ENUMERATION DEGREES AND NON-METRIZABLE TOPOLOGY

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Abstract. The enumeration degrees of sets of natural numbers can be identified with the degrees of difficulty of enumerating neighborhood bases of points in a universal second-countable T_0-space (e.g. the ω-power of the Sierpiński space). Hence, every represented second-countable T_0-space determines a collection of enumeration degrees. For instance, Cantor space captures the total degrees, and the Hilbert cube captures the continuous degrees by definition. Based on these observations, we utilize general topology (particularly non-metrizable topology) to establish a classification theory of enumeration degrees of sets of natural numbers.

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Contents

1. Introduction

1.1. **Context.** The notion of an enumeration degree was introduced by Friedberg and Rogers [26] in 1950s to estimate the degree of difficulty of enumerating a given set of natural numbers. Roughly speaking, given sets $A, B \subseteq \omega$, A is *enumeration reducible to* B (written $A \leq_e B$) if there is a computable procedure that, given an enumeration of B, returns an enumeration of A. The pre-ordering $\leq_e$ induces a degree structure called the *enumeration degrees*, or e-degrees for short. Since then, the study of enumeration degrees have been one of the most important subjects in computability theory.

Nevertheless, only a few subcollections of enumeration degrees have been isolated. Some prominent isolated properties are totality, semirecursiveness [41], and quasi-minimality [51]. Recently, the notion of cototality has also been found to be important and robust; see Andrews et al. [1], Jeandel [40], and McCarthy [48]. Our aim is to understand the profound structure of enumeration degrees by isolating further subcollections of enumeration degrees and then establish a "zoo" of enumeration degrees.

To achieve our objective, we pay attention to a topological perspective of enumeration degrees. The enumeration degrees can be identified with the degrees of difficulty of enumerating neighborhood bases of points in a universal second-countable T_0-space (e.g. the ω-power of the Sierpiński space). Hence, every represented second-countable T_0-space determines a collection of enumeration degrees.

This is exactly what Miller [54] did for metric spaces. Miller introduced the notion of *continuous degrees* as the degree structure of a universal separable metric space, and he described how this new notion can be understood as a substructure of enumeration degrees. Subsequently, Kihara-Pauly [43] noticed that the *total degrees* are the enumeration degrees of neighborhood bases of points in (sufficiently effective) countable dimensional separable metric spaces. This observation eventually led them to an application of continuous degrees in other areas outside of computability theory, such as descriptive set theory and infinite dimensional topology. Other applications of continuous degrees can also be found in Day-Miller [14] and Gregoriades-Kihara-Ng [32].

The connection to topology, and the results obtained through it, have already led to a flurry of recent activity in the study of enumeration degrees [2, 55, 29, 48].

In this article, we further develop and deepen these connections. We utilize general topology (particularly non-metrizable topology) to establish a classification theory of enumeration degrees. For instance, we will examine which enumeration degrees can be realized as points in T_0 (Kolmogorov), T_1 (Fréchet), T_2 (Hausdorff), $T_{2.5}$ (Urysohn), and submetrizable spaces. Furthermore, we will discuss the notion of T_i-quasi-minimality. We will also provide a characterization of the notion of cototality in terms of computable topology.

Our work reveals that general topology (non-metrizable topology) is extremely useful to understand the highly intricate structure of subsets of the natural numbers.

1.2. **Summary.** In Section 3, we reveal what substructures are captured by the degree structures of individual represented cb_0 spaces (some of which are quasi-Polish). For instance, we define various subcollections of e-degrees, and then show the following.

- We construct a represented, decidable, T_1, non-T_2, quasi-Polish space $\mathcal{X}$ such that the $\mathcal{X}$-degrees are precisely the telograph-cototal degrees (Proposition 3.21).
- We construct a represented, decidable, T_2, non-$T_{2.5}$, quasi-Polish space $\mathcal{X}$ such that the $\mathcal{X}$-degrees are precisely the doubled co-d-CEA degrees (Theorem 3.25).
- We construct a represented, decidable, $T_{2.5}$, non-submetrizable, quasi-Polish space $\mathcal{X}$ such that the $\mathcal{X}$-degrees are precisely the Arens co-d-CEA (the Roy halfgraph-above) degrees (Theorems 3.33 and 3.36).
- We construct a represented, decidable, submetrizable, non-metrizable, quasi-Polish space $\mathcal{X}$ such that the $\mathcal{X}$-degrees are precisely the co-d-CEA degrees (Proposition 3.48).
- Given a countable pointclass Γ, there is a computable extension γ of the standard representation of Cantor space (hence, it induces a submetrizable topology) such that the $(2^\omega, \gamma)$-degrees are exactly the Γ-above degrees (Proposition 3.39).
- Every e-degree is an $\mathcal{X}$-degree for some decidable, submetrizable, cb_0 space $\mathcal{X}$ (Theorem 3.40). In particular, every e-degree is the degree of a point of a decidable $T_{2.5}$ space.

For the details of the above results, see Section 3. In Section 3.8, we emphasize the importance of the notion of a network (which has been extensively studied in general topology). A G_δ-space is a topological space in which every closed set is G_δ. A second-countable T_0-space $\mathcal{X}$ is a G_δ-space if and only if $\mathcal{X}$ has a countable closed network (Proposition 3.56). The following is an unexpected characterization of cototality.

- An e-degree is cototal if and only if it is an $\mathcal{X}$-degree of a computably G_δ, cb_0 space $\mathcal{X}$ (Theorem 3.60).
- There exists a decidable, computably G_δ, cb_0 space $A^{\mathrm{co}}_{\max}$ such that the $A^{\mathrm{co}}_{\max}$-degrees are exactly the cototal e-degrees (Theorem 3.61).

We also show several separation results for specific degree-notions. For instance,

- There are an n-semirecursive e-degree $\mathbf{c} \leq \mathbf{0}''$ and a total e-degree $\mathbf{d} \leq \mathbf{0}''$ such that the join $\mathbf{c} \oplus \mathbf{d}$ is not $(n+1)$-semirecursive (Theorem 5.21).
- For any $n \in \omega$, an n-semirecursive e-degree is either total or a strong quasi-minimal cover of a total e-degree (Theorem 5.23).
- For any n, there is an $(n+1)$-cylinder-cototal e-degree which is not n-cylinder-cototal (Theorem 3.15).
- There is a co-d-CEA set $A \subseteq \omega$ such that A is not cylinder-cototal (Proposition 3.13).
- Every semirecursive, non-Δ^0_2 e-degree is quasi-minimal w.r.t. telograph-cototal e-degrees (Theorem 5.24).
- There is a semirecursive set $A \subseteq \omega$ which is quasi-minimal, but not quasi-minimal w.r.t. telograph-cototal e-degrees (Theorem 5.25).
- There is a cylinder-cototal e-degree which is quasi-minimal w.r.t. telograph-cototal e-degrees (Theorem 5.36).

- By $(\omega^\omega)_{GH(n)}$ we denote the set ω^ω endowed with the Σ^1_n-Gandy-Harrington topology. For any distinct numbers $n, m \in \omega$, there is no e-degree which is both an $(\omega^\omega)_{GH(n)}$-degree and an $(\omega^\omega)_{GH(m)}$-degree (Theorem 5.56).
- There is a continuous degree which is neither telograph-cototal nor cylinder-cototal (Proposition 5.43).

Moreover, we introduce the notion of a regular-like network, and give a characterization in the context of a closure representation, which plays a key role in Section 5. By using these notions, we will show the following separation results.

- Let $\mathcal{T}$ be a countable collection of second-countable T_1 spaces. Then, there is a $\mathcal{T}$-quasi-minimal semirecursive e-degree (Theorem 5.14).
- Let $\mathcal{T}$ be a countable collection of second-countable T_1 spaces. Then, there is an $(n+1)$-semirecursive e-degree which cannot be written as the join of an n-semirecursive e-degree and an $\mathcal{X}$-degree for $\mathcal{X} \in \mathcal{T}$ (Theorem 5.19).
- For any represented Hausdorff space $\mathcal{X}$, there is a cylinder-cototal e-degree which is not an $\mathcal{X}$-degree (Theorem 5.31).
- There is a cylinder-cototal e-degree which is $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimal (Theorem 5.35).
- Given any countable collection $\{S_i\}_{i\in\omega}$ of effective T_2 spaces, there is a telograph-cototal e-degree which is S_i-quasi-minimal for any $i \in \omega$ (Theorem 5.38).
- For any represented $T_{2.5}$-space $\mathcal{X}$, there is an $(\mathbb{N}_{\rm rp})^\omega$-degree which is not an $\mathcal{X}$-degree (Theorem 5.46).
- There is an $(\mathbb{N}_{\rm rp})^\omega$-degree which is $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimal (Theorem 5.47).
- Let $\mathcal{X} = (X, \mathcal{N})$ be a regular Hausdorff space with a countable cs-network. Then there is an $(\omega^\omega)_{GH}$-degree which is not an $\mathcal{X}$-degree (Theorem 5.53).
- The Gandy-Harrington space has no point of $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-degree (Theorem 5.54).

2. Preliminaries

2.1. **Notations.** We use A^{c} to denote the complement of A, and $\overline{U}$ always means the topological closure of U. We also use the standard notations in computability theory: For $X, Y \subseteq \omega$, let $X \oplus Y$ be the Turing join of X and Y; that is, $(X \oplus Y)(2n) = X(n)$ and $(X \oplus Y)(2n+1) = Y(n)$. For strings $\sigma, \tau \in \omega^{<\omega}$, let $\sigma^\frown\tau$ be the concatenation of σ and τ, and we mean by $\sigma \preceq \tau$ that σ is an initial segment of τ. For a string $\sigma \in \omega^{<\omega}$, let $|\sigma|$ be the length of σ, and $[\sigma] = \{X \in \omega^\omega : X \succ \sigma\}$. By $x \upharpoonright s$ we mean the restriction of x up to s.

2.2. **General topology.** We first review some basic concepts from general topology (see also Steen-Seebach [74]). In most parts of this paper, we only deal with second-countable T_0-spaces. However in Section 4 we also consider (non-second-countable) spaces which have a countable cs-network, e.g. the Kleene-Kreisel space $\mathbb{N}^{\mathbb{N}^\mathbb{N}} := C(\mathbb{N}^\mathbb{N}, \mathbb{N})$. A normal space having a countable cs-network is known as an $\aleph_0$-space (see [52, 35]). A space X is T_0 (*Kolmogorov*) if any two distinct points are topologically distinguishable. We are only concerned with T_0 spaces in this paper. A space X is T_1 (*Fréchet*) if every singleton is closed. A space X is T_2 (*Hausdorff*) if the diagonal is closed. A space X is $T_{2.5}$ (*Urysohn*) if any two distinct points are separated by their closed neighborhoods. A space X is T_D if every singleton is the intersection of an open set and a closed set. A space X is *completely Hausdorff* if any two distinct points are separated by a continuous

$[0,1]$-valued function. A space X is *submetrizable* if it admits a continuous metric. We have the following implications.

$$\text{metrizable} \Rightarrow \text{submetrizable} \Leftrightarrow {}^1\text{completely Hausdorff} \Rightarrow T_{2.5} \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_D \Rightarrow T_0.$$

A space X is *regular* if the closed neighborhoods of a point x form a local network at the point x, that is, every neighborhood of a point contains a closed neighborhood of the same point. In the category of second-countable T_0 spaces, by Urysohn's metrization theorem, the property being T_3 (regular Hausdorff) is equivalent to metrizability. A space in which every closed set is G_δ is called a *G_δ-space*. Every metrizable space is G_δ (see [74, Part III]), and every T_0, G_δ-space is T_1 (see Section 3.8.1).

2.3. **Computability theory.**

2.3.1. *Enumeration and Medvedev reducibility.* We review the definition of enumeration reducibility (see also Odifreddi [59, Chapter XIV], Cooper [11] & [12, Chapter 11]). Let $(D_e)_{e\in\omega}$ be a computable enumeration of all finite subsets of ω. Given $A, B \subseteq \omega$, we say that A *is enumeration reducible to* B (written $A \leq_e B$) if there is a c.e. set Φ such that

$$n \in A \iff (\exists e)\ [\langle n, e\rangle \in \Phi \text{ and } D_e \subseteq B].$$

The Φ in the above definition is called an *enumeration operator*. An enumeration operator induces a computable function on ω^ω, and indeed, $A \leq_e B$ iff there is a computable function $f : \omega^\omega \to \omega^\omega$ such that given an enumeration p of A, $f(p)$ returns an enumeration of B, where we say that $p \in \omega^\omega$ is an enumeration of A if $A = \{p(n)-1 : p(n) > 0\}$ ($p(n) = 0$ indicates that we enumerate nothing at the n-th step).

Each equivalence class under the e-equivalence $\equiv_e := \leq_e \cap \geq_e$ is called an *enumeration degree* or simply *e-degree*. The e-degree of a set $A \subseteq \omega$ is written as $\deg_e(A)$. The e-degree structure forms a upper semilattice, where the join is given by the disjoint union $A \oplus B = \{2n : n \in A\} \cup \{2n + 1 : n \in B\}$. We use the symbol $\mathcal{D}_e$ to denote the set of all e-degrees. Whenever we introduce a property for sets, we can naturally extend the property to degrees, by saying that an e-degree has the property iff it contains a set with the same property.

For $P, Q \subseteq \omega^\omega$, we say that P *is Medvedev reducible to* Q (written $P \leq_M Q$, [51]) if there is a partial computable function $\Psi :\subseteq \omega^\omega \to \omega^\omega$ such that for any $q \in Q$, $\Psi(q) \in P$. There is a natural embedding of the enumeration degrees into the Medvedev degrees of the Baire space, by taking a set A to the class of all enumerations of A.

2.4. **Represented spaces.**

The central objects of study in computable analysis are the represented spaces, which allow us to make sense of computability for most space of interest in everyday mathematics.

Definition 2.1. A *represented space* is a set X together with a partial surjection $\delta :\subseteq \omega^\omega \to X$. We often write $\mathcal{X}$ for a represented space.

[1]Being submetrizable and being completely Hausdorff coincides for spaces with hereditarily Lindelöf squares, which includes all spaces with countable networks, hence all spaces relevant for our purposes. We are grateful to Taras Banakh for pointing this out to us on mathoverflow (`https://mathoverflow.net/questions/280359/does-second-countable-and-functionally-hausdorff-imply-submetrizable`).

We say that $p \in \omega^\omega$ is a *δ-name of x* if $x = \delta(p)$. We use $\mathrm{Name}_\delta(x)$ to denote the set of all δ-names of x, or just write $\mathrm{Name}(x)$, if the space is clear from the context. Hereafter, by a *point*, we mean a pair of a point $x \in X$ and the underlying represented space $\mathcal{X} = (X, \delta)$, denoted by $x \colon \mathcal{X}$ or simply $x \colon \delta$. We call a point $x \colon \delta$ *computable* if it has a computable δ-name. Similarly we extend the notion to define C-computable points relative to an arbitrary oracle C.

A partial function $F :\subseteq \omega^\omega \to \omega^\omega$ is called a *realizer* of a partial function $f :\subseteq \mathcal{X} \to \mathcal{Y}$, if $\delta_{\mathcal{Y}}(F(p)) = f(\delta_{\mathcal{X}}(p))$ for any $p \in \mathrm{dom}(f\delta_{\mathcal{X}})$. We then say that f is computable (respectively continuous), if f has a computable (respectively continuous) realizer.

If γ and δ are representations, we say that γ *is (computably) reducible to* δ or γ *is (computably) finer than* δ if there is a continuous (computable) function Φ such that $\gamma = \delta \circ \Phi$. It is equivalent to saying that the identity map $\mathrm{id} \colon (X, \gamma) \to (X, \delta)$ is continuous (computable). If γ is (computably) reducible to δ and δ is (computably) reducible to γ, we call γ and δ (computably) equivalent. This corresponds to $\mathrm{id} \colon (X, \gamma) \to (X, \delta)$ being a (computable) isomorphism.

In the topological terminology, γ is reducible to δ if and only if $\tau_\delta \subseteq \tau_\gamma$, where τ_γ and τ_δ are the quotient topologies given by γ and δ, respectively. If X is equipped with a topology τ, then δ is continuous if and only if $\tau \subseteq \tau_\delta$.

2.4.1. *Representation via a countable basis.* A *represented cb space* is a pair $(\mathcal{X}, \beta)$ of a second-countable space $\mathcal{X}$ and an enumeration $\beta = (\beta_e)_{e \in \omega}$ of a countable open subbasis of $\mathcal{X}$. Here, "cb" stands for "countably based". If a represented cb space is T_0, then it is also called a represented cb_0 space. The enumeration β is called a *cb representation of $\mathcal{X}$*.

One of the key observations is that specifying a cb representation β of a second-countable T_0 space $\mathcal{X}$ is the same thing as specifying an embedding of $\mathcal{X}$ into the power set $\mathcal{P}\omega$ of ω endowed with the Scott topology (that is, basic open sets are $\{X \subseteq \omega : D \subseteq X\}$ where D ranges over finite subsets of ω). Hence, a cb_0 representation β (and the induced embedding) determines how a point $x \in \mathcal{X}$ is identified with a subset of the natural numbers. This observation entails the known fact that the Scott domain $\mathcal{P}\omega$ is a universal second-countable T_0 space, that is, every second-countable T_0 space embeds into $\mathcal{P}\omega$. We describe how an embedding : $\mathcal{X} \hookrightarrow \mathcal{P}\omega$ is induced from a representation β. One can identify a point x in a represented cb_0 space $(\mathcal{X}, \beta)$ with the coded neighborhood filter

$$\mathrm{Nbase}_\beta(x) = \{e \in \omega : x \in \beta_e\}.$$

It is not hard to see that $\mathrm{Nbase}_\beta \colon \mathcal{X} \hookrightarrow \mathcal{P}\omega$ is a topological embedding. An enumeration of $\mathrm{Nbase}_\beta(x)$ is called a *β-name of x*, that is, for a $p : \omega \to \omega$,

$$p \text{ is a } \beta\text{-name of } x \iff \mathrm{rng}(p) = \mathrm{Nbase}_\beta(x),$$

where one can assume $\beta_0 = \mathcal{X}$ without loss of generality.

