INCOMPARABILITY IN LOCAL STRUCTURES OF s-DEGREES AND Q-DEGREES

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ABSTRACT. We show that for every intermediate Σ_2^0 s-degree (i.e. a nonzero s-degree strictly below the s-degree of the complement of the halting set) there exists an incomparable Π_1^0 s-degree. (The same proof yields a similar result for other positive reducibilities as well, including enumeration reducibility.) As a consequence, for every intermediate Π_2^0 Q-degree (i.e. a nonzero Q-degree strictly below the Q-degree of the halting set) there exists an incomparable Σ_1^0 Q-degree. We also show how these results can be applied to provide proofs or new proofs (essentially already known, although some of them not explicitly noted in the literature) of upper density results in local structures of s-degrees and Q-degrees.

1. INTRODUCTION

The reducibility known as s-reducibility is a restricted version of enumeration reducibility. We recall that any computably enumerable (or, c.e.) set W defines an *enumeration operator* (for short: *e-operator*), i.e. a mapping Φ_W from the power set of ω to the power set of ω (where ω denotes the set of natural numbers) such that, for $A \subseteq \omega$,

$$\Phi_W(A) = \{x : (\exists u)\, [\langle x, u\rangle \in W \text{ and } D_u \subseteq A]\,\},$$

where D_u is the finite set with canonical index u: throughout the rest of the paper, we will often identify finite sets with their canonical indices, thus writing, for instance, $\langle x, D\rangle$ instead of $\langle x, u\rangle$ if $D = D_u$. If $A = \Phi(B)$ for some e-operator Φ then we say that A is *enumeration reducible* to B (or, more simply, A is *e-reducible* to B; in symbols: $A \leqslant_e B$) *via* Φ. An e-operator Φ is said to be an *s-operator*, if Φ is defined by a c.e. set W such that

$$(\forall \text{ finite } D)(\forall x)[\langle x, D\rangle \in W \Rightarrow \text{card}(D) \leqslant 1]$$

(where the symbol $\text{card}(X)$ denotes the cardinality of a given set X). Following [4] we say that A is *s-reducible* to B (in symbols: $A \leqslant_s B$) if $A = \Phi(B)$, for some s-operator Φ. We refer the reader to [7] for an introduction to s-reducibility.

It is easy to see (see e.g. [7]) that there is a least s-degree (denoted by $\mathbf{0}_s$, consisting of all c.e. sets) and that $A \in \Sigma_2^0$ if and only if $A \leqslant_s \overline{K}$, where K is the halting set, and for a given set X, the

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2010 *Mathematics Subject Classification.* 03D25, 03D30.

Key words and phrases. Enumeration reducibility, s-reducibility, s-degree, Q-reducibility, Q-degree.

Chitaia's work was supported by Shota Rustaveli National Science Foundation of Georgia (SRNSFG) [Grant number: YS-18-168].

Ng was partially supported by the grants MOE-RG131/17 and MOE2015-T2-2-055.

Sorbi was partially supported by PRIN 2017 Grant "Mathematical Logic: models, sets, computability". Sorbi is a member of INDAM-GNSAGA.

Yang's research was partially supported by NUS grant R-146-000-231-114.

symbol $\overline{X}$ denotes the complement of X. Let us denote by $\mathbf{0}'_s$ the s-degree of $\overline{K}$: the collection of s-degrees below $\mathbf{0}'_s$ is called the *local structure* of the s-degrees.

The reducibility known as Q-reducibility (introduced by Tennenbaum, see [8, p. 159]) is an isomorphic copy of s-reducibility: a set A is *Q-reducible* to a set B (in symbols: $A \leqslant_Q B$) if there exists a computable function f such that

$$(\forall x)[x \in A \Leftrightarrow W_{f(x)} \subseteq B].$$

(It is convenient to exclude ω from the universe of the reducibility, as $A \leqslant_Q \omega$ if and only if $A = \omega$.) Both s-reducibility and Q-reducibility have been frequently and successfully applied to computability theory, abstract complexity theory, group theory and word problems: we refer the reader to Omanadze's paper [6] for an exhaustive survey of applications of Q-reducibility. It is easy to see that $A \leqslant_s B$ if and only if there exists a computable function f such that (if $\overline{B} \neq \omega$)

$$x \in \overline{A} \Leftrightarrow W_{f(x)} \subseteq \overline{B};$$

in other words $A \leqslant_s B$ if and only if $\overline{A} \leqslant_Q \overline{B}$. Hence the Q-degrees are order isomorphic with the s-degrees. The *local structure* of the Q-degrees consists of the Q-degrees of the Π^0_2 sets. For every set A, we have that $A \in \Pi^0_2$ if and only if $A \leqslant_Q K$: thus the Q-degree of K is the greatest element of this local structure, whereas the least element is comprised of all decidable sets (with the exclusion of ω).

We prove in this paper (Theorem 2.3) that for every intermediate s-degree of the local structure of the s-degrees (i.e. for every $\mathbf{a}$ with $\mathbf{0}_s <_s \mathbf{a} <_s \mathbf{0}'_s$) there exists an incomparable Π^0_1 s-degree (i.e. an s-degree containing some Π^0_1 set), or, otherwise stated, for every Σ^0_2 set A, with A not c.e. and $\overline{K} \nleqslant_s A$, there is a Π^0_1 set B such that $A|_s B$ (where, for a given reducibility $\leqslant_r$, the symbol $|_r$ denotes incomparability with respect to $\leqslant_r$). As a consequence, via the isomorphism between s- and Q-degrees, for every Π^0_2 set A, with A undecidable and $K \nleqslant_Q A$, there exists a c.e. set B such that $A|_Q B$. A straightforward modification of the proof shows that for every Σ^0_2 set A such that $\emptyset <_e A <_e \overline{K}$ there is a Π^0_1 set B such that $A|_e B$ (in e-degrees this has the additional interest that every Π^0_1 e-degree contains a total function). Different straightforward modifications of the proof show that the same incomparability result holds of other positive reducibilities, stronger than $\leqslant_s$, with the corresponding result holding, via the isomorphism provided by complements of sets, of the corresponding sub-reducibilities of Q-reducibility. Finally we point out how to use our incomparability results for $\leqslant_s$ and $\leqslant_Q$ to prove in one shot some upper density results (essentially already known in the literature, although some of them never explicitly noticed), namely that the s-degrees lying in the class $\mathcal{X}$ (for $\mathcal{X} \in \{\Pi^0_1, \Delta^0_2, \Sigma^0_2\}$) are upwards dense, and the Q-degrees lying in $\mathcal{X}$ (for $\mathcal{X} \in \{\Sigma^0_1, \Delta^0_2, \Pi^0_2\}$) are upwards dense.

