Degrees of Weakly Computable Reals*

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1 Introduction

A real α is *left-recursively enumerable* (left-r.e. for short) if we can effectively generate α from below. That is, the left Dedkind cut of α, $L(\alpha) = \{q \in \mathbb{Q} : q \leq \alpha\}$, forms a r.e. set. Equivalently, a real α is left-r.e. if it is the limit of a converging recursive increasing sequence. *If we can also compute the radius of convergence effectively, then α is recursive.*

Left-r.e. reals are the *measures* of the domains of prefix-free Turing machines, or halting probabilities. These reals occupy a central place in the study of algorithmic randomness in the same way as recursively enumerable *sets* occupy a central place in classical recursion theory. However, the collection of left-r.e. reals does not behave well algebraically since it is not closed under subtraction. Because of this, in [1], Ambos-Spies, Weihrauch and Zheng introduced the collection of weakly computable reals, where a real α is *weakly computable* if there are left-r.e. reals β and γ such that $\beta - \gamma$ equals to α. Ambos-Spies, Weihrauch and Zheng [1] proved that the collection of weakly computable reals is closed under the arithmetic operations, and hence forms a field. The following proposition gives an analytical characterization of weakly computable reals:

Theorem 1.1 [1]**.** *(Ambos-Spies, Weihrauch and Zheng) A real number x is weakly computable reals iff there is a recursive sequence $\{x_s\}_{s\in\mathbb{N}}$ of rational numbers which converges to x such that $\sum_{s\in\mathbb{N}} |x_s - x_{s+1}| \leq c$ for a constant c.*

In this paper, we will study the Turing degrees of weakly computable reals reals. The following is known:

Theorem 1.2 [4]**.** *(Downey, Wu and Zheng) (1)Any ω-c.e. degree contains a weakly computable real. (2) There are Turing degrees below $\mathbf{0}'$ containing no weakly computable reals.*

In this paper, we first introduce a generalized notion of those degrees constructed in [4] (Theorem 1.2 (2)). Say that a nonzero degree $\mathbf{a}$ is *nonbounding*

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if every nonzero degree $\leq \mathbf{a}$ contains no weakly computable reals. The existence of such nonbounding degrees can be proved by an oracle construction.

Theorem 1.3. *There is a degree below $\mathbf{0}'$ such that every nonzero degree below it contains no weakly computable reals.*

Our construction can be easily modified to make the nonbounding degrees 1-generic. However, if we let $\mathbf{c}$ be any r.e. and strongly contiguous degree, then every degree below $\mathbf{c}$ is ω-r.e., and hence contains a weakly computable real (by Theorem 1.2 (1)). Now by the fact that any nonzero r.e. degree bounds a 1-generic degree, there are 1-generic degrees below $\mathbf{0}'$ not nonbounding.

The notion of $(\emptyset', f)$-genericity will be introduced, and an alternative proof of Theorem 1.3 by using $(\emptyset', f)$-genericity will be given. This proof can be modified to prove that there are degrees $\mathbf{a}$ below $\mathbf{0}'$ such that those degrees containing weakly computable reals comparable with $\mathbf{a}$ can only be $\mathbf{0}$ and $\mathbf{0}'$. This latter result improves Yates' result in [15].

We will also consider those Turing degrees on the other extreme, those degrees containing only weakly computable reals. A Turing degree is called *completely weakly computable* if every set in this degree is weakly computable. Our result involves the notion of array recursive degrees.

Recall that a sequence of finite sets is called a *strong array* $\{F_n\}_{n\in\mathbb{N}}$ if there is a recursive function f such that $F_n = D_{f(n)}$ for every $n \in \mathbb{N}$. In [7], Downey, Jockusch and Stob defined a *very strong array* $\{F_n\}_{n\in\mathbb{N}}$ as a strong array satisfying the following three properties: (a) $\cup_{n\in\mathbb{N}} F_n = \mathbb{N}$, (b) if $n_1 \neq n_2$, then $F_{n_1} \cap F_{n_2} = \emptyset$, (c) for all $n \in \mathbb{N}$, $0 < |F_n| < F_{n+1}|$. Given a very strong array $\{F_n\}_{n\in\mathbb{N}}$, an r.e. set is called *array nonrecursive with respect to* $\{F_n\}_{n\in\mathbb{N}}$ if for any e, there is some n such that

$$W_e \cap F_n = A \cap F_n.$$

An r.e. set is said to be array nonrecursive if A is array nonrecursive with respect to some very strong array $\{F_n\}_{n\in\mathbb{N}}$. As usual, a Turing degree is called *array nonrecursive* if it contains an array nonrecursiv r.e. set, and is called *array recursive* if it contains no array nonrecursiv r.e. set. Downey, Jockusch and Stob proved in [7] the following: (1) array nonrecursive degrees are closed upwards in the r.e. degrees; (2) all nonlow$_2$ r.e. degrees are array nonrecursive; (3) array nonrecursive degrees can be low. The following characterizations of array nonrecursive degrees, and hence of array recursive degrees, are also from [7]:

Theorem 1.4 [7]**.** *(Downey, Jockusch and Stob) For any r.e. set A, the following are equivalent:*

(1) A has array nonrecursive degree,
(2) there are disjoint r.e. sets B and C, each of which is reducible in A, such that $B \cup C$ is coinfinite and no set of degree $\mathbf{0}'$ separates B and C,
(3) there is a Martin Pour-El theory T of degree $\mathbf{a}$.

We will provide in this paper another characterizations of array recursive degrees:

Theorem 1.5. *Let A be any r.e. set. Then the Turing degree of A is completely weakly computable if and only if any set Turing reducible to A is weakly computable if and only if A has array recursive degree.*

Our notation and terminology are standard and generally follow Soare [14].

2 Nonbounding Degrees

In this section, we prove Theorem 1.3. That is, we will construct a real A such that weakly computable reals Turing reducible to it are all recursive. A is constructed satisfying the following requirements:

$\mathcal{P}_e$: $A \neq \{e\}$;
$\mathcal{R}_{e,i,j}$: if $\{e\}^A$ is total, then either $\{e\}^A$ is recursive or $\{e\}^A \neq \alpha_i - \alpha_j$,

where $\{\alpha_i\}_{i\in\mathbb{N}}$ is an effective list of all left-r.e. reals.

$\mathcal{P}_e$ requirement can be satisfied by the Kleene-Post's diagonalization. That is, at stage s, given a finite approximation σ_s, we can ask whether there is some number $m > |\sigma_s|$ such that $\{e\}(m)$ converges. This is a Σ_1 question, and we can get the answer from oracle $\emptyset'$. If the answer is "yes", then we can define σ_{s+1} as an extension of σ_s such that $|\sigma_{s+1}| = m+1$ and $\sigma_{s+1}(m) \neq \{e\}(m)$. Otherwise, we extend σ_s to σ_{s+1} by just letting $\sigma_{s+1} = \sigma_s{}^\frown 0$. Obviously, $\mathcal{P}_e$ is satisfied in both cases.

Now we describe the strategy satisfying the requirement $\mathcal{R}_{e,i,j}$. For convenience, we omit the subscript, and it will not cause any confusion. Suppose that at stage $s+1$, σ_s is given, and we want to satisfy $\mathcal{R}$. We ask $\emptyset'$ whether there is a number n and strings τ_1, τ_2 extending σ_s such that for all $m \leq n$, $\{e\}^{\tau_1}(m), \{e\}^{\tau_2}(m)$ converge and that $|\{e\}^{\tau_1} \upharpoonright (n+1) - \{e\}^{\tau_2} \upharpoonright (n+1)| \geq 2^{-(n-1)}$.

If the answer is "no", then we claim that if $\{e\}^A$ is total, then $\{e\}^A$ is recursive. To see this, for any n, to calculate $\{e\}^A(n)$, we find a string τ extending σ_s such that for all $m \leq n+2$, $\{e\}^\tau(m)$ converges (by the assumption that $\{e\}^A$ is total, such a τ exists). Then $\{e\}^A(n) = \{e\}^\tau(n)$, because by our assumption, $|\{e\}^A \upharpoonright (n+3) - \{e\}^\tau \upharpoonright (n+3)|$ is less than $2^{-(n+1)}$. In this case, we let $\sigma_{s+1} = \sigma_s{}^\frown 0$.

On the other hand, if the answer is "yes", then for any real number x,

$$|\{e\}^{\tau_1} \upharpoonright (n+1) - x \upharpoonright (n+1)| + |\{e\}^{\tau_2} \upharpoonright (n+1) - x \upharpoonright (n+1)|$$

is bigger than $|\{e\}^{\tau_1} \upharpoonright (n+1) - \{e\}^{\tau_2} \upharpoonright (n+1)|$ and hence is bigger than $2^{-(n-1)}$. As a consequence, one of $|\{e\}^{\tau_1} \upharpoonright (n+1) - x \upharpoonright (n+1)|$ and $|\{e\}^{\tau_2} \upharpoonright (n+1) - x \upharpoonright (n+1)|$ must be bigger than 2^{-n}. If we know that $|\{e\}^{\tau_1} \upharpoonright (n+1) - x \upharpoonright (n+1)| > 2^{-n}$, then we can define σ_{s+1} as τ_1, and we will have $\{e\}^A \upharpoonright (n+1) = \{e\}^{\tau_1} \upharpoonright (n+1)$. As a consequence, $\{e\}^A$ differs from x in the first $n+1$ digits.

