise non-uniform proofs as presented in Theorems 2.25 and 2.26. In some sense, the proofs presented in Section 2.3 are 'closer' to uniform reversals to $\mathtt{WKL}_0$ than the proof presented above. In order to analyse the level of (non-)uniformity of the reversals, we apply the tools of Weihrauch reductions to investigate where this non-uniformity is necessary.

3. Weihrauch Reducibility

In this section, we shall attempt to show where non-uniformity is necessary. Indeed, we shall succeed in doing so for the principles $\mathtt{COL}(n)$, $\mathtt{ConnCOL}(n)$ and $\mathtt{COL}^*(n)$ whenever $n \geq 7$. We leave the cases for $n = 5, 6$ open.

The following definition is from [7], following ideas from [18, 2].

Definition 3.1. Let P and Q be Π^1_2 statements of second-order arithmetic. We say that

- P is Weihrauch reducible to Q, $P \leq_W Q$, if there exists Φ, Ψ where Φ, Ψ are Turing reductions such that whenever A is an instance of P, $B = \Phi(A)$ is an instance of Q and if T is a solution to B, then $S = \Psi(T \oplus A)$ is a solution of P.
- P is strongly Weihrauch reducible to Q, $P \leq_{sW} Q$, if there exists Φ, Ψ where Φ, Ψ are Turing reductions such that whenever A is an instance of P, $B = \Phi(A)$ is an instance of Q and if T is a solution to B, then $S = \Psi(T)$ is a solution of P.

It follows from the definition that certain parallels can be drawn between Weihrauch reducibility and reverse mathematics. If $P \leq_W Q$, then in most cases $\mathtt{RCA}_0 \vdash Q \to P$. The uniformity of proofs can be formalised via the notion of Weihrauch reductions as follows. $P \leq_W Q$ represents a uniform way of proving $Q \to P$ if they are Π^1_2 sentences. Conversely, if $P \not\leq_W Q$, then $Q \to P$ cannot be proved in a uniform way.

Referring to the proofs presented in Sections 2.2 and 2.3, one can easily obtain the following positive results summarised in Fig. 5.

- $\mathtt{COL}(4), \mathtt{ConnCOL}(4), \mathtt{COL}^*(4)$ and $\mathtt{WKL}_0$ are all strongly Weihrauch equivalent.
- $\mathtt{DNR}(3) \leq_{sW} \mathtt{COL}^*(5)$ and $\mathtt{DNR}(3) \leq_{sW} \mathtt{ConnCOL}(5)$.
- $\mathtt{DNR}(4) \leq_{sW} \mathtt{COL}^*(6)$.
- $\mathtt{DNR}(6) \leq_{sW} \mathtt{ConnCOL}(6)$.
- $\mathtt{DNR}(8) \leq_{sW} \mathtt{COL}(7)$.

The rest of this section will be dedicated to providing proofs of the negative results presented in Fig. 5, and comparing the (strong) Weihrauch degrees of the different colouring principles.

3.1. Basic colouring principles.

We first analyse the uniformity of reductions between the $\mathtt{DNR}$ principles and $\mathtt{COL}$ principles. Recall (Definition 2.4) that we use $G \subseteq G'$ to denote that G is an induced subgraph of G', equivalently, we say G' is an extension of G. We shall continue to use this notation for the rest of this section.

Theorem 3.2. $\mathtt{DNR} \not\leq_W \mathtt{COL}(8)$.

Proof. Suppose to the contrary that $\mathtt{DNR} \leq_W \mathtt{COL}(8)$. Let $f = 0^\omega$ and for each s, let $f_s = 0^s$. Then there exists some Φ such that $\Phi(f)$ produces (the code) of some countable computable planar graph G. Furthermore, there is some Ψ such that for any 8-colouring h of G, $\Psi(h \oplus f)$ satisfies $\mathtt{DNR}$.

Define $\psi^f(x)$ as follows. Search for an s and σ such that σ is a 4-colouring of $\Phi(f_s) \subseteq G$ and $\Psi(\sigma \oplus f_s)(x)[s]\downarrow$. Then $\psi^f(x)$ outputs the value of the computation. Since $G_s = \Phi(f_s)$ is computable in f, and also a finite graph, then for each s, we can search through all possible 4-colourings, σ, of G_s and check if $\Psi(\sigma \oplus f_s)(x)[s]\downarrow$. By the recursion theorem, we may assume that the index of ψ is x, i.e. $\psi^f = \varphi^f_x$. We claim that $\psi^f(x)\downarrow$. Since $G = \Phi(f)$ is planar, there should exists some 4-colouring g of $\Phi(f)$. Then, there must be some s such that $\Psi(g \upharpoonright s \oplus f_s)(x)[s]\downarrow$ as $\Psi(g \oplus f)$ is assumed to be total provided g is an 8-colouring of $\Phi(f)$. Thus, $\psi^f(x)$ will converge by the time it finds $g \upharpoonright s$.

Evidently, the σ found by $\psi^f(x)$ might not extend to a 4-colouring of G; in particular σ need not be $g \upharpoonright s$. In order to preserve the computation $\Psi(\sigma \oplus f_s)(x)$, we instead extend σ to an 8-colouring of G as follows. Since $G \setminus G_s$ is also planar, then there exists some 4-colouring c of $G \setminus G_s$. It follows that

$$h(v) = \begin{cases} \sigma(v), & \text{if } v \in G_s \\ c(v) + 4, & \text{otherwise} \end{cases}$$

is an 8-colouring of G which extends σ. As a result, we obtain that

$$\varphi^f_x(x) = \psi^f(x) = \Psi(\sigma \oplus f_s)(x)[s]\downarrow = \Psi(h \oplus f)(x).$$

A contradiction to the assumption that $\Psi(h \oplus f)$ satisfies DNR for any 8-colouring h of $\Phi(f)$. □

Theorem 3.3. For any $n > 0$ and for any $i < 4$, $\texttt{COL}(4n + i) \not\leq_W \texttt{COL}(4(n + 1))$.