2. Main Result

Although stated for $\leqslant_s$, via the isomorphism between s-degrees and Q-degrees our main result (Theorem 2.3) can be viewed as a generalization of the following Fact 2.1: our generalization goes from c.e. sets to all Π^0_2 sets.

Fact 2.1 (Folklore)**.** *Given any undecidable c.e. set A such that $A <_Q K$, there exists a c.e. set B such that $A \,|_Q\, B$.*

Proof. The idea is the following. Let A be an undecidable c.e. set such that every c.e. set B is Q-comparable with A. If A is T-incomplete then there exists a c.e. set B such that A and B are

T-incomparable, so A and B are Q-incomparable as $X \leqslant_Q Y$ implies $X \leqslant_T Y$ for c.e. sets. Thus assume now that A is T-complete. For each c.e. set W_e by Dekker's deficiency set construction one can construct a hypersimple $W_{f(e)}$ of the same Turing degree as W_e such that $W_{f(e)}$ is semirecursive. Now $A \leqslant_Q W_{f(e)}$ if and only if $W_{f(e)}$ is T-complete, as every semirecursive and hypersimple T-complete set is also Q-complete. This would give a Σ^0_4 predicate recognizing T-incompleteness of $W_{f(e)}$ and hence also T-incompleteness of W_e. However, the index set $\{e : W_e\ T\text{-incomplete}\}$ is Π^0_4-complete. $\square$

On the other hand it also generalizes the following result:

Fact 2.2. [1, Theorem 13] *For every intermediate Π^0_2 Q-degree $\mathbf{a}$ there is a Δ^0_2 Q-degree $\mathbf{b}$ such that $\mathbf{a}|_Q\mathbf{b}$.*

In fact, it is shown in [1, Theorem 13] that the Δ^0_2 Q-degree $\mathbf{b}$ can be chosen to form a minimal pair with $\mathbf{a}$ in the c.e. Q-degrees. So, our Theorem 2.3 would follow from this, should it be true that every nonzero Δ^0_2 Q-degree bounds a nonzero c.e. Q-degree: but this is not certainly the case since it is known that there are quasiminimal Δ^0_2 e-degrees, which implies that there are nonzero Δ^0_2 s-degrees which do not bound any nonzero Π^0_1 s-degree, by the already mentioned fact that every Π^0_1 e-degree contains a total function.

Theorem 2.3. *Given any $A \in \Sigma^0_2 \smallsetminus \Sigma^0_1$ such that $A <_s \overline{K}$, there exists a Π^0_1 set B such that $A \mid_s B$, in fact $A \nleqslant_e B$ and $B \nleqslant_s A$.*

Proof. Let $A \in \Sigma^0_2 \smallsetminus \Sigma^0_1$ be given, with $A <_s \overline{K}$. The proof turns out to be nonuniform: we will reach the conclusion by distinguishing the cases $A \in \Sigma^0_2 \smallsetminus \Delta^0_2$, and $A \in \Delta^0_2 \smallsetminus \Sigma^0_1$.

The requirements. Fix effective lists $\{\Phi_e : e \in \omega\}$ and $\{\Gamma_e : e \in \omega\}$ of the e-operators and the s-operators, respectively: we may also assume (see e.g. [5] or [7]) that we have uniform computable approximations $\{\Phi_{e,s} : e, s \in \omega\}$ and $\{\Gamma_{e,s} : e, s \in \omega\}$ to these operators, viewed as c.e. sets, so that each $\Phi_{e,s}$ and $\Gamma_{e,s}$ is a finite set given by its canonical index. The reason for taking one of the two lists consisting of the e-operators is due to the fact that this will allow to derive $A \nleqslant_e B$, and the more general statement in Corollary 3.1.

We need to build a Π^0_1 set B (which will be given through a Π^0_1-approximation $\{B_s : s \in \omega\}$ starting with $B_0 = \omega$) so as to satisfy the following requirements:

$$N_e : (\exists x)\,[A(x) \neq \Phi_e(B)(x)] \text{ or } A \in \mathcal{X}$$
$$P_e : (\exists x)\,[B(x) \neq \Gamma_e(A)(x)] \text{ or } \overline{K} \leqslant_s A.$$

where, in N_e, $\mathcal{X} = \Sigma^0_1$ if $A \in \Delta^0_2 \smallsetminus \Sigma^0_1$, and $\mathcal{X} = \Delta^0_2$ if $A \in \Sigma^0_2 \smallsetminus \Delta^0_2$.

Strategy for N_e in isolation. We start with the more difficult case $A \in \Sigma^0_2 \smallsetminus \Delta^0_2$, and we fix a Σ^0_2-approximation to A, i.e. a strong array of finite sets $\{A_s : s \in \omega\}$ such that $A = \{x : (\exists t)(\forall s \geqslant t)[x \in A_s]\}$. In this case, the N_e-requirement becomes:

$$(\exists x)\,[A(x) \neq \Phi_e(B)(x)] \text{ or } A \in \Delta^0_2.$$

We organize the possible outcomes of the N_e-strategy in the chain of order type $\omega + 1$:

$$0 < 1 < 2 < \cdots < x < \cdots < \mathtt{fin}.$$

We define an auxiliary set V aiming to show that if $A = \Phi_e(B)$ then $A = V$ and V is Δ^0_2.

Suppose that we are working at stage $s+1$.

Starting from $x=0$, we work in cycles $x=0,1,\ldots$, until $x=s+1$, or until giving outcome x or `fin`. The case $x \in A$ is recognized in a Σ^0_2 *way* (this will be made precise in the construction); the case $x \in \Phi_e(B)$ denotes that there is an axiom $\langle x, D\rangle \in \Phi_{e,s}$ and $D \subseteq B_s$, where B_s is the cofinite approximation to B built by the end of stage s. (For simplicity, in describing the strategy below if there is no mention of the stage at which a given parameter is assumed to be approximated, then it is understood that this stage is s.)

