Computability of Polish spaces up to homeomorphism

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Abstract

We study computable Polish spaces and Polish groups up to homeomorphism. We prove a natural effective analogy of Stone duality, and we also develop an effective definability technique which works up to homeomorphism. As an application, we show that there is a Δ^0_2 Polish space not homeomorphic to a computable one. We apply our techniques to build, for any computable ordinal α, an effectively closed set not homeomorphic to any $0^{(\alpha)}$-computable Polish space; this answers a question of Nies. We also prove analogous results for compact Polish groups and locally path-connected spaces.

1 Introduction

In this article we focus on the following general problem fundamental to computable mathematics:

Describe computably presentable mathematical structures.

Of course, to formally clarify the problem we need to restrict it to some natural class of mathematical structures and agree on what we mean by a computable presentation for such structures. For instance, Turing [Tur36, Tur37] suggested the following formal definition of a computable real: A real r is computable if there is an effective procedure (Turing machine) which, on input s, outputs a rational q such that $|q-r| < 2^{-s}$. Turing's definition has a natural generalisation to functions. Similarly, we say that a function $f : [0,1] \to \mathbb{R}$ is computable if there is an effective procedure which, on input s, outputs a tuple of rationals $\langle q_0, \dots, q_n \rangle$ such that $\sup_{x \in [0,1]} \{|f - \sum_{i=0}^{n} q_i x^i|\} < 2^{-s}$. Using the formal notion of computability for functions, we can use tools of computability theory to attack the general problem of computable presentability informally stated above. For instance, Myhill [Myh71] showed that there exists a computable function which is continuously differentiable, but its derivative is not computable. In contrast, Pour-El and Richards [PER83] showed that if the second derivative of a computable function f exists (but is not necessarily computable), then the derivative of f has a computable presentation. Results of this kind belong to a field of mathematics called *computable analysis*; see books [PER89, BHW08].

The ideas of Turing can be naturally extended beyond the space of reals to define the classical notion of a computable Polish space [Wei00]. Recall that a metrized Polish space (M, d)

has a computable Polish presentation if there exists a countable metric space $((x_i)_{i\in\omega}, \tilde{d})$ whose completion is *isometrically isomorphic* to (M, d) and, given i, j and n, we can compute $\tilde{d}(x_i, x_j)$ with precision 2^{-n}. In the case of a separable Banach space we also assume that the standard Banach space operations are computable; we omit the definition (see [PER89]). The study of computable presentations of metrized separable spaces has been central to computable analysis for decades. See books [PER89, Wei00], a tutorial survey [BHW08], and also, e.g., [BG09, NS15, GW07, CMS19, HRSS19] for recent results on computable Polish and Banach spaces.

As was first noted in [Mel13], the study of separable spaces up to isometric isomorphism can be viewed as a generalisation of (discrete) computable algebra [AK00, EG00]. With some effort, the techniques and ideas from computable algebra can be adjusted to separable spaces. Beginning with [Mel13] there have been several successful applications of effective algebraic techniques to separable spaces; see [CMS19, MN13, NS15, MS19, GMKT18, GMNT18, McN17, MN16], a PhD thesis [Bro19], and a recent survey [DM20].

In the case of Polish *groups* the situation becomes more complex. We of course require the standard group operations to be computable with respect to the computable dense set (to be clarified), but this is not what makes the case of Polish groups different from Banach spaces. Since topological groups are typically studied up to topological isomorphism, we require that the completion of the computable presentation of G is merely algebraically *homeomorphic* to G. The relaxation of isometry to homeomorphism makes it essentially impossible to apply methods developed for Banach spaces and metric spaces up to isometry to Polish groups.

There were however some notable exceptions. For instance, working under the supervision of Nerode, La Roche [LR81] proved that the correspondence between computable algebraic number field extensions and profinite groups is uniformly effective. In particular, computable presentability of a profinite group is completely reduced to the similar problem for the corresponding field extension. Quite interestingly, the algorithmic techniques developed in [LR81] allowed La Roche to prove a theorem on free profinite groups that was new even in the purely algebraic (non-computable) setting, see [Jar74] for the earlier (and weaker) purely algebraic result. Based on the work of La Roche, Smith [Smi81, Smi79] studied "recursive presentations" of profinite groups; these are computable linear inverse systems of finite groups. These results and notions are of course naturally limited to the class of profinite groups, and there had been very little progress in computable topological groups theory for several decades (but see [GR93]).

Beginning with [MM18], there have been a few successful applications of effective algebraic techniques to topological group theory beyond profinite groups. If a group is not profinite then we follow [MM18] and define its computable presentation to be a computable Polish presentation of the underlying space which makes the group operations $\cdot$ and $^{-1}$ computable; see Definition 2.3 for formal details[1]. The main difficulty in such investigations is that there is still no general machinery, so every result seems to require a new method. For example, Melnikov [Mel18] used Pontryagin duality, computable pregeometries, and a result of Dobrica [Dob83] to partially reduce the study of computable compact topological abelian

[1]We note here that every "recursive" profinite group (in the sense of computable inverse limits) can naturally be viewed as a computably metrized Polish one, and passing from a computable Polish presentation to a "recursive" one requires $0'$, and this is sharp [Mel18].

groups to the theory of computable discrete abelian groups (see surveys [Mel14, Khi98]). Greenberg, Melnikov, Nies and Turetsky [GMNT18] used ideas from descriptive set theory, the above-mentioned result of La Roche [LR81], methods of higher recursion theory [Sac90], and the jump inversion technique from effective algebra to study computable totally disconnected groups.

One of the key obstacles here is that essentially nothing is known about computable presentability of Polish *spaces* up to homeomorphism. The little that was known before the publication of this paper can be found in the very recent short survey [Sel20]. For instance, Selivanov [Sel20] introduced the notion of the degree spectrum of a Polish space up to homeomorphism; however, Selivanov's results for algebraic domains does not have implications for Polish spaces since the notion of computability there is rather different. There are of course some results in the literature on computable *topological* spaces (e.g. [WG09]) which naturally hold up to homeomorphism, such as the effective metrization theorem [GW07], but until very recently computable presentability of metrized Polish spaces up to homeomorphism remained completely unexplored.

The main purpose of this paper is to establish the foundations of this new subject. Working simultaneously and independently, Kihara, Hoyrup and Selivanov [HKS] have recently proven a number of important and fundamental results on degree spectra of Polish spaces up to homeomorphism. We will indicate the connections between our results and [HKS] below.

We will prove several general results and will develop elements of definability which work up to homeomorphism. Although these are only the first steps, our new techniques will allow us to answer several fundamental questions, including:

(1) Is there a Δ^0_2-presented Polish space not homeomorphic to a computable one?

(2) Is every effectively closed set homeomorphic to a computable Polish space?[2]

We will then apply our techniques to answer similar questions for Polish groups. Using different methods, Takayuki Kihara, Mathieu Hoyrup and Victor Selivanov [HKS] have suggested an independent solution to the first question for Polish spaces.

As we mentioned above, very little is known about computable presentability up to homeomorphism. Nonetheless, the reader will perhaps be surprised that the two main questions above were open since they are so fundamental and basic. Of course, the analogy of the first question up to isometry is trivial: Just take two points at distance a real number α coding the halting set. But the question is no longer straightforward up to homeomorphism. For instance, Greenberg and Montalbán [GM08] showed that every hyperarithmetical compact countable Polish space has a computable copy. The proof relies on the computability-theoretic analysis of the Cantor-Bendixson process due to Friedman (unpublished notes); in particular, for a countable space the Cantor-Bendixson rank must be hyperarithmetical, and since a countable compact space is homeomorphic to an ordinal the result of Greenberg and Montalbán follows from the similar classical result for ordinals (see, e.g., [AK00]). (We note that [HKS] contains a detailed proof of this fact.) Thus, if we want to get an example of a Δ^0_2 Polish space not homeomorphic to a computable one, the space must contain a non-trivial

[2]Thanks to Andre Nies for asking this question. We also thank Alex Galicki for several discussions related to this topic. Alex is involved in a related project which is still in progress.

perfect kernel. Regarding the second question, we will see that computable presentability of an effectively closed set is related with the ability to decide whether a given open set intersects it.

Now to the results.

Stone spaces. Recall that compact and totally disconnected Polish spaces are also called profinite spaces or Stone spaces. The well-known Stone duality states that a countable discrete Boolean algebra B is dual to the profinite space $\widehat{B}$ of its ultrafilters, in the sense that $B \cong_{iso} C$ iff $\widehat{B} \cong_{hom} \widehat{C}$. Greenberg and Montalbán [GM08] (essentially) observed that Stone duality holds arithmetically. Passing from a computably metrized Stone space to the respective Boolean algebra is of course the harder direction, and it seems to require at least one Turing jump. Interestingly, we discovered that the duality holds computably:

Theorem 1.1. *Let B be a (countable, discrete) Boolean algebra. Then the following are equivalent:*

(1) B has a computable copy;

(2) the Stone space $\widehat{B}$ of B has a computable Polish presentation.