This theorem is a natural extension of Theorem 3.2 (take $i = 0$ and $n = 1$). We extend the ideas used in the proof earlier as follows. Define a graph G using the same gadgets as in Theorem 2.8, and colour some initial part of $H = \Phi(G)$ with some 4-colouring h. Once $g = \Psi(h \oplus G)$ has converged on the initial part of the gadget, then we extend the gadget in the same way as before, ensuring that g now has 4 less colours available to it to colour the rest of the gadget. In order to preserve this initial part of g, we then extend h to a 8 colouring. As long as we are always able to extend h 'more than' g, then g must fail as a colouring. We formalise this idea with the lemma below.

Lemma 3.4. Let H be a finite planar graph. If h is a k-colouring of H, then for any countable planar graph $H' \supseteq H$, there is some $k + 4$-colouring h' of H' such that $h' \supseteq h$.

Proof of Lemma 3.4. Let H be a finite planar graph with a k-colouring h. Suppose that H' is some countable planar graph with H as a subgraph. Since H' is planar, then so is $H' \setminus H$ and it must thus have a 4-colouring c. We then define a $k + 4$-colouring h' of H' as follows.

$$h'(v) = \begin{cases} h(v), & \text{if } v \in H \\ c(v) + k, & \text{if } v \notin H. \end{cases}$$

It is clear that $h' \supseteq h$ is a valid $k + 4$-colouring of H'. □

Proof of Theorem 3.3. Suppose to the contrary that $\texttt{COL}(4n + i) \leq_W \texttt{COL}(4(n + 1))$ for some $i < 4$. We aim to construct a graph G such that for some $4(n + 1)$-colouring h of $H = \Phi(G)$, $g = \Psi(h \oplus G)$ fails to be a $4n + i$-colouring of G. We will follow the proof of Theorem 2.8 closely. Begin by enumerating sufficiently many K_4 graphs to construct a W_m (as described in Section 2.1) where m is the least such that $m \geq \lceil (4n + i)/4 \rceil$.

During the construction, we will attempt to define some appropriate colouring σ of $H_s = \Phi(G_s)$ while waiting for $\Psi(\sigma \oplus G_s)$ to converge on certain vertices within G_s. For stages during which we are waiting, let G_{s+1} be the union of the graph G_s with a single isolated vertex. It is clear that if we wait forever, $\bigcup_s G_s$ is a countable planar graph and thus so is $H = \Phi\left(\bigcup_s G_s\right)$. Fix some 4-colouring h_0 of H. By the assumption that $\texttt{COL}(4n+i) \leq_W \texttt{COL}(4(n+1))$ is witnessed by Φ, Ψ, $g = \Psi(h_0 \oplus G)$ must be some $4n+i$ colouring of G. In particular, since we are only waiting for g to converge on the finitely many vertices of W_m enumerated thus far, there must be some finite stage at which it happens. Let this stage be s_0 and let G_{s_0+1} be the graph where the appropriate K_4 graphs (those coloured the same 4 colours) are extended into $K_4(W_1^g)$ graphs (recall Notation 2.6 and 2.7; W_1 depends on the colouring g). Furthermore, let $\sigma_0 = h_0 \upharpoonright \Phi(G_{s_0})$ be the finite initial segment such that $\Psi(\sigma_0 \oplus G_{s_0})$ converges on the required vertices. Obviously, for any extension $h \supseteq \sigma_0$ and $G \supseteq G_{s_0}$, we will have that $\Psi(h \oplus G)$ also extends $\Psi(\sigma_0 \oplus G_{s_0})$.

Suppose recursively that σ_l, s_l have been defined such that

- σ_l is a $4l$-colouring of $\Phi(G_{s_l})$, and
- s_l is the stage such that $\Psi(\sigma_l \oplus G_{s_l})$ converges on the finitely many vertices of W_m enumerated thus far.

Let G_{s_l+1} be the graph where the gadget W_m has been extended to the $l+1$-th layer (see Section 2.1) based on the colouring $\Psi(\sigma_l \oplus G_{s_l})$. Now let G be the union of G_{s_l+1} with countably many isolated vertices and consider $H = \Phi(G)$. By Lemma 3.4, there is a $4(l+1)$-colouring h_{l+1} of H extending σ_l. Clearly, as long as $l < m \leq n+1$, there exists some finite stage s_{l+1} such that $\Psi(\sigma_{l+1} \oplus G_{s_{l+1}})$ converges on the $l+1$-th layer of W_m, where $\sigma_{l+1} = h_{l+1} \upharpoonright \Phi(G_{s_{l+1}})$. Furthermore, on the vertices up to the l^{th} layer of W_m, $\Psi(\sigma_{l+1} \oplus G_{s_{l+1}})$ agrees with $\Psi(\sigma_l \oplus G_{s_l})$.

Repeat this procedure until W_m has been successfully constructed and let h_m be the $4m$-colouring of $\Phi(G)$ where G is the union of $G_{s_{m-1}+1}$ with countably many isolated vertices. Since $m \leq n+1$, then h_m is a $4(n+1)$-colouring of $H = \Phi(G)$. However, $\Psi(h_m \oplus G)$ has the property that for each layer j of W_m, $\Psi(h_m \oplus G)$ colours the vertices of the j^{th} layer of W_m in the same way as $\Psi(\sigma_j \oplus G_{s_j})$. In other words, $W_m = W_m^{\Psi(h_m \oplus G)}$ (recall Notation 2.7) and thus cannot be $(4n+i)$-coloured by $\Psi(h_m \oplus G)$. □

Considering the proofs of Theorems 2.17, 2.19, 2.21, we propose the following lemma that characterises the existence of a uniform reduction of a DNR principle to COL principle as a (finite) graph theoretic property.

Lemma 3.5. The following are equivalent.

(1) $\texttt{DNR}(k) \leq_{sW} \texttt{COL}(n)$.
(2) $\texttt{DNR}(k) \leq_W \texttt{COL}(n)$.
(3) There exists finite planar graph $G, G_0, G_1, \ldots, G_{k-1}$ such that for each $i < k$, $G_i \supseteq G$ and for any n-colouring h of G, there is some $i < k$ such that h does not extend to a n-colouring of G_i.