Then, how can we decide which one of τ_1 and τ_2 is the one we want to satisfy $\mathcal{R}$? Since α_i and α_j are left-r.e. reals, there are effective approximations of α_i, α_j from the left, $\{\alpha_{i,s}\}_{s\in\mathbb{N}}$, $\{\alpha_{j,s}\}_{s\in\mathbb{N}}$ say, and hence, we can use $\emptyset'$ as oracle to find a stage s such that $\alpha_{i,s} \upharpoonright (n+3) = \alpha_i \upharpoonright (n+3)$, $\alpha_{j,s} \upharpoonright (n+3) = \alpha_j \upharpoonright (n+3)$. Thus $\alpha_i \upharpoonright (n+3) - \alpha_{i,s} \upharpoonright (n+3) \leq 2^{-(n+2)}$, $\alpha_j \upharpoonright (n+3) - \alpha_{j,s} \upharpoonright (n+3) \leq 2^{-(n+2)}$, and hence $|(\alpha_i - \alpha_j) \upharpoonright (n+3) - (\alpha_{i,s} - \alpha_{j,s}) \upharpoonright (n+3)| \leq 2^{-(n+1)}$. Now if we let x above be $\alpha_{i,s} - \alpha_{j,s}$, then we can know which one of $|\{e\}^{\tau_1} \upharpoonright (n+1) - (\alpha_{i,s} - \alpha_{j,s}) \upharpoonright (n+1)|$ and $|\{e\}^{\tau_2} \upharpoonright (n+1) - (\alpha_{i,s} - \alpha_{j,s}) \upharpoonright (n+1)|$ is bigger than 2^{-n}. Suppose that $|\{e\}^{\tau_1} \upharpoonright (n+1) - (\alpha_{i,s} - \alpha_{j,s}) \upharpoonright (n+1)| \geq 2^{-n}$. Then $|\{e\}^{\tau_1} \upharpoonright (n+1) - (\alpha_i - \alpha_j) \upharpoonright (n+1)| \geq ||\{e\}^{\tau_1} \upharpoonright (n+1) - (\alpha_{i,s} - \alpha_{j,s}) \upharpoonright (n+1)| - |(\alpha_i - \alpha_j) \upharpoonright (n+1) - (\alpha_{i,s} - \alpha_{j,s}) \upharpoonright (n+1)|| \geq 2^{-n} - 2^{-(n+1)} = 2^{-(n+1)}$. Therefore we can satisfy $\mathcal{R}$ by extending σ_s to τ_1 (that is, define $\sigma_{s+1} = \tau_1$).

The whole construction of A is a finite extension argument, with $\emptyset'$ as oracle, where at each stage, one requirement is satisfied. ∎

3 $(\emptyset', f)$-generic degrees

In [15], Yates proved that there are degrees $\mathbf{d}$ below $\mathbf{0}'$ such that the r.e. degrees comparable with $\mathbf{d}$ are exactly $\mathbf{0}$ and $\mathbf{0}'$. Actually, as noticed later, Yates' degree $\mathbf{d}$ can be 1-generic, and can be minimal. In [16], Wu proved that Yates' degree $\mathbf{d}$ can appear in every jump class. In this section, we construct a degree $\mathbf{a}$ below $\mathbf{0}'$ such that the degrees containing weakly computable reals which are comparable with $\mathbf{a}$ are exactly $\mathbf{0}$ and $\mathbf{0}'$.

We need the following notion of $(\emptyset', f)$-genericity.

Definition 3.1. (1) *A set A is called $(\emptyset', f)$-generic iff for each $e \in \mathbb{N}$, if there are infinitely many m such that $W^{\emptyset'}_{e,f(m)}$ contains an extension of $A(0)A(1)\cdots A(m)$, then there is an n such that $W^{\emptyset'}_e$ contains $A(0)A(1)\cdots A(n)$.*

(2) *A set A is called $(\emptyset', f)$-semigeneric iff for each e, if for almost all m, $W^{\emptyset'}_{e,f(m)}$ contains an extension of $A(0)A(1)\cdots A(m)$, then there is an n such that $A(0)A(1)\cdots A(n) \in W^{\emptyset'}_e$.*

Here $W^{\emptyset'}_e$ is the set of all strings enumerated by the e-th algorithm using the oracle $\emptyset'$ and $W^{\emptyset'}_{e,f(m)}$ is the set of those strings in $W^{\emptyset'}_e$ which are enumerated in time $f(m)$.