In cycle x we try to recognize a winning outcome $A(x) \neq \Phi_e(B)(x)$. Whenever we see $x \in \Phi_e(B)$ we act to restrain $x \in \Phi_e(B)$ and we define $V(x)=1$: in case A makes useless our efforts to diagonalize $x \in \Phi_e(B) \smallsetminus A$ by ending up with the Σ^0_2 event $x \in A$ we have at least that $V(x)$ gives the correct $A(x)=V(x)=1$. Likewise, when we are not able to force $x \in \Phi_e(B)$ because we are short of opportunities of restraining x in $\Phi_e(B)$, then, whenever we see this, we define $V(x)=0$: should A refuse to diagonalize at x (such a diagonalization would make `fin` with parameter x the winning Σ^0_2 outcome) we still end up with the correct guess of the value $V(x)$, i.e. $A(x)=V(x)=0$.

If we see $x \in \Phi_e(B) \smallsetminus A$ then we give outcome x, a Π^0_2 outcome since infinitely many times we may see $x \in A$, although $x \notin A$; if we see $x \in A \smallsetminus \Phi_e(B)$ then we give outcome `fin` with parameter x: as the event $x \notin \Phi_e(B)$ will turn out to be Σ^0_2 (we try to restrain $x \in \Phi_e(B)$ whenever this appears to be the case, and we could certainly be able to do so, should this appear infinitely often) this is a winning Σ^0_2 outcome.

Whether we give outcome x or `fin` we move on to next requirement; in the other cases (i.e. when it appears $A(x)=\Phi_e(B)(x)$) we move to cycle $x+1$ (unless already $x=s$)

If $A=\Phi_e(B)$ then we can easily argue that V is in the end a Δ^0_2 set (in fact, as we will see at the end of the proof, a c.e. set if $A \in \Delta^0_2$), so that there is some least x such that $A(x) \neq V(x)$, and x is the winning outcome (recognized infinitely often) if $x \notin A$, or `fin` with parameter x is the winning outcome if $x \in A$.

We summarize the cases into the following table

	$x \notin A_s$	$x \in A_s$	$V_s(x)$
$x \in \Phi_e(B)[s]$	(1.1); outcome x	(2.1); outcome $\rightarrow$	1
$x \notin \Phi_e(B)[s]$	(1.2); outcome $\rightarrow$	(2.2); outcome `fin` with parameter x	0

where we assume that "outcome $\rightarrow$" means that we move to $x+1$, if $x<s$. (In the expression $\Phi_e(B)[s]$ we use the suffix $[s]$ to denote that both Φ_e and B in the expression are evaluated at stage s.)

Strategy for P_e in isolation. We now describe P_e in isolation. We work with a fixed Π^0_1-approximation $\{\overline{K}_s : s \in \omega\}$ to $\overline{K}$. We have movable markers $b^e_0, b^e_1, \ldots$ (for simplicity we will write $b_i = b^e_i$) aiming at B, and targeted for P_e. The plan is to code $\overline{K}$ into B (if i stays in $\overline{K}$, then we keep b_i into B; when i enters K we extract b_i from B) and threaten to have $i \in \overline{K}$ if and only if $i \in \Gamma(A)$, in the case that $\Gamma_e(A)(b_i)=B(b_i)$: here Γ is the s-operator consisting of axioms $\langle i, D\rangle \in \Gamma$ if and only if $\langle b_i, D\rangle \in \Gamma_e$, so if $B=\Gamma_e(A)$ then this would give $\overline{K} \leqslant_s A$, a contradiction.

In the following we measure the value $\Gamma_e(A)(b)$ in a Σ^0_2 way, as will be made precise in the construction. (Again, for simplicity, if there is no mention of the stage at which a given parameter is assumed to be approximated, then it is understood that this stage is s.)

Suppose we are working at stage $s+1$. Starting from $i=0$, we work in cycles $i=0,1,\ldots$, until $i=s+1$ or we give outcome i or `fin`: In cycle i, we try to recognize a winning outcome $\Gamma_e(A)(b_i) \neq B(b_i)$. If we see $i \in \overline{K}$ (hence $b_i \in B$), but $b_i \notin \Gamma_e(A)$ then we give outcome i, a winning Π^0_2 outcome. If we see $i \notin \overline{K}$ (hence $b_i \notin B$), but $b_i \in \Gamma_e(A)$ then we give outcome `fin` with parameter i, a winning Σ^0_2 outcome. In the previous two cases we move on to next requirement; in all other cases we move on to next cycle (unless already $i=s$).

We summarize the cases into the following table:

	$i \in \overline{K}_s$ and $b_i \in B_s$	$i \notin \overline{K}_s$ and $b_i \notin B_s$
$b_i \notin \Gamma_e(A)[s]$	(1.1); outcome i	(2.1); outcome $\rightarrow$
$b_i \in \Gamma_e(A)[s]$	(1.2); outcome $\rightarrow$	(2.2); outcome `fin` with parameter i.

Should there be no least i such that i, or `fin` with parameter i, is the final winning outcome (recognized infinitely often) then we would end up with $\Gamma_e(A)(b_i) = B(b_i)$ for every i. But then the s-operator Γ (as described earlier) would give $\overline{K} = \Gamma(A)$, contrary to the assumption that $\overline{K} \nleq_s A$.

Interactions between strategies. We now briefly discuss the interactions between the strategies. Although the tree of strategies is formally introduced only in next section, nevertheless for the ease of the reader who is acquainted with priority arguments using trees of strategies we will make in this section occasional references to the tree of strategies.

N-strategies below a P-strategy. Now an adjustment to the strategy of P_e must be made. If i is the leftmost outcome to be played infinitely often, then it can be the case that at infinitely many times we play outcomes to the right of i: this could be a problem for N-strategies below outcome i of P_e, as they might require to restrain in B markers of the form $b_j (= b^e_j)$, with $j > i$ (again, in referring to these markers, we will omit to specify the superscript e), that at the moment are still in B, but later may be requested to be extracted from B by P_e, in response to P_e working in cycle j, and j entering K. So we must take measures to prevent this from happening: when we play at $s+1$ outcome $i \in \omega$ we shall extract the current values, if defined, of the markers b_j from B for every $j > i$, and later choose a fresh value for b_j when needed.

Observation 1. This move of pulling out of B the current values of the markers b_j will make it possible for the N-requirements below outcome i for P_e to choose elements to restrain which are different from these discarded markers and for which there is no danger that P_e will later demand to have them out of B. So if P_e is located at node τ in the tree of outcomes, N_k is located at node $\sigma \supseteq \tau^\frown\langle i\rangle$ in the tree of outcomes, then no injury is made by τ to σ when τ is working at cycles $j > i$.