Proof. (1) implies (2) is obvious. We first prove the implication from (2) to (3). Suppose that $\mathtt{DNR}(k) \leq_W \mathtt{COL}(n)$. Then there exists Φ, Ψ such that for any oracle f, $\Phi(f) = H$ is a countable planar graph and given a n-colouring h of H, $\Psi(h \oplus f) = g$ satisfies $\mathtt{DNR}(k)$.

Consider a tree T defined as follows. T contains elements of the form $\langle \sigma, f \rangle$ where σ is a finite n-ary string and f is a finite binary string of the same length. In addition, the ordering between two elements is given by $\langle \sigma, f \rangle \leq \langle \sigma', f' \rangle$ iff $\sigma \subseteq \sigma'$ and $f \subseteq f'$. Now generate T recursively as follows.

- $\langle\rangle \in T$.
- If $\langle \sigma, f \rangle \in T$, then $\langle \tau, f^\frown j \rangle \in T$ provided $\tau \supseteq \sigma$ and τ is a valid n-colouring of $\Phi(f^\frown j)$.[3] We also require that τ has length the number of vertices enumerated by $\Phi(f^\frown j)$.

Given any n-colouring σ of $\Phi(f)$, there is only a finite number of colourings $\tau \supseteq \sigma$ that colours $\Phi(f^\frown j)$. Futhermore, such τ can be computably determined. Therefore, T is a computable finite branching tree. For a given e, define the subtree $T_e \subseteq T$ as containing all nodes $\langle \sigma, f \rangle \in T$ for which $\Psi(\sigma \oplus f)(e)[|f|]\uparrow$. Since T is finite branching, if $T_e \subseteq T$ is infinite, then there must be a path through T_e. Let this path be given by $\langle \sigma, f \rangle$, where f is an infinite binary string. Observe that σ, f, has the following properties.

- Since Φ is total, then $\Phi(f)$ must be a countable planar graph. Therefore, σ must also be of infinite length by the second condition in the definition of T. Furthermore, σ is an n-colouring of $\Phi(f)$.
- For each s, $\Psi(\sigma \oplus f)(e)[s]\uparrow$. That is, $\Psi(\sigma \oplus f)$ is not a total function and cannot possibly satisfy $\mathtt{DNR}(k)$.

This contradicts the assumption that Φ, Ψ witnesses $\mathtt{DNR}(k) \leq_W \mathtt{COL}(n)$. Thus T_e must be finite. In other words, there can only be finitely many nodes of T for which $\Psi(\sigma \oplus f)(e)[|f|]\uparrow$. Therefore, there is a finite s, such that for every f of length s, and every possible n-colouring σ of $\Phi(\tau)$, $\Psi(\sigma \oplus \tau)(e)[s]\downarrow$. Furthermore, we note that such an s can be found recursively in e. For any given input e, we denote such an s as s_e.

Consider the partial computable function $\psi^f(x)$ defined as follows. On any input x, ψ outputs the number of consecutive zeros between the first two non-zero bits of f. Let the index of ψ be e, that is, $\psi^f = \varphi_e^f$. Fix such an e and let $G = \Phi(0^{s_e})$. Note that there are only finitely many possible n-colourings of G. We list these colourings as $\sigma_0, \sigma_1, \ldots, \sigma_m$. For each $i < k$, let $r_i \geq 0$ be the least number satisfying the following. For any $l \leq m$, if $\Psi(\sigma_l \oplus 0^{s_e})(e)[s_e] = i$ (note that $\Psi(\sigma_l \oplus 0^{s_e})(e)[s_e]\downarrow$ by choice of s_e), then for any $\tau \supseteq \sigma_l$, $\langle \tau, 0^{s_e}10^i10^{r_i} \rangle \notin T$.

Suppose for a contradiction that r_i does not exist. That is, for every $j \in \omega$, there exists some $\tau \supseteq \sigma_l$ such that $\langle \tau, 0^{s_e}10^i10^j \rangle \in T$. Then, the subtree consisting of nodes $\langle \tau, 0^{s_e}10^i10^j \rangle$ must be infinite, and thus have a path, say $\langle \hat{\tau}, 0^{s_e}10^i10^\omega \rangle$, where $\hat{\tau} \supseteq \sigma_l$. As explained previously, since Φ is total, then $\hat{\tau}$ must be of infinite length and a n-colouring

[3] As usual, we assume that $\Phi(f^\frown j)$ only enumerates the vertices whose edge relations with existing vertices have been completely decided. Furthermore, note that it could be that $\Phi(f) = \Phi(f^\frown j)$, in which case $\langle \sigma, f^\frown j \rangle$ would also be in T.

of $\Phi(0^{s_e}10^i10^\omega)$. Hence, we obtain that

$$\Psi(\hat{\tau} \oplus 0^{s_e}10^i10^\omega)(e) = \Psi(\sigma_l \oplus 0^{s_e})(e)[s_e]\downarrow = i = \psi^{0^{s_e}10^i10^\omega}(e) = \varphi_e^{0^{s_e}10^i10^\omega}(e).$$

Contradicting the assumption that Φ, Ψ witnesses $\mathtt{DNR}(k) \leq_W \mathtt{COL}(n)$. Thus r_i must exist; there is some finite depth such that the subtree extending $\langle \sigma_l, 0^{s_e}10^i1\rangle$ is witnessed to be finite. For each $i < k$, define $G_i = \Phi(0^{s_e}10^i10^{r_i})$. Evidently, for any colouring h of G, $h = \sigma_l$ for some l. Let $\Psi(\sigma_l \oplus 0^{s_e})(e) = i$. Then h cannot extend to an n-colouring of G_i, otherwise there must be some τ such that $\langle \tau, 0^{s_e}10^i10^{r_i}\rangle \in T$ where $\tau \supseteq \sigma_l$.

It remains to prove the implication from (3) to (1). Suppose that there is some finite planar graph H, and finite planar extensions of H, $H_0, H_1, \ldots, H_{k-1}$ such that for any k-colouring h of H, there is some $i < k$ where h does not extend to a n-colouring of H_i. The idea is to build G in diagonalisation components, one for each e. Each component will initially consist of H and potentially be extended to one of the H_i. It is important to note that H and H_i are completely independent of f (the oracle to $\mathtt{DNR}(k)$), and thus we may fix (beforehand) some encoding of the initial vertices of H in the e^{th} component.