A $(\emptyset', f)$-generic set A forces membership in $W^{\emptyset'}_e$ only if for infinitely many prefixes $A(0)A(1)\cdots A(m)$ of A an extension in $W^{\emptyset'}_e$ can be found within time $f(m)$, so this notion differs from the 1-genericity (see [10]) by having an oracle and bounding the search. Nevertheless, if f is sufficiently fast growing then $(\emptyset', f)$-genericity implies 1-genericity.

We note that for a real α, the notions of computable (by approximation) and recursive (by computing all digits) coincides. But this does no longer hold for sequences of reals as one can have that they are uniformly computable in the sense that there is a function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that $|\alpha_i - g(i,j)| < 2^{-j}$ for all i, j while they are not uniformly recursive in the sense that the function $i, j \to \alpha_i(j)$ which computes the digit $j+1$ of α_i after the dot is not computable.

This fact relativizes to the oracle $\emptyset'$. While there is a uniformly $\emptyset'$-recursive sequence of all left-r.e. reals, there is no uniformly $\emptyset'$-recursive sequence containing all weakly computable reals. But there is still an enumeration $\alpha_0, \alpha_1, \alpha_2, \cdots$ of all weakly computable reals and an $\emptyset'$-recursive function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that the approximation condition $|\alpha_i - g(i,j)| < 2^{-j}$ holds for all i, j.

Theorem 3.2. *Assume that for $\alpha_0, \alpha_1, \alpha_2, \cdots$ is a list of weakly computable reals such that there is a $\emptyset'$-recursive function $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ with $\forall i, j\, (|\alpha_i - g(i,j)| < 2^{-j})$. Then there is a function $f \leq_T \emptyset'$ such that: (1) every $(\emptyset', f)$-generic set A is 1-generic, (2) for all i, if $\alpha_i \leq_T A$ then $\alpha_i \equiv_T \emptyset$, and (3) for all i, if $\alpha_i \geq_T A$ then $\alpha_i \equiv_T \emptyset'$. Furthermore, one can choose A such that $A \leq_T \emptyset'$ and hence A can be chosen to be low.*

Proof. Below in Propositions 3.3, 3.4 and 3.5, we will construct functions $f_1, f_2, f_3 \leq_T \emptyset'$ respectively such that: every $(\emptyset', f_1)$-semigeneric set satisfies (1), every $(\emptyset', f_2)$-semigeneric set satisfies (2), and every $(\emptyset', f_3)$-generic set satisfies (3).

Let f be defined as $f(n) = f_1(n) + f_2(n) + f_3(n)$ for all n. Then every $(\emptyset', f)$-generic set A is $(\emptyset', f_1)$-semigeneric, $(\emptyset', f_2)$-semigeneric and $(\emptyset', f_3)$-generic, and hence A satisfies all three statements simultaneously. ∎

Proposition 3.3. *There is a function $f_1 \leq_T \emptyset'$ such that every $(\emptyset', f_1)$-semigeneric set is also 1-generic.*

Proof. In an acceptable numbering, there are indice for algorithms and not only for sets. Thus every r.e. set has an index in the enumeration $W_0^{\emptyset'}, W_1^{\emptyset'}, \cdots$ *(the oracle is not accessed during the enumeration of this r.e. set)*. So it is reasonable to make the following definition.

Let $f_1(n)$ be the time needed to find for every $e \leq n$ and every string $\sigma \in \{0,1\}^*$ with $|\sigma| \leq n+1$ a string $\tau \succeq \sigma$ which is enumerated into $W_e^{\emptyset'}$ without having accessed the oracle $\emptyset'$ whenever such a τ exists.

As the search in an $W_e^{\emptyset'}$ is aborted for this e whenever the oracle $\emptyset'$ is accessed, the oracle $\emptyset'$ does not play any role in the definition of f_1 and so $f_1 \leq_T \emptyset'$.

Now let A be any $(\emptyset', f_1)$-generic set and consider any r.e. set V of strings. There is an index e such that $W_e^{\emptyset'} = V$ and the enumeration procedure does not access the oracle $\emptyset'$ at all. Suppose that there are infinitely many n for which $A(0)A(1)\cdots A(n)$ has an extension in V. Then by the definition of f_1, it is easy to see that for all n, there is an extension of $A(0)A(1)\cdots A(n)$ in $W_{e,f_1(n)}^{\emptyset'}$. Since A is $(\emptyset', f_1)$-semigeneric, there is an m with $A(0)A(1)\cdots A(m) \in W_{e,f_1(m)}^{\emptyset'}$. Thus A meets V and hence, A is 1-generic. ∎

Proposition 3.4. *Let $\alpha_0, \alpha_1, \alpha_2, \cdots$ and g be the same as in Theorem 3.2. Then there is a function $f_2 \leq_T \emptyset'$ such that for every $(\emptyset', f_2)$-semigeneric set A and every i, if $\alpha_i \leq_T A$ then α_i is recursive.*