Clearly, a cb-representation β always induces a representation δ_β defined by $\delta_\beta(p) = x$ iff p is a β-name of x (i.e., p enumerates $\mathrm{Nbase}_\beta(x)$). This entails that $\mathrm{Nbase}_\beta(x)$ is c.e. iff $x \colon \mathcal{X}$ is computable. In situations where no confusion is expected, we may speak of a cb representation and its induced representation interchangeably. For a given cb space $\mathcal{X}$ there is usually a canonical cb-representation. If β is clear from the context, we will write $\mathrm{Nbase}_{\mathcal{X}}(x)$ or simply $\mathrm{Nbase}(x)$ instead of $\mathrm{Nbase}_\beta(x)$.

We can also express computability of partial functions between represented cb spaces equivalently as a special case of computability on represented spaces, or in the language of enumeration reducibility: Saying that $f :\subseteq \mathcal{X} \to \mathcal{Y}$ is computable is equivalent to saying that there is a single enumeration operator Ψ such that

$$(\forall x \in \mathrm{dom}(f))\ [\mathrm{Nbase}(f(x)) \leq_e \mathrm{Nbase}(x) \text{ via } \Psi].$$

Remark 2.2. It is known that the Scott domain $\mathcal{P}\omega$ is homeomorphic to the ω-power $\mathbb{S}^\omega$ of the Sierpiński space, where $\mathbb{S} = \{0,1\}$ which has the three open sets $\emptyset$, $\{1\}$, and $\mathbb{S}$ (see [15, 65]).

Remark 2.3. In Weihrauch-Grubba [79], a represented cb_0 space is called an *effective topological space*. However we prefer to emphasize second-countability (cb) since the range of computability theory is far larger than second-countable spaces (see e.g. [65, 66, 70, 69, 20, 61, 18, 60, 67]). We also avoid the use of the terminology "effective" since the definition of a represented cb_0 space does not involve any effectivity.

2.4.2. *Changing representations.* We have introduced a cb representation as a countable subbasis $(\beta_e)_{e\in\omega}$, but without loss of generality, we can always assume that it is actually a countable basis. To see this, let $\beta = (\beta_e)_{e\in\omega}$ be a countable subbasis of a cb_0 space $\mathcal{X}$. Then we get a basis β^+ of $\mathcal{X}$ by defining $\beta^+_\sigma = \bigcap_{i<|\sigma|}\beta_{\sigma(i)}$ for $\sigma \in \omega^{<\omega}$. Note that there is no difference between β and β^+ from the computability-theoretic perspective:

$$\begin{aligned} e \in \mathrm{Nbase}_\beta(x) &\iff \langle e\rangle \in \mathrm{Nbase}_{\beta^+}(x),\\ \sigma \in \mathrm{Nbase}_{\beta^+}(x) &\iff \{\sigma(0), \dots, \sigma(|\sigma|-1)\} \subseteq \mathrm{Nbase}_\beta(x). \end{aligned}$$

In other words, $\mathrm{Nbase}_\beta(x)$ is e-equivalent to $\mathrm{Nbase}_{\beta^+}(x)$ in a uniform manner. Indeed, we find that every subbasis β is computably equivalent to the induced basis β^+ (in the sense of represented spaces).

We observe that translations between cb_0 representations have a particular convenient form: Let β and γ be cb_0 representations of $\mathcal{X}$. We see that β *is computably reducible to* γ (written $\beta \leq \gamma$) if there is a single enumeration operator witnessing the reduction $\mathrm{Nbase}_\beta(x) \leq_e \mathrm{Nbase}_\gamma(x)$ for any $x \in \mathcal{X}$. It is equivalent to saying that any β-basic open set is γ-c.e. open in an effective manner, that is, there is a computable function h such that

$$\beta_e = \bigcup\{\gamma^+_\sigma : \sigma \in W_{h(e)}\},$$

where γ^+ is a basis for $\mathcal{X}$ defined as above.

Computable topological spaces. We will consider the following additional effective properties for represented cb spaces (X, β).

(I) There is a c.e. set S such that $\beta_i \cap \beta_j = \bigcup\{\beta_e : (i,j,e) \in S\}$.
(E) $\{e : \beta_e \neq \emptyset\}$ is c.e.

In Weihrauch-Grubba [79], a represented cb_0 space with (I) is called a *computable topological space*. In Kurovina-Kudinov [45], a represented cb space with (I) and (E) is called a *effectively enumerable topological space*.

Moreover, if every positive finite Boolean operation on β is computable, then we say that (X, β) is *decidable*.

Proposition 2.4. *Let β, γ be representations of X such that $\gamma \equiv \beta$. If (X, β) is computable, so is (X, γ).*

Proof. For computability, since $\gamma \leq \beta$, given d and e, γ_d and γ_e can be written as β-c.e. open set U_d and U_e. Thus, $\gamma_d \cap \gamma_e = U_d \cap U_e$. One can easily find a β-index of the β-c.e. open set $U_d \cap U_e$, that is, $U_d \cap U_e = \bigcup_n \beta_{f(n,d,e)}$. Since $\beta \leq \gamma$, we also have a γ-index of $U_d \cap U_e$, that is, $U_d \cap U_e = \bigcup_n \gamma_{g(n,d,e)}$. Hence, $\gamma_d \cap \gamma_e = \bigcup_n \gamma_{g(n,d,e)}$, that is, γ is computable. □

2.4.3. *Multi-representations.* As a technical took, we will rarely make use of multi-representations. A *(multi-)representation* is a multi-valued partial surjection $\delta :\subseteq \omega^\omega \rightrightarrows \mathcal{X}$. We say that $p \in \omega^\omega$ is a *δ-name of x* if $x \in \delta(p)$. Notions such as realizer, computable functions between multi-represented spaces, etc, are all defined analogously to the case of ordinary representations. In the context of computable analysis, multi-representations were introduced by Schröder [64]. They are an instance of assemblies from realizability theory [75].

2.4.4. *Admissible representation.* For a topological space $\mathcal{X} = (X, \tau)$, we say (following Schröder [65]) that $\delta \colon \omega^\omega \to \mathcal{X}$ is *admissible* if it is $\leq$-maximal among continuous representations of X, that is, it is continuous, and every continuous representation of $\mathcal{X}$ is reducible to δ. Equivalently, a representation $\delta :\subseteq \omega^\omega \to \mathcal{X}$ is admissible if it is continuous, and for any continuous representation $\gamma : \omega^\omega \to \mathcal{X}$, the identity map $(\mathcal{X}, \gamma) \to (\mathcal{X}, \delta)$ is continuous. Note also that admissible representations are the ones which realize the coarsest quotient topology refining τ.

Observation 2.5. *The representation δ_β induced from a cb-representation is always admissible.*

In fact, if $\mathcal{N} = (N_e)_{e \in \omega}$ is a countable cs-network (see Section 4) for a T_0 space $\mathcal{X}$, Schröder [65] showed that the following map $\delta_\mathcal{N}$ always gives an admissible representation of $\mathcal{X}$:

$$\delta_\mathcal{N}(p) = x \iff \{N_{p(n)} : n \in \omega\} \text{ is a strict network at } x \text{ (see Definition 4.1).}$$

We call $\delta_\mathcal{N}$ the *induced ω^ω-representation of $\mathcal{X}$* (obtained from $\mathcal{N}$). We also use the symbol $\mathrm{Name}_\mathcal{N}(x)$ to denote the set of all enumerations of a strict subnetwork of $\mathcal{N}$ at x, that is,

$$\mathrm{Name}_\mathcal{N}(x) = \delta_\mathcal{N}^{-1}\{x\} = \{p \in \omega^\omega : \delta_\mathcal{N}(p) = x\}$$

If $\mathcal{N}$ is clear from the context, we also use $\mathrm{Name}(x)$ instead of $\mathrm{Name}_\mathcal{N}(x)$. See Section 4 for more details.

2.5. **Quasi-Polish spaces.** A *quasi-Polish space* is a second-countable space which is Smyth-completely quasi-metrizable [15]. Recall that a set in a space is $\mathbf{\Pi}^0_2$ if it is the intersection of countably many constructible sets, where a constructible set is a finite Boolean combination of open sets (see [15, 71]). De Brecht [15, Theorem 24] showed that a space is quasi-Polish if and only if it is homeomorphic to a $\mathbf{\Pi}^0_2$ subset of $\mathbb{S}^\omega$. Be careful that it is not always the case that $\mathbf{\Pi}^0_2 = G_\delta$. Indeed, $\mathbf{\Pi}^0_2 = G_\delta$ holds if and only if the underlying space is a G_δ space (see Section 3.8.1).

Assume that $\mathcal{X}$ is a represented cb_0 space. Consider the set of all names of points in $\mathcal{X}$:

$$\mathrm{Name}(\mathcal{X}) = \{p \in \omega^\omega : (\exists x \in \mathcal{X})\ \mathrm{rng}(p) = \mathrm{Nbase}(x)\}.$$

Essentially, $\mathrm{Name}(\mathcal{X})$ is the domain of an admissible representation of the space $\mathcal{X}$. For a pointclass Γ, we say that $\mathcal{X}$ is Γ-*named* if $\mathrm{Name}(\mathcal{X})$ is Γ.

Proposition 2.6 (De Brecht [15]). *A represented cb_0 space $\mathcal{X}$ is quasi-Polish if and only if $\mathcal{X}$ is $\mathbf{\Pi}^0_2$-named.*

Proof. Note that $\mathrm{Nbase} : x \mapsto \mathrm{Nbase}(x)$ is an embedding of $\mathcal{X}$ into $\mathbb{S}^\omega$. If $\mathcal{X}$ is quasi-Polish, then so is the homeomorphic image $\mathrm{Nbase}[\mathcal{X}]$. By de Brecht [15, Theorem 21], $\mathrm{Nbase}[\mathcal{X}]$ is $\mathbf{\Pi}^0_2$ in $\mathbb{S}^\omega$. Note that $\mathrm{Name}(\mathcal{X}) = \mathrm{rng}^{-1}[\mathrm{Nbase}[\mathcal{X}]]$. Since $\mathrm{rng} : \omega^\omega \to \mathbb{S}^\omega$ is clearly continuous, $\mathrm{Name}(\mathcal{X})$ is $\mathbf{\Pi}^0_2$, that is, $\mathcal{X}$ is $\mathbf{\Pi}^0_2$-named.

Conversely, if $\mathcal{X}$ is $\mathbf{\Pi}^0_2$-named, then $\mathrm{Name}(\mathcal{X})$ is Polish, and in particular, quasi-Polish. Note that $\mathrm{rng} : \mathrm{Name}(\mathcal{X}) \to \mathrm{Nbase}[\mathcal{X}]$ is an open continuous surjection. Hence, by de Brecht [15, Theorem 40], $\mathrm{Nbase}[\mathcal{X}]$ is quasi-Polish. Consequently, $\mathcal{X}$ is quasi-Polish since $\mathcal{X}$ is homeomorphic to $\mathrm{Nbase}[\mathcal{X}]$. □

Remark 2.7. Let δ be an admissible representation of a space $\mathcal{X}$. Then, consider

$$\mathrm{Eq}(\mathcal{X}, \delta) = \{(p, q) \in \omega^\omega : p, q \in \mathrm{dom}(\delta) \text{ and } \delta(p) = \delta(q)\}.$$

Generally, de Brecht et al. [20] has studied the classification of spaces based on the complexity of $\mathrm{Eq}(\mathcal{X}, \delta)$.

The following fact is useful to show that a space is quasi-Polish.

Fact 2.8 (De Brecht [15, Theorems 40 and 41]). *A T_0 space $\mathcal{X}$ is quasi-Polish if and only if there is an open continuous surjection from a Polish space onto $\mathcal{X}$.* □

Conversely, de Brecht has generalized the Hurewicz dichotomy from Polish to quasi-Polish spaces in [16] to yield the following:

Theorem 2.9 (De Brecht [16, Theorem 7.2]). *A $\mathbf{\Pi}^1_1$-subspace of a quasi-Polish is not quasi-Polish if and only if it contains a homeomorphic copy of one of the following spaces as a $\mathbf{\Pi}^0_2$-subspace:*

(1) $\mathbb{Q}$ *with the subspace topology inherited from* $\mathbb{R}$
(2) ω_{cof}*, the integers with the cofinite topology*
(3) $\omega_<$*, the lower integers*
(4) S_0*, with underlying set $\omega^{<\omega}$ equipped with the lower topology, where the basic closed sets are of the form $\uparrow p := \{q \in \omega^{<\omega} \mid p \preceq q\}$ (here $\preceq$ denotes the prefix relation)*

Of these, we make use of $\mathbb{Q}$ and ω_{cof} to show that particular spaces are not quasi-Polish. We refer the reader to [19, 39] for a possible definition of a *computable quasi-Polish space*. We do not need it for our purposes, but we shall point out that whenever we are arguing that a particular space is quasi-Polish, it will in fact be computably quasi-Polish.

3. Enumeration degree zoo

3.1. **Definitions and overview.** In this section, we focus on the degree structures of second-countable spaces. Our objective of this section is to see that general topology is surprisingly useful for investigating the enumeration degrees.

3.1.1. *The degree structure of a space.* We now introduce one of the key notions in this article. To each point $x\colon \mathcal{X}$, we assign the *degree of difficulty of calling a name of* x.

Definition 3.1 (see Kihara-Pauly [43])**.** Let $x\colon \mathcal{X}$ and $y\colon \mathcal{Y}$ be points. Then, we define

$$x\colon \mathcal{X} \leq_{\mathbf{T}} y\colon \mathcal{Y} \overset{\text{def}}{\iff} \mathrm{Name}_{\mathcal{X}}(x) \leq_M \mathrm{Name}_{\mathcal{Y}}(y).$$

One can see that if $\mathcal{X}$ and $\mathcal{Y}$ are represented cb_0 spaces, then

$$\mathrm{Name}_{\mathcal{X}}(x) \leq_M \mathrm{Name}_{\mathcal{Y}}(y) \iff \mathrm{Nbase}_{\mathcal{X}}(x) \leq_e \mathrm{Nbase}_{\mathcal{Y}}(y)$$

Therefore, reducibility between points can be defined in the following manner:

$$x\colon \mathcal{X} \leq_{\mathbf{T}} y\colon \mathcal{Y} \iff \mathrm{Nbase}_{\mathcal{X}}(x) \leq_e \mathrm{Nbase}_{\mathcal{Y}}(y)$$

We now describe how we classify the e-degrees by using topological notions. Let $\mathcal{X}$ be a represented cb_0 space. We say that an enumeration degree $\mathbf{d}$ is an $\mathcal{X}$*-degree* if $\mathrm{Nbase}(x) \in \mathbf{d}$ for some $x \in \mathcal{X}$. By $\mathcal{D}_{\mathcal{X}}$, we denote the set of all $\mathcal{X}$-degrees. In other words,

$$\mathcal{D}_{\mathcal{X}} = \{\deg_e(\mathrm{Nbase}_{\mathcal{X}}(x)) : x \in \mathcal{X}\}.$$

A key observation is that every represented cb_0 space determines a subset $\mathcal{D}_{\mathcal{X}}$ of the e-degrees $\mathcal{D}_e$.

Example 3.2. Cantor space 2^ω, Baire space ω^ω, Euclidean n-space $\mathbb{R}^n$, and Hilbert cube $[0,1]^\omega$ are represented in a standard manner.

1. If $X \in \{2^\omega, \omega^\omega, \mathbb{R}^n\}$, then $\mathcal{D}_X$ is exactly the total degrees $\mathcal{D}_T$ (for the definition, see 3.1.2).
2. $\mathcal{D}_{[0,1]^\omega}$ exactly the continuous degrees $\mathcal{D}_r$ (see Miller [54]).
3. Let $\mathbb{R}_<$ be the real numbers endowed with the lower topology, and represented by $\beta_e = (q_e, \infty)$, where q_e is the e-th rational. Then, $\mathcal{D}_{\mathbb{R}_<}$ is exactly the semirecursive e-degrees (see Kihara-Pauly [43]).

For (1), Kihara-Pauly [43] showed that total e-degrees are characterized by countable dimensionality, that is, a separable metrizable space X is countable dimensional iff, for any representation β of X, there is an oracle C such that every (X,β)-degree is total relative to C. For (2), one can also easily see that metrizability captures continuous degrees by universality of the Hilbert cube. These are what we indicated by our slogan "*utilizing general topology to classify e-degrees*".

In classical computability theory, Medvedev [51] introduced the notion of quasi-minimality. An e-degree $\mathbf{a}$ is *quasi-minimal* if for every total degree $\mathbf{b} \leq_e \mathbf{a}$, we have $\mathbf{b} = \mathbf{0}$. It is equivalent to saying that there is $A \in \mathbf{d}$ such that

$$(\forall x \in 2^\omega)\ [\mathrm{Nbase}_{2^\omega}(x) \leq_e A \implies \mathrm{Nbase}_{2^\omega}(x) \text{ is c.e.}]$$

We introduce a topological version of quasi-minimality.