For any oracle f, define G as follows.

Stage s**:** Enumerate a new copy of H into G and refer to this as the *initial part* of the s^{th} component. If $\varphi_e^f(e)[s]\downarrow$ for some $e < s$ and the e^{th} component has not yet acted, then extend H to H_i provided $\varphi_e^f(e)[s] = i < k$. Once H has been extended to H_i, we say that the e^{th} component has acted.

Now let h be any n-colouring of G as defined above. In order to define $\Psi(h)(e)$, we consider the e^{th} component. Wait for h to converge on all the vertices of H in the e^{th} component. Once it does so, we know that there exists some i such that h cannot be extended to a n-colouring of H_i. Since there are only finitely many possible colourings of each H_j for $j < k$, we can find such an i computably. If there are multiple such i, pick the least one and let $\Psi(h)(e) = i$. Then $\Psi(h)(e)\downarrow \neq \varphi_e^f(e)$, otherwise h cannot be a n-colouring of G. In particular, it fails to colour the e^{th} component of G. □

Applying Lemma 3.5, we can easily obtain that for any planar graph G and any number of planar extensions $G_0, \ldots, G_k$, there is an 8-colouring of G (use only 4 colours) that can be extended to an 8 colouring of any G_i (Lemma 3.4). Thus, for any k, $\mathtt{DNR}(k) \not\leq_W \mathtt{COL}(8)$. This is not too surprising considering the stronger result in Theorem 3.2. We can however obtain an improvement to the result in Theorem 3.2, in the sense that the non-uniformity is necessary even for $\mathtt{RCA}_0 + \mathtt{COL}(7) \vdash \mathtt{WKL}_0$.

Theorem 3.6. $\mathtt{DNR}(2) \not\leq_W \mathtt{COL}(7)$.

Proof. By Lemma 3.5, it suffices to show that for any finite planar graph G, and finite planar extensions G_0, G_1, there exists g, a 7-colouring of G such that g extends to a 7-colouring of both G_0 and G_1. Since G_0 and G_1 are both finite planar extensions of G, then it must be that there are 4-colourings g_0, g_1 of G_0, G_1 respectively. Now we construct a

7-colouring, h, of G as follows.

$$h(v) = \begin{cases} g_0(v), & \text{if } g_0(v) \in \{0,1,2\} \\ g_1(v)+3, & \text{if } g_0(v) = 3 \end{cases}$$

It follows that $h(v) < 7$ for each v. We check that h is a 7-colouring of G. Let $u, v \in G$ be arbitrarily chosen such that $\{u,v\} \in E_G$. Suppose to the contrary that $h(u) = h(v)$. Then either $g_0(u) = g_0(v)$ or $g_1(u) + 3 = g_1(v) + 3$, in either cases, it implies that g_0 or g_1 is not a colouring of G, which cannot be the case. since $G \subset G_i$ for each $i < 2$. Thus h is a 7-colouring of G. It remains to show that h can be extended to a 7-colouring h_i of G_i, for each $i = 0, 1$.

$$h_0(v) = \begin{cases} h(v), & \text{if } v \in G \\ g_0(v), & \text{if } v \in G_0 \setminus G \end{cases}$$

and

$$h_1(v) = \begin{cases} h(v), & \text{if } v \in G \\ g_1(v)+3, & \text{if } v \in G_1 \setminus G \end{cases}$$

It remains to check that each h_i is a 7-colouring of G_i. We first check for $i = 0$. Let $u, v \in G_0$ such that $\{u,v\} \in E_{G_0}$. The case for $u, v \in G$ has already been checked, since h is a valid colouring of G. Also, if both $u, v \in G_0 \setminus G$, $h_0(u) = g_0(u) \neq g_0(v) = h_0(v)$. So we may assume without loss of generality that $u \in G$ and $v \in G_0 \setminus G$ with the property that $h_0(u) = h_0(v)$. Following the definition of h_0, since $v \in G_0 \setminus G$, then $h_0(v) < 4$. Then we must also have that $h_0(u) < 4$. It is evident that if $h_0(u) = 0, 1, 2$, then $h_0(u) = g_0(u)$. On the other hand, if $h_0(u) = 3$, then it can only be that $g_0(u) = 3$ and $g_1(u) = 0$. In particular, if $h_0(u) < 4$, then $h_0(u) = g_0(u)$. Therefore, we obtain $g_0(v) = h_0(v) = h_0(u) = g_0(u)$, contradicting the fact that g_0 is a 4-colouring of G_0.

For the case $i = 1$, let $u, v \in G_1$ be such that $\{u,v\} \in E_{G_1}$. Once again, if both $u, v \in G$ or both $u, v \notin G$, then it must be that $h_1(u) \neq h_1(v)$, otherwise g_0 or g_1 respectively fails to be a colouring. Thus, without loss of generality, we may assume that $u \in G$ and $v \notin G$ such that $h_1(u) = h_1(v)$. This implies that $h_1(u) = 3 = h_1(v)$. Applying the definition of h allows us to conclude that if $h(u) = 3$, then $g_1(u) = 0$. Since $h_1(v) = 3 = g_1(v) + 3$, we obtain $g_1(u) = g_1(v)$, contradicting the assumption that g_1 is a colouring.

Thus, for any G, G_0, G_1 where $G_i \supset G$ for each $i < 2$, there is some 7-colouring h of G such that h can be extended to a 7-colouring of G_0 and G_1. Then by Lemma 3.5, $\mathtt{DNR}(2) \not\leq_W \mathtt{COL}(7)$. □

3.2. **Extending colourings.** The proofs of Theorem 3.3 and Lemma 3.5 seem to suggest that the question of Weihrauch reducibility between the colouring principles and DNR boils down to the ability to extend colourings. The task now is to prove an analogous result as in Lemma 3.4 for the principles ConnCOL and COL*. It turns out that with the additional assumptions, we have a slightly stronger result for ConnCOL (Lemma 3.7) and COL* (Lemma 3.13).