Proof. Given any binary string τ, for any e, let $\sigma_{\tau,e,0}$, $\sigma_{\tau,e,1}$, $k_{\tau,e}$ and t be the first data found such that (1) $\sigma_{\tau,e,0}$, $\sigma_{\tau,e,1}$ extend τ and have length t; (2) $\{e\}_t^{\sigma_{\tau,e,0}}(m)$ and $\{e\}_t^{\sigma_{\tau,e,1}}(m)$ are defined for all $m < k_{\tau,e}$; (3) there are at least two binary strings of length $k_{\tau,e}$ lexicographically between $\{e\}_t^{\sigma_{\tau,e,0}}(m) \upharpoonright k_{\tau,e}$ and $\{e\}_t^{\sigma_{\tau,e,1}}(m) \upharpoonright k_{\tau,e}$.

Let h be a recursive function such that $W_{h(e,i)}^K$ contains one of $\sigma_{\tau,e,0}$ and $\sigma_{\tau,e,1}$, $\sigma_{\tau,e,j}$ say, such that α_i does not extend $\{e\}_t^{\sigma_{\tau,e,j}}(m)$, if the above search terminates for e, i, τ *(such a string can be found since the third condition guarantees that the restriction of $g(i, k_{\tau,e}+2)$ to its $k_{\tau,e}$ first bits cannot be a identical with or a neighbour of both computed strings)*. In other words, the function h searches for the "real number variant" of an e-splitting, as described in Theorem 1.3.

Let $f_2(n)$ be the time needed to find with oracle $\emptyset'$ for each $e, i \leq n$ and each $\tau \in \{0,1\}^{n+1}$ an extension of τ in $W_{h(e,i)}^{\emptyset'}$ whenever $\sigma_{\tau,e,0}, \sigma_{\tau,e,1}$ exist.

Assume that A is $(\emptyset', f_2)$-semigeneric and $\alpha_i = \{e\}^A$. If there is an n with $A(0)A(1)\cdots A(n) \in W_{h(e,i)}^{\emptyset'}$ then it would follow that every extension of $\{e\}_n^A$ differs from α_i as a real, contradicting to the assumption. Thus there is an $n \geq e+i$ such that no extension of $A(0)A(1)\cdots A(n)$ is in $W_{h(e,i),f_2(n)}^{\emptyset'}$. Then there is also no extension of $A(0)A(1)\cdots A(n)$ in $W_{h(e,i)}^{\emptyset'}$ and α_i is the unique real such that for every $\eta \succeq A(0)A(1)\cdots A(n)$ there is a binary representation of α_i extending $\{e\}_{|\eta|}^{\eta}$, which means that α_i is recursive. ∎

Proposition 3.5. *Let $\alpha_0, \alpha_1, \alpha_2, \cdots$ and g be the same as in Theorem 3.2. Then there is a function $f_3 \leq_T \emptyset'$ such that for every $(\emptyset', f_3)$-generic set A and every i, if $\alpha_i \geq_T A$ then $\emptyset' \leq_T \alpha_i$.*

Proof. Let $c(n)$ be the convergence module of $\emptyset'$, that is, the time to enumerate all elements of $\{0,1,\cdots,n\} \cap \emptyset'$ into $\emptyset'$. Now let $\tilde{h}(e,i)$ be a recursive function such that $W_{\tilde{h}(e,i)}^{\emptyset'}$ contain all strings σ of length $n+1$ for which there are m, j, η such that (1) $m < n$, $j < c(n)$, (2) η is the binary representation of the first $j+3$ bits of $g(i, j+4)$, (3) $\eta(j) = 0$ and $\eta(j+1) = 1$, and (4) $\{e\}_j^{\eta}(m)$ converges to a value different from $\sigma(m)$ without querying the oracle at j or beyond.

Since the sets $W_{\tilde{h}(e,i)}^{\emptyset'}$ are uniformly $\emptyset'$-recursive, there is an $\emptyset'$-recursive function f_3 such that $f_3(n)$ is the time needed to enumerate relative to $\emptyset'$ all members of length $\leq n+2$ of sets $W_{\tilde{h}(e,i)}^{\emptyset'}$ with $e, i \leq n$.

Let A be a $(\emptyset', f_3)$-generic set. Suppose that α_i is irrational and that $A = \{e\}^{\alpha_i}$. Then there are only finitely many n such that some string $A(0)A(1)\cdots A(n)b$ is in $W_{\tilde{h}(e,i),f_3(n)}^{\emptyset'}$ since whenever such an extension is in then $A(n+1) \neq b$. By the choice of f_3, the extension is also not in $W_{\tilde{h}(e,i)}^{\emptyset'}$. Now let $u(n)$ be the first j such that for all $m \leq n$, $\alpha_i(j) = 0$, $\alpha_i(j+1) = 1$, and $\{e\}^{\alpha_i}(m)$ converges within j steps without querying the oracle at j or above.