In this updated version of the P_e-strategy, now b_i may change depending on the stage, and so one should rather talk about $\lim_s b_{i,s}$ (whenever this limit exists), where $b_{i,s}$ denotes the value of b_i at s, rather than a single e-marker b_i for i, appointed once for all. Then the above considerations for P_e in isolation work for $b_i = \lim_s b_{i,s}$. But this introduces new outcomes, with respect to the ones shown above: if at $s+1$ we discard $b_{i,s}$ which is thus moved out of B, then if the reduction given by Γ is to fail because of $b_{i,s}$ (before discarding $b_{i,s}$ we may have added axioms $\langle i, D\rangle \in \Gamma$ with $\langle b_{i,s}, D\rangle \in \Gamma_e$, and now we may have $i \in \Gamma(A)$ even if $i \notin \overline{K}$), it must be via (2.2) and outcome `fin` will detect this (we shall say in this case that the outcome is `fin` with parameter (i,u), where u

denotes that $b_{i,s}$ is the u-th different value of b_i chosen after last initialization of P_e). This is a Σ^0_2-outcome for P_e. Otherwise, the burden of coding $\overline{K}(i)$ is resting only on the limit value b_i for which we can repeat the analysis of the outcomes of cycle i for P_e in isolation (with the only difference that what was therein called "outcome `fin` with parameter i", will now be called "outcome `fin` with parameter (i,u)", where u denotes that the final b_i is the u-th different value of b_i chosen after last initialization of P_e).

P-strategies below an N-strategy. As we have seen, P-requirements below outcome x for N_e may need to extract markers from B. Assume that x is the true outcome of N_e because of (2.2): due to the infinitary nature of the construction, infinitely many times we may have outcomes $y > x$, for which N_e would like to restrain elements in B, but we cannot give to this restraints priority σ (where we assume that N_e is located at σ in the tree of strategies) since moving infinitely many times to the right of x could entail an infinite amount of restraint located at σ, which would spoil the strategies of the P-requirements below outcome x for N_e. In order to overcome this problem, we give priority $\sigma^\frown\langle y\rangle$ to the restraint requested by y, so that this restraint will not be obeyed by the P-strategies below $\sigma^\frown\langle x\rangle$, with the agreement that when N_e plays outcome x, we will reset *all* outcomes $y > x$ by cancelling their restraints. Hence all those $y > x$ for which we previously had $\Phi_e(B)(y) = 1$ and which were restraining some number z_y in B will now release the restraint they held on z_y.

Observation 2. This allows P nodes extending N_e below outcome x to use these numbers z_y for their markers. So no injury will be made by N_e to a P-node $\tau \supseteq \sigma^\frown\langle x\rangle$.

The construction. As for the description of the strategy of N_e in isolation, we deal first with the more difficult case, i.e. $A \in \Sigma^0_2 \smallsetminus \Delta^0_2$, and we work with our fixed Σ^0_2-approximation to A. For the construction we use the tree of outcomes $T = (\omega \cup \{\mathtt{fin}\})^*$, i.e. the set of finite strings over the set $\omega \cup \{\mathtt{fin}\}$ ordered by

$$0 < 1 < 2 < \cdots < x < \cdots < \mathtt{fin}.$$

Notations and terminology for trees of outcomes are standard, see e.g. [9]: in particular, λ denotes the empty string; if $\sigma, \tau \in T$, then σ is *to the left of* τ if σ precedes τ in the lexicographical order (we use the symbol $\sigma <_L \tau$); and we say that σ has *higher priority than* τ if $\sigma \subset \tau$ or $\sigma <_L \tau$. To each $\sigma \in T$ is assigned a requirement $R(\sigma)$, by

$$R(\sigma) = \begin{cases} N_e, & \text{if } |\sigma| = 2e, \\ P_e, & \text{if } |\sigma| = 2e+1. \end{cases}$$

If $R(\sigma) = N_e$ for some e, then we say that σ is an *N-node*, or an *N-strategy*; otherwise, σ is a *P-node*, or a *P-strategy*. The construction uses several parameters: if σ is an N-node, then it uses the parameters (all depending on the stages) V^σ, and $\{z^\sigma_x : x \in \omega\}$; if σ is a P-node, then it uses markers $\{b^\sigma_i : i \in \omega\}$ (these markers are *targeted* for σ, i.e. distinct P-nodes pick markers from disjoint sets), and the s-operator Γ^σ; together with each current value of b^σ_i we have a sequence $b^\sigma(i,0), \dots, b^\sigma(i, n^\sigma_i)$ (which introduces a new parameter n^σ_i): these are the old chosen versions of the marker, appointed after last initialization of σ, which have been later abandoned (except for the last one, $b^\sigma_i = b^\sigma(i, n^\sigma_i)$) due to procedure of resetting its markers performed by the P-node σ to deal with its interactions with lower priority N-strategies, as explained in the previous section. All parameters, including markers, Γ^σ and n^σ_i, depend of course on the stages.

We say that a strategy σ is *initialized at stage s* if at s we set $V^\sigma = \Gamma^\sigma = \varnothing$, and all its other parameters are set to be undefined.

At stage s together with the values of the parameters, we also define a finite string δ_s, of length $\leqslant s$. A stage s is a *σ-stage* if $\sigma \subseteq \delta_s$. It is understood that at the end of stage $s+1$ each parameter keeps the same value as at s unless it is explicitly redefined, or initialized, during stage $s+1$.

Stage 0. Initialize all strategies. Let $B_0(x) = 1$ for all x.

Stage $s+1$. We define δ_{s+1} by substages: if stage $s+1$ has not been already stopped, then at substage $t < s+1$ we define σ_t extending all σ_r with $r < t$.

If $t = 0$ then we define $\sigma_0 = \lambda$, and we move to substage 1.

At substage $t+1$, if $t+1 = s+1$ then we go directly to stage $s+2$.

Otherwise (i.e. $t+1 < s+1$) denote $\sigma = \sigma_t$. In the following description of the actions at substage $t+1$, for simplicity we do not specify σ when referring to the various parameters (assumed to be approximated as at the end of the previous substage, or at the end of stage s if $t = 1$), thus writing for instance $V = V^\sigma$, $z_x = z_x^\sigma$, etc. The string $\sigma_{t+1} = \sigma^\frown\langle o\rangle$, for some outcome o, ends the stage, or is the string which will be *let to act next.* The action to be taken depends of course on whether σ is an N-node or a P-node.