Lemma 3.7. For a given finite connected planar graph G, and a countable connected planar graph $G' \supset G$, if c is a k-colouring ($k \geq 3$) of G, then there is some $\hat{c}$ a $k+3$-colouring of G' that extends c.

We follow the proof of Theorem 5 in [1] closely. In the paper, the author considers recursive colourings of connected highly recursive graphs; graphs where $N(v)$, the set of neighbours of a vertex v, are uniformly computable. The same technique can essentially be applied assuming that we have access to $N(v)$. We note here that every computable graph is $\emptyset'$-highly recursive. In any case, we do not require that the colourings c and $\hat{c}$ be effective in any way. One might wonder if the proof would work to show an analagous result for the basic colouring principle. Unfortunately, this technique seems to fail in the case that G has infinitely many disconnected components because we no longer are able to guarantee that the 'layers' (to be specified in the proof) are sufficiently disjoint.

Proof of Lemma 3.7. Let $G_0 = G$. Iteratively define G_{n+1} as the induced subgraph of G' such that $V_{G_{n+1}} = \bigcup_{v \in V_{G_n}} N(v) \cup G_n$. Also define $\overline{G_0} = G_0$, and $\overline{G_{n+1}} = G_{n+1} \setminus G_n$. Note that the vertex sets of the graphs $\overline{G_n}$ are mutually disjoint, and form a partition of G'. Furthermore, for any n,

- $N\left(V_{\overline{G_n}}\right) = V_{\overline{G_{n-1}}} \cup V_{\overline{G_{n+1}}}$
- $N\left(V_{\overline{G_n}}\right) \cap V_{\overline{G_m}} = \emptyset$ and $N\left(V_{\overline{G_m}}\right) \cap V_{\overline{G_n}} = \emptyset$, for any m where $|m-n| > 1$.

We will attempt to $k+3$-colour G' as follows. First we k-colour $G_0 = G$ using c. Now we 3-color each subsequent *layer* $\overline{G_n}$ where $n > 0$. If n is odd, then we use the colours $k, k+1, k+2$, otherwise we use the colours $0, 1, 2$. The properties listed above ensure that the even and odd layers are 'sufficiently' disjoint, and ensures that this strategy results in a $k+3$-colouring of G'. We provide the details below.

For each $n > 0$, define an equivalence relation on the vertices of G_n; $u \sim_n v$ iff $u, v \in G_{n-1}$. Then let H_n be the graph with vertices representing the distinct equivalence classes of G_n (denoted $\mathbf{u}$) and having edge relation $\{\mathbf{u}, \mathbf{v}\} \in E_{H_n}$ iff there exists $u \in \mathbf{u}$ and $v \in \mathbf{v}$ such that $\{u, v\} \in E_{G_n}$. H_n can then be seen as the graph $\overline{G_n}$ with an additional vertex say w_n (representing the equivalence class containing all the vertices of G_{n-1}) that is connected to all other vertices in $\overline{G_n}$. We first prove that H_n is indeed planar.

Suppose to the contrary that H_n is not planar, then H_n must have K_5 or $K_{3,3}$ as a minor. That is, there is some finite sequence of contractions that can be used to obtain a graph isomorphic to $K_{3,3}$ or K_5. If such a sequence of contractions do not include the vertex w_n, then we note that the same sequence of contractions can be done on G_n to show that G_n is not planar. We may thus suppose that the sequence of contractions does include w_n. In particular, there are two vertices $u, v \in \overline{G_n}$ such that $\{u, w_n\}, \{v, w_n\} \in E_{H_n}$. But that means that there exists $u^*, v^* \in G_{n-1}$ where $\{u, u^*\}, \{v, v^*\} \in E_{G_n}$. Since G_n is connected, then u^*, v^* must be connected via some finite path. In other words, by replacing the contraction which removes w_n with this finite sequence of contractions that removes the path between u^* and v^*, we are again able to obtain a graph that is isomorphic to $K_{3,3}$ or K_5, contradicting the assumption that G_n is planar.

Since H_n is planar, then H_n can be 4-coloured. Furthermore, in any colouring of H_n, w_n must be a different colour from any of the other vertices in $\overline{G_n}$. We may thus assume that the 4-colouring h of H_n assigns the colours $0, 1, 2$ to the vertices in $\overline{G_n}$ and the colour 3 to w_n. Then given a $k+3$-colouring g of G_{n-1}, we extend it to a $k+3$-colouring g' of G_n as follows.

$$g'(u) = \begin{cases} g(u), & \text{if } u \in G_{n-1} \\ k + h(u), & \text{if } u \notin G_{n-1} \text{ and } n \text{ is odd.} \\ h(u), & \text{if } u \notin G_{n-1} \text{ and } n \text{ is even.} \end{cases}$$

Then repeat the process inductively to obtain a colouring for G'. To see that it is indeed a $k+3$-colouring, we use the fact that for any n, for any $u \in \overline{G_n}$ and any $v \in \overline{G_{n+2}}$, $\{u, v\} \notin E$. Together with the fact that all vertices in the odd layers are coloured using $k, k+1, k+2$ while vertices in the even layers utilising colours $< k$, we obtain that the colouring defined is indeed a $k+3$-colouring. □

In order to prove a similar lemma for `COL`*, it should be clear that we have to utilise the plane diagram (recall Definition 1.2) in some way. The rough idea is to consider *good* colourings (to be defined later) of the graph and show that they can be extended. This class of colourings is characterised by their behaviour on the faces of a diagram. Since we need to consider the plane diagrams, a notion for diagram extensions is required.

Definition 3.8. Let D be a plane diagram of G. A plane diagram D' is an extension of D, written $D' \supseteq D$ if the following holds.

- D' is a plane diagram of $G' \supseteq G$ (recall this means that G is an induced subgraph of G').
- D' restricted to G is exactly D.

Intuitively, this means that if a diagram D has exhibited that two vertices are not adjacent, then no extension of D can add an edge between the two vertices. Using this property, once we know how D embeds the vertices and edges of G into $\mathbb{R}^2$, then we have a good idea about which vertices can later share some common neighbour. Using this information, we can be more efficient on the usage of colours.