Since α_i is is assumed to be irrational, the function u is total. Note that $u \leq_T \alpha_i$. Since for almost all n, there are no strings of length n in $W_e^{\emptyset'}$, it follows that $u(n) \geq c(n)$ for these n. Thus $\emptyset' \leq_T \alpha_i$.

The case that α_i is rational is trivial. ∎

Proposition 3.6. *If $f \leq_T \emptyset'$ then there is a set $A \leq_T \emptyset'$ such that A is $(\emptyset', f)$-generic.*

Proof. We assumes that f is strictly monotonically increasing; if not one replaces f by $\hat{f}$ with $\forall n\,(\hat{f}(n) = f(0) + f(1) + \ldots + f(n) + n)$ and use that every $(\emptyset', \hat{f})$-generic set is also $(\emptyset', f)$-generic.

First define a partial $\emptyset'$-recursive function h with $\emptyset'$-recursive domain such that, for all σ, e, if $W^{\emptyset'}_{e,f(e+|\sigma|)}$ contains a proper extension of σ then $h(e,\sigma)$ is that proper extension of σ which is enumerated into $W^{\emptyset'}_{e}$ first, and if $W^{\emptyset'}_{e,f(e+|\sigma|)}$ contains no proper extension of σ then $h(e,\sigma)$ is undefined.

Obviously, $\tau \prec \sigma \prec h(e,\tau)$ implies that $h(e,\sigma) = h(e,\tau)$, because $h(e,\tau)$ is the first element of the enumeration of $W^{\emptyset'}_{e}$ properly extending τ and is also the first element of this enumeration properly extending σ.

Now we construct set A relative to oracle $\emptyset'$. Assume that $A(m)$ for all $m < n$ is already defined and let σ be the string $A(0)A(1)\ldots A(n-1)$ (we let σ be the empty string if $n = 0$). Now $A(n)$ is defined as follows: find the least e such that $h(e, A(0)A(1)\cdots A(n))$ is defined and there is no $m < n$ with $h(e, A(0)A(1)\cdots A(m)) \preceq A(0)A(1)\cdots A(n)$ and define

$$A(n) = h(e, A(0)A(1)\cdots A(n))(n).$$

The first step in the definition of $A(n)$ can be satisfied since there are a t and infinitely many programs e with $W^{\emptyset'}_{e} = W^{\emptyset'}_{e,t} = \{0,1\}^{n+2}$. The second step is again satisfied since $n = |\sigma|$ and $h(e,\sigma) \succ \sigma$. Thus $h(e,\tau)$ is so long that the bit $h(e,\tau)(n)$ exists and can be copied into $A(n)$. Therefore, the definition above never runs into an undefined place. Obviously, $A \leq_T \emptyset'$.

Now we verify that A is $(\emptyset', f)$-generic. For the sake of contradiction, assume that there is an index e such that there are infinitely many n such that $W^{\emptyset'}_{e,f(n)}$ contains an extension of $A(0)A(1)\cdots A(n)$ but no element of $W^{\emptyset'}_{e}$ is of the form $A(0)A(1)\cdots A(n)$. Let e be the least such index. Then there is a length n for which all $e' < e$ satisfy that $h(e, A(0)A(1)\cdots, A(n))$ is defined and either there is an $m < n$ with $A(0)A(1)\cdots A(m) \in W^{\emptyset'}_{e',f(n+e')}$ or there is no $m \geq n$ such that $W^{\emptyset'}_{e',f(n+e')}$ contains a proper extension of $A(0)A(1)\cdots A(m)$. At any m with $n < m < |h(e, A(0)A(1)\cdots A(n))|$, the first step of the algorithm takes e as the parameter of the same name and then assigns to $A(m)$ the value $h(e, A(0)A(1)\ldots A(n))(m)$. This is done since $h(e, A(0)A(1)\ldots A(m-1)) = h(e, A(0)A(1)\cdots A(n))$. Thus if $k = |h(e, A(0)A(1)\cdots A(n))|$ then

$$A(0)A(1)\cdots A(k-1) = h(e, A(0)A(1)\cdots A(n))$$

and this string is a member of $W^{\emptyset'}_{e}$, contradicting our assumption on e. Hence A is $(\emptyset', f)$-generic and the theorem is proved. ∎

Following from Theorem 3.2 and Theorem 3.6, we have:

Theorem 3.7. *There is degree* $\mathbf{a}$ *below* $\mathbf{0}'$ *such that* $\mathbf{0}$ *and* $\mathbf{0}'$ *are the only degrees containing weakly computable reals and comparable with* $\mathbf{a}$.