The proof of the theorem is not difficult but is subtle, and it makes an essential use of the well-known result of Downey and Jockusch [DJ94]. The proof of Theorem 1.1 implies that every computable Stone space has a computable effectively compact presentation; recall that a space is effectively compact if for every i we can computably cover the space with balls having radii $\leq 2^{-i}$. This is because every computable Boolean algebra has a computable presentation with a tree-basis represented as a computable binary subtree of $2^{<\omega}$ (folklore); see Subsection 4.1 for more about spanning trees. It follows that Theorem 1.1 still holds if in (2) we additionally require the space to be effectively compact; see [HKS] for an independent proof of this fact that does not use Downey and Jockusch [DJ94].

Combine Theorem 1.1 with the classical theorem of Feiner [Fei70] to obtain:

Corollary 1.2. *There exists a Δ^0_2-presented profinite Polish space not homeomorphic to any computable Polish space.*

This answers the first main question. Theorem 1.1 essentially completely reduces the theory of computable Stone spaces to computable Boolean algebras; see book [Gon97] for an excellent but somewhat dated exposition of the latter. We state only one of the many corollaries: Every low_4 profinite Polish space is homeomorphic to a computable one (follows from [KS00]).

Effectively closed subspaces. Recall that a closed subspace of a computable Polish space is Π^0_1 or *effectively closed* if its complement is computably enumerable; i.e., there is a computably enumerable set of open balls which makes up the complement. Effectively closed subspaces of 2^ω are called Π^0_1-classes and have been studied extensively. However, not much is known about effectively closed spaces of arbitrary spaces; see, e.g., [CR99, YMT99] for a few results. Note that an effectively closed space does not have to contain a dense set computable in the ambient space. If an effectively closed C does contain a dense uniformly computable sequence then C is called *effectively overt*. Equivalently, a closed set is effectively overt if the set of basic open balls intersecting the set is computably enumerable (folklore; see, e.g., [HRSS19, Mel18] for details). If a set is effectively overt then of course it has a computable presentation under the induced metric. It turns out that effective overtness is essential in producing a computable presentation of a Π^0_1 closed set.

Theorem 1.3. *For each computable ordinal α, there is a computable Polish space M and a Π^0_1 subspace C of M such that C is not homeomorphic to any $0^{(\alpha)}$-computable metric space.*

This answers the second main question raised above in a strong way. It also follows from Theorem 1.3 that there exists a Π^0_1 subspace C which is not $0^{(\alpha)}$-overt. We leave open:

Question 1. Is there an effectively closed X not homeomorphic to any hyperarithmetically represented space?

Locally connected spaces. Although we settled both main questions in general, we would like to test these ideas on spaces from other natural classes. After all, Stone spaces are very specific topological spaces, they are just representations of the well-studied class of countable Boolean algebras. One would naturally expect that techniques required to build examples of Stone spaces will likely be too specific to be of any use outside of this narrow class of spaces.

In particular, the classes of connected and locally connected spaces seem to be on the other end of the technical "spectrum". Recall that there is no logical implication between connectedness and local connectedness. We focus on the locally connected case. As we mentioned above, methods developed in the proof of Theorem 1.1 seem to be of little help. However, the definability techniques developed in the proof of Theorem 1.3 for Π^0_1 sets are more versatile and will allow us to prove:

Theorem 1.4. *There is a Δ^0_2 locally compact and locally path-connected space which is not homeomorphic to any computable Polish space.*

In contrast with Corollary 1.2, the proof of the theorem above is much more direct, in the sense that shows how to build such spaces "by hand" without outsourcing to effective algebraic techniques. Its proof is a priority construction combined with definability, and the complexity of guessing in the proof is at the level of Π^0_3. We conjecture that any construction of a locally connected space witnessing the theorem above must involve a difficult guessing. This is because definability up to homeomorphism in such spaces seems to require at least three quantifiers. The proof of the theorem above is the first example of a $0'''$ argument in computable analysis that we are aware of, but the good news is that there will not be much injury and therefore no special training in $0'''$ arguments is necessary to understand the proof. We also strongly conjecture that one can modify the proof of Theorem 1.4 to get a locally compact locally path-connected subspace of $\mathbb{R}^2$ witnessing the theorem; see Remark 4.10. In fact, the space can be realised as an effectively closed Δ^0_2-overt subset of $\mathbb{R}^2$ thus also witnessing Theorem 1.3 for $\alpha = 0$. We leave open:

Question 2. Is there a Δ^0_2 connected Polish space not homeomorphic to a computable one? What about compact connected spaces?

Topological groups. We test our methods on the class of Polish groups. We apply techniques similar to those used in Theorem 1.1 for Stone spaces to prove:

Theorem 1.5. *There exists a Δ^0_2 compact Polish abelian group not topologically isomorphic to any computable Polish group.*

The group witnessing the theorem is not profinite but all the information is coded into the profinite factor. In particular, the group is not connected. We strongly suspect that with some extra work one can design a profinite group witnessing the theorem above, but we leave it for future work. The connected case seems more challenging.

Question 3. Is there a Δ^0_2 connected Polish group not topologically isomorphic to any computable Polish group?

Finally, we finish the introduction with the theorem which is a version of the second main result Theorem 1.3 for Polish groups.

Theorem 1.6. *For every computable α there exists an effectively closed compact (thus, profinite) subgroup of S_∞ not homeomorphic to a Δ^0_α Polish group.*

The proof of this theorem extends an argument from [GMNT18] and should not be hard to understand to anyone familiar with the standard techniques of computable structure theory [AK00].

2 Formal Definitions

Recall that a real α is *computable* (Turing [Tur36, Tur37]) if there exists a Turing machine that, given $n \in \mathbb{N}$, outputs a rational r within 2^{-n} of α. A Polish space (M, d) is *computable* if there exists a sequence $(\alpha_i)_{i\in\mathbb{N}}$ of M-points which is dense in M and such that, for every $i, j \in \mathbb{N}$, the distance $d(\alpha_i, \alpha_j)$ is a computable real, uniformly in i and j [Wei00]. Given a computable presentation of a Polish space, we call the points $(\alpha_i)_{i\in\mathbb{N}}$ *special points.*

Definition 2.1. Let f be a continuous function between Polish metric spaces M and N. A *name* of f is any collection of pairs of basic open balls (B, C) such that $f(B) \subseteq C$, and for every $x \in M$ and every $\epsilon > 0$ there exists $(B, C) \in \Psi$ such that $B \ni x$ and $r(C) < \epsilon$.

Definition 2.2. A function $f : M \to N$ between computably presented Polish spaces M, N is *computable* if it possesses a c.e. name.

A function is continuous iff it has an X-c.e. name for some oracle X.

In a metric space, we say that a Cauchy sequence (x_i) is *fast* if $d(x_i, x_{i+1}) < 2^{-i-1}$. The above definition of a computable map is equivalent to saying that f is represented by a Turing functional that maps fast Cauchy sequences to fast Cauchy sequences (folklore). The definition below was first suggested in [MM18].

Definition 2.3. A computable Polish group is a triple (G, Φ, Ψ), where G is a computable Polish presentation of the underlying metric space and Φ and Ψ are (indices for) c.e. names of group-operations $\cdot$ and $^{-1}$ upon G.

3 Effectively closed subspaces

Theorem 3.1. *For each computable ordinal α, there is a computable Polish space M and a Π^0_1 subspace C of M such that C is not homeomorphic to any $0^{(\alpha)}$-computable Polish space.*

Proof. The proof relies on a definability technique. To develop the technique we first need to prove several lemmas. Some of these lemmas (such as the lemma below) are really folklore, but for completeness sake we include their proofs. The problem with dealing with a computable presentation of a Polish space is that we only really have access to a dense subset of special points, so we cannot for example talk about a path from one point to another. We can however approximate such a path by taking a discrete series of short steps.

Definition 3.2. Let (M, d) be a Polish space. Given special points x, y, an ϵ-path from x to y is a sequence of special points $x = u_0, u_1, \dots, u_n = y$ such that $d(u_i, u_{i+1}) < \epsilon$.

The existence of a path yields the existence of an ϵ-path for every $\epsilon > 0$.

Lemma 3.3. *Let (M, d) be a Polish space with special points $\langle q_i \rangle_{i\in\omega}$. Suppose that there is a path between special points r and s. Then for every $\epsilon > 0$, there is an ϵ-path from r to s.*

Proof. Let $[0, 1]$ be the unit line, let $f\colon [0, 1] \to M$ be a continuous path from r to s. Then f is uniformly continuous. So for a sufficiently large rational q, we have that for each i,

$$d\left(f\left(\frac{i}{q}\right), f\left(\frac{i+1}{q}\right)\right) < \frac{\epsilon}{4}.$$

Then choose $x_0 = r$, $x_q = s$, and for each $i = 1, \dots, q-1$, choose a special point x_i with $d(x_i, f(i/q)) < \epsilon/4$. Then $r = x_0, \dots, x_q = s$ is an ϵ-path from r to s. □

The converse is not true in general. For example there might be two points r and s at distance 1 from each other, such that for each $n \in \mathbb{N}$ there is a discrete set of n points at distance $1/n$ from each other forming an $1/n$-path from r to s; but no continuous path from r to s. However, this is only possible due to a failure of compactness for the path-components of r and s.