Definition 3.3. Let $\mathcal{T}$ be a collection of represented cb_0 spaces. We say that an e-degree $\mathbf{d}$ is *$\mathcal{T}$-quasi-minimal* if there is $A \in \mathbf{d}$ such that

$$(\forall \mathcal{X} \in \mathcal{T})(\forall x \in \mathcal{X})\ [\mathrm{Nbase}_{\mathcal{X}}(x) \leq_e A \implies \mathrm{Nbase}_{\mathcal{X}}(x) \text{ is c.e.}]$$

If $\mathcal{X}$ is a represented cb_0 space then we say that an e-degree $\mathbf{d}$ is *$\mathcal{X}$-quasi-minimal* if there is $A \in \mathbf{d}$ such that

$$(\forall x \in \mathcal{X})\ [\mathrm{Nbase}_{\mathcal{X}}(x) \leq_e A \implies \mathrm{Nbase}_{\mathcal{X}}(x) \text{ is c.e.}]$$

3.1.2. *Enumeration degree zoo.* Our aim of this section is to investigate $\mathcal{X}$-degrees for specific represented cb_0-spaces $\mathcal{X}$. Surprisingly, we will see that for most $\mathcal{X}$, the $\mathcal{X}$-degrees have very simple descriptions.

A set A is *total* if $A^{\mathsf{c}} \leq_e A$, and *cototal* [1, 40] if $A \leq_e A^{\mathsf{c}}$. For a total function $g : \omega \to \omega$ and $b \in \omega$, we define the graph $\mathrm{Graph}(g)$, the *cylinder-graph* $\mathrm{CGraph}(g)$, and the *b-telograph* $\mathrm{TGraph}(g)$ of g as follows:

$$\begin{aligned} \mathrm{Graph}(g) &= \{\langle n, m\rangle : g(n) = m\}, \\ \mathrm{CGraph}(g) &= \{\sigma \in \omega^{<\omega} : \sigma \prec g\}, \\ \mathrm{TGraph}_b(g) &= \{\langle n, m\rangle : g(n) = m \text{ and } m \geq b\}. \end{aligned}$$

Definition 3.4. Let $\mathbf{a}$ be an enumeration degree.

(1) We say that $\mathbf{a}$ is *graph-cototal* [1, 73] if $\mathbf{a}$ contains the complement $\mathrm{Graph}(g)^{\mathsf{c}}$ of the graph of a total function g.
(2) We say that $\mathbf{a}$ is *cylinder-cototal* if $\mathbf{a}$ contains the complement $\mathrm{CGraph}(g)^{\mathsf{c}}$ of the cylinder graph of a total function g. We also say that $\mathbf{a}$ is *n-cylinder-cototal* if it is the join of n many cylinder-cototal e-degrees.
(3) We say that $\mathbf{a}$ is *telograph-cototal* if $\mathbf{a}$ contains the join $\mathrm{Graph}(g)^{\mathsf{c}} \oplus \mathrm{TGraph}_b(g)$ for some total function $g : \omega \to \omega$ and $b \in \omega$.

Recall that a subset of ω is *d-c.e.* if it is the difference of two c.e. sets, and co-d-c.e. if it is the complement of a d-c.e. set, that is, the union $A \cup P$ of a c.e. set P and co-c.e. set A such that A and P are disjoint. Note that an enumeration degree contains a co-d-c.e. set if and only if it contains a 3-c.e. set.

Definition 3.5. Let $\mathbf{a}$ be an enumeration degree.

(1) We say that $\mathbf{a}$ is *co-d-CEA* if $\mathbf{a}$ contains a set of the form $(X \oplus X^{\mathsf{c}}) \oplus (A \cup P)$ for some $X, A, P \subseteq \omega$ such that P and A^{c} are X-c.e., and A and P are disjoint.
(2) Generally, we say that $\mathbf{a}$ is *Γ-above* if $\mathbf{a}$ contains a set of the form $(X \oplus X^{\mathsf{c}}) \oplus Z$ such that Z is Γ in X.
(3) We say that $\mathbf{a}$ is *doubled co-d-CEA* if $\mathbf{a}$ contains a set of the form

$$(X \oplus X^{\mathsf{c}}) \oplus (A \cup P) \oplus (B \cup N)$$

for some $X, A, B, P, N \subseteq \omega$ such that P, N, and $(A \cup B)^{\mathsf{c}}$ are X-c.e., and that A, B, P and N are pairwise disjoint.

In Sections 3.6.1 and 3.6.2, we will introduce further variants of co-d-CEA degrees. We call a degree *doubled co-d-c.e.* if it contains a set which is co-d-CEA with $X = \emptyset$.

$$\begin{tikzcd}
 & \text{total} \arrow[dl] \arrow[dr] & \\
\text{cylinder-cototal} \arrow[d] & & \text{co-d-CEA} \arrow[d] \\
n\text{-cylinder-cototal} \arrow[d] & & \text{doubled co-d-CEA} \arrow[d, "\text{(Prop. 3.26)}"] \\
(n+1)\text{-cylinder-cototal} \arrow[dr] & & \text{telograph-cototal} \arrow[dl, "\text{(Obs. 3.20)}"] \\
 & \text{graph-cototal} &
\end{tikzcd}$$

FIGURE 1. A zoo of enumeration degrees I

We will see that a co-d-CEA e-degree can be described using a Medvedev degree of separability. Given $S, A, B \subseteq \omega$, consider the following notions:

$$\mathrm{Enum}(S) = \{p \in \omega^\omega : \mathrm{rng}(p) = S\},$$
$$\mathrm{Sep}(A, B) = \{C \subseteq \omega : A \subseteq C \text{ and } B \cap C = \emptyset\}.$$

Note that an enumeration degree $\mathbf{a}$ is total if and only if $\mathbf{a}$ contains a set S such that $\mathrm{Enum}(S)$ is Medvedev equivalent to $X \oplus X^{\mathsf{c}} \oplus \mathrm{Sep}(A, B)$ for some $X, A, B \subseteq \omega$ such that A and B are disjoint and X-co-c.e. (If Z is a set and $\mathcal{C}$ is a class then $Z \oplus \mathcal{C} = \{Z \oplus C \mid C \in \mathcal{C}\}$).

Definition 3.6. An enumeration degree $\mathbf{a}$ is $[\Gamma_0, \Gamma_1; \Gamma_2]$-*separating-above* ($[\Gamma_0, \Gamma_1; \Gamma_2]$-SepA) if $\mathbf{a}$ contains a set S such that $\mathrm{Enum}(S)$ is Medvedev equivalent to $X \oplus X^{\mathsf{c}} \oplus \mathrm{Sep}(A, B)$ for some $X, A, B \subseteq \omega$ such that A and B are disjoint, $A \in \Gamma_0^X$, $B \in \Gamma_1^X$, and $A \cup B \in \Gamma_2^X$.

In this terminology, an enumeration degree $\mathbf{a}$ is total if and only if $\mathbf{a}$ is $[\Pi^0_1, \Pi^0_1; \Pi^0_1]$-SepA. We use $*$ to denote the pointclass containing all sets. Then, for instance, we will see that an enumeration degree $\mathbf{a}$ is co-d-CEA if and only if $\mathbf{a}$ is $[*, \Pi^0_1; \Pi^0_1]$-SepA.

We will identify the above classes of e-degrees as the degree structures of certain second-countable T_0 spaces.

co-d-CEA	the ω-power of the irregular lattice space (submetrizable)
Arens co-d-CEA	the ω-power of the quasi-Polish Arens square ($T_{2.5}$)
Roy halfgraph-above	the ω-power of the quasi-Polish Roy space ($T_{2.5}$)
doubled co-d-CEA	the ω-power of the double origin space (T_2)
telograph-cototal	the ω-power of the telophase space (T_1)
n-cylinder-cototal	the n-power of the cocylinder topology on ω^ω (T_1)
graph-cototal	the ω-power of the cofinite topology on ω (T_1)

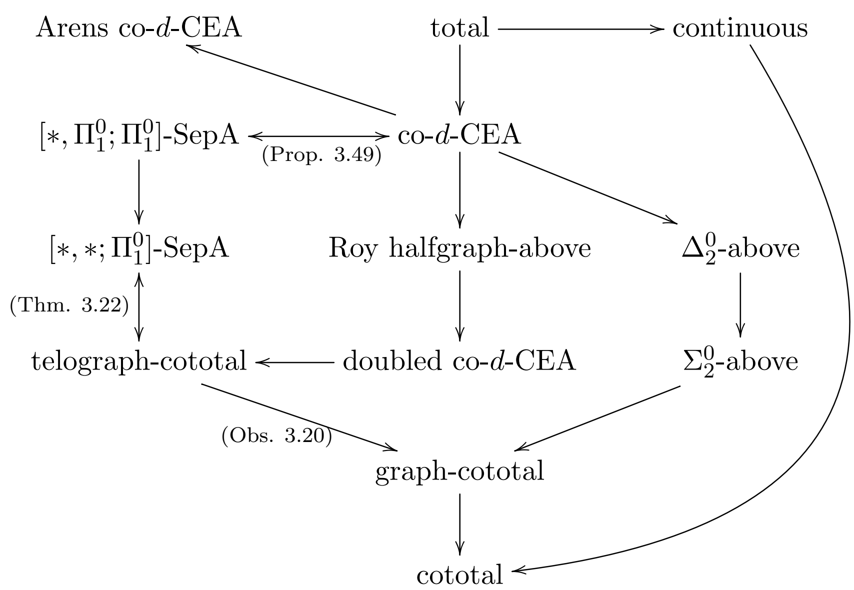

FIGURE 2. A zoo of enumeration degrees II

For non-implications, there is a co-d-CEA e-degree which is not cylinder-cototal (Proposition 3.13), there is a cylinder-cototal e-degree which is quasi-minimal w.r.t. telograph-cototal e-degrees (Theorem 5.36), there is a telograph-cototal e-degree which is quasi-minimal w.r.t. doubled co-d-CEA e-degrees (Theorem 5.38), there is a quasi-minimal co-d-CEA e-degree (see Cooper [11]; Theorem 3.50). For any n, there is an $(n+1)$-cylinder-cototal e-degree which is not n-cylinder-cototal (Theorem 3.15). We also show that there is a continuous degree which is neither telograph-cototal nor cylinder-cototal (Theorem 5.43).

Andrews et al. [1] showed that every Σ^0_2 e-degree is graph-cototal, while we will see that there is a Σ^0_2 e-degree which is quasi-minimal w.r.t. telograph-cototal e-degrees (Theorem 5.24).

3.2. **Degrees of points: T_0-topology.** Let $\mathbb{R}_<$ be the space of all reals equipped with the lower topology generated by $\rho_< = ((q, \infty) : q \in \mathbb{Q})$. Note that $\mathbb{R}_<$ is a T_0 space which is not T_1. Fixing a bijection $e \mapsto q_e : \omega \to \mathbb{Q}$ gives us a representation of $\mathbb{R}_<$ by setting $\beta_e = (q_e, \infty)$. Then, the coded neighborhood filter of x is given by $\mathrm{Nbase}(x) = \{e \in \omega : q_e < x\}$. For notational simplicity, hereafter we fix a standard effective indexing $e \mapsto q_e$, and always assume that every $q \in \mathbb{Q}$ is coded by a natural number without explicitly mentioning $e \mapsto q_e$. Then, for a point $x \in \mathbb{R}_<$, the coded neighborhood filter of x is just given as follows:

$$\mathrm{Nbase}_{\mathbb{R}_<}(x) = \{q \in \mathbb{Q} : q < x\}.$$

For notational simplicity, for a given $x \in \mathbb{R}$, we assume that $\mathrm{Nbase}(x)$ always means $\mathrm{Nbase}_{\mathbb{R}}(x)$, and $\mathrm{Nbase}_<(x)$ always means $\mathrm{Nbase}_{\mathbb{R}_<}(x)$. We say that a real $x \in \mathbb{R}$ is *left-c.e.* if $\mathrm{Nbase}_<(x)$ is c.e. Similarly, a real x is *right-c.e.* if $\mathrm{Nbase}_<(-x)$ is c.e. Recall that $A \subseteq \omega$ is *semirecursive* [41], if there is a computable function $s : \omega \times \omega \to \{0, 1\}$ such that if $\{n_0, n_1\} \cap A \neq \emptyset$, then $n_{s(n_0,n_1)} \in A$. As pointed out by Kihara-Pauly [43], the $\mathbb{R}_<$-degrees are exactly the semirecursive e-degrees . In other words:

$$\mathcal{D}_{\mathbb{R}_<} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \text{ is semirecursive}\}.$$

In Section 5.1.1, we investigate how $\mathbb{R}_<$-degrees behave. Moreover, for instance, in Section 5.1.3, we will see that almost no $\mathbb{R}_<$-degrees are realized by a point in a T_1 space. Thus, the results on $\mathbb{R}_<$-degrees describe the behavior of typical non-T_1-degrees.

3.3. **Degrees of points: T_D-topology.** A topological space is T_D if every singleton can be written as the intersection of an open set and a closed set (see [4]). Note that a space is T_D iff every singleton is $\mathbf{\Delta}^0_2$ in the sense of the non-metrizable Borel hierarchy [15, 71]: A set in a space is $\mathbf{\Pi}^0_2$ if it is the union of countably many constructible sets, where a constructible set (in the sense of classical algebraic geometry) is a finite Boolean combination of open sets. Note that $\mathbf{\Delta}^0_2$ (i.e., $\mathbf{\Sigma}^0_2 \cap \mathbf{\Pi}^0_2$) is not necessarily equal to $F_\sigma \cap G_\delta$ (see Section 3.8.1). Recall that a space is T_1 if every singleton is closed.

Proposition 3.7. *For every represented, T_D, cb_0 space $\mathcal{X}$, there is a represented T_1, cb_0 space $\mathcal{Y}$ whose degree structure is the same as that of $\mathcal{X}$, i.e., $\mathcal{D}_\mathcal{X} = \mathcal{D}_\mathcal{Y}$.*

Proof. Since $\mathcal{X} = (X, \beta)$ is T_D, for any $x \in \mathcal{X}$, there is an open set $\beta_{e(x)}$ such that $\{x\} = \beta_{e(x)} \cap F_x$ for some closed set F_x. Consider $\mathcal{Y} = \{(j, x) \in \omega \times \mathcal{X} : e(x) = j\}$, whose representation is induced from $\mathcal{X}$, that is, $\gamma_{d,e} = \mathcal{Y} \cap (\{d\} \times \beta_e)$. Then $\mathcal{Y}$ is T_1 since $\{(e(x), x)\} = \{e(x)\} \times F_x$. Moreover, $\langle j, k\rangle \in \mathrm{Nbase}_\mathcal{Y}(e(x), x)$ iff $e(x) = j$ and $k \in \mathrm{Nbase}_\mathcal{X}(x)$. Hence, $\mathrm{Nbase}_\mathcal{Y}(e(x), x)$ is e-equivalent to $\mathrm{Nbase}_\mathcal{X}(x)$. □

One can also consider a uniform version of being T_D, that is, having a $\mathbf{\Delta}^0_2$-diagonal. Again, be careful that such a space does not necessarily have a G_δ-diagonal. Recall that a space is Hausdorff if it has a closed diagonal. Following [61], we say that a represented space $\mathcal{X}$ is computably Hausdorff if the diagonal $\Delta_\mathcal{X}$ is Π^0_1. For computability on topological separation axioms, see also Weihrauch [77, 78].

Proposition 3.8. *If $\mathcal{X}$ is a represented cb_0 space which has a Δ^0_2-diagonal, then there is a computably Hausdorff cb_0 space $\mathcal{Y}$ such that $\mathcal{D}_\mathcal{X} = \mathcal{D}_\mathcal{Y}$.*

Proof. Assume that $\mathcal{X}$ has a Δ^0_2 diagonal. In particular, its diagonal is written as $\Delta_\mathcal{X} = \bigcup_n D_n$ where $D_n = G_n \cap F_n$ for some computable sequences of Σ^0_1 sets $(G_n)_{n\in\omega}$ and Π^0_1 sets $(F_n)_{n\in\omega}$. Consider $\mathcal{X}_n = \{x \in \mathcal{X} : (x, x) \in D_n\}$. Let $(B_k)_{k\in\omega}$ be a countable open basis of $\mathcal{X}$. If (x, x) is contained in an open set G in $\mathcal{X}^2$, then there is k such that $x \in B_k$ and $B_k^2 \subseteq G$. In particular, for every $x \in \mathcal{X}_n$ there is $k(x) \in \omega$ such that $(x, x) \in B^2_{k(x)}$ and $\Delta_{\mathcal{X}_n} \cap B^2_{k(x)} = B^2_{k(x)} \cap F_n$. Define $\mathcal{X}_{n,k} = \{x \in \mathcal{X}_n : k(x) = k\}$. Then the diagonal on $\mathcal{X}_{n,k}$ is the restriction of F_n on $\mathcal{X}_{n,k}$. This is because, for any $x, y \in \mathcal{X}_{n,k}$, we have $(x, y) \in B^2_k$. Therefore $x = y$ if and only if $(x, y) \in F_n$. Consequently, the diagonal on $\mathcal{X}_{n,k}$ is Π^0_1. Then, define $\mathcal{Y} = \{(n, k, x) : x \in \mathcal{X}_{n,k}\}$ as a subspace of $\omega^2 \times \mathcal{X}$. Note that $(n, k, x) = (m, \ell, y)$ iff $n = m$, $k = \ell$ and $(x, y) \in F_n$; hence, the diagonal $\Delta_\mathcal{Y}$ is Π^0_1. One can easily see that $\mathrm{Nbase}_\mathcal{Y}(n, k, x)$ is e-equivalent to $\mathrm{Nbase}_\mathcal{X}(x)$ for any $x \in \mathcal{X}_{n,k}$, so $\mathcal{D}_\mathcal{Y} \subseteq \mathcal{D}_\mathcal{X}$. Conversely, for any $x \in \mathcal{X}$ there exists $n, k \in \omega$ such that $x \in \mathcal{X}_{n,k}$ (where we do not need to compute such n, k from x), so $\mathcal{D}_\mathcal{X} \subseteq \mathcal{D}_\mathcal{Y}$. □

For Propositions 3.7 and 3.8, the same observation is independently made by de Brecht. Roughly speaking, these Propositions show that the T_D-degrees are exactly the T_1-degrees, and the Δ^0_2-diagonal-degrees are exactly the T_2-degrees. Thus, we do not need to consider the T_D-separation axiom (and any notion between T_D and T_1 such as $T_{\frac{1}{2}}$), and its uniform version any more.