Actually, the degree **a** in Theorem 3.7 occurs in every jump class. This genearlize results in Wu [16] and Yates [15].

We end this section by stating the following result:

Theorem 3.8. *If $A, f \leq_T \emptyset'$ and A is nonlow$_2$, then there are $(\emptyset', f)$-semigeneric sets A_0, A_1 such that $A \equiv_T A_1 \oplus A_2$. Particularly, every nonlow$_2$ degree below $\mathbf{0}'$ is the join of nonbounding degrees.*

4 Completely weakly computable degrees

In this section, we prove Theorem 1.5. First we need some background of Chaitin's Ω numbers.

In [2], Chaitin introduced Ω as the halting probability of a universal prefix-free machine and Kučera and Slaman [11] showed these Ω-numbers cover indeed all the left-r.e. Martin-Löf random sets. Indeed, it is sufficient for the further investigations and definitions to fix Ω as one of these possible numbers as the notions defined below turn out to be the same, independently of the choice of Ω. Ω has the following properties:

- Ω has a recursive approximation $\Omega_0, \Omega_1, \cdots$ from the left as it is left-r.e..
- The convergence module c_Ω defined as

$$c_\Omega(n) = \min\{s : \forall m \leq n\, (\Omega_s(m) = \Omega(m))\}$$

  dominates all total-recursive functions and furthermore $c_\Omega(n)$ is larger than the time for any terminating computation of the underlying universal machine needs on any input of length n or less.
- There are nonrecursive sets A such that Ω is random relative to A. These sets are called low for Ω.

In particular the subclass of those sets low for Ω which are reducible $\emptyset'$ has several natural characterizations [6,12]. Downey, Hirschfeldt, Miller and Nies [5, Corollary 8.6] showed that every Δ^0_2 degree low for Ω is completely weakly computable, and that such degrees can be nonrecursive.

Theorem 4.1 [5]**.** *(Downey, Hirschfeldt, Miller and Nies) If a set $A \leq_T \emptyset'$ is low for Ω then it is weakly computable.*

One could generalize the notion "low for Ω" to the notion that c_Ω dominates every A-recursive function. This class of degrees is indeed an old friend and there are several characterizations for it [6,9,5], one of which adapts "Ω is Martin-Löf random relative to A" to "Ω is Schnorr random relative to A". So for every r.e. set A the following statements are equivalent:

- c_Ω dominates every A-recursive function;
- Ω is Schnorr random relative to A;
- the Turing degree of A is array recursive;

– the Turing degree of A has a strong minimal cover;
– A is r.e. traceable, that is, for every $f \leq_T A$ and almost all n, the Kolmogorov complexity of $f(n)$ is at most n.

Theorem 1.5 provides another characterization of array recursive degrees.

Theorem 1.5. *For any r.e. set A, the following are equivalent:*

1. *The Turing degree of A is array recursive;*
2. *Every $B \leq_T A$ is weakly computable;*
3. *The Turing degree of A is completely weakly computable.*

Proof. $(1 \Rightarrow 2)$: Assume that A is r.e. and domination low for Ω. Let $B \leq_T A$. B has a recursive approximation $\beta_0, \beta_1, \cdots$ of rationals such that the convergence module

$$c_B(n) = \min\{t : \forall s \geq t \forall m \leq n\,(\beta_s(m) = \beta_t(m))\}$$

is A-recursive. Furthermore define the recursive function g inductively by $g(0) = 0$ and $g(s+1)$ being the minimal element of $\Omega_{s+1} - \Omega_s$ which, by choice, is indeed the first element where these two numbers differ; without loss of generality such an element always exists. Since c_Ω dominates the function $n \to c_B(2n)$, with a change of finitely many β_n, one can achieve that c_Ω actually majorizes this function. Now define a subsequence $\gamma_0, \gamma_1, \cdots$ of $\beta_0, \beta_1, \cdots$ by the following recursive algorithm:

1. Let $t = 0$. Let $s = 0$.
2. Let $\gamma_t = \beta_s$.
3. While $\exists m \leq 2g(s+1) - 2\,(\beta_{s+1}(m) \neq \gamma_t(m))$ Do $s = s + 1$.
4. Let $t = t + 1$. Let $s = s + 1$. Goto 2.

First one shows by induction that for all n and all $s = c_\Omega(n)$ there is a t with $\gamma_t = \beta_s$.

If $c_{\Omega(0)} = 0$ then $\gamma_0 = \beta_0$ and the assumption holds; if $c_{\Omega(0)} > 0$ then $\Omega(0) = 1$, $g(c_\Omega(n)) = 0$ and the existential quantifier in step 3 becomes false when $s = c_\Omega(0) - 1$, thus the while loop stops and $\beta_{c_\Omega(0)}$ is added into the γ_e.