Lemma 3.4. *Let (M, d) be a Polish space with special points $\langle q_i \rangle_{i\in\omega}$. Suppose that each path-component of M is compact and open. If special points r, s are not in the same path-component, then there is an ϵ such that there is no ϵ-path between r and s.*

Proof. Let C be the path-component of r. Since C is open, its compliment is closed, and since C is compact there is a distance ϵ between C and C^c. Then there is no $\epsilon/2$-path from r to s, as given any path $r = u_0, u_1, \dots, u_n = s$ there must be a first i such that $u_i \in C$ and $u_i \notin C$, and so $d(u_i, u_{i+1}) \geq \epsilon$. □

It is important that the path-components be open. If they were just compact, then for sufficiently small ϵ one could not make an ϵ-path that went directly from the path-component of r to the path-component of s, but one might be able to find an ϵ-path that travels via some third path-component, with a different third path-component for each value of ϵ. The main coding components in our proof will be designed using *k-stars* which are defined below.

Definition 3.5. A k-star is a topological space homeomorphic to k copies of the interval $[0,1]$ all joined at one end in a single point. Note that a k-star is not homeomorphic to a k'-star for $k \neq k'$.

We say that a component is a star if it is a k-star for some k. The value of k will always be finite, so that every star is compact. Stars with infinite branching would not allow us to use Lemma 3.4.

We are ready to state and prove the main definability lemma.

Lemma 3.6. *Let (M,d) be a Polish space with special points $\langle q_i \rangle_{i \in \omega}$. Suppose that M is homeomorphic to the disjoint union of stars, each of which is compact and open. A special point r is contained within an n-star with $n \geq \ell$ if and only if*

(∗) *there are distinct points $p_1, p_2, \ldots, p_\ell$ in the same path-component as r and a $\delta > 0$ such that for each i,j,k and for every $\epsilon < \delta$ there is an ϵ-path $p_i = u_0, u_1, \ldots, u_n = p_j$ from p_i to p_j such that $u_0, \ldots, u_n \notin \bar{B}_\delta(p_k) = \{x \in M : d(x, p_k) \leq \delta\}$.*

Moreover, the witnesses $p_1, \ldots, p_\ell$ are all on different arms of the star.

Proof. First we show that this is true of an n-star, $n \geq \ell$. Let $p_1, \ldots, p_\ell$ be points on different arms of the stars. Given p_i, p_j, and p_k, let δ be sufficiently small that $\bar{B}_\delta(p_k)$ does not intersect the arms containing p_i and p_j, and also does not intersect the complement of the star. Then there is a path between p_i and p_j in $M - \bar{B}_\delta(p_k)$, so by Lemma 3.3, for every $\epsilon < \delta$ there is an ϵ-path from p_i to p_j which avoids $\bar{B}_\delta(x_k)$.

Let S be an n-star in M. If $n < \ell$, then given any distinct special points $p_1, \ldots, p_\ell$, two of them are on the same arm of the star (or one of these points is the center of the star). So we can choose p_i, p_j, p_k such that by removing p_k, the star divides into two connected components, one containing p_i, and the other containing p_j. We must show that for every δ, there is an $\epsilon < \delta$ such that for every ϵ-path $p_i = u_0, u_1, \ldots, u_n = p_j$ from p_i to p_j, there is some $u_i \in \bar{B}_\delta(p_k)$. We may assume that δ is sufficiently small that $p_i, p_j \notin \bar{B}_\delta(p_k)$ (otherwise it is trivial).

Now we can write S as a disjoint union $C_i \cup C_j \cup B_\delta(p_k)$ where C_i and C_j are closed sets containing p_i and p_j respectively. Then C_i and C_j are compact, and so we can choose ϵ smaller than the distance between C_i and C_j, and also smaller than the distance between S and the compliment of S. Then any ϵ-path $p_i = u_0, \ldots, u_n = p_j$ in M must have $u_0, \ldots, u_n \in S$ (since the distance between S and the compliment of S is greater than ϵ). Also, since $u_0 \in C_i$, $u_n \in C_j$, and the distance between C_i and C_j is greater than ϵ, for some s, $u_s \in B_{\delta(p_k)} \subseteq \bar{B}_\delta(p_k)$. So there is no ϵ-path from p_i to p_j avoiding $\bar{B}_\delta(p_k)$. □

For the computability-theoretic part of our proof we rely on the following simple lemma.

Lemma 3.7. *Let R be a hyperarithmetic relation. Then there is a computable sequence of trees $T_n \subseteq \omega^{<\omega}$ such that if $n \notin R$, then T_n has a single path, and if $n \in R$, then T_n has no path.*

Proof. Let α be such that R is $0^{(\alpha)}$-computable. It is well-known that for each computable ordinal α, there is a computable tree T with a single path $f \equiv_T 0^{(\alpha)}$. (By Proposition II.4.1 of [Sac90], $0^{(\alpha)}$ is a Π^0_2 singleton, and following Theorem 3.1 of [JM69] we can replace $0^{(\alpha)}$

by a lexicographically least Skolem function $f \equiv_T 0^{(\alpha)}$ such that f is a Π^0_1 singleton.) Let Φ be a Turing functional such that $R = \Phi^f$. Then for each n, let T_n be a computable tree with $g \in [T_n]$ if and only if $g \in [T]$ and either $\Phi^g(n)\uparrow$ or $\Phi^g(n) = 0$. Then T_n has at most one path since T has at most one path, and if T_n has a path, that path is f. If $n \notin R$, then f is still a path of T_n; and if $n \in R$, then $\Phi^f(n) = 1$, and so $f \notin [T_n]$. □

We return to the proof of the theorem. We will build M as a computable Polish space with special points $(q_i)_{i\in\omega}$ and metric d. The Π^0_1 subspace C of M will be the disjoint union of stars, each of which is compact and open.

Given a presentation $(X, d, (r_i)_{i\in\omega})$ of a Polish space which is a disjoint union of compact open stars, we claim that the set $S(X) = \{n : X \text{ has an } n+3\text{-star}\}$ of sizes of stars in X is Σ^0_4 relative to this presentation of X. Indeed, $n \in S(X)$ if and only if X contains a special point r satisfying $(*)$ of Lemma 3.6 for $\ell = n+3$, but not satisfying $(*)$ of Lemma 3.6 for $\ell = n+4$. (In $(*)$, we ask that $p_1, \ldots, p_\ell$ are in the same path component as r; we express this by saying that for each i and for every ϵ there is an ϵ-path from r to p_i, as in Lemma 3.4.)

Let R be a relation which is not $\varnothing^{(\alpha+4)}$-computable. Using Lemma 3.7, let $T_n \subseteq \omega^{<\omega}$ be a computable sequence of trees such that if $n \in R$ then T_n has no path, but if $n \notin R$ then T_n has a unique path. Let M be the disjoint union, over $n \in \omega$, of $\omega^\omega \times S_{n+3}$ where S_{n+3} is a particular chosen computable presentation of an $n+3$-star. Set each component $\omega^\omega \times S_{n+3}$ to be at distance 1 from each other such component. As a metric space, we use the sum metric on each component $\omega^\omega \times S_{n+3}$, i.e., if (f,x) and (g,y) are in the same component, we set $d((f,x),(g,y)) = d_{\omega^\omega}(f,g) + d_{S_{n+3}}(x,y)$.

Now we will define the Π^0_1 set C. Whenever we see $\sigma \notin T_n$, put $[\sigma] \times S_{n+3} \notin C$. To see that we can do this effectively, we must note that we can write $[\sigma] \times S_{n+3}$ as an effective union of basic open balls

$$\bigcup_{n\in\omega} \bigcup_{p \in S_{n+3}} B_{2^{-|\sigma|}}(\sigma\hat{\ } n, p)$$

where p ranges over special points in the chosen computable presentation of S_{n+3}. If $n \in R$, then C is disjoint from $\omega^\omega \times S_{n+3}$, and so does not have an $n+3$-star; and if $n \notin R$, then $C \cap \omega^\omega \times S_{n+3} = \{f\} \times S_{n+3}$ where f is the path through T_n, and so C has an $n+3$-star. Thus C is homeomorphic to the the disjoint union of an $n+3$-star for $n \notin R$, and each of these stars is compact and open. We have $S(C) = R$.

We claim that C is not homeomorphic to any $0^{(\alpha)}$-computable metric X. Indeed, suppose to the contrary that X was homeomorphic to C. Then $R = S(C) = S(X)$ which is Σ^0_4 relative to $0^{(\alpha)}$, contradicting the choice of R. □

4 Δ^0_2 Polish spaces with no computable homeomorphic copies

4.1 Totally disconnected spaces

Recall that a Polish space is profinite if it is compact and totally disconnected. The well-known Stone duality states that a countable discrete Boolean algebra B is dual to the profinite space $\widehat{B}$ of its ultrafilters, in the sense that $B \cong_{iso} C$ iff $\widehat{B} \cong_{hom} \widehat{C}$; that is, the isomorphism type of a countable discrete Boolean algebra is uniquely determined by the homeomorphism type of its dual profinite space.

Theorem 4.1. *Let B be a (countable, discrete) Boolean algebra. Then the following are equivalent:*

(1) B has a computable copy;

(2) the Stone space $\widehat{B}$ of B has a computable Polish presentation.

Proof. Given a computable Boolean algebra B, produce a computable tree-basis of B [Gon97]. Recall that a tree basis is the set of generators of B which form a tree under the standard $\leq$ with root $\mathbf{1}$, such that every generator y in the tree has either no children (and then it is an atom) or exactly two children whose disjoint union is equal to y. Interpret the tree basis as a dense subset of closed subspace of 2^ω under the usual ultrametric. This gives a computable Polish presentation of $\widehat{B}$.