3.4. **Degrees of points: T_1-topology.** A topological space is T_1 if every singleton is closed. We first consider the cofinite topology, which is one of the most basic constructions obtaining a non-Hausdorff (actually non-sober) T_1-topology. Here, we consider the cofinite topology on ω, that is, a basis is given by $(\omega \setminus D_e : e \in \omega)$, where D_e is the e-th finite subset of ω. By $\omega_{\rm cof}$, we denote the natural numbers ω endowed with the cofinite topology and the above representation. As before, we will not mention a fixed canonical indexing $e \mapsto D_e$, and we treat a finite set $D \subset \omega$ as if it were a natural number.

The space $\omega_{\rm cof}$ itself is countable, and thus, not degree-theoretically interesting. Instead, we consider the ω-power $(\omega_{\rm cof})^\omega$, and then for any $x \in (\omega_{\rm cof})^\omega$, we have

$$\mathrm{Nbase}(x) = \{\langle n, D\rangle : x(n) \notin D\},$$

where D ranges over finite subsets of ω. It is not hard to see that the $(\omega_{\rm cof})^\omega$-degrees are exactly the graph-cototal e-degrees (Definition 3.4), that is,

$$\mathcal{D}_{(\omega_{\rm cof})^\omega} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \text{ is graph-cototal}\}.$$

For basic properties on graph-cototal e-degrees, see Andrews et al. [1]. In this section, we will isolate other proper subcollections of graph-cototal e-degree as degrees of points of specific non-Hausdorff T_1-spaces.

3.4.1. *Cocylinder topology.* For a represented cb$_0$-space $\mathcal{X} = (X, \beta)$, recall that β is an enumeration of a countable open basis of a cb$_0$ space X. We introduce the co-representation $\beta^{\rm co}$ of X by $\beta^{\rm co}_e = X \setminus \beta_e$. We write $\mathcal{X}_{\rm co} = (X, \beta^{\rm co})$.

Example 3.9. We define $\lambda_{\langle n,m\rangle} = \{x \in \omega^\omega : x(n) = m\}$ for any $n, m \in \omega$. Then, λ is a representation of the Baire space ω^ω. It is not hard to see that $(\omega^\omega, \lambda^{\rm co})$ is computably homeomorphic to $(\omega_{\rm cof})^\omega$.

Example 3.10. We define $\gamma_\sigma = \{x \in \omega^\omega : \sigma \prec x\}$ for any $\sigma \in \omega^{<\omega}$. Then, γ is a representation of the Baire space ω^ω. It is easy to see that γ is computably equivalent to the representation λ in Example 3.9, and therefore (ω^ω, λ) and (ω^ω, γ) are computably homeomorphic. However, Proposition 3.13 shows that $(\omega^\omega, \lambda^{\rm co})$ and $(\omega^\omega, \gamma^{\rm co})$ are not homeomorphic! This indicates that the topology on X induced from $\beta^{\rm co}$ heavily depends on the choice of the representation β of X.

Remark 3.11. A better-behaved "co-topology" is known as the *de Groot dual* (after [21, 22]). It only depends on the topology on X, but not on its representation. Unfortunately, the de Groot dual of a cb$_0$ space is not necessarily second-countable, and therefore it exceeds the scope of this section. However, it is worth mentioning that it does NOT exceed the scope of computability theory (see Section 4 and also [44]).

Hereafter, by $\omega^\omega_{\rm co}$ we always mean the cocylinder space $(\omega^\omega, \gamma^{\rm co})$. As usual, via a fixed canonical bijection between ω and $\omega^{<\omega}$, we treat a string $\sigma \in \omega^{<\omega}$ as if it were a natural number. Then, for any $x \in \omega^\omega_{\rm co}$,

$$\mathrm{Nbase}(x) = \{\sigma \in \omega^{<\omega} : \sigma \not\prec x\}.$$

By definition, it is clear that the $\omega^\omega_{\rm co}$-degrees are exactly the cylinder-cototal e-degrees (Definition 3.4), that is,

$$\mathcal{D}_{\omega^\omega_{\rm co}} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \text{ is cylinder-cototal}\}.$$

Recall that a join of n cylinder-cototal e-degrees is called n-cylinder-cototal. In other words, the n-cylinder-cototal e-degrees are exactly the $(\omega^\omega_{\rm co})^n$-degrees.

Observation 3.12. *Every n-cylinder-cototal e-degree is graph-cototal.*

Proof. It suffices to show that each product cocylinder space $(\omega^\omega_{\rm co})^n$ is effectively embedded into $(\omega_{\rm cof})^\omega$. To see this, given $x = (x_m)_{m<n}$, consider $\check{x}(n) = \bigoplus_{m<n} x_m \restriction n$. It is not hard to check that $x \mapsto \check{x}$ gives a desired computable embedding. It is also clear that $(\omega^\omega_{\rm co})^\omega$ is effectively homeomorphic to $(\omega_{\rm cof})^\omega$. □

We first show that cylinder-cototal e-degrees form a proper subclass of graph-cototal e-degrees. As shown by Andrews et al. [1], every Σ^0_2-above degree is graph-cototal. It is not true for cylinder-cototal e-degrees.

Proposition 3.13. *There is a co-d-CEA set $A \subseteq \omega$ such that A is not cylinder-cototal.*

The proof of Proposition 3.13 requires some preparation, so we will postpone it until Section 5.2. The above proposition is useful for separating cylinder-cototal degrees and other degrees, since we will see that most collections of degrees obtained from represented cb_0 spaces in this article are larger than the co-d-CEA e-degrees. We also show that $\omega^\omega_{\rm co}$ has a nontrivial Σ^0_2 e-degree. An e-degree $\mathbf{d}$ is *proper*-Σ^0_2 if $\mathbf{d}$ contains a Σ^0_2 set, but no Δ^0_2 set. The following shows that the standard construction of a proper-Σ^0_2 e-degree is doable in the cocylinder space.

Proposition 3.14. *There is a proper-Σ^0_2 cylinder-cototal degree.*

Proof (Sketch). In this proof, we assume that the reader is familiar with a priority argument. We will construct a point x in the cocylinder space fulfilling the following requirements:

$$\mathcal{P}_{D,\Phi,\Psi} : \Psi(\mathrm{Nbase}(x)) \neq D \text{ or } \Phi(D) \neq \mathrm{Nbase}(x) \text{ or } D \text{ is not } \Delta^0_2.$$

where D ranges over all Σ^0_2 subsets of ω, and Φ and Ψ range over all enumeration operators. The following describe the action of a $\mathcal{P}_{D,\Phi,\Psi}$-strategy ξ:

(1) Choose σ_ξ, and enumerate σ_ξ into $A = \mathrm{Nbase}(x)$.
(2) Wait for $\Phi(D)(\sigma_\xi) = A(\sigma_\xi) = 1$ by $\Phi(D)$ enumerating σ_ξ with some use $F \subseteq D$.
(3) Wait for $F \subseteq \Psi(A)$ with some use $G \subseteq A$.
(4) Then the strategy ξ declares that we decided to enumerate $G \setminus \{\sigma_\xi\}$ into A forever.
(5) Remove σ_ξ from A.
(6) Wait for $\Phi(D)(\sigma_\xi) = A(\sigma_\xi) = 0$ being recovered by $\Phi(D)$ removing σ_ξ. This forces $F \not\subseteq D$.
(7) Then enumerate σ_ξ into A. This recovers $G \subseteq A$, and therefore forces $F \subseteq \Psi(A)$ and thus $\Psi(A) \restriction F \neq D \restriction F$.
(8) Wait for $F \subseteq D$ being recovered. This may recover $\Psi(A) \restriction F = D \restriction F$, but this forces $\Phi(D)(\sigma_\xi) = 1$. Then go back to Step 5.

For each stage reaching at Step 5, the strategy ξ returns the infinitary outcome ∞. Otherwise, the strategy ξ returns the finitary outcome $\mathtt{f}$. It should be careful about the choice of σ_ξ. Let ζ be the maximal string such that $\zeta^\frown\infty \preceq \xi$.

(1) Let σ_ξ be the lexicographically least immediate successor of σ_ζ which is neither chosen by any strategy nor declared to be determined.
(2) Enumerate all strings incomparable with σ_ζ into A. Remove all initial segments of σ_ξ from A. □

3.4.2. *Products of cocylinder topology.* In this section, we will see that there is a hierarchy of graph-cototal e-degrees. We write $\mathcal{D}(\mathcal{X})$ for the substructure of $\mathcal{D}_e$ consisting of e-degrees of neighborhood bases of points in $\mathcal{X}$. We will then have the following proper hierarchy of degree structures:

$$\mathcal{D}_T = \mathcal{D}(\omega^\omega) \subsetneq \mathcal{D}(\omega^\omega_{\mathrm{co}}) \subsetneq \cdots \subsetneq \mathcal{D}((\omega^\omega_{\mathrm{co}})^n) \subsetneq \mathcal{D}((\omega^\omega_{\mathrm{co}})^{n+1}) \subsetneq \cdots \subsetneq \mathcal{D}((\omega_{\mathrm{cof}})^\omega)$$

Theorem 3.15. *For any n, there is an $(n+1)$-cylinder-cototal e-degree which is not n-cylinder-cototal, that is,*

$$\mathcal{D}_{(\omega^\omega_{\mathrm{co}})^{n+1}} \not\subseteq \mathcal{D}_{(\omega^\omega_{\mathrm{co}})^n}.$$

Proof. First note that a basic open set in $(\omega^\omega_{\mathrm{co}})^n$ is of the form $\prod_{k<n}[D_k]^{\mathsf{c}}$ for some collection $(D_k)_{k<n}$ of finite sets of strings. Here we write $[D] = \{X \subseteq \omega^\omega : \sigma \prec X$ for some $\sigma \in D\}$ if $D \subseteq \omega^{<\omega}$. We code the set $\prod_{k<n}[D_k]^{\mathsf{c}}$ by $(D_k)_{k<n}$.

Therefore, an enumeration operator from $(\omega^\omega_{\mathrm{co}})^m$ to $(\omega^\omega_{\mathrm{co}})^n$ is a c.e. set Ψ of collection of $(n+m)$-tuples of the form $((\sigma_k)_{k<n}, (D_\ell)_{\ell<m})$. For each such enumeration operator Ψ, we write $\Psi^{[k]}$ for its k-th section, that is, a collection of tuples of the form $(\sigma_k, (D_\ell)_{\ell<m})$.

We will construct a tuple $\boldsymbol{x} = (x_k)_{k\le n} \in (\omega^\omega_{\mathrm{co}})^{n+1}$ fulfilling the following requirements:

$$R_{\langle d,e\rangle} : [(\exists \boldsymbol{y} \in (\omega^\omega_{\mathrm{co}})^n\ \Phi_d^{\mathrm{Nbase}(\boldsymbol{x})} = \mathrm{Nbase}(\boldsymbol{y})] \implies \Psi_e^{\mathrm{Nbase}(\boldsymbol{y})} \neq \mathrm{Nbase}(\boldsymbol{x}).$$

Let $s = \langle d, e\rangle$. Inductively we assume that $x_k \upharpoonright s$ is constructed for every $k \le n$. We also assume that we have constructed a collection $(E_{k,s})_{k\le n}$ of finite sets of strings such that $\sigma \notin E_{k,s}$ for any $\sigma \preceq x_k \upharpoonright s$, where $E_{k,0} = \emptyset$. At stage s, proceed the following strategy:

(1) Choose an $(n+1)$-tuple $(a_k)_{k\le n}$ of large numbers which are not mentioned by $E_{k,s}$.
(2) Ask whether there exists a finite collection $\boldsymbol{D} = (D_\ell)_{\ell<n}$ of finite sets of strings such that $\langle ((x_k \upharpoonright s)^\frown a_k)_{k\le n}, (D_\ell)_{\ell<n}\rangle \in \Psi_e$.
(3) If there is no such $\boldsymbol{D}$, for each $k \le n$, choose a large number $a'_k \neq a_k$ not mentioned in $E_{k,s}$, and define $x_k(s) = a'_k$, and $E_{k,s+1} = E_{k,s}$. Go to stage $s+1$.
(4) If such a $\boldsymbol{D}$ exists, we say that a finite set G of strings is k-*avoidable* if G has no initial segment of $(x_k \upharpoonright s)^\frown a_k$. Then we say that a $(n+1)$-tuple $(G_k)_{k\le n}$ of finite sets of strings is *avoidable except for* j if G_k is k-avoidable for any $k \neq j$, and that $(G_k)_{k\le n}$ is *all-but-one avoidable* if it is avoidable except for at most one j.
(5) Ask whether for any $\ell < n$ there is an all-but-one avoidable tuple $(G^\ell_k)_{k\le n}$ such that for any $\sigma \in D_\ell$, $\langle \sigma, (G^\ell_k)_{k\le n}\rangle \in \Phi_d^{[\ell]}$.
(6) If yes, define $E_{k,s+1} = E_{k,s} \cup \bigcup_{\ell<n} G^\ell_k$. Note that there is $m \le n$ such that G^ℓ_m is m-avoidable for any $\ell < n$ since $((G^\ell_k)_{k\le n})_{\ell<n}$ is an n-tuple of all-but-one avoidable $(n+1)$-tuples. For such m, we put $a'_m = a_m$, and for each $k \neq m$,

choose a large $a'_k \neq a_k$ not mentioned in $E_{k,s+1}$. Then define $x_k(s) = a'_k$. Go to stage $s+1$.

(7) If no with $\ell < n$, note that if G_0 and G_1 are k-avoidable, so is $G_0 \cup G_1$. Therefore, for any $j \leq n$, there is $\sigma_j \in D_\ell$ such that if $(G_k)_{k\leq n}$ is avoidable except for j, we have $\langle \sigma_j, (G_k)_{k\leq n}\rangle \notin \Phi_d^{[\ell]}$.

(8) If σ_i and σ_j are incomparable for some $i \neq j$, define $x_k(s) = a_k$ and $E_{k,s+1} = E_{k,s}$. Go to stage $s+1$.

(9) If $(\sigma_j)_{j\leq n}$ is pairwise comparable, then let $\sigma_{\boldsymbol{D}}$ be the shortest one.

Case 1. We reach Step (3). In this case, note that for any $\boldsymbol{y} \in (\omega^\omega_{\mathrm{co}})^n$, $\Psi_e^{\mathrm{Nbase}(\boldsymbol{y})}$ does not enumerate $((x_k \upharpoonright s)^\frown a_k)_{k\leq n}$, that is, if $\Psi_e^{\mathrm{Nbase}(\boldsymbol{y})}$ is a neighborhood basis of a point $(z_k)_{k\leq n} \in (\omega^\omega_{\mathrm{co}})^{n+1}$, then z_k must extend $(x_k \upharpoonright s)^\frown a_k$ for some $k \leq n$. Since our action at Step (3) ensures that $x_k \upharpoonright s+1$ is incomparable with $(x_k \upharpoonright s)^\frown a_k$ for every $k \leq n$, the requirement $R_{\langle d,e\rangle}$ is fulfilled.

Case 2. Otherwise, for $\boldsymbol{a} = (a_k)_{k\leq n}$, let $\mathcal{D}_{\boldsymbol{a}}$ be the set of all $\boldsymbol{D}$'s witnessing that the question in Step (2) is true. Consider the case that we reach Step (6) or (8) with some $\boldsymbol{D} \in \mathcal{D}_{\boldsymbol{a}}$.

Assume that we reach Step (6) with a collection $((G_k^\ell)_{k\leq n})_{\ell<n}$ of avoidable tuples. Fix $m \leq n$ such that G_m^ℓ is m-avoidable. Then $\bigcup_\ell G_m^\ell$ is also m-avoidable, and moreover, $E_{m,s}$ is m-avoidable by our choice of a_m. Therefore, $E_{m,s+1}$ is also m-avoidable, and therefore, $E_{m,s+1}$ has no initial segment of $x_m \upharpoonright s+1 = (x_m \upharpoonright s)^\frown a_m$. Moreover, for each $k \neq m$, by our choice of a'_k, $E_{k,s+1}$ has no initial segment of $x_k \upharpoonright s+1 = (x_k \upharpoonright s)^\frown a'_k$. Since $G_k^\ell \subseteq E_{k,s+1}$ for any $\ell < n$ and $k \leq n$, given $\boldsymbol{x} = (x_k)_{k\leq n}$, if x_k is an extensions of $x_k \upharpoonright s+1$, then $(G_k^\ell)_{k\leq n} \subseteq \mathrm{Nbase}(\boldsymbol{x})$, and therefore, the ℓ-th section of $\Phi_d^{\mathrm{Nbase}(\boldsymbol{x})}$ enumerates all strings in D_ℓ for any $\ell < n$. However, since we have $\langle ((x_k \upharpoonright s)^\frown a_k)_{k\leq n}, (D_\ell)_{\ell<n}\rangle \in \Psi_e$, if $\Psi_e\Phi_d^{\mathrm{Nbase}(\boldsymbol{x})}$ enumerates a neighborhood basis of a point $(z_k)_{k\leq n} \in (\omega^\omega_{\mathrm{co}})^{n+1}$, z_k cannot extend $(x_k \upharpoonright s)^\frown a_k$ for any $k \leq n$. Since $x_m \upharpoonright s+1 = (x_m \upharpoonright s)^\frown a_m$, we must have $\Psi_e\Phi_d^{\mathrm{Nbase}(\boldsymbol{x})} \neq \mathrm{Nbase}(\boldsymbol{x})$. Thus, the requirement $R_{\langle d,e\rangle}$ is fulfilled.