Fact 3.9. Given a finite graph G with plane diagram D, $\mathbb{R}^2 \setminus D$ is a disjoint union $\bigsqcup_{i<m} U_i$ where each U_i is homeomorphic to an open disc with at most finitely many holes.

Proof sketch. Since D can be expressed as a union of finitely many compact line segments (each representing an edge) such that two distinct line segments possibly intersect only at the endpoints, an inductive argument can be done to show the desired result. We present some representative cases in Fig. 10. Each subdiagram corresponds to some configuration of how the $n+1$-th line segment could be positioned relative to one of the connected components of $\mathbb{R}^2$ without the first n many line segments. It follows that in each case, the hypothesis is still satisfied after removing the $n+1$-th line. □

Consider some extension $D' \supseteq D$. Let u, v be two vertices which are adjacent and are added only in D'. Applying Fact 3.9, we can conclude that u, v are both within the same

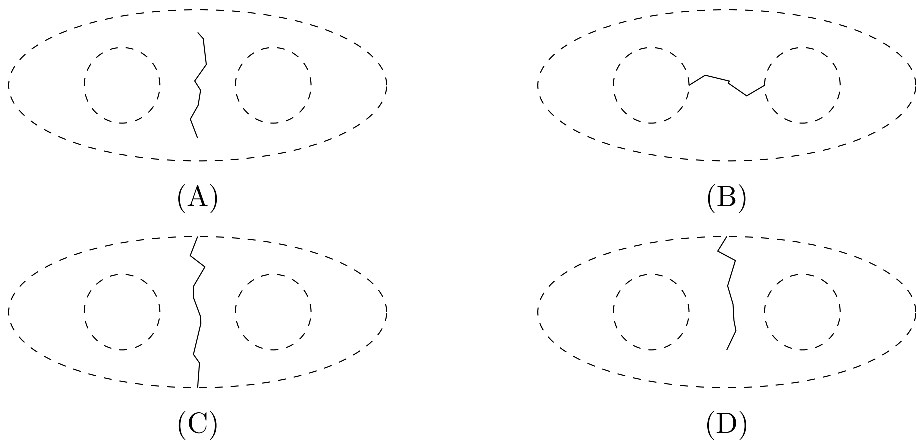

FIGURE 10. Some possible configurations of a line segment and a connected component.

connected component. Otherwise, D' cannot be a planar diagram as the embedding of the edge between u, v necessarily crosses some part of D. Intuitively, the connected components of D influence which vertices added later may be adjacent. This motivates the following definition.

Definition 3.10. Let G be a finite planar graph with plane diagram D and let $\{U_i\}_{i \leq m}$ be a partition of $\mathbb{R}^2 \setminus D$ such that each U_i is an open connected region. Then we say that c is a good $(3k+1)$-colouring of D if c is a $(3k+1)$-colouring of G and for each U_i, c uses $3k$ colours to colour all the vertices on ∂U_i. Note that the $3k$ colours can differ between different U_i.

In what follows, since we will be working mainly with the diagrams, we shall simply write $u \in X$ for some $X \subseteq \mathbb{R}^2$ to mean that u is embedded into X by D.

Lemma 3.11. For any finite planar graph G with plane diagram D, there is a good 4-colouring of D.

Proof. Let G be some planar graph with plane diagram D and let $\mathbb{R}^2 \setminus D = \bigsqcup_{i \leq m} U_i$. For each such open connected set, enumerate a vertex v_i and connect it to every vertex u on ∂U_i. We claim that this new graph G' is also planar.

Applying Fact 3.9, one easily obtains that for any point $v \in U_i$ and a finite collection of points $v_0, \ldots, v_l$ in ∂U_i, there are paths $p_0, \ldots, p_l : [0,1] \to \mathbb{R}^2$ such that the following holds.

- $p_j(0) = v$ and $p_j(1) = v_j$.
- Each p_j is injective and the sets $p_j((0,1))$ are pairwise mutually disjoint.

In particular, G' is planar as it has a plane diagram. Then G' has some 4-colouring c. It remains to check that $c \upharpoonright G$ is a good 4-colour of D. Fix some U_i, a connected component of $\mathbb{R}^2 \setminus D$. By construction of G', there is some vector v connected to all vertices u in ∂U_i. Since c is a 4 colouring, then c can only use 3 colours for the vertices in ∂U_i. □

Lemma 3.12. Let G be a finite planar graph with plane diagram D. Also let D' be some finite extension D' of D. If D has a good $(3k+1)$-colouring c, then c can be extended to a good $(3(k+1)+1)$-colouring of D'.

Proof. Let D be a plane diagram of some finite planar graph G and let c be a good $(3k+1)$-colouring of D. Fix some finite planar extension G' with plane diagram D' extending D. Consider the diagram D'' which draws only the induced subgraph of the vertices in $G' \setminus G$. Clearly, $D'' \subseteq D'$ (in the sense of Definition 3.8). Applying Lemma 3.11, there is a good 4-colouring c' of D''. Let $\mathbb{R}^2 \setminus D = \bigsqcup_{i \leq m} U_i$ with the properties as stated in Fact 3.9. Define a $(3(k+1)+1)$-colouring, d, of D' as follows.

- For any $v \in G$, let $d(v) = c(v)$.
- For any $v \in G' \setminus G$, let U_i be the open connected region of $\mathbb{R}^2 \setminus D$ such that D' embeds v into U_i. Since c is good, c only uses $3k$ colours for the vertices in ∂U_i and thus there are four remaining colours that d can use. Let these colours be i_0, i_1, i_2, i_3, and define $d(v) = i_{c'(v)}$ for any $v \in G' \setminus G$.