Assume now that the assumption is true for n, so there is a t' with $\gamma_{t'} = \beta_{c_\Omega(n)}$. If $c_\Omega(n+1) = c_\Omega(n)$ then there is nothing to show. If $c_\Omega(n+1) > c_\Omega(n)$ then let s, t be the values of the variables of the same name in the algorithm when $s = c_\Omega(n+1) - 1$. In this case $g(s+1) = n+1$, $t \geq t'$ and $\gamma_t(m) = B(m)$ for all $m \leq 2n = g(n+1) - 2$ since γ_t equals to some $\beta_{s''}$ with $s'' \geq c_B(2n)$. So $\beta_{s+1}(m) = \gamma_t(m)$ for all $m \leq g(s+1)$ and the loop in Step 3 terminates. Thus β_{s+1} will become γ_{t+1} and the assumption is verified again.

So it follows that the sequence of all γ_t is infinite and converges to B. Furthermore, for every t there is unique s with $\gamma_{t+1} = \beta_{s+1}$ and $\forall m \leq g(s+1) - 2\,(\gamma_{t+1}(m) = \gamma_t(m))$. Thus $|\gamma_{t+1} - \gamma_t| \leq 2^{3-g(s+1)}$ and $\sum_{t \in \mathbb{N}} |\gamma_{t+1} - \gamma_t| \leq \sum_{s \in \mathbb{N}} 2^{g(s+1)-2} \leq \sum_{n \in \mathbb{N}} 2^n \cdot 2^{3-2n} \leq 16$ and the sequence $\gamma_0, \gamma_1, \cdots$ witnesses that B is weakly computable.

($2 \Rightarrow 3$): Obviously.

($3 \Rightarrow 1$): Let $f \leq_T A$ be a strictly increasing function. Define a set $B \equiv_T A$ as follows: $B(0) = 0$; $B(2^n) = A(n)$; $B(2^n + m) = \Omega_{f(2^{n+3})}(2^{n+1} + m)$ for $m = 1, 2, \cdots, 2^n - 1$.

Since $f \leq_T A$, it is obvious that $B \equiv_T A$. Thus B is weakly computable. Suppose that $B = \beta_0 - \beta_1$ for two left-r.e. reals β_0 and β_1. Now one has the following four facts:

(1) $\exists c_0 \forall n\, C(\beta_0(0)\beta_0(1)\cdots\beta_0(n)|\Omega(0)\Omega(1)\cdots\Omega(n)) \leq c_0$;
(2) $\exists c_1 \forall n\, C(\beta_1(0)\beta_1(1)\cdots\beta_1(n)|\Omega(0)\Omega(1)\cdots\Omega(n)) \leq c_1$;
(3) $\exists c_2 \forall n\, C(B(0)B(1)\cdots B(n)|\Omega(0)\Omega(1)\cdots\Omega(n)) \leq c_2$;
(4) $\forall c_3 \forall^\infty n\, (\Omega(2^{n+1}+1)\Omega(2^{n+1}+2)...\Omega(2^{n+1}+2^n-1)|\Omega(0)\Omega(1)...\Omega(2^n)) > c_3$.

Here the first two statements follow from the fact that Ω is complete among the left-r.e. sets with respect the so called rK-reducibility, that third statement follows from the fact that $B = \beta_0 - \beta_1$ and the fourth statement from the fact that Ω is Martin-Löf random and that the digits between 2^{n+1} and $2^{n+1} + 2^n$ cannot be predicted from those up to 2^{n+1}. Therefore, for almost all n, there is an $m \in \{1, 2, \cdots, 2^n - 1\}$ with $B(2^n + m) \neq \Omega(2^{n+1} + m)$. Thus, for almost all n, $f(2^{n+3}) < c_\Omega(2^{n+1} + 2^n - 1)$. The fact that both functions are monotone gives the following inequality:

$$\forall^\infty m \exists n\, (2^{n+2} \leq m < 2^{n+3} \text{ and } f(m) < f(2^{n+3}) < c_\Omega(2^{n+1}+2^n-1) < c_\Omega(m)).$$

So c_Ω dominates f. ∎

There are r.e. sets such that their Turing degree is array-recursive and low$_2$ but not low; thus these sets are not low for Ω. So one has that the above characterization shows that there are more completely weakly computable degrees than those found in [5].

Corollary 4.2. *There is a completely weakly computable and r.e. Turing degree which is not low for Ω.*

We note that in the proof of Theorem 1.5, the proof of the direction ($1 \Rightarrow 2$) needs that A has r.e. Turing degree, while the directions ($2 \Rightarrow 3$) and ($3 \Rightarrow 1$) work for all sets $A \leq_T \emptyset'$.

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