Assume that we reach Step (8) with ℓ and incomparable σ_i and σ_j. Later we will also use the symbol $\ell_{\boldsymbol{D}}$ to specify this ℓ. In this case, our action ensures that $x_k \upharpoonright s+1 = (x_k \upharpoonright s)^\frown a_k$, and therefore, if $(G_k)_{k\leq n} \subseteq \mathrm{Nbase}(\boldsymbol{x})$, then G_k must be k-avoidable for any $k \leq n$. In particular, for any $k \leq n$, $(G_k)_{k\leq n}$ is avoidable except for k. Thus, we have $\langle \sigma_k, (G_k)_{k\leq n}\rangle \notin \Phi_d^{[\ell]}$. Since $(\sigma_k)_{k\leq n}$ contains an incomparable pair, the ℓ-th section $\Phi_d^{\mathrm{Nbase}(\boldsymbol{x})}$ does not define a point in $\omega^\omega_{\mathrm{co}}$. Thus, the requirement $R_{\langle d,e\rangle}$ is fulfilled.

Case 3. Assume that we reach Step (9) for any $\boldsymbol{D} \in \mathcal{D}_{\boldsymbol{a}}$. We say that $\boldsymbol{z} = (z_k)_{k\leq n} \in (\omega^\omega_{\mathrm{co}})^{n+1}$ is *all-but-one good* (for $\boldsymbol{a} = (a_k)_{k\leq n}$) if z_k extends $(x_k \upharpoonright s)^\frown a_k$ for all but one $k \leq n$. If $\boldsymbol{z}$ is all-but-one good, any $\boldsymbol{G} \subseteq \mathrm{Nbase}(\boldsymbol{z})$ is all-but-one avoidable. Therefore, if $\Phi_d^{\mathrm{Nbase}(\boldsymbol{z})}$ enumerates a neighborhood basis of a point $(y_\ell)_{\ell<n}$, y_ℓ must extend $\sigma_{\boldsymbol{D}}$, where $\ell = \ell_{\boldsymbol{D}}$.

If there are $\boldsymbol{D}, \boldsymbol{D}' \in \mathcal{D}_{\boldsymbol{a}}$ such that $\ell_{\boldsymbol{D}} = \ell_{\boldsymbol{D}'}$ and that $\sigma_{\boldsymbol{D}}$ is incomparable with $\sigma_{\boldsymbol{D}'}$, then this means that $\Phi_d^{\mathrm{Nbase}(\boldsymbol{z})}$ is not a neighborhood basis of a point for any all-but-one good tuple $\boldsymbol{z}$. In this case, by putting $x_k(s) = a_k$, and $E_{k,s+1} = E_{k,s}$ for each $k \leq n$, the requirement $R_{\langle d,e\rangle}$ is fulfilled. Then go to stage $s+1$.

Case 4. Otherwise, for any $\boldsymbol{D}, \boldsymbol{D}' \in \mathcal{D}_{\boldsymbol{a}}$, if $\ell_{\boldsymbol{D}} = \ell_{\boldsymbol{D}'}$, then $\sigma_{\boldsymbol{D}}$ and $\sigma_{\boldsymbol{D}'}$ are comparable. Then we define $\sigma_\ell^{\boldsymbol{a}}$ for any $\ell < n$ and large tuple $\boldsymbol{a}$ not mentioned in $E_{k,s}$ as follows:

$$\sigma_\ell^{\boldsymbol{a}} = \bigcup \{\sigma_{\boldsymbol{D}} : \boldsymbol{D} \in \mathcal{D}_{\boldsymbol{a}} \text{ and } \ell_{\boldsymbol{D}} = \ell\}.$$

If there is a tuple $(a_k)_{k \le n}$ such that, whenever x_k extends $(x_k \upharpoonright s)^\frown a_k$ for any $k \le n$, $\Phi_d^{\mathrm{Nbase}(\boldsymbol{x})}$ does not enumerate a neighborhood basis of a point in $(\omega^\omega_{\mathrm{co}})^n$, then we just put $x_k(s) = a_k$ and $E_{k,s+1} = E_{k,s}$ for each $k \le n$, and then go to stage $s+1$.

Therefore, we can assume that for any $(a_k)_{k \le n}$, there are $\boldsymbol{x} = (x_k)_{k \le n}$ such that x_k extends $(x_k \upharpoonright s)^\frown a_k$ and that $\Phi_d^{\mathrm{Nbase}(\boldsymbol{x})}$ enumerates a neighborhood basis of a point in $(\omega^\omega_{\mathrm{co}})^n$. Under this assumption, we show the following claim.

Claim. For any $(n+1)$-tuples $\boldsymbol{a}, \boldsymbol{b} \in \omega^{n+1}$ of large numbers and $\ell < n$, $\sigma_\ell^{\boldsymbol{a}}$ and $\sigma_\ell^{\boldsymbol{b}}$ are comparable.

Proof. Given large $\boldsymbol{a}$, assume that $\boldsymbol{z}$ is an all-but-one good tuple (for $\boldsymbol{a}$), and that $\Phi_d^{\mathrm{Nbase}(\boldsymbol{z})}$ defines a neighborhood basis of a point $\boldsymbol{y} \in (\omega^\omega_{\mathrm{co}})^n$. For any t, there is $\boldsymbol{D} \in \mathcal{D}_{\boldsymbol{a}}$ such that $\ell_{\boldsymbol{D}} = \ell$ and $\sigma_\ell^{\boldsymbol{a}} \upharpoonright t \preceq \sigma_{\boldsymbol{D}}$. As mentioned in Case 3, for any such $\boldsymbol{D}$, y_ℓ extends $\sigma_{\boldsymbol{D}}$. In particular, we have $\sigma_\ell^{\boldsymbol{a}} \upharpoonright t \preceq y_\ell$. Since t is arbitrary, we get $\sigma_\ell^{\boldsymbol{a}} \preceq y_\ell$. Note that $\boldsymbol{x} \upharpoonright s$ followed by $\boldsymbol{a} = (a_0, a_1, a_2, \dots, a_n)$ is all-but-one good for $\boldsymbol{a}[b_0/a_0] := (b_0, a_1, a_2 \dots, a_n)$, $\boldsymbol{x} \upharpoonright s$ followed by $\boldsymbol{a}[b_0/a_0]$ is all-but-one good for $\boldsymbol{a}[b_0/a_0, b_1/a_1] := (b_0, b_1, a_2 \dots, a_n)$, and so on. Therefore, for any $\boldsymbol{x} = (x_k)_{k \le n}$, if x_k extends $(x_k \upharpoonright s)^\frown a_k$ for every k, and if $\Phi_d^{\mathrm{Nbase}(\boldsymbol{x})}$ defines a neighborhood basis of a point $(y_\ell)_{\ell < n} \in (\omega^\omega_{\mathrm{co}})^n$, then y_ℓ extends $\sigma_\ell^{\boldsymbol{a}}$, $\sigma_\ell^{(b_0, a_1, a_2, \dots, a_n)}$, $\sigma_\ell^{(b_0, b_1, a_2, \dots, a_n)}$, and so on. This implies that all of these strings are comparable. By our assumption, for any large $\boldsymbol{a}$, there is such $\boldsymbol{x}$, and therefore, for any large $\boldsymbol{a}$ and $\boldsymbol{b}$, and any $\ell < n$, $\sigma_\ell^{\boldsymbol{a}}$ and $\sigma_\ell^{\boldsymbol{b}}$ are comparable. Thus, our claim is verified. $\square$

Now, for any $\boldsymbol{a} \in \omega^{n+1}$, we define $r(\boldsymbol{a}) \in (\omega+1)^n$ as follows:

$$r(\boldsymbol{a})(\ell) = |\sigma_\ell^{\boldsymbol{a}}|.$$

Consider any infinite sequence $(\boldsymbol{a}^i)_{i \in \omega}$ of large $(n+1)$-tuples $\boldsymbol{a}^i = (a_k^i)_{k \le n}$ such that if $i \neq j$ then $a_k^i \neq a_k^j$ for any $k \le n$. Let $\le^n$ be the product order on $(\omega+1)^n$. Since $\omega + 1$ is a well quasi order, so is $((\omega+1)^n, \le^n)$ by Dickson's lemma. Thus, there are $j < i$ such that $r(\boldsymbol{a}^j) \le^n r(\boldsymbol{a}^i)$. Put $\boldsymbol{a} = \boldsymbol{a}^i$ and $\boldsymbol{b} = \boldsymbol{a}^j$, and define $x_k(s) = a_k$ for each $k \le s$.

Assume that x_k extends $x_k \upharpoonright s+1$ for any $k \le n$. Then, $(x_k \upharpoonright s)^\frown b_k)_{k \le n} \in \mathrm{Nbase}(\boldsymbol{x})$ since $x_k(s) = a_k \neq b_k$ for any $k \le n$. If $\Phi_d^{\mathrm{Nbase}(\boldsymbol{x})}$ defines a neighborhood basis of a point $\boldsymbol{y} = (y_\ell)_{\ell < n} \in (\omega^\omega_{\mathrm{co}})^n$ then we must have $y_\ell \succeq \sigma_\ell^{\boldsymbol{a}}$ for every $\ell < n$. Since $r(\boldsymbol{b}) \le^n r(\boldsymbol{a})$, we get $y_\ell \succeq \sigma_\ell^{\boldsymbol{b}}$ for every $\ell < n$. However, by our choice of $\sigma_\ell^{\boldsymbol{b}}$, if $\langle ((x_k \upharpoonright s)^\frown b_k)_{k \le n}, \boldsymbol{D} \rangle \in \Psi_e$, then for $\ell = \ell_{\boldsymbol{D}}$, there is $\sigma \in D_\ell$ such that $\sigma \preceq \sigma_\ell^{\boldsymbol{b}}$. Therefore, D_ℓ contains an initial segment of y_ℓ, and thus $D \not\subseteq \mathrm{Nbase}(\boldsymbol{y})$. Hence, we have $\Psi_e^{\mathrm{Nbase}(\boldsymbol{y})} \neq \mathrm{Nbase}(\boldsymbol{x})$, that is, $\Psi_e \Phi_d^{\mathrm{Nbase}(\boldsymbol{x})} \neq \mathrm{Nbase}(\boldsymbol{x})$. Put $E_{k,s+1} = E_{k,s}$ for each $k \le n$, and then go to stage $s+1$. Then, the requirement $R_{\langle d,e \rangle}$ is fulfilled. $\square$

3.4.3. *Telophase topology.* Let $\mathcal{L}$ be a linearly ordered set. The order topology on $\mathcal{L}$ is generated by the subbasis $(\{x : a < x\}, \{x : x < a\} : a \in \mathcal{L})$. Assume that $\mathcal{L}$ has a countable basis $\mathcal{B}$, that is, there is a countable set $\mathcal{B} \subseteq \mathcal{L}$ such that for any $a, b \in \mathcal{L}$, if

$a < b$ then there are $c, d \in \mathcal{B}$ such that $a \leq c < d \leq b$. Then the order topology on $\mathcal{L}$ is separable and metrizable.

We now assume that $\mathcal{L}$ has the greatest element $\mathbf{1}$. Choose $\mathbf{1}_\star \notin \mathcal{L}$. Then $\mathcal{L} \cup \{\mathbf{1}_\star\}$ forms a partial order by adding the relation $a < \mathbf{1}_\star$ for each $a \in \mathcal{L}$ with $a \neq \mathbf{1}$. Roughly speaking, $\mathcal{L} \cup \{\mathbf{1}_\star\}$ is almost linear ordered except that it has two maximal elements $\mathbf{1}$ and $\mathbf{1}_\star$. The *telophase space* $\mathcal{L}_{TP}$ is defined as the set $\mathcal{L} \cup \{\mathbf{1}_\star\}$ equipped with the Lawson topology, that is, generated by the following subbasis:

$$\{\{x : a \not\leq x\} : a \in \mathcal{L} \cup \{\mathbf{1}_\star\}\} \cup \{\{x : a < x\} : a \in \mathcal{L}\}.$$

If $\mathcal{L}$ has a countable basis $\mathcal{B}$, the following gives us a countable subbasis of $\mathcal{L}_{TP}$:

$$\{\{x : x < a\} : a \in \mathcal{B}\} \cup \{\{x : x \leq a\} : a \in \{\mathbf{1}, \mathbf{1}_\star\}\} \cup \{\{x : a < x\} : a \in \mathcal{B}\}$$

Example 3.16 (see Steen-Seebach [74, II.73])**.** Consider the Cantor space $\mathcal{C} = 2^\omega$, which is linearly ordered by defining $x \leq_{\text{left}} y$ if and only if there is $n \in \omega$ such that $x \upharpoonright n = y \upharpoonright n$ and $x(n) < y(n)$. Then, the greatest element $\mathbf{1}$ is the sequence 1^ω consisting only of 1's. One can easily see that the following gives us a countable subbasis of $\mathcal{C}_{TP}$:

$$\{[\sigma] : \sigma \in 2^{<\omega}\} \cup \{[\sigma]_\star : \sigma \prec 1^\omega\},$$

where $[\sigma]_\star = ([\sigma] \setminus \{\mathbf{1}\}) \cup \{\mathbf{1}_\star\}$. Unfortunately, the degree structure of $\mathcal{C}_{TP}$ is not so interesting since every $\mathcal{C}_{TP}$-degree is total. This is because, this construction adds only one new point, and thus, it is clear that $\mathrm{Nbase}_{\mathcal{C}_{TP}}(x) \equiv_e \mathrm{Nbase}_{\mathcal{C}}(x)$ for any $x \in \mathcal{C}$, and that $\mathbf{1}_\star$ is computable, that is, $\mathrm{Nbase}_{\mathcal{C}_{TP}}(\mathbf{1}_\star)$ is c.e.

Nevertheless, we will see that the degree structure of the ω-power $(\mathcal{C}_{TP})^\omega$ is pretty interesting. Note that for each $x \in (\mathcal{C}_{TP})^\omega$, its coded neighborhood filter is given as

$$\mathrm{Nbase}(x) = \{\langle n, 0, \sigma\rangle : \sigma \prec x(n)\} \cup \{\langle n, 1, k\rangle : 1^k \prec x(n) \text{ and } x(n) \neq 1^\omega\}.$$

Example 3.17. Consider the one-point compactification $\hat{\omega} = \omega \cup \{\infty\}$ of ω, which is naturally linear ordered with the greatest element ∞. The telophase space $\hat{\omega}_{TP}$ looks like a "two-point compactification" of ω. The topology is generated by

$$\{\{m\}, [m, \infty], [m, \infty_\star] : m \in \omega\}.$$

Here, $[a, b]$ is the interval $\{c : a \leq c \leq b\}$. Then, for each $x \in (\hat{\omega}_{TP})^\omega$, its coded neighborhood filter is given as

$$\begin{aligned}\mathrm{Nbase}(x) = \{\langle n, 0, m\rangle : x(n) = m\} &\cup \{\langle n, 1, m\rangle : m \leq x(n) \leq \infty\} \\ &\cup \{\langle n, 2, m\rangle : m \leq x(n) \leq \infty_\star\},\end{aligned}$$

where m and n range over ω.

It is easy to check that the spaces in Examples 3.16 and 3.17 are T_1 but not Hausdorff (since $\mathbf{1}$ and $\mathbf{1}_\star$ cannot be separated by open sets). Next we see the following:

$$\begin{aligned}\mathcal{D}_{(\mathcal{C}_{TP})^\omega} = \mathcal{D}_{(\hat{\omega}_{TP})^\omega} &= \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \text{ is telograph-cototal}\} \\ &= \{\mathbf{d} \in \mathcal{D}_e : [*, *, \Pi^0_1]\text{-separating-above}\}.\end{aligned}$$

Here, recall from Definitions 3.4 and 3.6 for the above notions.

Proposition 3.18. *$\mathcal{C}_{TP}$ computably embeds into $(\hat{\omega}_{TP})^\omega$. Hence, $\mathcal{D}_{(\mathcal{C}_{TP})^\omega} = \mathcal{D}_{(\hat{\omega}_{TP})^\omega}$.*

Proof. We define a function $c : \mathcal{C}_{TP} \to 2^\omega$ by $c(x) = 1^\omega$ if $x = \mathbf{1}_\star$; otherwise $c(x) = x$. This function is computable, i.e., $\mathrm{Nbase}(c(x)) \leq_e \mathrm{Nbase}(x)$ for any $x \in \mathcal{C}_{TP}$, since $\{\sigma \in 2^{<\omega} : x \in [\sigma] \text{ or } x \in [\sigma]_\star\}$ is a neighborhood basis of $c(x) \in 2^\omega$.

Given $x \in \mathcal{C}_{TP}$, we define $h(x)(n+1) = c(x)(n)$. We define $h(x)(0) = n$ if we find that $1^n 0 \prec x$. Otherwise, $x \in \{\mathbf{1}, \mathbf{1}_\star\}$. If $x = \mathbf{1}$, then define $h(x)(0) = \infty$, and if $x = \mathbf{1}_\star$, then define $h(x)(0) = \infty_\star$. One can see that $h\colon \mathcal{C}_{TP} \to (\hat{\omega}_{TP})^\omega$ is an embedding, that is, there are enumeration operators Φ, Ψ witnessing that $\mathrm{Nbase}(x) \equiv_e \mathrm{Nbase}(h(x))$ for any $x \in \mathcal{C}_{TP}$. The construction of a witness for $\mathrm{Nbase}(h(x)) \leq_e \mathrm{Nbase}(x)$ is the same as above. To see $\mathrm{Nbase}(x) \leq_e \mathrm{Nbase}(h(x))$, wait for a neighborhood of $h(x)(0)$ to be enumerated. If it is of the form $\{m\}$ or $[m, \infty]$, then $\{[\sigma] : c(x) \in [\sigma]\}$ is a neighborhood basis of x, otherwise $\{[\sigma]_\star : c(x) \in [\sigma] \text{ and } \sigma \prec 1^\omega\} \cup \{[\sigma] : c(x) \in [\sigma] \text{ and } \sigma \not\prec 1^\omega\}$ is a neighborhood basis of x.