Let $u, v \in G'$ be given such that u, v are adjacent. If both $u, v \in G$, then $d(u) = c(u)$ and $d(v) = c(v)$ and thus $d(u) \neq d(v)$ as c is a colouring of G. If both $u, v \in G' \setminus G$, then there exists some U_i such that $u, v \in U_i$. This is because $D' \supseteq D$, meaning that the embedding of the edge connecting u, v cannot intersect with any part of D. Then $d(u) = i_{c'(u)}$ and $d(v) = i_{c'(v)}$ where i_0, i_1, i_2, i_3 are the 4 colours unused by c to colour ∂U_i. Since c' is a colouring of $G' \setminus G$, then $d(u) \neq d(v)$. Finally, consider the case that $u \in G$ and $v \in G' \setminus G$. Once again, since D' is a plane diagram of G', it must be that there exists some U_i such that $v \in U_i$ and $u \in \partial U_i$, otherwise the embedding of the edge connecting u, v must intersect D. In such a case, $d(u) = c(u)$ and $d(v) = i_{c'(v)}$, where i_0, i_1, i_2, i_3 are all different from $c(u)$. Therefore, $d(u) \neq d(v)$. Thus d is a $(3(k+1)+1)$-colouring of G'.

Now we prove that d is a good colouring of D'. The idea is as follows. We shall first show that d is a good colouring of $D \cup D''$. This diagram $D \cup D''$ is a plane drawing of the disjoint union of the graph G and the induced subgraph $G' \setminus G$. In other words, the edges between the vertices in $G' \setminus G$ and the vertices in G are not shown in the diagram $D \cup D''$. However, the intuition is that it is 'easier' for some fixed colouring to be good when there are more edges; if d is a good colouring of $D \cup D''$, then it must also be a good colouring of D'. We provide the details below.

Using Fact 3.9, let $\mathbb{R}^2 \setminus D = \bigsqcup_{i \leq m} U_i$ and $\mathbb{R}^2 \setminus D'' = \bigsqcup_{j \leq m''} U''_j$. Fix some V_l, an open connected region of $\mathbb{R}^2 \setminus (D \cup D'')$. It is evident that there exists unique i, j, such that $V_l \subseteq U_i$ and $V_l \subseteq U''_j$. We show now that $\partial V_l \subseteq \partial U_i \cup \partial U''_j$. Let $x \in \partial V_l$ be given. That is, x is a limit point of V_l, and therefore is also a limit point of U_i. In other words, $x \in U_i$ or $x \in \partial U_i$. By repeating the same argument with U''_j, we also obtain that $x \in U''_j$ or $x \in \partial U''_j$. Applying the fact that $\partial V_l \subseteq D \cup D''$, we thus obtain that $x \in \partial U_i \cup \partial U''_j$. In addition, since $x \in \partial U_i$ or $x \in U_i$ for each $x \in \partial V_l$, and $D \cap D'' = \emptyset$, we also have that $\partial V_l \cap D'' \subseteq U_i$.

Partition the vertices in ∂V_l into those contained in G and those contained in $G' \setminus G$. Since $d \restriction G = c$ is a good $(3k+1)$-colouring and $\partial V_l \cap D \subseteq \partial U_i$, we have that all vertices in ∂V_l from G are coloured with at most $3k$ colours. Now consider the vertices in $\partial V_l \cap D''$. We know that all such vertices v are contained within U_i and thus, $d(v) = i_{c'(v)}$ where i_0, i_1, i_2, i_3 are the 4 colours unused by c in the colouring of ∂U_i, and c' is a good 4-colouring of D''. Since c' is a good 4-colouring of D'', then it uses at most 3 colours to colour the vertices on $\partial U''_j$, say $0, 1, 2$. As a result, d colours the vertices in $\partial V_l \cap D'' \subseteq \partial U''_j \cap U_i$ using only the colours i_0, i_1, i_2. Therefore, d uses at most $3k+3$ colours to colour the vertices in ∂V_l and is thus a good $(3(k+1)+1)$-colouring of $D \cup D''$. Finally, to see that d is a good $(3(k+1)+1)$-colouring of D', recall that $D \cup D''$ is the drawing of G' without the edges between vertices in $G' \setminus G$ and G. In other words, for every connected open region of $\mathbb{R}^2 \setminus D'$, there is a connected open region of $\mathbb{R}^2 \setminus (D \cup D'')$ such that every vertex in the boundary of the former is contained in the boundary of the latter. It follows immediately that d is a good $(3(k+1)+1)$-colouring of D'. □

Lemma 3.13. Let G be a finite planar graph with plane diagram D. Let $G' \supseteq G$ be a countable planar graph with some plane diagram $D' \supseteq D$. If D has a good $(3k+1)$-colouring c, then c can be extended to a $(3(k+1)+1)$-colouring of G'.

Proof. We apply a standard compactness argument. Let a finite planar graph G with plane diagram D be given. Also let σ be a good $(3k+1)$-colouring of D. Consider the tree T defined as follows.

- $\langle \sigma, G, D \rangle \in T$.
- If $\langle \xi, H, P \rangle \in T$, then $\langle \tau, H', P' \rangle \in T$ provided that all of the following holds. $H' \subseteq G'$ and $P' \subseteq D'$. $\tau \supseteq \xi$ is a good $(3(k+1)+1)$-colouring of $P' \supseteq P$. P' is a plane diagram of H'. And H' extends H by exactly one vertex.

It is clear that T is finitely branching. To see that it is infinite, since G' is an infinite graph with diagram D', then it remains to argue that there always exists some $(3(k+1)+1)$-colouring τ of $D' \restriction H$ for each finite induced subgraph H where $G \subseteq H \subseteq G'$. Since $H \supseteq G$ is a finite planar graph with a plane diagram $D' \restriction H \supseteq D$, then applying Lemma 3.12 allows us to conclude that there is some good $(3(k+1)+1)$-colouring $\tau \supseteq \sigma$. Thus there must be a path through T and the first coordinate of this path provides a $(3(k+1)+1)$-colouring of G'. □

Applying the Lemmas 3.4, 3.7 and 3.13, we can now complete the picture illustrated in Fig. 5 and also compare the Weihrauch degrees of the different colouring principles.

Theorem 3.14. $\texttt{DNR} \not\leq_W \texttt{ConnCOL}(7)$ and $\texttt{DNR} \not\leq_W \texttt{COL}^*(7)$.

Proof. We adopt a similar idea as in the proof of Theorem 3.2. Roughly speaking, Lemma 3.7 and 3.13 allows us to extend 4-colourings into 7-colourings. We will present the proof for $\texttt{DNR} \leq_W \texttt{COL}^*(7)$ as it has some slight intricacies not required for $\texttt{DNR} \leq_W \texttt{ConnCOL}(7)$, but essentially the same proof can be used to show the latter.