Now suppose the Stone space $\widehat{B}$ of B has a computable Polish presentation. We effectivize the standard proof of Stone duality. The key lemma is:

Lemma 4.2. *Suppose M is a computable compact Polish metric space. If M is not connected then $0'$ can produce a splitting of M into two disjoint clopen components. Furthermore, $0'$ can compute two representations for each of the two components: one via a finite union of basic* open *balls, and the other via a finite union of basic* closed *balls. Moreover, if x and y are special points in distinct connected components of M, we can find a splitting with x in one component and y on the other.*

Proof. Suppose M_0 and M_1 are two non-intersecting clopen components of M such that $M_0 \cup M_1 = M$. Since M is compact and each M_i is closed, M_i is compact. Since M_i is open and compact it is equal to a finite union of open balls, say $M_0 = \bigcup_{i=1,\dots,k} B_i$ and $M_1 = \bigcup_{j=1,\dots,n} D_i$, where B_i, D_j are basic open balls. Write $\bar{B}_i$ and $\bar{D}_i$ for the corresponding closed balls, which are contained in but may not be equal to the closures of the open balls. Then $M_0 = cl(M_0) = \bigcup_{i=1,\dots,k} \bar{B}_i$, and similarly for M_1. Identify M_i with the respective finite cover by *closed* balls.

Now we will show that $0'$ can search for M_1 and M_2, as represented above. Suppose we have two subsets M_1 and M_2 of M, represented by finite unions of balls B_i and D_j respectively, but we do not know if M_1 and M_2 are disjoint and we do not know whether they cover M.

We first claim that the property $M_0 \cup M_1 = M$ becomes Π^0_1. Indeed, to see if $M_0 \cup M_1 = M$ it is sufficient to search for a special point in the open set $M \setminus (M_0 \cup M_1)$, i.e., outside of all the finitely many closed balls. This is a Σ^0_1 process which is of course uniform in the finite tuple describing the balls B_i and D_j.

To guarantee that M_0 and M_1 are also disjoint we must check whether

$$\bigcup_{i=1,\dots,k} \bar{B}_i \cap \bigcup_{j=1,\dots,n} \bar{D}_j = \varnothing$$

which is reduced to verifying finitely many statements of the form $\bar{B}_i \cap \bar{D}_j = \varnothing$. We suppress i and j. Let B^ϵ be the basic *open* ball with the same centre as B but having radius $r(B) + \epsilon$, where ϵ is a positive rational number. Define D^ϵ similarly. We claim that $\bar{B} \cap \bar{D} \neq \varnothing$ is equivalent to

$$(\forall \epsilon > 0)\, B^\epsilon \cap D^\epsilon \neq \varnothing.$$

One implication is trivial. For the other implication, assume x_ϵ is a point witnessing non-emptiness for ϵ. By compactness, $(x_{2^{-m}})_{m \in \mathbb{N}}$ has a converging subsequence. The limit of this sequence will be a point witnessing $\bar{B} \cap \bar{D} \neq \varnothing$. We have just shown that $\bar{B} \cap \bar{D} \neq \varnothing$ is a Π^0_2-property, which makes $M_0 \cap M_1 = \varnothing$ a Σ^0_2-property.

It follows that $0'$ can search for finitely many basic open B_i and D_j as above. If *some* decomposition of M exists then we will eventually find (perhaps, some other) decomposition of M. (For the moreover clause, we search for a decomposition containing x on one side and y on the other.) Furthermore, for this fixed decomposition $0'$ will be able to see the first found ϵ for which the property $(\forall \epsilon > 0)\, B^\epsilon \cap D^\epsilon \neq \varnothing$ fails. Then the clopen components can be represented as unions of B^ϵ_i and D^ϵ_j rather than B_i and D_j. □

We now return to the proof of the theorem. The idea is to iterate the lemma above to get a $0'$-computable presentation of B with a $0'$-computable set of atoms. This is done as follows. Given a computable Polish presentation of $\widehat{B}$ and using $0'$, initiate the procedure of splitting $\widehat{B}$ into clopen disjoint subsets. At every stage we will have a finite collection of clopen subsets of $\widehat{B}$, and each of these sets will be represented as a union of finitely many open balls as well as a finite union of closed balls. In particular, although the procedure of splitting the space is merely $0'$-computable, at every stage each of the components will naturally be a computable Polish space. To see why, list all special points of the ambient space which belong to the finitely many open balls describing the space. Since the space is also closed, the completion of this set of points will be equal to the whole component. Therefore, we can apply the lemma again to each of the components. At the ith stage of iterating this process, we must make sure that we separate the ith pair of special points of $\widehat{B}$ into separate components, if they have not been already.

The iterated procedure ensures that the set-theoretic inclusion between the produced clopen components is decidable relative to $0'$. Since the space $\mathcal{B}$ is totally disconnected, each pair of points must belong to disjoint clopen sets. In particular, whenever we are given a clopen subspace X represented as a finite union of basic balls, X is an isolated point iff it contains exactly one special point. This property can be decided using $0'$.

It follows that we can produce a Δ^0_2 copy of the Boolean algebra of the clopen subsets of $\widehat{B}$, which furthermore has the atom relation of complexity Δ^0_2. It remains to apply the well-known theorem of Downey and Jockusch [DJ94] who showed that every Δ^0_2 Boolean algebra with Δ^0_2 atom relation is isomorphic to a computable one. □

Thus we obtain:

Corollary 4.3. *There exists a Δ^0_2-presented profinite Polish space not homeomorphic to any computable Polish space.*

Proof. By Feiner [Fei70], there is a Δ^0_2 Boolean algebra B with no computable copy. By the previous theorem, the Stone space $\hat{B}$ of B is a Δ^0_2-presented Polish space not homeomorphic to any computable Polish space. □

4.2 The locally path-connected case

We say that a Polish space is an LCPC space if it is both locally compact and locally path-connected. Such spaces are more reflective of physical geometry and are in some sense the opposite of totally disconnected spaces. We prove:

Theorem 4.4. *There is a Δ^0_2 LCPC space which is not homeomorphic to any computable Polish space.*

We begin by proving a computational lemma about building computable spaces depending on the answer to a Π^0_3 question.

Lemma 4.5. *Let R and S be Π^0_3 sets and $(k_n)_{n\in\omega}$ a computable sequence of natural numbers ≥ 3. Then there is, uniformly in n, a Δ^0_2 separable metric space (M_n, d_n) such that:*

- *if $n \notin R$, then (M_n, d_n) is the disjoint union of a point and $k_n + 1$ closed line segments;*
- *if $n \in R$ and $n \notin S$, then (M_n, d_n) is the disjoint union of a k_n-star and a closed line segment;*
- *if $n \in R$ and $n \in S$, then (M_n, d_n) is a $(k_n + 1)$-star.*

Proof. Let

$$n \in R \iff \forall x \exists y \forall z R^*(x, y, z, n).$$

We may assume that for each x, there is at most one y with $\forall z R^*(x, y, z, n)$, and that if $x' < x$, and there is no witness y for x', then there is no witness y for x (folklore). Similarly, let

$$n \in R \cap S \iff \forall x \exists y \forall z S^*(x, y, z, n).$$

Fix n and let $k = k_n$. We will define $M = M_n$ as a subspace of a $(k + 1)$-star. Let the $(k+1)$-star have a center point c and arcs $A_1, \ldots, A_{k+1}$ with $A_i = f_i[0, 1]$ and $f_i(0) = c$. Then we let

$$M = \{c\} \cup \bigcup \{f_i[1/x, 1] : 1 \leq i \leq k,\ x \in \mathbb{N}, \text{ and } \exists y \forall z R^*(x, y, z, n)\}$$
$$\cup \bigcup \{f_{k+1}[1/x, 1] : x \in \mathbb{N} \text{ and } \exists y \forall z S^*(x, y, z, n)\}.$$

More formally, this description of M gives a Σ^0_2 way of deciding which special points of the $(k + 1)$-star to include in M; then as M is Δ^0_2, we can build a computable copy of this subspace of $(k + 1)$-star. □

Now we prove the theorem.

Proof of Theorem 4.4. Let $(M_n, d_n)_{n\geq 1}$ be a list of all the (possible partial) computable Polish spaces. We will construct a Δ^0_2 Polish space (M^*, d^*) while diagonalizing against each $(M_n, d_n)_{n\in\omega}$. The space (M^*, d^*) will be the disjoint union of infinitely many stars, points, and line segments, each separated from the others by open sets. (For example, for any star, there will be an open set containing it and nothing else.) We will make sure that (M^*, d^*) is not isomorphic to (M_n, d_n) by either having, for some k, a k-star in (M^*, d^*) when there is no k-star in (M_n, d_n), or vice versa.

The space (M^*, d^*) will be built entirely by constructing Π^0_3 sets R and S, and a computable sequence k_n, and then letting (M^*, d^*) be the disjoint union of the sequence obtained by Lemma 4.5, with each space in the union set at distance one from the others. The resulting space will be LCPC since each component is.