For the latter assertion, the function $i\colon \hat{\omega}_{TP} \to \mathcal{C}_{TP}$ defined by $i(n) = 1^n 0$, $i(\infty) = \mathbf{1}$ and $i(\infty_\star) = \mathbf{1}_\star$ is a computable embedding. Moreover, it is clear that $\mathcal{X}^{\omega\times\omega}$ is computably homeomorphic to $\mathcal{X}^\omega$, so $(\mathcal{C}_{TP})^\omega$ computably embeds into $(\hat{\omega}_{TP})^\omega$, and therefore $\mathcal{D}_{(\mathcal{C}_{TP})^\omega} = \mathcal{D}_{(\hat{\omega}_{TP})^\omega}$. □

We also see that the hierarchy of telograph-cototal e-degrees collapses. For $b \in \omega$, we say that an enumeration degree $\mathbf{a}$ is *b-telograph-cototal* if it contains $\mathrm{Graph}(g)^{\mathsf{c}} \oplus \mathrm{TGraph}_b(g)$ for some total function $g : \omega \to \omega$.

Proposition 3.19. *The 1-telograph-cototal e-degrees are exactly the total degrees. For any natural numbers $b, c > 1$, the b-telograph-cototal e-degrees are exactly the c-telograph-cototal e-degrees.*

Proof. For the first assertion, given $g : \omega \to \omega$, consider

$$G = \{\langle n, m, 0\rangle : g(n) \neq m+1\} \cup \{\langle n, m, 1\rangle : g(n) = m+1\}.$$

Clearly, G is total, and one can check that $G \equiv_e \mathrm{Graph}(g)^{\mathsf{c}} \oplus \mathrm{TGraph}_1(g)$.

One can easily check that every b-telograph-cototal e-degree is $(b+1)$-telograph-cototal by considering $\tilde{g}(n) = g(n) + 1$. To see that every b-telograph-cototal e-degree is 2-telograph-cototal, given $g : \omega \to \omega$ and $i < b$, consider the following:

$$g_i(n) = \begin{cases} 0 & \text{if } g(n) = i, \\ 1 & \text{if } g(n) < b \text{ and } g(n) \neq i, \\ g(n) - b + 2 & \text{if } g(n) \geq b. \end{cases}$$

Define $\tilde{g}$ by $\tilde{g}(bn+i) = g_i(n)$. Then, we claim that $\mathrm{Graph}(\tilde{g})^{\mathsf{c}} \oplus \mathrm{TGraph}_2(\tilde{g}) \equiv_e \mathrm{Graph}(g)^{\mathsf{c}} \oplus \mathrm{TGraph}_b(g)$. It is straightforward to see the following for $n, m \in \omega$ and $i < b$,

$$\begin{aligned} \langle bn+i, 0\rangle \in \mathrm{Graph}(\tilde{g})^{\mathsf{c}} &\iff \langle n, i\rangle \in \mathrm{Graph}(g)^{\mathsf{c}}, \\ \langle bn+i, 1\rangle \in \mathrm{Graph}(\tilde{g})^{\mathsf{c}} &\iff (\forall j < b)\ j = i \text{ or } \langle n, j\rangle \in \mathrm{Graph}(g)^{\mathsf{c}}, \\ \langle bn+i, m+2\rangle \in \mathrm{Graph}(\tilde{g})^{\mathsf{c}} &\iff \langle n, m+b\rangle \in \mathrm{Graph}(g)^{\mathsf{c}}, \\ \langle bn+i, m+2\rangle \in \mathrm{TGraph}_2(\tilde{g}) &\iff \langle n, m+b\rangle \in \mathrm{TGraph}_b(g). \end{aligned}$$

The reduction $\leq_e$ clearly follows from the above equivalences. For the reduction $\geq_e$, see the first, third, and forth equivalences. □

Observation 3.20. *Every telograph-cototal e-degree is graph-cototal.*

Proof. Given $h\colon \omega \to \omega$, define $g_h(n,k) = 1$ if $h(n) = k+2$; otherwise $g_h(n,k) = 0$. Then $\mathrm{Graph}(h \oplus g_h)^{\mathsf{c}} \equiv_e \mathrm{Graph}(h)^{\mathsf{c}} \oplus \mathrm{TGraph}_2(h)$. To see this, observe that $g_h(n,k) \neq 1$ if and only if $h(n) \neq k+2$, and that $g_h(n,k) \neq 0$ if and only if $h(n) = k+2$ if and only if $\langle n, k+2\rangle \in \mathrm{TGraph}_2(h)$. □

We show that the $(\hat{\omega}_{TP})^\omega$-degrees are characterized by the telograph-cototal e-degrees (Definitions 3.4).

Proposition 3.21. *The $(\hat{\omega}_{TP})^\omega$-degrees are exactly the telograph-cototal e-degrees.*

Proof. By Proposition 3.19, it suffices to show that the $(\hat{\omega}_{TP})^\omega$-degrees are exactly the 2-telograph-cototal e-degrees. Given a point $x \in (\hat{\omega}_{TP})^\omega$, consider the following g_x:

$$g_x(n) = \begin{cases} 0 & \text{if } x(n) = \infty, \\ 1 & \text{if } x(n) = \infty_\star, \\ x(n)+2 & \text{if } x(n) \in \omega. \end{cases}$$

We claim that $\mathrm{Nbase}(x) \equiv_e \mathrm{Graph}(g_x)^{\mathsf{c}} \oplus \mathrm{TGraph}_2(g_x)$. Recall from Example 3.17 the definition of $\mathrm{Nbase}(x)$ in the telophase space $(\hat{\omega}_{TP})^\omega$. To verify the reduction $\leq_e$, we claim that

$$\begin{aligned} \langle n,0,m\rangle \in \mathrm{Nbase}(x) &\iff \langle n, m+2\rangle \in \mathrm{TGraph}_2(g_x), \\ \langle n,1,m\rangle \in \mathrm{Nbase}(x) &\iff \langle n,1\rangle, \langle n,2\rangle, \dots, \langle n,m+1\rangle \in \mathrm{Graph}(g_x)^{\mathsf{c}}, \\ \langle n,2,m\rangle \in \mathrm{Nbase}(x) &\iff \langle n,0\rangle, \langle n,2\rangle, \dots, \langle n,m+1\rangle \in \mathrm{Graph}(g_x)^{\mathsf{c}}. \end{aligned}$$

This is because, for the first equivalence, if $x(n) = m \in \omega$, then $g_x(n) = m+2 \geq 2$, and thus $\langle n, m+2\rangle$ is enumerated into $\mathrm{TGraph}_2(g_x)$. For the second equivalence, $m \leq x(n) \leq \infty$ if and only if $x(n) \neq \infty_\star$ and $x(n) \notin \{0,\dots,m-1\}$. This means that $\langle n,1\rangle \notin \mathrm{Graph}(g_x)$ and $\langle n,2\rangle,\dots,\langle n,m-1+2\rangle \notin \mathrm{Graph}(g_x)$. The last equivalence holds by a similar reason. The above three equivalences give us a reduction witnessing $\mathrm{Nbase}(x) \leq_e \mathrm{Graph}(g_x)^{\mathsf{c}} \oplus \mathrm{TGraph}_2(g_x)$.

For the reduction $\geq_e$, first note that $g_x(n) \neq 0$ if and only if $m \leq x(n) \leq \infty_\star$ for some $m \in \omega$. Similarly, $g_x(n) \neq 1$ if and only if $m \leq x \leq \infty$ for some $m \in \omega$. Therefore, for $i < 2$,

$$\langle n,i\rangle \in \mathrm{Graph}(g_x)^{\mathsf{c}} \iff \langle n, 2-i, m\rangle \in \mathrm{Nbase}(x) \text{ for some } m.$$

For $m \in \omega$, $g_x(n) \neq m+2$ if and only if either $x(n) = k$ for some $k < m$ or $k \leq x(n)$ for some $k > m$. Therefore, for $m \in \omega$,

$$\begin{aligned} \langle n, m+2\rangle \in \mathrm{Graph}(g_x)^{\mathsf{c}} \iff & (\exists k < m)\ \langle n,0,k\rangle \in \mathrm{Nbase}(x) \\ & \text{or } (\exists k > m)(\exists i < 2)\ \langle n, i+1, k\rangle \in \mathrm{Nbase}(x). \end{aligned}$$

Finally, it is clear that for any $n, m \in \omega$,

$$\langle n,m\rangle \in \mathrm{TGraph}_2(g_x) \iff m \geq 2 \text{ and } \langle n, 0, m-2\rangle \in \mathrm{Nbase}(x).$$

The above equivalences give us a reduction witnessing $\mathrm{Graph}(g_x)^{\mathsf{c}} \oplus \mathrm{TGraph}_2(g_x) \leq_e \mathrm{Nbase}(x)$. Moreover, one can easily see that for any function $g\colon \omega \to \omega$, there is $x \in (\hat{\omega}_{TP})^\omega$ such that $g_x = g$, so that every 2-telograph-cototal degree is realized in $(\hat{\omega}_{TP})^\omega$. This completes the proof. □

We give another characterization of $(\hat{\omega}_{TP})^\omega$-degrees in terms of separating sets. For any $A, B \subseteq \omega$, recall that $\mathrm{Sep}(A, B)$ is the collection of sets separating A and B:

$$\mathrm{Sep}(A, B) = \{C \subseteq \omega : A \subseteq C \text{ and } B \cap C = \emptyset\}.$$

We also recall that $\mathrm{Enum}(E)$ is the set of all enumerations of $E \subseteq \omega$, that is,

$$\mathrm{Enum}(E) = \{p \in \omega^\omega : \mathrm{rng}(p) = E\}.$$

To make our argument simple, we always assume that E is nonempty. Note that $D \equiv_e E$ if and only if $\mathrm{Enum}(D) \equiv_M \mathrm{Enum}(E)$.

Theorem 3.22. *The $(\hat{\omega}_{TP})^\omega$-degrees (hence the telograph-cototal e-degrees) are exactly the $[*, *, \Pi^0_1]$-separating-above e-degrees. In other words, a nonempty set $E \subseteq \omega$ is e-equivalent to* $\mathrm{Nbase}(x)$ *for some $x \in (\hat{\omega}_{TP})^\omega$ if and only if there are $X, A, B \subseteq \omega$ such that $A \cup B$ is X-co-c.e., $A \cap B = \emptyset$, and*

$$\mathrm{Enum}(E) \equiv_M \{X\} \times \mathrm{Sep}(A, B).$$

Proof. Given a point $x \in (\hat{\omega}_{TP})^\omega$, we define

$$X = \{2\langle n, m\rangle : x(n) = m\} \cup \{2\langle n, m\rangle + 1 : x(n) \neq m\},$$

where m ranges over ω. Clearly, X is total. We define $A = \{n \in \omega : x(n) = \infty\}$ and $B = \{n \in \omega : x(n) = \infty_\star\}$. It is clear that $A \cap B = \emptyset$. Note that $A \cup B$ is co-c.e. relative to X, since $n \in A \cup B$ if and only if $2\langle n, m\rangle + 1 \in X$ for any $m > 0$.

It is clear that $X \oplus X^{\mathsf{c}} \leq_e \mathrm{Nbase}(x)$. To see $\mathrm{Sep}(A, B) \leq_M \mathrm{Enum}(\mathrm{Nbase}(x))$, given $n \in \omega$, wait for the first triple $\langle n, i, m\rangle$ to be enumerated into $\mathrm{Nbase}(x)$. If $i = 1$ (then, $x(n) \neq \infty_*$), set $C(n) = 1$. If $i = 2$ (then, $x(n) \neq \infty$), set $C(n) = 0$.If $i = 0$ (then, $x(n) \in \omega$), set $C(n) = 1$.The constructed set C satisfies that $C \in \mathrm{Sep}(A, B)$.

Conversely, assume that $C \in \mathrm{Sep}(A, B)$ is given. For each n, if $n \in C$, then $x(n) \neq \infty_\star$. Thus, enumerate $\langle n, 1, m\rangle$ into $\mathrm{Nbase}(x)$ (which indicates that $m \leq x(n) \leq \infty$) if $2\langle n, k\rangle + 1$ is enumerated into X for all $k < m$. If $n \notin C$, then $x(n) \neq \infty$. Thus, enumerate $\langle n, 2, m\rangle$ into $\mathrm{Nbase}(x)$ (which indicates that $m \leq x(n) \leq \infty_\star$) if $2\langle n, k\rangle + 1$ is enumerated into X for all $k < m$. Moreover, if $2\langle n, m\rangle$ is enumerated into X, then enumerate $\langle n, 0, m\rangle$ and $\langle n, i + 1, k\rangle$ into $\mathrm{Nbase}(x)$ for each $i < 2$ and $k \leq m$. It is not hard to check that this procedure eventually enumerates $\mathrm{Nbase}(x)$.

Now let A, B be a pair of disjoint sets such that $A \cup B$ is X-co-c.e. We construct a point $x \in (\hat{\omega}_{TP})^\omega$ such that $\{X\} \times \mathrm{Sep}(A, B)$ is equivalent to $\mathrm{Enum}(\mathrm{Nbase}(x))$. We define $x(2n)$ to be $X(n)$. Fix an X-computable enumeration of the complement of $A \cup B$. We define $x(2n + 1)$ as follows:

$$x(2n+1) = \begin{cases} \infty \text{ if } n \in A, \\ \infty_\star \text{ if } n \in B, \\ s \text{ if we see } n \notin A \cup B \text{ at stage } s. \end{cases}$$

As in the above argument, it is not hard to see that $\{X\} \times \mathrm{Sep}(A, B)$ is Medvedev-equivalent to $\mathrm{Enum}(\mathrm{Nbase}(x))$. ∎

3.5. **Degrees of points: T_2-topology.** A topological space $\mathcal{X}$ is T_2 or *Hausdorff* if any distinct points $x \neq y \in \mathcal{X}$ are separated by open sets, that is, there are disjoint open sets $U, V \subseteq \mathcal{X}$ such that $x \in U$ and $y \in V$. It is equivalent to saying that the diagonal $\Delta_{\mathcal{X}} = \{(x, x) : x \in \mathcal{X}\}$ is closed in $\mathcal{X}^2$.

3.5.1. *Double Origin Topology.* Let $\mathcal{L}_0$ and $\mathcal{L}_1$ be linearly ordered sets. Consider the product $\mathcal{L} = \mathcal{L}_0 \times \mathcal{L}_1$. Let $\tau_{\mathcal{L}}$ be the product of the order topologies on $\mathcal{L}_0$ and $\mathcal{L}_1$. Fix an element $\mathbf{0} = (\mathbf{o}_0, \mathbf{o}_1) \in \mathcal{L}$, and $\mathbf{0}_\star \notin \mathcal{L}$. The *double origin space* $\mathcal{L}_{DO}$ is defined as the set $\mathcal{L} \cup \{\mathbf{0}_\star\}$ equipped with the topology generated by the following subbasis:

$$\{U \setminus \{\mathbf{0}\} : U \in \tau_{\mathcal{L}}\} \cup \{\{\mathbf{0}\} \cup \{(x,y) : a < x < b \text{ and } \mathbf{o}_1 < y < c\} : a < \mathbf{o}_0 < b, c > \mathbf{o}_1\}$$
$$\cup \{\{\mathbf{0}_\star\} \cup \{(x,y) : a < x < b \text{ and } c < y < \mathbf{o}_1\} : a < \mathbf{o}_0 < b, c < \mathbf{o}_1\},$$

where a, b, x range over $\mathcal{L}_0$ and y ranges over $\mathcal{L}_1$. We observe that $\mathcal{L}_{DO}$ is Hausdorff; in fact, effectively so.