Suppose to the contrary that $\mathtt{DNR} \leq_W \mathtt{COL}^*(7)$. Then, there are Turing reductions Φ, Ψ such that $\Phi(f)$ produces a planar graph with a plane diagram D given any oracle f. And given any 7-colouring h of G, $\Psi(h \oplus f)$ satisfies DNR.

Let f be the empty oracle; $f_s = 0^s$ for each $s \in \omega$. Define $\psi^f(x) = \Psi(h \oplus f)(x)$ in a similar way as in the proof of Theorem 3.2. At each stage s, compute the graph and diagram G_s, D_s given by $\Phi(f_s)$. Instead of directly searching for a colouring of G_s, we adopt the idea presented in Lemma 3.11 and search instead for a 4-colouring of an extended graph $G'_s \supseteq G_s$ which enumerates a new vertex for each open connected component of $\mathbb{R}^2 \setminus D_s$ and connects the vertex to all vertices on the boundary of the open connected component. As explained in the lemma, G'_s is planar and must thus have some 4-colouring h_s. Furthermore, $h_s \upharpoonright G_s$ is a good 4-colouring of D_s. There must thus exist some stage s such that $\Psi(h_s \upharpoonright G_s \oplus f_s)(x)[s]\downarrow$, otherwise by a compactness argument, there is some h a 4-colouring of $\Phi(f)$ such that $\Psi(h \oplus f)(x)\uparrow$. By the recursion theorem, there exists x such that $\varphi^f_x(x) = \psi^f(x)$. We thus have that for some s and some 4-colouring h_s of $G'_s \supseteq \Phi(f_s)$,

$$\psi^f(x) = \Psi(h_s \upharpoonright G_s \oplus f_s)(x)[s]\downarrow.$$

Furthermore, $h_s \upharpoonright G_s$ is a good 4-colouring of D_s. Now applying Lemma 3.13 allows us to conclude that for any planar extension G of G_s, there is some h a 7-colouring of G which extends $h_s \upharpoonright G_s$. In particular,

$$\Psi(h \oplus f)(x) = \Psi(h_s \upharpoonright G_s \oplus f_s)(x)[s]\downarrow = \psi^f(x) = \varphi^f_x(x).$$

A contradiction to the assumption that Φ, Ψ witnesses $\mathtt{DNR} \leq_W \mathtt{COL}^*(7)$. □

Theorem 3.15. For any $n > 0$ and for any $i < 4$,

- $\mathtt{COL}(4n+i) \not\leq_W \mathtt{ConnCOL}(3(n+1)+1)$, and
- $\mathtt{COL}(4n+i) \not\leq_W \mathtt{COL}^*(3(n+1)+1)$.

Proof. We follow the idea presented in the proof of Theorem 3.3. Once again, we will only present the proof for the slightly more complicated $\mathtt{COL}^*$. Suppose to the contrary that $\mathtt{COL}(4n+i) \leq_W \mathtt{COL}^*(3(n+1)+1)$. That is, there exists Φ, Ψ, Turing reductions such that $\Phi(G)$ produces a countable planar graph H with a diagram D and given any $(3(n+1)+1)$-colouring h of H, $\Psi(h \oplus G)$ is a $(4n+i)$-colouring of G. Start by enumerating sufficiently many K_4 graphs to construct a W_m (recall the proof of Theorem 2.8) where m is the least such that $m \geq \lceil (4n+i)/4) \rceil$.

Using the same procedure as described in the proof of Theorem 3.3 to define h, a colouring of H and build W^g_m (recall Notation 2.7) where $g = \Psi(h \oplus G)$. The only difference is that each time we extend h_l to h_{l+1} for $l < m-1$, we use Lemma 3.12 in place of Lemma 3.4. That is, for each $l < m$, h_l is a good $(3l+1)$-colouring of some finite restriction of H. Finally, we use Lemma 3.13 to extend h_{m-1} to h a $(3m+1)$-colouring of H. Since $m \leq n+1$, h is a $(3(n+1)+1)$-colouring of H but H contains a $W^{\Psi(h\oplus G)}_m$ and thus cannot be $(4n+i)$-coloured by $\Psi(h \oplus G)$. □

From the theorem, we have the following interesting result.

Corollary 3.16. For all $n \geq 16$,

- $\mathtt{ConnCOL}(n) <_W \mathtt{COL}(n)$, and
- $\mathtt{COL}^*(n) <_W \mathtt{COL}(n)$.

4. Further Questions

We have shown that for each n, the principles $\mathtt{COL}(n), \mathtt{ConnCOL}(n)$ and $\mathtt{COL}^*(n)$ are equivalent to $\mathtt{WKL}_0$ over $\mathtt{RCA}_0$. However, for the cases of $n > 4$, non-uniform proofs were used to obtain the reversals. We also showed that this non-uniformity is necessary for $n \geq 7$ but leave the cases for $n = 5, 6$ open. A possible way to tackle this question is to consider the uniformity of reductions between the colouring principles. It is trivial to see that $\mathtt{COL}(7) \leq_{sW} \mathtt{COL}(6) \leq_{sW} \mathtt{COL}(5) \leq_{sW} \mathtt{COL}(4)$. But it is still unknown which of the reductions (other than $\mathtt{COL}(4) \not\leq_W \mathtt{COL}(7)$) can be made strict. Similarly, we leave open the question of whether or not the implications presented in Fig. 5 can be made strict. A deeper analysis of the Weihrauch degrees of the different colouring principles can also be done. We note that each of the principles $\mathtt{COL}(n), \mathtt{ConnCOL}(n)$, and $\mathtt{COL}^*(n)$ are all *parallelizable* but we do not know if they are *cylinders* (see [2] for definitions). In Remark 2.9 we mentioned the principle "every planar graph G has some n-colouring". We know that it is not provable in $\mathtt{RCA}_0$ and clearly provable in $\mathtt{WKL}_0$, but leave open the question of whether or not it is equivalent to $\mathtt{WKL}_0$ over $\mathtt{RCA}_0$.

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