σ is an N-strategy. Suppose $R(\sigma) = N_e$. We first give two definitions. Given a number x we say that *we see $x \in A$ at $\sigma, s+1$* if, letting s^- be the last σ-stage at which we processed cycle x, if any, and $s^- = 0$ otherwise, we have that $x \in A_v$ for all v with $s^- \leqslant v \leqslant s$. We also say that *we see $x \in \Phi_e(B)$ at $s+1$* to denote that there exists $\langle x, D\rangle \in \Phi_{e,s}$ and $D \subseteq B^-_{s+1}$, where B^-_{s+1} is B_s minus the elements y which have been extracted from B by the P-strategies that have acted at the previous substages $0, 1, \dots, t$ of the current stage $s+1$.

Observation 3. An easy argument shows that if there are infinitely many stages v at which we process cycle x for σ, then $x \in A$ if and only if at cofinitely many such stages v we see $x \in A$ at σ, v.

We start the cycles.

Cycle x. Suppose we have dealt with the cycles $y < x$ all ending with "outcome"$\rightarrow$, i.e. all ending with neither y nor `fin`. We now *process* cycle x if $x < s+1$, otherwise if $x = s+1$ we stop the stage with outcome x. If $x < s+1$ we distinguish the following cases.

Case 1. We see $x \in A$ at $\sigma, s+1$.

Subcase 1.1. We do not see $x \in \Phi_e(B)$ at $s+1$. Let the outcome be `fin` with parameter x, hence $\sigma^\frown\langle \texttt{fin}\rangle$ is eligible to act next. Define $V(x) = 0$. Go to next requirement.

Subcase 1.2. We see $x \in \Phi_e(B)$ at $s+1$. Define $V(x) = 1$ and add a restraint on B (with priority $\sigma^\frown\langle x\rangle$) to keep in B from now on some D such that $\langle x, D\rangle \in \Phi_{e,s}$ and $D \subseteq B^-_{s+1}$: in fact, take the least such D if no such restraint has been made after the last initialization of $\sigma^\frown\langle x\rangle$, otherwise keep the same D as already chosen at the previous $\sigma^\frown\langle x\rangle$-stage. If we restrain this D for the first time then stop the stage with outcome x, otherwise go to cycle $x+1$.

Case 2. We do not see $x \in A$ at $\sigma, s+1$.

Subcase 2.1. We do not see $x \in \Phi_e(B)$ at $s+1$. Define $V(x) = 0$. Go to cycle $x+1$.

Subcase 2.2. We see $x \in \Phi_e(B)$ at $s+1$. Define $V(x) = 1$. Let the outcome be x and add a restraint on B (with priority $\sigma^\frown\langle x\rangle$) to keep in B from now on the least D such that $\langle x, D\rangle \in \Phi_{e,s}$ and $D \subseteq B^-_{s+1}$: in fact, take the least such D if no such restraint has been made after the last initialization of $\sigma^\frown\langle x\rangle$, otherwise keep the same D as already chosen at the previous $\sigma^\frown\langle x\rangle$-stage. If we restrain D for the first time then stop the stage with outcome x.

Notice that in both subcases 1.2 and 2.2, if we restrain D for the first time then we stop the stage: since lower priority P-strategies are reset by our stopping the stage and will have to choose their (fresh) markers again out of D, the restraint of $D \subseteq B$ will be preserved unless we move in the future to the left of $\sigma^\frown\langle x\rangle$ again.

The case of a P-node σ. Suppose $R(\sigma) = P_e$. We start the cycles. Suppose we have dealt with the cycles $j < i$ all ending with "outcome" $\rightarrow$, i.e. all ending with neither outcome j nor `fin`. We now *deal with* cycle i, if $i < s+1$, otherwise if $i = s+1$ then we stop the stage with outcome i. For simplicity, when mentioning any defined marker b_j^σ or $b^\sigma(j,u)$, we will omit the superscript σ. In the following, when writing $i \in \overline{K}$ we mean that $i \in \overline{K}_s$, where $\{\overline{K}_s : s \in \omega\}$ is our fixed Π^0_1-approximation to $\overline{K}$; we write $i \in K$ to denote that $i \notin \overline{K}_s$.

Preparing cycle i. We first define b_i if it is undefined, and update the s-operator Γ, as follows.

- if b_i is undefined, then pick and appoint a new targeted $b_i = b_{i,s+1} \in B_s$ (not currently restrained out of B by higher priority nodes); this also defines the new value of n_i; stop the stage.
- Otherwise b_i is defined: if $i \in \overline{K}$ then keep $B(b_i) = 1$, and if $i \in K$ then let $B(b_i) = 0$. *Update* Γ, i.e. add an axiom $\langle i, D\rangle \in \Gamma_{s+1}$ for each axiom $\langle b_i, D\rangle \in \Gamma_{e,s}$, where b_i is the current value of the i-marker for σ.

Now, in the case b_i is defined, let

$$b(i,0), b(i,1), \dots, b(i,n_i)$$

(with $b_i = b(i,n_i)$) be the markers for i defined after last initialization of σ and later abandoned (except for the last one) with $b(i,u) = b_{i,s_u}$, where s_u is the stage at which $b(i,u)$ has been appointed, and $s_0 < s_1 < \cdots < s_{n_i}$. (The construction has ensured that $b(i,u) \notin B$, for all $u < n_i$.)

Starting from our fixed Σ^0_2-approximation $\{A_s : s \in \omega\}$ to A and our uniform c.e. approximation $\{\Gamma_{e,s} : e, s \in \omega\}$ to the s-operators, we can build (see e.g. [5] and [7]) a computable relation $R(e,x,s)$ such that

$$x \in \Gamma_e(A) \Leftrightarrow (\exists t)(\forall s \geqslant t) R(e,x,s).$$

Within cycle i we will work in subcycles (i,u) with $u \leqslant n_i$. We stipulate that *we see* $b(i,u) \in \Gamma_e(A)$ *at* $\sigma, s+1$ if, letting s^- be the last σ-stage at which we processed subcycle (i,u) within cycle i, if any, and $s^- = 0$ otherwise, we have that

$$(\forall v)[s^- \leqslant v \leqslant s \Rightarrow R(e, b(i,u), v)].$$

Observation 4. For $u \leqslant n_i$ it is easy to see that if there are infinitely many σ-stages at which we process subcycle (i,u) then $b(i,u) \in \Gamma_e(A)$ if and only if we see $b(i,u) \in \Gamma_e(A)$ at cofinitely many such stages.