Example 3.23 (see Steen-Seebach [74, II.74])**.** For each $i < 2$, let $\mathcal{Q}_i$ be the unit open rational interval, that is, $\mathcal{Q}_i = \mathbb{Q} \cap (-1, 1)$, equipped with the canonical ordering, and put $\mathbf{o}_0 = \mathbf{o}_1 = 0$. Then, a countable subbasis of $\mathcal{Q}_{DO}$ is given as follows:

$$\{((p,q) \times (r,s)) \setminus \{\mathbf{0}\} : p, q, r, s \in \mathbb{Q} \cap (-1,1)\}$$
$$\cup \{((-k^{-1}, k^{-1}) \times (0, \ell^{-1})) \cup \{\mathbf{0}\} : k, \ell \in \omega\}$$
$$\cup \{((-k^{-1}, k^{-1}) \times (-\ell^{-1}, 0)) \cup \{\mathbf{0}_\star\} : k, \ell \in \omega\}.$$

Clearly $\mathcal{Q}_{DO}$ is countable, and so its degree structure is not interesting. Instead, we consider the ω-power $(\mathcal{Q}_{DO})^\omega$. We treat each $z \in (\mathcal{Q}_{DO})^\omega$ as if it were a pair (x, y). If $z(n) \neq \mathbf{0}_\star$ for all $n \in \omega$, it is actually a pair (x, y) given by $z(n) = \langle x(n), y(n)\rangle$ for any $n \in \omega$. If $z(n) = \mathbf{0}_\star$ for some $n \in \omega$, one may put $\mathbf{0}_\star = (0_\star, 0_\star)$ by choosing a new symbol $0_\star$, where we assume that $0_\star$ has no relationship with other rationals. Then, for any point $(x, y) \in (\mathcal{Q}_{DO})^\omega$, its coded neighborhood filter is given as follows:

$$\mathrm{Nbase}(x,y) = \{\langle n, 0, p, q, r, s\rangle : p < x(n) < q,\ r < y(n) < s, \text{ and } (x(n), y(n)) \notin \{\mathbf{0}, \mathbf{0}_\star\}\}$$
$$\cup \{\langle n, 1, k, \ell\rangle : (|x(n)| < k^{-1} \text{ and } 0 < y(n) < \ell^{-1}) \text{ or } (x(n), y(n)) = \mathbf{0}\},$$
$$\cup \{\langle n, 2, k, \ell\rangle : (|x(n)| < k^{-1} \text{ and } -\ell^{-1} < y(n) < 0) \text{ or } (x(n), y(n)) = \mathbf{0}_\star\}.$$

Example 3.24. Define $\mathcal{P}_0 = \hat{\omega} \simeq \omega + 1$, $\mathcal{P}_1 = \omega + 1 + \omega^*$, and $\mathcal{P} = \mathcal{P}_0 \times \mathcal{P}_1$. Here, recall that $\hat{\omega} = \omega \cup \{\infty\}$ is a one-point compactification of ω, and ω^* is the reverse order of ω. More precisely, $\mathcal{P}_1$ is the set $\{n : n \in \omega\} \cup \{*\} \cup \{\overline{n} : n \in \omega\}$ ordered as follows:

$$0 < 1 < \cdots < n < n+1 < \cdots < * < \cdots < \overline{n+1} < \overline{n} < \cdots < \overline{1} < \overline{0}.$$

Then, define $\mathbf{o}_0 = \infty \in \mathcal{P}_0$ and $\mathbf{o}_1 = * \in \mathcal{P}_1$. A countable subbasis of $\mathcal{P}_{DO}$ is given as follows:

$$\{[n, \infty] \times \{m\} : n \in \omega,\ m \neq *\} \cup \{\{(n, m)\} : n \in \omega,\ m \in \mathcal{P}_1\} \cup \{\{n\} \times [n, \overline{m}] : n, m \in \omega\}$$
$$\cup \{([n, \infty] \times (*, \overline{n}]) \cup \{\mathbf{0}\} : n \in \omega\} \cup \{([n, \infty] \times [n, *)) \cup \{\mathbf{0}_\star\} : n \in \omega\}.$$

It is clear that $\mathcal{P}_{DO}$ embeds into $\mathcal{Q}_{DO}$. To see this, consider embeddings $j_0 : \mathcal{P}_0 \to \mathcal{Q}_0$ and $j_1 : \mathcal{P}_1 \to \mathcal{Q}_1$ defined by $j_0(n) = 2^{-n}$, $j_0(\infty) = 0$, $j_1(n) = -2^{-n}$, $j_1(*) = 0$, and $j_1(\overline{n}) = 2^{-n}$. Then $j_0 \times j_1$ clearly induces an embedding of $\mathcal{P}_{DO}$ into $\mathcal{Q}_{DO}$. An advantage of using $\mathcal{P}$ is that $(\mathcal{P}_{DO})^\omega$ is quasi-Polish (see Proposition 3.71) while $(\mathcal{Q}_{DO})^\omega$ is not (since it clearly contains $\mathbb{Q}$ as an open subspace; so apply Theorem 2.9).

We will show the following characterization of double origin spaces.

$$\mathcal{D}_{(\mathcal{Q}_{DO})^\omega} = \mathcal{D}_{(\mathcal{P}_{DO})^\omega} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \text{ is doubled co-d-CEA}\}.$$

Here, recall from Definition 3.5 for the above notion.

Theorem 3.25. *The $(\mathcal{P}_{DO})^\omega$ and the $(\mathcal{Q}_{DO})^\omega$-degrees each are exactly the doubled co-d-CEA degrees. In other words, an e-degree $\mathbf{d}$ is a $(\mathcal{Q}_{DO})^\omega$-degree if and only if there are $X, A, B, P, N \subseteq \omega$ such that A, B, P and N are pairwise disjoint, P, N, and $(A \cup B)^{\mathsf{c}}$ are X-c.e., and*

$$X \oplus X^{\mathsf{c}} \oplus (A \cup P) \oplus (B \cup N) \in \mathbf{d}.$$

Proof. We first show the "only if" part, where we only need to consider $(\mathcal{Q}_{DO})^\omega$. To simplify the argument, we treat $\mathbf{0}_\star$ as a pair $(0_\star, 0_\star)$, where $0_\star \notin \mathcal{Q}_i = \mathbb{Q} \cap (-1, 1)$. Define $c\colon \mathcal{Q}_i \cup \{0_\star\} \to \mathcal{Q}_i$ by $c(x) = x$ if $x \neq 0_\star$; otherwise $c(x) = 0$. Given $(x, y) = (x_n, y_n)_{n\in\omega} \in (\mathcal{Q}_{DO})^\omega$, define $c_{(x,y)}(2n) = c(x_n)$ and $c_{(x,y)}(2n+1) = c(y_n)$.

We observe that $(x, y) \mapsto c_{(x,y)} : (\mathcal{Q}_{DO})^\omega \to \mathbb{Q}^\omega$ is the computable quotient map that merges the two origins. For computability, if $\langle n, 0, p, q, r, s\rangle \in \mathrm{Nbase}(x, y)$ then $(p, q) \times (r, s)$ is a neighborhood of $(c(x_n), c(y_n))$, and if $\langle n, i, k, \ell\rangle \in \mathrm{Nbase}(x, y)$ for $i \in \{1, 2\}$ then $(-k^{-1}, k^{-1}) \times (-\ell^{-1}, \ell^{-1})$ is a neighborhood of $(c(x_n), c(y_n))$. It is easy to check that this process enumerates a neighborhood basis of $(c(x_n), c(y_n)) \in \mathbb{Q}^2$.

Note that $\mathbb{Q}$ embeds into $\mathbf{2}^\omega$ (as the subspace of eventually constant sequences), which is inherited by $\mathbb{Q}^\omega$. Thus, $c_{(x,y)}$ has total degree, and we pick a suitable total representative X. Moreover, we define A, B, P, N as follows:

$$\begin{aligned}
A &= \{n \in \omega : (x_n, y_n) = \mathbf{0}\},\\
B &= \{n \in \omega : (x_n, y_n) = \mathbf{0}_\star\},\\
P &= \{n \in \omega : y_n > 0\},\\
N &= \{n \in \omega : y_n < 0\}.
\end{aligned}$$

Clearly, A, B, P, N are pairwise disjoint, $A \cup B$ is X-co-c.e., and P, N are X-c.e. Observe that $X \oplus X^{\mathsf{c}} \le_e \mathrm{Nbase}(x, y)$ by computability of $(x, y) \mapsto c_{x,y}$. Note that $\langle n, 1, 1, 1\rangle$ is enumerated into $\mathrm{Nbase}(x, y)$ if and only if n is enumerated into $A \cup P$. Similarly, $\langle n, 2, 1, 1\rangle$ is enumerated into $\mathrm{Nbase}(x, y)$ if and only if n is enumerated into $B \cup N$. Therefore, we get $X \oplus X^{\mathsf{c}} \oplus (A \cup P) \oplus (B \cup N) \le_e \mathrm{Nbase}(x, y)$.

Conversely, assume that an enumeration of $X \oplus X^{\mathsf{c}} \oplus (A \cup P) \oplus (B \cup N)$ is given. Then we proceed the following algorithm:

(I) If we see $n \notin A \cup B$ by using $X \oplus X^{\mathsf{c}}$, we start to enumerate all tuples of the form $\langle n, 0, p, q, r, s\rangle$ such that $p < c(x_n) < q$ and $r < c(y_n) < s$.
(II) if we see $n \in A \cup P$, by using $X \oplus X^{\mathsf{c}}$, we start to enumerate all tuples of the form $\langle n, 1, k, \ell\rangle$ such that $|c(x_n)| < k^{-1}$ and $c(y_n) < \ell^{-1}$.
(III) If we see $n \in B \cup N$, by using $X \oplus X^{\mathsf{c}}$, we start to enumerate all tuples of the form $\langle n, 2, k, \ell\rangle$ such that $|c(x_n)| < k^{-1}$ and $-\ell^{-1} < c(y_n)$.

Here, for (I), recall that $A \cup B$ is co-c.e. relative to X. We claim that the above procedure gives us an enumeration of $\mathrm{Nbase}(x, y)$. To show this claim, let $\mathrm{Nbase}_n(x, y)$ be the n-th section of $\mathrm{Nbase}(x, y)$, that is, $\mathrm{Nbase}_n(x, y) = \{\alpha : \langle n, \alpha\rangle \in \mathrm{Nbase}(x, y)\}$.

If $n \in A$, clearly, $\mathrm{Nbase}_n(x, y) = \{\langle 1, k, \ell\rangle : k, \ell \in \omega\}$. Since $n \in A$, our algorithm only proceeds (II), and since $c(x_n) = c(y_n) = 0$, we have $|c(x_n)| < k^{-1}$ and $c(y_n) < \ell^{-1}$ for all $k, \ell \in \omega$. Thus, the n-th section enumerated by the above algorithm is exactly $\mathrm{Nbase}_n(x, y)$. If $n \in B$, a similar argument holds. If $n \in P$, then $\mathrm{Nbase}_n(x, y)$ is the union of the set of all $\langle 0, p, q, r, s\rangle$ such that $p < x_n < q$ and $r < y_n < s$, and that of all $\langle n, 1, k, \ell\rangle$ such that $|x_n| < k^{-1}$ and $y_n < \ell^{-1}$. Since $n \in P$, our algorithm proceeds

(I) and (II), and clearly, $c(x_n) = x_n$ and $c(y_n) = y_n$. Thus, the n-th section enumerated by the above algorithm is exactly $\mathrm{Nbase}_n(x, y)$. If $n \in N$, a similar argument holds. Finally, assume that $n \notin A \cup B \cup P \cup N$. In this case, $\mathrm{Nbase}_n(x, y)$ is the set of all $\langle 0, p, q, r, s\rangle$ such that $p < x_n < q$ and $r < y_n < s$. Since $n \notin A \cup B \cup P \cup N$, our algorithm only proceeds (I), and clearly, $c(x_n) = x_n$ and $c(y_n) = y_n$. Thus, the n-th section enumerated by the above algorithm is exactly $\mathrm{Nbase}_n(x, y)$. This verifies the claim.

To show the "if" part, let X, A, B, P, N be such that A, B, P, and N are disjoint, and P, N, and $(A \cup B)^{\mathsf{c}}$ are X-c.e. We will construct a point $(x, y) = (x_n, y_n)_{n\in\omega} \in (\mathcal{Q}_{DO})^\omega$ such that $\mathrm{Nbase}(x, y) \equiv_e X \oplus X^{\mathsf{c}} \oplus (A\cup P) \oplus (B\cup N)$; where (x, y) will be in the image of the computale embedding of $(\mathcal{P}_{DO})^\omega$ into $(\mathcal{Q}_{DO})^\omega$ constructed in Example 3.24. Hence, (x, y) has even a $(\mathcal{P}_{DO})^\omega$-degree.

Fix X-computable enumerations of $(A \cup B)^{\mathsf{c}}$, P, and N. First, (x_{2n}, y_{2n}) is used to code $X \oplus X^{\mathsf{c}}$. For instance, put $(x_{2n}, y_{2n}) = (X(n)/2, 1/2)$. We first define $c(x_{2n+1})$ and $c(y_{2n+1})$ as follows:

$$c(x_{2n+1}) = \begin{cases} 2^{-s} & \text{if we see } n \notin A \cup B \text{ at stage } s, \\ 0 & \text{if } n \in A \cup B, \end{cases}$$

$$c(y_{2n+1}) = \begin{cases} 2^{-s} & \text{if we see } n \in P \text{ at stage } s, \\ -2^{-s} & \text{if we see } n \in N \text{ at stage } s, \\ 0 & \text{if } n \notin P \cup N, \end{cases}$$

Note that $n \in A\cup B$ implies $(c(x_{2n+1}), c(y_{2n+1})) = (0, 0)$ since A, B, P, N are disjoint. If $(c(x_{2n+1}), c(y_{2n+1})) = (0, 0)$, then define $(x_{2n+1}, y_{2n+1}) = \mathbf{0}$ if $x \in A$, and define $(x_{2n+1}, y_{2n+1}) = \mathbf{0}_\star$ if $x \in B$. We can then decode A, B, P, N as before, and we get $X \oplus X^{\mathsf{c}} \oplus (A \cup P) \oplus (B \cup N) \equiv_e \mathrm{Nbase}(x, y)$ as in the above argument. $\square$

A set is *d-c.e.* if it is a difference of two c.e. sets. A set is *co-d-c.e.* if it is the complement of a d-c.e. set, that is, the union of a c.e. set and a co-c.e. set. Clearly, double origin degrees include all co-d-c.e. degrees. It is known that there is a quasi-minimal co-d-c.e. degree (see Cooper [11]). Thus, there is a quasi-minimal double origin degree.

Moreover, one can see that the degree structure of the double origin space is included in that of the telophase space.

Proposition 3.26. *Every doubled co-d-CEA e-degree is telograph-cototal. Hence, we have* $\mathcal{D}_{(\mathcal{Q}_{DO})^\omega} \subseteq \mathcal{D}_{(\hat{\omega}_{TP})^\omega}$.

Proof. Every doubled co-d-CEA e-degree is of the form $Y = (C\oplus C^{\mathsf{c}})\oplus(A\cup P)\oplus(B\cup N)$ such that $P, N, (A \cup B)^{\mathsf{c}}$ are C-c.e. Put $Z = (A\cup P) \oplus (B\cup N)$. It suffices to construct a total function g such that $C\oplus C^{\mathsf{c}}\oplus Z \equiv_e C\oplus C^{\mathsf{c}}\oplus \mathrm{Graph}(g)^{\mathsf{c}}\oplus \mathrm{TGraph}_2(g)$ (since we can remove $C \oplus C^{\mathsf{c}}$ from the right-hand set by replacing g with $\tilde{g}$ such that $\tilde{g}(2n) = C(n)+2$ and $\tilde{g}(2n + 1) = g(n)$). Fix C-computable enumerations of P, N, and $(A \cup B)^{\mathsf{c}}$. For each n, let t^0_n (t^1_n, resp.) be the first stage such that n is enumerated into P (N, resp.) if

$n \in P \cup N$. Then we define g as follows.

$$g(2n) = \begin{cases} 0, & \text{if } n \notin A \cup P, \\ 1, & \text{if } n \in A, \\ t_n^0 + 2, & \text{if } n \in P, \end{cases}$$

$$g(2n+1) = \begin{cases} 0, & \text{if } n \notin B \cup N, \\ 1, & \text{if } n \in B, \\ t_n^1 + 2, & \text{if } n \in N, \end{cases}$$

Clearly, g is total. As A, B, P, N are disjoint, trivially we have

$$A = (A \cup B) \setminus (B \cup N), \text{ and } B = (A \cup B) \setminus (A \cup P).$$

Thus, $n \in A$ if and only if $n \in A \cup B$ and $2n+1 \in Z^{\mathsf{c}}$ (i.e., $n \notin B \cup N$). Similarly, $n \in B$ if and only if $n \in A \cup B$ and $2n \in Z^{\mathsf{c}}$ (i.e., $n \notin A \cup P$). Thus, we note that

$$\begin{aligned} g(2n) \neq 0 &\iff 2n \in Z, \\ g(2n) \neq 1 &\iff n \in (A \cup B)^{\mathsf{c}} \text{ or } 2n+1 \in Z, \\ g(2n) = m+2 &\iff m = t_n^0. \end{aligned}$$

Here, note that the equality $m = t_n^0$ is C-computable. Similarly,

$$\begin{aligned} g(2n+1) \neq 0 &\iff 2n+1 \in Z, \\ g(2n+1) \neq 1 &\iff n \in (A \cup B)^{\mathsf{c}} \text{ or } 2n \in Z, \\ g(2n+1) = m+2 &\iff m = t_n^1. \end{aligned}$$

The above equivalences clearly witness that $C \oplus C^{\mathsf{c}} \oplus Z$ is e-equivalent to $C \oplus C^{\mathsf{c}} \oplus \mathrm{Graph}(g)^{\mathsf{c}} \oplus \mathrm{TGraph}_2(g)$. Consequently, Y has a telograph-cototal e-degree. $\square$

3.6. **Degrees of points: $T_{2.5}$-topology.** A topological space $\mathcal{X}$ is $T_{2.5}$ if any distinct points $x \neq y \in \mathcal{X}$ are separated by closed neighborhoods, that is, there are open sets $U, V \subseteq \mathcal{X}$ such that $x \in \overline{U}$, $y \in \overline{V}$, and $\overline{U} \cap \overline{V} = \emptyset$. A topological space $\mathcal{X}$ is *completely Hausdorff* or *functionally Hausdorff* if any distinct points $x \neq y \in \mathcal{X}$ are separated by a function, that is, there exists a continuous function $f : \mathcal{X} \to [0,1]$ with $f(x) = 0$ and $f(y) = 1$. Note that every metrizable space is completely Hausdorff, and every completely Hausdorff space is $T_{2.5}$, but none of the converse holds. In this section, we examine the degree structure of a $T_{2.5}$ space which is not completely Hausdorff (hence, not submetrizable).

3.6.1. *Arens square.* We would like to know a typical degree-theoretic behavior of a space which is second-countable, $T_{2.5}$, but not completely Hausdorff. As such an example, Steen-Seebach [74, II.80] mentioned the Arens square; however there it has been found[2] that their definition contains an error, that is, the Arens square defined in [74, II.80] is not $T_{2.5}$. We here construct a simple example of a space which fulfills the above required property by modifying the definition of Arens square. Rather than describing

[2]The problem was observed by Martin Sleziak on math.stackexchange.com. A direct fix was then proposed by Brian M. Scott (`https://math.stackexchange.com/questions/1715435/is-arens-square-a-urysohn-space`). Our modification can be seen as an abstraction of the one proposed by Scott.

the space as a subset of the rational unit square, we observe that the crucial ideas of the construction are all order-theoretic in nature and thus use corresponding language.