Recall from Lemma 3.6 that, in a space homeomorphic to a disjoint union of stars, points, and line segments, each of which is compact and open, a special point r is contained within an n-star with $n \geq \ell$ if and only if:

($*$) there are distinct points $p_1, p_2, \dots, p_\ell$ in the same path-component as r and a $\delta > 0$ such that for each i, j, k and for every $\epsilon < \delta$ there is an ϵ-path $p_i = u_0, u_1, \dots, u_n = p_j$ from p_i to p_j such that $u_0, \dots, u_n \notin B_\delta(p_k)$.

We call such a space *nice*.

Saying that $p_1, \dots, p_\ell$ are in the same path-component as r is Π_2, as we must say that for every ϵ there is an ϵ-path from r to these points. Thus for a fixed ℓ, asking whether a point r is contained within an n-star with $n \geq \ell$ is Σ^0_3.

We write $\Theta_{\geq \ell}(p_1, \dots, p_k)$ for the relation which holds if there are distinct points $p_{k+1}, \dots, p_\ell$, also distinct from $p_1, \dots, p_k$, and a $\delta > 0$ such that for each i, j, k and for every $\epsilon < \delta$ there is an ϵ-path $p_i = u_0, u_1, \dots, u_n = p_j$ from p_i to p_j such that $u_0, \dots, u_n \notin B_\delta(p_k)$. This relation is Σ^0_3. It expresses (in nice spaces) that $p_1, \dots, p_k$ are distinct arms of a $(\geq \ell)$-star. With no parameters, e.g. $\Theta_{\geq \ell}(-)$, it expresses that there is a $(\geq \ell)$-star.

We also write $\Gamma_{\geq \ell}(p_1, \dots, p_\ell, \delta)$ for the relation which holds if for each i, j, k and for every $\epsilon < \delta$ there is an ϵ-path $p_i = u_0, u_1, \dots, u_n = p_j$ from p_i to p_j such that $u_0, \dots, u_n \notin B_\delta(p_k)$. This relation is Π^0_2. It expresses (in nice spaces) that $p_1, \dots, p_\ell$ are distinct arms of a $(\geq \ell)$-star, with parameter δ. We have that $p_1, \dots, p_\ell$ are distinct arms of a $(\geq \ell)$-star if and only if there is δ such that $\Gamma_{\geq \ell}(p_1, \dots, p_\ell, \delta)$.

To begin, we will describe how to make (M^*, d^*) non-homeomorphic to a single computable metric space (M, d). It will be easiest to think about (M, d) being a nice space; if it is not nice then it cannot be homeomorphicto the nice space (M^*, d^*). If (M, d) is not nice, then we follow the same procedure interpreting the predicates Γ and Θ literally even though they perhaps do not have their intended meaning in (M, d). For instance, we say "(M, d) has a (≥ 3)-star" but we really mean that the respective predicate holds in (M, d).

First, ask whether (M, d) has a (≥ 3)-star at all, i.e., whether $\Theta_{\geq \ell}(-)$ holds in (M, d). In the Π^0_3 case where $\Theta_{\geq \ell}(-)$ does not hold, using Lemma 4.5 we can build a 3-star in (M^*, d^*), and so (M^*, d^*) is not homeomorphic to (M, d); if $\Theta_{\geq \ell}(-)$ does hold and (M, d) does have such a star, then Lemma 4.5 builds a point and three line segments. This is enough to make (M^*, d^*) non-homeomorphic to (M, d), but we want our diagonalization to be more robust; for, if (M, d) has a (≥ 3)-star, it would be very limiting to say that we can never build any star in (M^*, d^*).

So we need to do some additional work in the case where (M, d) has a (≥ 3) star. The idea is to guess at where the (≥ 3)-star is, and then diagonalize against it. List the tuples $(x^1_i, x^2_i, x^3_i, \delta_i)_{i \in \omega}$ of three points from (M, d) and a rational $\delta_i > 0$; these are guesses at (≥ 3)-stars.

For $i = 0$, using Lemma 4.5, build:

- a 3-star (and a line segment) if $\Gamma_{\geq 3}(x_0^1, x_0^2, x_0^3, \delta_0)$ and $\Theta_{\geq 4}(x_0^1, x_0^2, x_0^3)$;
- a 4-star if $\Gamma_{\geq 3}(x_0^1, x_0^2, x_0^3, \delta_0)$ but not $\Theta_{\geq 4}(x_0^1, x_0^2, x_0^3)$; and
- a point and 4 line segments if $\Gamma_{\geq 3}(x_0^1, x_0^2, x_0^3, \delta_0)$ does not hold.

(Note that we can do this because $\Theta_{\geq 4}$, which is the negation of S from Lemma 4.5, is Σ_3^0 and $\Gamma_{\geq 3}$, which is R, is Π_2^0.) So if x_0^1, x_0^2, x_0^3 are arms of a (≥ 3)-star as witnessed by δ_0 in (M, d), then: if they are part of a 3-star then (M^*, d^*) has a 4-star but no 3-star; and if they are part of a (≥ 4)-star then (M^*, d^*) has a 3-star but no (≥ 4)-star.

If we were in one of the first two cases for $x_0^1, x_0^2, x_0^3, \delta_0$, then we do not want to build stars for any other $x_i^1, x_i^2, x_i^3, \delta_i$; so if $\Gamma_{\geq 3}(x_0^1, x_0^2, x_0^3, \delta_0)$ holds, then when acting for the sake of $x_1^1, x_1^2, x_1^3, \delta_1$ we build a point and 4 line segments. Otherwise, we want to do the same thing for $x_1^1, x_1^2, x_1^3, \delta_1$ that we did for $x_0^1, x_0^2, x_0^3, \delta_0$. So we build, using Lemma 4.5:

- if $\Gamma_{\geq 3}(x_0^1, x_0^2, x_0^3, \delta_0)$ holds:
  - a point and 4 line segments
- if $\Gamma_{\geq 3}(x_0^1, x_0^2, x_0^3, \delta_0)$ does not hold:
  - a 3-star (and a line segment) if $\Gamma_{\geq 3}(x_1^1, x_1^2, x_1^3, \delta_1)$ and $\Theta_{\geq 4}(x_1^1, x_1^2, x_1^3)$;
  - a 4-star if $\Gamma_{\geq 3}(x_1^1, x_1^2, x_1^3, \delta_1)$ but not $\Theta_{\geq 4}(x_1^1, x_1^2, x_1^3)$; and
  - a point and 4 line segments if $\Gamma_{\geq 3}(x_1^1, x_1^2, x_1^3, \delta_1)$ does not hold.

(Here, the S of Lemma 4.5 is $\neg\Theta_{\geq 4}(x_1^1, x_1^2, x_1^3)$ and the R is the conjunction of Σ_2^0 and Π_2^0 predicates $\Gamma_{\geq 3}(x_0^1, x_0^2, x_0^3, \delta_0) \wedge \neg\Gamma_{\geq 3}(x_1^1, x_1^2, x_1^3, \delta_1)$.) Then we want to do the same thing for $x_2^1, x_2^2, x_2^3, \delta_2$, and so on, with each $x_i^1, x_i^2, x_i^3, \delta_i$ only having the potential to build a star if the previous ones did not. So, for example, for $x_2^1, x_2^2, x_2^3, \delta_2$, if either $\Gamma_{\geq 3}(x_0^1, x_0^2, x_0^3, \delta_0)$ or $\Gamma_{\geq 3}(x_1^1, x_1^2, x_1^3, \delta_1)$ holds, we will build a point and 4 line segments; otherwise we may build a star. There will be some least i, if any exist, with $\Gamma_{\geq 3}(x_1^1, x_1^2, x_1^3, \delta_1)$—the least witness to the existence of a (≥ 3)-star—and for this i we will build a star in (M^*, d^*).

If (M, d) is a nice space, then either:

- it has no (≥ 3)-star, in which case the very first action we took built a 3-star in (M^*, d^*); or
- for some least i, $x_i^1, x_i^2, x_i^3, \delta_i$ is part of a 3-star in (M, d) as witnessed by δ_i, and then we built no 3-star in (M^*, d^*); or
- x_i^1, x_i^2, x_i^3 is part of a (≥ 4)-star in (M, d), say a k-star, and we built only a 3-star in (M^*, d^*) and no k-star.

So (M^*, d^*) is not homeomorphic to (M, d).

In the first two cases above, there is not much additional complication in diagonalizing against more computable metric spaces, as long as we work with e.g. 5-stars and (≥ 6)-stars for the next computable metric space. The third case will require a little more work, because the diagonalization was due to not building a k-star in (M^*, d^*), but we do not actually know what the value of k is. We must make sure to never add a k-star to (M^*, d^*) by having other diagonalization modules guess at the value of k and avoid adding k-stars.

Now we are ready to describe the entire construction. To organize the Π^0_3 sets we feed into Lemma 4.5, we will build a tree. The nth level of the tree will contain attempts to diagonalize against (M_n, d_n). At the nth level of the tree, a node σ with predecessors $\sigma_1, \ldots, \sigma_{n-1}$, consists of the following:

(1) a label which is either ∞ or f (for finite);

(2) for each $i < n$ with σ_i labeled f, a value $k_\sigma[i]$ which is either the symbol "$\geq 4n$" or a natural number $3 \leq k_\sigma[i] < 4n$;

- if $k_\sigma[i]$ is "$\geq 4n$" then there are also elements $y^1_\sigma[i], \ldots, y^{4n}_\sigma[i] \in M_i$ and $\delta_\sigma[i] \in \mathbb{Q}$, and
- if $k_\sigma[i] \leq 4n$ then there are elements $y^1_\sigma[i], \ldots, y^{k_\sigma[i]}_\sigma[i] \in M_i$ and $\delta_\sigma[i] \in \mathbb{Q}$;

(3) a number ℓ_σ which is the least odd number $< 4n$ such that $\{\ell, \ell+1\}$ is disjoint from

$$\{\ell_\tau, \ell_\tau + 1 : \tau \text{ is a predecessor of } \sigma\} \cup \{k_\sigma[i] : i < n\};$$

(4) if the label is f, elements $x^1_\sigma, \ldots, x^{\ell_\sigma}_\sigma \in M_n$ and $\rho_\sigma \in \mathbb{Q}$.