Subcycle (i,u). Suppose we have already dealt with all subcycles (i,v) relative to $v < u \leqslant n_i$, and we now consider u. We distinguish the following cases.

Case $u < n_i$. We have two subcases:

Case 1. If we see $b(i,u) \in \Gamma_e(A)$ at $\sigma, s+1$ (remember that $b(i,u)$ has been extracted from B) then let the outcome be $\mathtt{fin}$ with parameter (i,u), hence $\sigma^\frown\langle\mathtt{fin}\rangle$ is eligible to act next.

Case 2. otherwise, we move on to subcycle $(i, u+1)$.

Case $u = n_i$. We now consider the case $u = n_i$, i.e. $b(i, n_i) = b_i$, and we process cycle (i, n_i).

Case 1. $i \in \overline{K}$ (then, currently, $B(b_i) = 1$).

Subcase 1.1. We see $b_i \notin \Gamma_e(A)$ at $\sigma, s+1$. Let the outcome be i. Extract b_j from B by defining $B(b_j) = 0$ for the current values of all defined b_j with $j > i$.

Subcase 1.2. We see $b_i \in \Gamma_e(A)$. Then go to cycle $i+1$.

Case 2. $i \notin \overline{K}$. Let $B(b_i) = 0$.

Subcase 2.1. We see $b_i \notin \Gamma_e(A)$ at $\sigma, s+1$. Then go to cycle $i+1$.

Subcase 2.2. We see $b_i \in \Gamma_e(A)$ at $\sigma, s+1$. Let the outcome be $\mathtt{fin}$ with parameter (i, n_i), hence $\sigma^\frown\langle\mathtt{fin}\rangle$ is eligible to act next.

At the end of the stage, *initialize* all strategies τ with $\tau > \delta_{s+1}$, where $\delta_{s+1} = \sigma_t$ for the last substage t of stage $s+1$.

At the end of the construction if σ is initialized only finitely often, then take Γ^σ to be the s-operator consisting of all axioms $\langle i, D\rangle \in \Gamma$ which have been added to Γ after last initialization of σ.

The verification. It is immediate to see that B is Π^0_1, as we start with $B_0 = \omega$ and no number is ever re-enumerated in B after being extracted from B. The rest of the verification relies on the following lemma.

Lemma 2.4. *There exists an infinite path f through T, called the* true path*, such that (letting f_n denote $f{\upharpoonright}n$, i.e. the initial segment of f having length n) for every n there exists a least stage s_n for which*

(1) for all $s \geqslant s_n$ if $f_n \subseteq \delta_s$ then $f_n \subset \delta_s$;
(2) for all $s \geqslant s_n$, $\delta_s \not<_L f_n$;
(3) there exist infinitely many s such that $f_n \subseteq \delta_s$;
(4) $R(f_n)$ is satisfied.

Proof. The proof is by induction on n. We show in fact that if (1), (2), (3) are true of n then (4) is also true of n, and (1), (2), (3) are true of $n+1$.

Clearly (1), (2), (3) hold of $n = 0$, as $f_0 = \lambda$, which never ends the stage.

Assume now that claims (1), (2) and (3) hold of n, and let s_n be as in the statement of the lemma. For simplicity, in the rest of the proof we will omit to specify f_n when referring to the parameters relative to f_n, thus writing, if appropriate, $V = V^{f_n} = \bigcup_{s \geqslant s_n} V^{f_n}_s$, and similarly for the other parameters.

<u>Case $n = 2e$, i.e. $R(f_n) = N_e$.</u> Let $T^N(x)$ be the following predicate: "There is a stage $t_x \geqslant s_n$ such that for all $s > t_x$ $f_n{}^\frown\langle x\rangle <_L \delta_s$, and infinitely many times when processing cycle x for f_n we get "outcome $\rightarrow$", i.e. we go on and process cycle $x+1$ for f_n, and the value $V(x)$ changes finitely often with $A(x) = V(x)$, where $V(x)$ is the limit value."

By induction on x we want to show:

(*) *If $T^N(y)$ holds of every $y < x$, then $T^N(x)$ or $A(x) \neq \Phi_e(B)(x)$ (this latter case implying that $R(f_n)$ is satisfied).*

Assume that $T^N(y)$ holds of all $y < x$. Notice that this implies that infinitely many times we process cycle x for f_n. For each $y < x$ pick a stage t_y as in the definition of $T^N(y)$, and let $u_x = \max\{t_y : y < x\}$.

Assume that at some stage $s+1 \geqslant u_x$ we define $V(x) = 1$: this depends on the fact that at $s+1$ we see an axiom $\langle x, D\rangle \in \Phi_{e,s}$ with $D \subseteq B^-_{s+1}$, so that we restrain D in B with priority $f_n{}^\frown\langle x\rangle$ in subcase 1.2. or 2.2.. But by our assumptions on u_x we will never visit in the future any P-strategy to the left of $f_n{}^\frown\langle x\rangle$, whereas by Observation 1 only P-strategies of this priority would be entitled to use elements in D as markers, and possibly extract them from B. So, this shows that $V(x)$ can change only finitely often, since we never go back to see $x \notin \Phi_e(B)$.

If $A(x) = \Phi_e(B)(x) = 0$ then any time we process cycle x for f_n after u_x we always see $x \notin \Phi_e(B)$, since if we ever see $x \in \Phi_e(B)$ then by the above argument we would be able to permanently restrain $x \in \Phi_e(B)$. For big enough $f_n{}^\frown\langle x\rangle$-stages when processing cycle x for f_n either we move to $x+1$ or have outcome `fin` with parameter x, but since by Observation 3 there are infinitely many f_n-stages at which we process cycle x and we see $A(x) = 0$, then infinitely many times we go to cycle $x+1$. In this case, the final value of $V(x)$ is $V(x) = 0$.

If $A(x) = \Phi_e(B)(x) = 1$ then from some stage on, when processing cycle x for f_n we always go to cycle $x+1$. In this case, for the final value of $V(x)$ we have $V(x) = 1$.

This shows that either $T^N(x)$ or $A(x) \neq \Phi_e(B)(x)$, and in the latter case $R(f_n)$ is satisfied. On the other hand there must be a least x such that $A(x) \neq \Phi_e(B)(x)$, otherwise it would be $T^N(x)$ for all x, but this would imply that $V \in \Delta^0_2$ and $A = V$, giving $A \in \Delta^0_2$, a contradiction. For this least x for which $A(x) \neq \Phi_e(B)(x)$, if $x \in \Phi_e(B)$ and $x \notin A$, then there exist infinitely many f_n-stages at which we give outcome x (thus $f_{n+1} = f_n{}^\frown\langle x\rangle$); if $\Phi_e(B)(x) = 0$ and $x \in A$, then at cofinitely many f_n-stages we give outcome `fin` with parameter x (thus $f_{n+1} = f_n{}^\frown\langle \texttt{fin}\rangle$). Finally, let u_x be the same stage as before: strategy f_{n+1} may stop the stage, at stages bigger than u_x and x, at most once more (namely, the first time when subcase 1.2. or 2.2 happens). Thus claims (1), (2), (3) hold of $n+1$.

<u>Case $n = 2e+1$, i.e. $R(f_n) = P_e$.</u> This time let $T^P(i)$ be the following predicate: "There is a stage t_i such that for all $s > t_i$ $f_n{}^\frown\langle i\rangle <_L \delta_s$, and $\lim_s b_{i,s} = b_i$ and $\lim_s n_{i,s} = n_i$ exist (where $n_{i,s}$ is the value of n_i at s), so that we can fix the finite sequence $b(i,0), \ldots, b(i,n_i)$, with $b(i,n_i) = b_i$, of i-markers appointed after last initialization of f_n, and later abandoned except for b_i; $B(b_i) = \Gamma_e(A)(b_i)$ and $\Gamma_e(A)(b(i,u)) = 0$ for every $u < n_i$ (remember that $b(i,u) \notin B$ for all $u < n_i$); infinitely many times when we process a subcycle (i,u) with $u < n_i$ for f_n we move on to subcycle $(i,u+1)$, and when we process subcycle (i,n_i) for f_n, we move on to cycle $i+1$."

We want to show:

(**) *If $T^P(j)$ holds of every $j < i$ then $T^P(i)$, or $B(b(i,u)) \neq \Gamma_e(A)(b(i,u))$ for some $u \leqslant n_i$ (in the latter case $R(f_n)$ is satisfied).*

Suppose that (**) is true of all $j < i$. By Observation 2 and the inductive assumption (2) on f_n, we have that $R(f_n)$ has to deal only with a finite amount of restraint posed by higher priority N-strategies, so that it can eventually pick its markers, $\lim_s b_{i,s} = b_i$ and $\lim_s n_{i,s} = n_i$ exist, and thus we may also fix the corresponding sequence $b(i,0),\dots,b(i,n_i)$ of i-markers previously defined, and later abandoned except for $b_i = b(i,n_i)$; when we choose the final value of b_i then initially $b_i \in B$, and b_i is not subject to be extracted from B by any other P-strategy as b_i is targeted for f_n, b_i is not subject to be restrained by any N-strategy to the left of f_n (by Observation 2, these are the only N-strategies that would be entitled to restrain b_i), as no such N-strategy will ever act again. Thus in the future b_i will be extracted from B only following $R(f_n)$'s own demands, in response to i being extracted from $\overline{K}$.

If $B(b(i,u)) = \Gamma_e(A)(b(i,u))$ for all $u \leqslant n_i$ then by an argument similar to the one for N-strategies (using this time Observation 4), we see that $T^P(i)$ holds. Indeed an easy induction on $u < n_i$ shows that at infinitely many stages at which we process subcycle (i,u) for f_n, with $u < n_i$, we move to subcycle $(i,u+1)$ for f_n (due to the existence of infinitely many f_n-stages at which we process (i,u) for f_n, when we see $\Gamma_e(A)(b(i,u)) = 0$). It follows also that at infinitely many times we process subcycle (i,n_i), and arguing as for the N-strategies infinitely many times after processing (i,n_i) we move on to cycle $i+1$.

On the other hand, if $T^P(i)$ holds for every i, then by definition of $\Gamma = \Gamma^\sigma = \bigcup_{s \geqslant \bar{s}} \Gamma^\sigma_s$ (where $\bar{s}$ is the least stage such that at no $s \geqslant \bar{s}$ is σ initialized) we see that for every i if $u < n_i$ then there is no axiom $\langle i, D\rangle \in \Gamma$, with $D \subseteq A$, which has been added because of an axiom $\langle b(i,u), D\rangle \in \Gamma_e$, as $b(i,u) \notin \Gamma_e(A)$; as a consequence, to see whether or not $i \in \Gamma(A)$, we need only look at the axioms $\langle i, D\rangle \in \Gamma$ which have been added because of an axiom $\langle b_i, D\rangle \in \Gamma_e$: but since $i \in \overline{K}$ if and only if $b_i \in B$ we easily conclude that

$$i \in \overline{K} \Leftrightarrow i \in \Gamma(A).$$

Since $\overline{K} \nleqslant_s A$, it follows that there exist a least i and a least $u \leqslant n_i$ so that $B(b(i,u)) \neq \Gamma_e(A)(b(i,u))$: thus $R(f_n)$ is satisfied. On the other hand, if $u < n_i$ and $\Gamma_e(A)(b(i,u)) = 1$ then when processing subcycle u we eventually give outcome `fin` with parameter (i,u); if $B(b_i) = 0 \neq 1 = \Gamma_e(A)(b_i)$ then we eventually give outcome `fin` with parameter (i,n_i); if $B(b_i) = 1 \neq 0 = \Gamma_e(A)(b_i)$ then there are infinitely many f_n-stages at which we give outcome i. This shows also that $f_{n+1} = f_n{}^\frown\langle o\rangle$ exists. Finally, since f_{n+1} never ends the stage at stages bigger than i and after the final value of b_i has been appointed, we conclude that claims (1), (2), (3) hold of $n+1$. ☐

This ends the proof of the case $A \in \Pi^0_2 \smallsetminus \Delta^0_2$.