We suppress σ in ℓ_σ and $k_\sigma[i]$. The number ℓ is the size of star that σ will be building; σ will try to build either an ℓ-star or an $(\ell+1)$-star, just as in the procedure for a single diagonalization described previously we built either a 3-star or a 4-star. The values $k[i]$ are the guesses at the sizes of the stars used to diagonalize against (M_i, d_i); and we must choose ℓ so that building an ℓ-star or an $(\ell+1)$-star for σ would not interfere with the diagonalization against (M_i, d_i). The label is a guess at whether (M_n, d_n) will have a star of size $\geq \ell$; the label ∞ corresponds to having no such star, and the label f corresponds to having such a star. The value $< 4n$ is simply chosen so that there will be some odd ℓ with $\ell < 4n$ such that $\{\ell, \ell+1\}$ is disjoint from

$$\{\ell_\tau, \ell_\tau + 1 : \tau \text{ is a predecessor of } \sigma\} \cup \{k_\sigma[i] : i < n\}.$$

For each node σ of the tree at level n, and each possible choice of these parameters at level $n+1$, there is a *single* child τ of σ at level $n+1$ with those parameters. At each level, order the children of each node from left to right, with order type ω (so that each node has finitely many other nodes to its left).

To each node σ^* which is a child of σ, we associate Π^0_3 predicates R_{σ^*} and S_{σ^*}; the definitions of these will depend on the label, from $\{\infty, f\}$, of σ^*. If σ^* has label ∞, then R_{σ^*} will be of the form $R_\sigma \wedge P_{\sigma^*} \wedge T_{\sigma^*}$ where P_{σ^*} and T_{σ^*} are Π^0_3 and R_σ is the predicate associated to the parent σ of σ^*. If σ^* has label f, then R_{σ^*} will be of the form

$$R_\sigma \wedge P_{\sigma^*} \wedge (\neg Q_{\tau^*_1}) \wedge \cdots \wedge (\neg Q_{\tau^*_t}) \wedge Q_{\sigma^*}$$

where P_{σ^*} is Π^0_3, Q_{σ^*} is Π^0_2, and $Q_{\tau^*_1}, \ldots, Q_{\tau^*_t}$ are the Π^0_2 predicates associated to the other children $\tau^*_1, \ldots, \tau^*_t$ of σ which are to the left of σ^* and which have the same parameters, except that they might have different values of $x^1, \dots, x^\ell, \rho^\ell$, as σ^*. The predicate P_{σ^*} is defined in the same way for nodes labeled ∞ and f, and does not depend on the values of $x^1_{\sigma^*}, \dots, x^{\ell_{\sigma^*}}_{\sigma^*}, \rho_{\sigma^*}$.

We must define these predicates inductively, defining the predicates associated to a parent before its children, and to nodes at a single level from left to right. We will also write down the interpretations of these predicates in nice spaces. Let σ^* be a node at level $n+1$, with predecessors $\sigma_1, \sigma_2, \dots, \sigma_n$ at levels $1, \dots, n$. We define, if σ^* has label ∞:

- P_{σ^*} is the Π^0_3 predicate that says:

  (1) for $i \leq n$ with σ_i labeled f, if $k_\sigma[i]$ is "$\geq 4n$" then:

    - in M_i, for every $\epsilon < \delta$, and each j, j' there is an ϵ-path from $x^j_{\sigma_i}$ to $y^{j'}_{\sigma^*}[i]$: these are all in the same connected component;
    - $\Gamma_{\geq 4n}(y^1_{\sigma^*}[i], \dots, y^{4n}_{\sigma^*}[i], \delta_{\sigma^*}[i])$ holds in M_i: $y^1_{\sigma^*}[i], \dots, y^{4n}_{\sigma^*}[i]$ are arms of a $(\geq 4n)$-star;

  (2) for $i \leq n$ with σ_i labeled f, if $k_\sigma[i] < 4n$ then:

    - in M_i, for every $\epsilon < \delta$, and each j, j' there is an ϵ-path from $x^j_{\sigma_i}$ to $y^{j'}_{\sigma^*}[i]$: these are all in the same connected component;
    - $\Gamma_{\geq k_\sigma[i]}(y^1_{\sigma^*}[i], \dots, y^{k_{\sigma^*}[i]}_{\sigma^*}[i], \delta_{\sigma^*}[i])$ holds in M_i: $y^1_{\sigma^*}[i], \dots, y^{k_{\sigma^*}[i]}_{\sigma^*}[i]$ are arms of a $(\geq k_\sigma[i])$-star;
    - $\neg\Theta_{\geq k_\sigma[i]+1}(y^1_{\sigma^*}[i], \dots, y^{k_{\sigma^*}[i]}_{\sigma^*}[i])$ holds in M_i: $y^1_{\sigma^*}[i], \dots, y^{k_{\sigma^*}[i]}_{\sigma^*}[i]$ are not arms of a $(> k_\sigma[i])$-star.

- T_{σ^*} is the Π^0_3 predicate that says that $\neg\Theta_{\geq \ell_{\sigma^*}}(-)$ holds in M_{n+1}: there is no $(\geq \ell_{\sigma^*})$-star.

- S_{σ^*} is $\perp$, i.e. always false.

If σ^* has label f, we define:

- P_{σ^*} is defined in the same way as above.

- Q_{σ^*} is the Π^0_2 predicate that says that $\Gamma_{\geq \ell_{\sigma^*}}(x^1_{\sigma^*}, \dots, x^{\ell_{\sigma^*}}_{\sigma^*}, \rho_{\sigma^*})$ holds in M_{n+1}: $x^1_{\sigma^*}, \dots, x^{\ell_{\sigma^*}}_{\sigma^*}$ are arms of a $(\geq \ell_{\sigma^*})$-star.

- S_{σ^*} is the Π^0_3 predicate that says that $\neg\Theta_{\geq \ell_{\sigma^*}+1}(x^1_{\sigma^*}, \dots, x^{\ell_{\sigma^*}}_{\sigma^*})$ holds in M_{n+1}: $x^1_{\sigma^*}, \dots, x^{\ell_{\sigma^*}}_{\sigma^*}$ are not arms of a $(> \ell_{\sigma^*})$-star.

Note that P_{σ^*} does not depend on the values of $x^1_{\sigma^*}, \dots, x^{\ell_{\sigma^*}}_{\sigma^*}, \rho_{\sigma^*}$. The predicates are all expressed using the formulas Θ and Γ and have their intended meaning in nice spaces.

Now let (M^*, d^*) be obtained from Lemma 4.5 using the Π^0_3 predicates R_σ and S_σ, and the sequence (ℓ_σ). Write M^*_σ for the component built for σ, so that:

- if R_σ is false, then M^*_σ is the disjoint union of a point and ℓ_σ line segments;

- if R_σ is true and S_σ is false, then M^*_σ is the disjoint union of an ℓ_σ-star and a line segment;

- if R_σ and S_σ are both true, then M^*_σ is an $(\ell_\sigma+1)$-star.

We prove that M^* is not homeomorphic to any M_n through the following sequence of claims.

Claim 4.6. *For each σ with R_σ true, R_{σ^*} is true for exactly one child σ^* of σ.*

Proof. Let n be the height of σ, and $\sigma_1,\dots,\sigma_{n-1}$ the predecessors of $\sigma=\sigma_n$.

First we argue that R_{σ^*} cannot be true for two different children σ^* of σ. First, if two children σ^* and σ^{**} disagree about the value of $k_{\sigma_i}[i]$ for some σ_i labeled f, then it cannot be that R is true for both of them; indeed, (1) of P is incompatible with (2) of P, and (2) cannot be true for two different values of k. So if R is true for both σ^* and σ^{**}, then they agree on the values of $k_{\sigma_i}[i]$ and hence also on ℓ. If both σ^* and σ^{**} are labeled f, then R can be true of only the one which is to the left. They cannot both be labeled ∞, as then they would have the same parameters, and there is only one child of σ with each set of parameters. Finally, if σ^* is labeled f and σ^{**} is labeled ∞, then Q_{σ^*} being true is incompatible with $T_{\sigma^{**}}$.

Now we will show that R_{σ^*} is true for at least one child σ^* of σ. For each i with σ_i labeled f, since R_{σ_i} is true, $\Gamma_{\ge\ell_{\sigma_i}}(x^1_{\sigma_i},\dots,x^{\ell_{\sigma_i}}_{\sigma_i},\rho_{\sigma_i})$ holds. Let $k[i]$ be "$\ge 4n$" if there are $y^1[i],\dots,y^{k[i]}[i],\delta[i]$ such that:

- in M_i, for every $\epsilon<\delta$, and each j,j' there is an ϵ-path from $x^j_{\sigma_i}$ to $y^{j'}[i]$;
- $\Gamma_{\ge 4n}(y^1[i],\dots,y^{4n}[i],\delta[i])$ holds in M_i.