We now consider briefly the case when $A \in \Delta^0_2 \smallsetminus \Pi^0_1$. In this case we work with some fixed Δ^0_2-approximation to A, so that the N-strategies are finitary since for every x, $\lim_s A_s(x)$ exists. Our tree of strategies is now organized so that any N-strategy σ has outcomes

$$0 < 1 < \cdots < x < \cdots$$

and any restraint requested by σ to fix $x \in \Phi_e(B)$ has now priority σ and not $\sigma{}^\frown\langle x\rangle$ so that even if in the future we have an outcome $y < x$ at σ, the restraint made for x is not abandoned. If σ is on the true path, then this entails that when we define $V(x) = 1$ depending on the fact that we see $x \in \Phi_e(B)$ then from now on we will always see $x \in \Phi_e(B)$ and thus we never go back to redefine $V(x) = 0$. Hence, if at the node σ on the true path where our N-strategy is located we have that $T^N(x)$ for all x, then we would have $A = V$ and thus $A \in \Sigma^0_1$ as $V \in \Sigma^0_1$, a contradiction: we rely of course on the fact that for the least x for which $T^N(x)$ does not hold, at cofinitely many σ-stages we get outcome x, and thus σ demands only a finite amount of restraint which does not prevent lower priority P-strategies from pursuing their strategies. ☐

3. Applications

If $A \leqslant_Q B$ via a computable function f such that for all x, y,

$$x \neq y \Rightarrow W_{f(x)} \cap W_{f(y)} = \varnothing,$$

then we say that A is Q_1-*reducible* to B, denoted by $A \leqslant_{Q_1} B$, *via* f. One can view Q_1-reducibility as the "injective" version of Q-reducibility. It is easy to isolate the subreducibility $\leqslant_{s_1}$ of $\leqslant_s$ which equals Q_1-reducibility on complements of sets. Precisely, $A \leqslant_{s_1} B$ if and only if $A = \Phi(B)$ via an s_1-*operator*, i.e. an s-operator such that:

(a) for all distinct x, y, $\{z : \langle x, \{z\}\rangle \in \Phi\} \cap \{z : \langle y, \{z\}\rangle \in \Phi\} = \varnothing$;
(b) there is no x with $\langle x, \varnothing\rangle \in \Phi$.

In the same vein, one can define $A \leqslant_{s_2} B$ if and only if $A = \Phi(B)$ for some s_2-*operator* Φ, i.e. an s-operator such that for all distinct x, y, $\{z : \langle x, \{z\}\rangle \in \Phi\} \cap \{z : \langle y, \{z\}\rangle \in \Phi\} = \varnothing$. The difference between s_1 and s_2 is that for s_2 we may have also axioms of the form $\langle x, \varnothing\rangle \in \Phi$. Clearly $\leqslant_{s_1} \subseteq \leqslant_{s_2}$: it is shown in [2] that inclusion is proper even at the level of Σ^0_2 sets.

Corollary 3.1. *If $r \in \{e, s, s_1, s_2\}$ then for every Σ^0_2 set A such that $\varnothing <_r A <_r \overline{K}$ there exists a Π^0_1 set B such that $A|_r B$. As a consequence, if $r \in \{Q_1, Q\}$ then for every Π^0_2 set A such that $\varnothing <_r A <_r K$ there exists a Σ^0_1 set B such that $A|_r B$.*

Proof. For $r \in \{e, s, s_1, s_2\}$ the claim follows immediately from the proof of Theorem 2.3 since $A \nleqslant_e B$ implies $A \nleqslant_r B$, and the fact (which is immediate to check) that if $\{\Gamma_e : e \in \omega\}$ is an effective list of all r-operators, then for every σ in the tree of strategies, the operator Γ^σ built by R_σ is an r-operator. The claim about Q-reducibility (and Q_1-reducibility, respectively) follows from the isomorphism between the Σ^0_2 s–degrees (s_1-degrees, respectively) and the Π^0_2 Q-degrees (Q_1-degrees, respectively). □

Finally, we show how Theorem 2.3 may be used to give rather uniform proofs of upper density for the local structures of s- and Q-reducibility. Both downwards density and upwards density of the Σ^0_2 s-degrees hold: downwards density follows from the proof of Gutteridge's celebrated theorem stating that there is no minimal enumeration degree; upper density was proved in [7]. However (with the exception of [3] which shows density of the c.e. Q-degrees, yielding also, via isomorphism, density of the Π^0_1 s-degrees) none of these papers dealing with upper density seems to care about distinguishing the three main arithmetical classes in which these local structures can be divided, namely the classes $\Sigma^0_2, \Delta^0_2, \Pi^0_1$ for s-reducibility, and $\Pi^0_2, \Delta^0_2, \Sigma^0_1$ for Q-reducibility: hopefully a nice feature of Theorem 2.3 is that, as a side effect, it yields in one shot upwards density for all the above arithmetical classes.

We recall that the s-degree of $\overline{K}$ (and hence the Q-degree of K) is join-irreducible, see [7].

Corollary 3.2. *The Σ^0_2 (respectively: Δ^0_2, Π^0_1) s-degrees are upwards dense. The Π^0_2 (respectively: Δ^0_2, c.e.) Q-degrees are upwards dense.*

Proof. We prove the claim for s-reducibility. Assume that $\varnothing <_s A <_s \overline{K}$. By Theorem 2.3 let B be a Π^0_1 set such that $A|_s B$. Then by join-irreducibility of the s-degree of $\overline{K}$, we have that $A <_s A \oplus B <_s \overline{K}$. Clearly if $A \in \Delta^0_2$ (respectively, A is Π^0_1) then so is $A \oplus B$. □

References

[1] M.M. Arslanov, I.I. Batyrshin, and R. Sh. Omanadze. Structural properties of q-degrees of n-c.e. sets. *Ann. Pure Appl. Logic*, 156:13–20, 2008.
[2] I. Chitaia, K.M. Ng, A. Sorbi, and Y. Yang. Minimal degrees and downwards density in some strong s-reducibilities and Q-reducibilities. To appear.
[3] R. G. Downey, G. Laforte, and A. Nies. Computably enumerable sets and quasi-reducibility. *Ann. Pure Appl. Logic*, 95:1–35, 1998.
[4] R. M. Friedberg and H. Rogers, Jr. Reducibility and completeness for sets of integers. *Z. Math. Logik Grundlag. Math.*, 5:117–125, 1959.
[5] K. McEvoy and S. B. Cooper. On minimal pairs of enumeration degrees. *J. Symbolic Logic*, 50:983–1001, 1985.
[6] R. Sh. Omanadze. Quasi–degrees of recursively enumerable sets. In S. B. Cooper and S. Goncharov, editors, *Computability and Models: Perspectives East and West*. Kluwer Academic/Plenum Publishers, New York, Boston, Dordrecht, London, Moscow, 2002.
[7] R. Sh. Omanadze and A. Sorbi. Strong enumeration reducibilities. *Arch. Math. Logic*, 45(7):869–912, 2006.
[8] H. Rogers, Jr. *Theory of Recursive Functions and Effective Computability*. McGraw-Hill, New York, 1967.
[9] R. I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic, Omega Series. Springer-Verlag, Heidelberg, 1987.

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