Otherwise, let $k[i]<4n$ be greatest such that there are $y^1[i],\dots,y^{k[i]}[i],\delta[i]$ such that:

- in M_i, for every $\epsilon<\delta$, and each j,j' there is an ϵ-path from $x^j_{\sigma_i}$ to $y^{j'}[i]$;
- $\Gamma_{\ge k_\sigma[i]}(y^1[i],\dots,y^{k[i]}[i],\delta[i])$ holds in M_i;

In the second case, by choice of $k[i]$, we also have

- $\neg\Theta_{\ge k_\sigma[i]+1}(y^1[i],\dots,y^{k[i]}[i])$ holds in M_i.

These are just the conditions from (1) and (2) of P_{σ^*}, so (1) and (2) of P_{σ^*} are true of any σ^* with these parameters.

Let ℓ be the least odd number $\le 4n$ such that $\{\ell,\ell+1\}$ is disjoint from

$$\{\ell_\tau,\ell_\tau+1:\tau \text{ is a predecessor of } \sigma^*\}\cup\{k[i]:i<n\}.$$

If $\Theta_{\ge\ell}(-)$ does not hold in M_{n+1} (there is no $(\ge\ell)$-star), then the node σ^* with the parameters described above and labeled ∞ has R_{σ^*} true. Otherwise, $\Theta_{\ge\ell}(-)$ holds in M_{n+1}; let $x^1,\dots,x^\ell,\rho$ be a witness to this such that the corresponding child σ^* of σ is the leftmost such child. Then Q_{σ^*} is true, but Q is not true of any child to the left of σ^*. □

Let $\sigma_1,\sigma_2,\sigma_3,\dots$ be the sequence of nodes at level 1, 2, and so on for which R holds. We call σ the true path.

Claim 4.7. *Fix $n < m$. Suppose that M_n is homeomorphic to the disjoint union of stars, points, and line segments, each of which is compact and open. Suppose that σ_n is labeled f, and $m > n$. Then if $k_{\sigma_m}[n]$ is "$\geq 4m$" if and only if $x^1_{\sigma_n}, \ldots, x^{\ell_{\sigma_n}}_{\sigma_n}$ is part of a $(\geq 4m)$-star, and otherwise $x^1_{\sigma_n}, \ldots, x^{\ell_{\sigma_n}}_{\sigma_n}$ is part of a $k_{\sigma_m}[n]$-star.*

Proof. This is immediate from the definitions of the predicate P. □

Claim 4.8. *Fix n. Suppose that M_n is homeomorphic to the disjoint union of stars, points, and line segments, each of which is compact and open. Then:*

- *if σ_n has label ∞, then M_n has no $\geq \ell$-star but M^* has an ℓ-star.*
- *if σ_n has label f, then:*
  - *if $x_1, \ldots, x_k$ are arms of an ℓ-star, then M^* has no ℓ-star;*
  - *if $x_1, \ldots, x_k$ are arms of a $(> \ell)$-star, then M^* has no star of the same size.*

Proof. First suppose that σ_n has label ∞. Then from T_{σ_n} we see that M_n does not have an r-star, $r \geq \ell_{\sigma_n}$. But $M^*_{\sigma_n}$ is an $(\ell_{\sigma_n} + 1)$-star.

Now suppose that σ_n has label f. The stars present in M^* are as follows, and no more: for each m, either an ℓ_{σ_m}-star or an $(\ell_{\sigma_m} + 1)$-star, where ℓ_{σ_m} is the least odd number $< 4m$ such that $\{\ell_{\sigma_m}, \ell_{\sigma_m} + 1\}$ is disjoint from

$$\{\ell_{\sigma_i}, \ell_{\sigma_i} + 1 : i < m\} \cup \{k_{\sigma_m}[i] : i < m\}.$$

Since Q_{σ_n} is true, $x^1_{\sigma_n}, \ldots, x^{\ell_{\sigma_n}}_{\sigma_n}$ are arms of a $(\geq \ell_{\sigma_n})$-star in M_n.

If they are arms of an ℓ_{σ_n}-star, then S_{σ_n} is true, and $M^*_{\sigma_n}$ is an $\ell_{\sigma_n} + 1$-star; for any i, j, $\{\ell_{\sigma_i}, \ell_{\sigma_i} + 1\}$ and $\{\ell_{\sigma_j}, \ell_{\sigma_j} + 1\}$ are disjoint, and so M^* does not have an ℓ_{σ_n}-star.

If $x^1_{\sigma_n}, \ldots, x^{\ell_{\sigma_n}}_{\sigma_n}$ are arms of a t-star for some $t > \ell_{\sigma_n}$, then S_{σ_n} is false, and $M^*_{\sigma_n}$ is an ℓ_{σ_n}-star. We claim that $t \notin \{\ell_{\sigma_i}, \ell_{\sigma_i} + 1\}$ for any $i \neq n$, so that M^* does not have a t-star. For $i < n$, $\ell_{\sigma_i} + 1 < \ell_{\sigma_n} < t$. For $i > n$, $\ell_{\sigma_i} < 4i$ is chosen so that if $k_{\sigma_m}[i] < 4i$, then $k_{\sigma_m}[i] \notin \{\ell_{\sigma_i}, \ell_{\sigma_i} + 1\}$. □

It follows from this claim that M^* is not homeomorphic to M_n for any n; indeed, if M^* was homeomorphic to M_n, then M_n would be the disjoint union of stars, points, and line segments, each of which is compact and open. Then either M_n has no $(\geq \ell)$-star but M^* does, or M_n has an ℓ-star but M^* does not, or M_n has a t-star for some $t > \ell$ but M^* does not have a t-star. □

Remark 4.9. The reader perhaps suspects that we could simplify the proof if we used "infinite stars". Indeed, we can make "infinite stars" compact by making the nth branch twice shorter than the $(n+1)$th branch and then putting them together carefully into a star-like object (we omit details). If we could use infinite stars then we would not have to worry about correcting errors too much; we could introduce an infinitary outcome under which an infinite star would be produced. This would significantly simplify the recursion-theoretic combinatorics of the proof by absorbing or completely eliminating some of the complex outcomes we we use in our proof. It seems that the resulting space will no longer be path-connected, but perhaps this construction (if it worked) would have some independent value.

Unfortunately, there is a fundamental technical issue with this idea which is actually quite subtle. More specifically, if we allow infinite stars then the analogy of Lemma 4.5 can potentially produce a finite star which is not open in the ambient space. (This will also make the space not locally path-connected.) This is because the "junk" left over from unsuccessful approximations to the infinitely many potential branches will be left arbitrarily close to the resulting finite star. The proof of the main definability Lemma 3.6 relies on Lemma 3.4 which requires each finite star to be also open. Informally, we could have the undesired situation when an ϵ-path jumps off a star to one of the junk elements and then returns back to some other star or to some other part of the same star; for a smaller ϵ_0 some other ϵ_0-path would jump off to a different piece of junk, etc. This completely breaks down both the intuition and the mathematical arguments used to justify the ϵ-paths technique. Even though the definability techniques developed in this paper can perhaps be adjusted to cover spaces with infinite stars which are not open, at the moment it is not clear.

Remark 4.10. With a bit of extra care, the space constructed in the theorem above can be realised as an effectively closed Δ^0_2-overt subset of $\mathbb{R}^2$ thus also witnessing Theorem 1.3 for $\alpha = 0$. (Recall that the space witnessing $\alpha = 0$ of Theorem 1.3 was not even Δ^0_4-overt.) For that, work inside $\mathbb{R}^2$ and use a careful Δ^0_2-approximation of the space. Make sure that every part of a component which is ever erased stays out of the space; if we ever have to reintroduce points to the space, we can use a new version of this erased part instead of reintroducing the old version which was erased. We leave the precise details to the reader.

Furthermore, by taking Alexandroff one-point extension of the space, we can produce a compact space witnessing the theorem. It will not be locally path-connected around the extra point adjoined in the process of compactification, but in contrast with Remark 4.9 the definability technique will still work. More specifically, we can arrange the construction so that all components are located inside the unit disc in $\mathbb{R}^2$. The components introduced later in the construction will be located closer to the centre of the disc. Of course, when we take the completion it will also force the centre of the disc to be in the space; the centre is the "one point" in the above-mentioned one-point compactification. It is not difficult to see that every point of the space, with the exception of the centre, belongs to a clopen star, while the connected component of the centre is a singleton which is not open. Although the resulting space is not nice according to the terminology introduced in the proof, we strongly conjecture that the definability technique that we developed is still sufficient to diagonalise against all computable spaces. This is because, e.g., in $\Theta_{\geq \ell}(p_1, \ldots, p_k)$ points $p_1, \ldots, p_k$ cannot all lie in the singleton connected component of the centre-point; we can of course assume $k > 1$. Then at least one point p_i must belong to a clopen compact star C, and if some other p_j happens to be the centre-point then, for ϵ smaller than the distance between C and $M^* \setminus C$, there cannot be an ϵ-path between p_i and p_j. We leave the exact details to the reader.

5 Groups

Theorem 5.1. *There exists a $0'$-computable compact Polish group not topologically isomorphic to any computable Polish group.*

Proof. The following lemma is a consequence of Lemma 4.2 in a compact group.

Lemma 5.2. *There exists a uniformly $0'$-computable procedure which, on input a computable compact Polish group G enumerates (the Cayley/multiplication table of) all finite groups K of the form G/N, where N ranges over (clopen) normal subgroups of A.*

Proof. The proof is similar to the proof of Cor. 4.8 from [Mel18]. We give details.

By Lemma 4.2, with the help of $0'$ we can enumerate all clopen subsets of G. Furthermore, these clopen subsets will be represented as finite unions of basic open balls. It also follows from the proof of Lemma 4.2 that the same clopen set N will also be described by a finite union of slightly smaller closed basic balls, and $0'$ can uniformly produce both the closed and the open description of N.

Since both operations $\cdot$ and $^{-1}$ are computable in G and N has two descriptions, checking whether N forms a normal subgroup requires merely $0'$. For example, $(\exists x)(x \in N \& x^{-1} \notin N)$ is a Σ^0_1-property because we can use the open description of N to check $x \in N$ and the closed description to verify $x^{-1} \notin N$. Note that we can restrict the quantifier to special points because $x \in N \& x^{-1} \notin N$ describes an open set. Using a similar trick we can see that normality, emptiness, and that N is closed under $\times$ are also $0'$-decidable properties.

Since G is compact, for a fixed clopen N the quotient group G/N is finite. Note that every coset of the form $\tilde{x}N$ is open and thus contains a special point x, thus is of the form xN for some special x. We claim that, given such an N, $0'$ can find finitely many special points $x_0 = e, x_1, \ldots, x_n$ such that $\{x_i N\}$ is a disjoint cover of G. To see why, note that $x_i N \cap x_j N \neq \varnothing$ iff for some special y, $y x_i^{-1} \in N$ and $y x_j^{-1} \in N$, both events are c.e. because N has a finite open description. Also, since left-translation is a self-homeomorphism of G onto itself and N is clopen, each $x_i N$ is clopen as well. Thus, $\{x_i N\}$ is a closed cover iff for every special y there is an i such that $y x_i^{-1} \in N$; if we view the latter as a finite union of *closed* balls then the statement becomes Π^0_1 and thus can be decided using $0'$. Similarly, the group structure upon $\{x_i\}$ mod N can be reconstructed effectively and uniformly, in N. Simply search for an x_k such that $x_i x_j x_k^{-1} \in N$ (this is an effective search in the open name of N) and then declare $x_i x_j =_N x_k$ in G/N. Note that the procedure above is uniform in the description of N. □

Consider the discrete countable group $G_S = \bigoplus_{p \in S} \mathbb{Z}_p \oplus \bigoplus_{i \in \omega} \mathbb{Z}$, where S is a set of primes (which is not necessarily the set of all primes). We use $\mathbb{Z}_p$ for the cyclic group of order p. The Pontryagin dual of G_S is the compact Polish group $A_S = \prod_{p \in S} \mathbb{Z}_p \times \prod_{i \in \omega} \mathbb{T}$, where $\mathbb{T}$ is the unit circle group. For this proof we only need one fact from Pontryagin duality: the finite quatients of A_S by its clopen subgroups are isomorphic to finite subgroups of G_S. This property can be seen directly for G_S and A_S, and therefore we will not give further details; see book [Pon66] for Pontryagin duality theory.

If we can list all finite factors of A_S, then we can also enumerate the set S. Using Lemma 5.2 above, to prove the theorem is sufficient to produce a $0'$-computable Polish presentation of A_S for a Π^0_2-complete set of primes S; then if A_S had a computable presentation, by Lemma 5.2 and the fact stated about Pontryagin duality we would be able to use $0'$ to enumerate the set S; but since S is not Σ^0_2, there is no enumeration of S relative to $0'$, and so A_S has no computable presentation.

Let $(p_i)_{i \in \omega}$ be the natural enumeration of all primes. Fix such a Π^0_2-complete set S and a computable predicate R such that $p_i \in S \iff \exists^\infty y R(i, y)$.

The $0'$-presentation of A_S will be a closed subgroup of the natural computable Polish presentation of $\prod_{i\in\omega}\mathbb{T}$. In this presentation, the special points are ω-tuples of rational numbers having finite support (i.e., zero almost everywhere).

For the i-th prime p_i, we reserve the i-th unit circle in $\prod_{i\in\omega}\mathbb{T}$. We intend to make the point $1/p_i$ in this circle isolated iff there are infinitely many y with $R(i,y)$. To do this, start with the set of fractions $\{1/p_j : j \geq i\}$. We will keep $1/p_i$ in the set regardless of the outcome. We will however extract $1/p_{i+1},\ldots,1/p_{i+t}$ from the set when we find that there are at least t numbers y with $R(i,y)$. If for i there are infinitely many such y, and thus $p_i \in S$, we will end up with $\{1/p_i\}$. Otherwise we will be left with $\{1/p_i\} \cup \{1/p_j : j \geq t\}$ for some t. Note that in the latter case the completion of the subgroup of $\mathbb{T}$ generated by the set is equal to the whole circle. In the former case the set generates a subgroup of $\mathbb{T}$ isomorphic to $\mathbb{Z}_{p_i}$.

For each fixed j, $0'$ can uniformly decide whether $1/p_j$ will be permanently kept in the set. Therefore, $0'$ can enumerate a dense subset of $\mathbb{T}$ whose completion is equal to $\mathbb{T}$ iff $p_i \notin S$, and furthermore the completion is isomorphic to $\mathbb{Z}_{p_i}$ iff $p_i \in S$. Since S is Π^0_2-complete it must be coinfinite, and therefore the product of the resulting uniformly $0'$-computable closed subgroups (sitting within their respective copies of the unit circle) will be isomorphic to $A_S = \prod_{p\in S}\mathbb{Z}_p \times \prod_{i\in\omega}\mathbb{T}$, as desired. This subgroup has a $0'$-computable Polish presentation which is naturally given by the $0'$-enumerable set of ω-tuples of special points with finite support, and with non-zero components corresponding to those fractions which have already been $0'$-effectively listed in the respective circle. □

Theorem 5.3. *For every computable α there exists an effectively closed compact (thus, profinite) subgroup of S_∞ not homeomorphic to any Δ^0_α Polish group.*

Proof. As in the previous theorem, a Δ^0_α-computable Polish presentation of the group $A_S = \prod_{p\in S}\mathbb{Z}_p$ gives rise to a $\Delta^0_{\alpha+1}$-enumeration of S. The theorem would follow if for each computable successor ordinal β we could produce an effectively closed subgroup of S_∞ isomorphic to A_S, where S is Σ^0_β-complete. (To witness the theorem for α take $\beta > \alpha + 1$.)

To do that, we will construct a computable structure $\mathcal{M}$ with domain ω and automorphism group $Aut(\mathcal{M}) \cong A_S$. Since the automorphism groups of such computable algebraic structures are effectively closed subgroups of S_∞ (see [GMNT18]), the theorem will follow.

The structure $\mathcal{M}$ will consist of infinitely many disjoint gadget-substructures $\mathcal{M}_i$, each working with the respective prime p_i. The automorphism group of $\mathcal{M}$ will naturally be the Cartesian product of the automorphism groups of all these $\mathcal{M}_i$.

It remains to produce a uniformly effective sequence of computable structures $(\mathcal{M}_i)_{i\in\omega}$ with the property:

$$Aut(\mathcal{M}_i) \cong \mathbb{Z}_{p_i} \iff i \in S,$$

where S is the Σ^0_β-complete set we fixed above, and $\mathcal{M}_i$ is rigid if $i \notin S$.

Using the standard technique due to Ash (see [Ash86] and, for the specific result, see [GK02]) on input i we can uniformly produce a computable ordinal γ_i such that $\gamma_i \cong \delta$ if $i \in S$ and $\gamma_i \cong \delta'$ if $i \notin S$, where $\delta \ncong \delta'$ are computable ordinals which depend on β. The structure $\mathcal{M}_i$ consists of a loop of size p_i realised using a unary function u:

$$u(x_{i,j}) = x_{i,j+1\, mod\, p_i}.$$

In M_i, each point $x_{i,j}$ will be computably associated with a "box"; more formally, using another unary function s we isolate the set $Y_{i,j} = \{y : s(y) = x_{i,j}\}$ which will be disjoint from

$Y_{i',j'}$ whenever $j \neq j'$. On each such set $Y_{i,0}$, using the aforementioned result of Ash and a special binary relational symbol, we will uniformly construct a computable well-ordering which will be isomorphic to δ if $i \in S$, and δ' if $i \notin S$. On each other set $Y_{i,j}$, $j \neq 0$, we construct a computable copy of δ.

For each fixed i, we will end up with all $x_{i,j}$ being automorphic to each other if and only if $i \in S$; if $i \in S$, then they each have a copy of δ in their associated "box", and if $i \notin S$, then $x_{i,0}$ has a copy of δ' while each other $x_{i,j}$ has a copy of δ. Furthermore, since ordinals are rigid, $Aut(M_i) \cong \mathbb{Z}_{p_i}$ if $i \in S$, and $\mathcal{M}_i$ is rigid otherwise, as desired. □

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