Example 3.27. Consider a linear ordering $\mathcal{L}$ of type $\omega + 1 + \zeta + 1 + \omega^*$, where ζ is the order type of the integers $\mathbb{Z}$. More precisely, consider the sets $\omega = \{n : n \in \omega\}$, $\omega^* = \{\overline{n} : n \in \omega\}$, and $\zeta = \{n_\zeta : n \in \mathbb{Z}\}$, and then, let $\mathcal{L}$ be the linear order consisting of the set $\omega \cup \{\infty\} \cup \zeta \cup \{\overline{\infty}\} \cup \omega^*$ ordered as follows.

$$0 < 1 < \dots < \infty < \dots < (-1)_\zeta < 0_\zeta < 1_\zeta < \dots < \overline{\infty} < \dots < \overline{1} < \overline{0}.$$

Consider the following subset I_x of the ordinal $\omega^3 + 1$ for each $x \in \mathcal{L}$.

$$\begin{aligned}
I_0 = I_{\overline{0}} &= \{\omega^3\},\ I_{0_\zeta} = \{\omega^2 \cdot (2j+1) : j \in \omega\},\\
I_\infty &= \{\omega^2 \cdot (2j) + \omega \cdot (2k+1) : j,k \in \omega\},\\
I_{\overline{\infty}} &= \{\omega^2 \cdot (2j) + \omega \cdot (2k+2) : j,k \in \omega\},\\
I_n &= \{\omega^2 \cdot (2j) + \omega \cdot (2k) + 2n - 1 : j,k \in \omega\},\\
I_{\overline{n}} &= \{\omega^2 \cdot (2j) + \omega \cdot (2k+1) + 2n - 1 : j,k \in \omega\},\\
I_{n_\zeta} &= \{\omega^2 \cdot (2j) + \omega \cdot (2k+1) + 2n : j,k \in \omega\},\\
I_{(-n)_\zeta} &= \{\omega^2 \cdot (2j) + \omega \cdot (2k) + 2n : j,k \in \omega\}.
\end{aligned}$$

where n ranges over $\omega^+ := \omega \setminus \{0\}$. Note that $(I_x : x \in \mathcal{L} \setminus \{0\})$ is a partition of $(\omega^3 + 1) \setminus \{0\}$. Moreover, $(I_x : x \in \mathcal{L}^+)$ is a partition of the nonzero successor ordinals $< \omega^3$, where $\mathcal{L}^+ = \mathcal{L} \setminus \{0, \overline{0}, 0_\zeta, \infty, \overline{\infty}\}$. Then define a modified Arens square (which we will call the *quasi-Polish Arens space*) $\mathcal{QA} \subseteq \mathcal{L} \times (\omega^3 + 1)$ as follows:

$$\mathcal{QA} = \{(x,y) : x \in \mathcal{L} \text{ and } y \in I_x\}.$$

Let τ be a topology on ω^3 (i.e., the collection of all open intervals). The set $\mathcal{QA}$ is topologized by declaring the following as an open basis.

$$\begin{aligned}
&\{\omega \times (\alpha, \omega^3],\ \omega^* \times (\alpha, \omega^3],\ \zeta \times U,\ [n, (-n)_\zeta] \times U,\ [n_\zeta, \overline{n}] \times U,\ \{x\} \times U\\
&\qquad\qquad\qquad\qquad\qquad\qquad : n \in \omega^+,\ x \in \mathcal{L}^+,\ \alpha < \omega^3,\ U \in \tau\}
\end{aligned}$$

For a visualization of this topology, we may refer to [74, II.80]. One can easily see that an open basis of this topology can also be given as follows:

$$\begin{aligned}
&\{\omega \times (\omega^2 \cdot j, \omega^3] : j \in \omega\} \cup \{\omega^* \times (\omega^2 \cdot j, \omega^3] : j \in \omega\}\\
&\cup \{\zeta \times (\omega^2 \cdot j + \omega \cdot n, \omega^2 \cdot (j+1)) : n, j \in \omega\}\\
&\cup \{[n, (-n)_\zeta] \times (\omega^2 j + \omega(2k) + 2n - 1, \omega^2 j + \omega(2k+1)] : n \in \omega^+,\ k, j \in \omega\}\\
&\cup \{[n_\zeta, \overline{n}] \times (\omega^2 j + \omega(2k+1) + 2n - 1, \omega^2 j + \omega(2k+2)] : n \in \omega^+,\ k, j \in \omega\}\\
&\cup \{\{(x,y)\} : x \in \mathcal{L}^+ \text{ and } y \in I_x\}.
\end{aligned}$$

Remark 3.28. Note that the second projection $\pi : \mathcal{QA} \to \omega^3 + 1$ given by $\pi(x,y) = y$ is continuous w.r.t. the order topology on the ordinal $\omega^3 + 1$. Hence, given a name of $(x,y) \in \mathcal{QA}$, one can compute a name of y w.r.t. a suitable representation of $\omega^3 + 1$. The computability is ensured by just considering a quotient admissible representation of

ω^3+1 induced from π, or equivalently, by considering an embedding $h:\omega^3+1\to[0,1]$ defined by

$$h(\omega^3)=0,\ h(\omega^3[j])=2^{-j},\ h(\omega^3[j+1][k])=2^{-j}(1+2^{-k}),$$
$$h(\omega^3[j+1][k+1][\ell])=2^{-j}(1+2^{-k}(1+2^{-\ell})),$$

where fundamental sequences are given by $\omega^{n+1}[j]=\omega^n\cdot j$ and $(\alpha+\beta)[k]=\alpha+\beta[k]$. Thus, one can consider the embedded image of ω^3+1 into the unit interval $[0,1]$. Note also that the ordinal space ω^3+1 is a (computably) zero-dimensional compact metrizable space. Hence, for any $x\in(\omega^3+1)^\omega$, $\mathrm{Nbase}(x)$ has a total e-degree.

Proposition 3.29. *The quasi-Polish Arens space $\mathcal{QA}$ is second-countable, and $T_{2.5}$, but not completely Hausdorff.*

Proof. It is clear that $\mathcal{QA}$ is second-countable. To see that $\mathcal{QA}$ is $T_{2.5}$, choose two distinct points $(x_0,y_0)\neq(x_1,y_1)$ in $\mathcal{QA}$. If $y_0\neq y_1$, then there are disjoint open sets $U,V\subseteq\omega^3+1$ such that $y_0\in\overline{U}$ and $y_1\in\overline{V}$ (w.r.t. the order topology on ω^3+1). Then, it is easy to see that $\mathcal{L}\times U$ and $\mathcal{L}\times V$ separate (x_0,y_0) and (x_1,y_1). If $y_0=y_1$, then we must have $\{x_0,x_1\}=\{0,\overline{0}\}$ and $y_0=y_1=\omega^3$. Assume that $x_0=0$ and $x_1=\overline{0}$. Then, it is easy to see that $\omega\times(\omega^3+1)$ and $\omega^*\times(\omega^3+1)$ separate (x_0,y_0) and (x_1,y_1).

Assume for the sake of a contradiction that there is a continuous function $f:\mathcal{QA}\to\mathbb{R}$ such that $f(0,\omega^3)=0$ and $f(\overline{0},\omega^3)=1$. As $\{\omega\times(\alpha,\omega^3]:\alpha<\omega^3\}$ (respectively $\{\omega^*\times(\alpha,\omega^3]:\alpha<\omega^3\}$) is a neighbourhood basis of $(0,\omega^3)$ (respectively of $(\overline{0},\omega^3)$), there is $\alpha<\omega^3$ such that $f(x,y)<1/4$ if $x\in\omega$ and $y>\alpha$, and that $f(x,y)>3/4$ if $x\in\omega^*$ and $y>\alpha$. Note that if $\beta\in I_\infty$ and $\beta>\alpha$, then $(\infty,\beta)\in\overline{\omega\times(\alpha,\omega^3]}$. This is because β is a limit ordinal, and hence β is an accumulation point of I_n for any $n\in\omega$. Hence, $f(\infty,\beta)\leq 1/4$. Similarly, $(\overline{\infty},\beta)\in\overline{\omega^*\times(\alpha,\omega^3]}$, and hence $f(\overline{\infty},\beta)\geq 3/4$. Choose $\gamma\in I_{0_\zeta}$ such that $\gamma>\alpha$, and consider the value $f(0_\zeta,\gamma)$. There is δ such that $\alpha<\delta<\gamma$ and $|f(x,y)-f(0_\zeta,\gamma)|<1/8$ for any $x\in\zeta$ and $\delta<y\leq\gamma$. Since γ is of the form $\omega^2\cdot j$, γ is an accumulation point of both I_∞ and $I_{\overline{\infty}}$. Hence, there are $\beta\in I_\infty$ and $\overline{\beta}\in I_{\overline{\infty}}$ such that $\alpha<\delta<\beta,\overline{\beta}<\gamma$. Then, we also have $(\infty,\beta),(\overline{\infty},\overline{\beta})\in\overline{\zeta\times(\delta,\gamma]}$. Hence,

$$\frac{3}{8}\geq f(\infty,\beta)+\frac{1}{8}\geq f(0_\zeta,\gamma)\geq f(\overline{\infty},\overline{\beta})-\frac{1}{8}\geq\frac{5}{8},$$

which is clearly a contradiction. □

From the descriptive set theoretic perspective, our modified Arens square $\mathcal{QA}$ is better behaved than the original one in a certain sense: The space $\mathcal{QA}$ is quasi-Polish (hence so is the ω-power $\mathcal{QA}^\omega$) as we will see in Proposition 3.71.

We proceed to examine the degree structure of the product Arens space $\mathcal{QA}^\omega$. For $z=(x(n),y(n))_{n\in\omega}\in\mathcal{QA}^\omega$, the coded neighborhood basis of z is given as follows:

$$\begin{aligned}
&\{\langle 0,n,j\rangle : x(n)\in\omega \text{ and } y(n)>\omega^2 j\}\cup\{\langle 1,n,j\rangle : x(n)\in\omega^* \text{ and } y(n)>\omega^2 j\}\\
&\cup\{\langle 2,n,j,k\rangle : x(n)\in\zeta \text{ and } \omega^2 j+\omega k<y(n)<\omega^2(j+1)\}\\
&\cup\{\langle 3,n,j,k,\ell\rangle : j\leq x(n)\leq(-j)_\zeta \text{ and } \omega^2 k+\omega(2\ell+1)+2j-1<y(n)\leq\omega^2 k+\omega(2\ell+2)\}\\
&\cup\{\langle 4,n,j,k,\ell\rangle : j_\zeta\leq x(n)\leq\overline{j} \text{ and } \omega^2 k+\omega(2\ell)+2j<y(n)\leq\omega^2 k+\omega(2\ell+1)\}\\
&\cup\{\langle 5,n,x,y\rangle : x(n)=x\in\mathcal{L}^+ \text{ and } y(n)=y\in I_x\}.
\end{aligned}$$

We will see that the degree structure of the product quasi-Polish Arens space $\mathcal{QA}^\omega$ can be considered as a variant of the co-d-CEA degrees. To the similarity, we provide a characterization of the co-d-CEA degrees similar to what we show below as Definition 3.31. Let $\mathcal{E}$ be the collection of e-degrees $\mathbf{d}$ which contain a set $S \subseteq \omega$ of the following form

$$S = X \oplus X^{\mathsf{c}} \oplus (A \cup P) \oplus (B \cup N)$$

for some $A, B, P, N, X \subseteq \omega$ such that P, N and $(A \cup B)^{\mathsf{c}}$ are c.e. in X, A, B, P, N are pairwise disjoint, and P and N are X-computably separated over $(A \cup B)^{\mathsf{c}}$. Here, we say that P and N are *X-computably separated over C* if there are disjoint X-c.e. sets $H_P, H_N \subseteq \omega$ such that $C = H_P \cup H_N$, $P \subseteq H_P$, and $N \subseteq H_N$.

Observation 3.30. *For an e-degree $\mathbf{d}$, $\mathbf{d} \in \mathcal{E}$ if and only if $\mathbf{d}$ is co-d-CEA.*

Proof. It is clear that if $\mathbf{d}$ is co-d-CEA, then $\mathbf{d} \in \mathcal{E}$. Conversely, assume that $S = X \oplus X^{\mathsf{c}} \oplus (A \cup P) \oplus (B \cup N)$ for some $A, B, P, N, X \subseteq \omega$ satisfying the above mentioned conditions. Define $Z = (A \cup P) \cup H_P$, and then $Z \leq_e S$ since H_P is X-c.e. One can see that $Z^{\mathsf{c}} = (B \cup N) \cup H_N$, and thus $Z^{\mathsf{c}} \leq_e S$. Then, we have $A \cup P = Z \cap ((A \cup B) \cup P)$ and $B \cup N = Z^{\mathsf{c}} \cap ((A \cup B) \cup N)$. Since $P, N, (A \cup B)^{\mathsf{c}}$ are X-c.e., the sets $A \cup P$ and $B \cup N$ are co-d-c.e. relative to $X \oplus Z$. Hence, $S \oplus Z \oplus Z^{\mathsf{c}} \equiv_e S$ is co-d-CEA. □

Definition 3.31. We say that an e-degree $\mathbf{d}$ is *Arens co-d-CEA* if $\mathbf{d}$ contains a set $S \subseteq \omega$ of the following form

$$S = Y \oplus Y^{\mathsf{c}} \oplus (L \cup J_L) \oplus (R \cup J_R) \oplus ((L \cup R \cup N)^{\mathsf{c}} \cup J_M)$$

for some $L, R, N, J_L, J_R, J_M, Y \subseteq \omega$ such that N, J_L, J_R, J_M and $(L \cup R)^{\mathsf{c}}$ are c.e. in Y, L, R, N are pairwise disjoint, and $J_L, J_R, J_M \subseteq N$ are pairwise disjoint, where J_L and J_R are Y-computably separated over N, that is, there is a Y-c.e. partition $\{H_L, H_R\}$ of N such that $J_L \subseteq H_L$ and $J_R \subseteq H_R$.

Observation 3.32. *Every co-d-CEA e-degree is Arens co-d-CEA.*

Proof. If an e-degree $\mathbf{d}$ is co-d-CEA, then there are $X, A, P \subseteq \omega$ such that P and A^{c} are X-c.e., A and P are disjoint, and $(X \oplus X^{\mathsf{c}}) \oplus (A \cup P) \in \mathbf{d}$. Put $Y = X$, $L = A$, $J_L = P$, $N = L^{\mathsf{c}}$, and $R = J_R = J_M = \emptyset$. This witnesses that $\mathbf{d}$ is Arens co-d-CEA. □

Theorem 3.33. *The degree structure of the product quasi-Polish Arens space $\mathcal{QA}^\omega$ consists exactly of Arens co-d-CEA e-degrees:*

$$\mathcal{D}_{\mathcal{QA}^\omega} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \textit{ is Arens co-d-CEA}\}.$$

Proof. Assume that $z = (x_n, y_n)_{n \in \omega} \in \mathcal{QA}^\omega$ is given. Consider the following sets:

$$\begin{aligned}
L &= \{n \in \omega : (x_n, y_n) = (0, \omega^3)\}, \\
R &= \{n \in \omega : (x_n, y_n) = (\overline{0}, \omega^3)\}, \\
M &= \{n \in \omega : x_n = 0_\zeta\} = \{n \in \omega : (\exists j \in \omega)\ y_n = \omega^2 \cdot j\}, \\
J_L &= \{n \in \omega : x_n \in \omega \setminus \{0\}\} = \{n \in \omega : (\exists k \in \omega \setminus \{0\})\ y_n \in I_k\}, \\
J_R &= \{n \in \omega : x_n \in \omega^* \setminus \{\overline{0}\}\} = \{n \in \omega : (\exists k \in \omega \setminus \{0\})\ y_n \in I_{\overline{k}}\}, \\
J_M &= \{n \in \omega : x_n \in \zeta \setminus \{0_\zeta\}\} = \{n \in \omega : (\exists k \in \zeta \setminus \{0\})\ y_n \in I_{k_\zeta}\}.
\end{aligned}$$

Let Y code the information on the second coordinate $(y_n)_{n\in\omega}$. Then Y is total as seen in Remark 3.28. The sets $L\cup R$, M, J_L, J_R, and J_M are characterized only by using $(y_n)_{n\in\omega}$. For instance, $n\in L\cup R$ iff $y_n=\omega^3$, which is co-c.e. condition relative to Y. One can also see that $n\in J_L$ iff $y_n\in I_k$ for some $k\in\omega$, which is a c.e. condition relative to Y, since I_k consists of successor ordinals, which are isolated in the space ω^3+1. Similarly, J_R, and J_M are c.e. in Y. Define $N=(L\cup R\cup M)^{\mathsf{c}}$. Then, $N=\{n: y(n)\notin\{\omega^3,\omega^2\cdot j: j\in\omega\}\}$, which is a c.e. condition relative to Y. Define $H_L=\{n: 0<x_n<0_\zeta\}$, and $H_R=\{n:0_\zeta<x_n<\overline{0}\}$. Clearly, $\{H_L,H_R\}$ is a partition of N. Note that $n\in H_L$ iff $y_n\in I_\infty\cup\bigcup_{k\in\omega^+}(I_k\cup I_{(-k)_\zeta})$, which is c.e. condition relative to Y. Similarly, H_R is c.e. in Y. Hence, $A=Y\oplus Y^{\mathsf{c}}\oplus(L\cup J_L)\oplus(R\cup J_R)\oplus((L\cup R\cup N)^{\mathsf{c}}\cup J_M)$ is of an Arens co-d-CEA e-degree.

We first check that $A\leq_e \mathrm{Nbase}(z)$. It is easy to see that

$$n\in L\cup J_L \iff (\exists j)\ \langle 0,n,j\rangle\in\mathrm{Nbase}(z),$$
$$n\in R\cup J_R \iff (\exists j)\ \langle 1,n,j\rangle\in\mathrm{Nbase}(z).$$

Moreover, one can see that $(L\cup R\cup N)^{\mathsf{c}}=M$, and therefore,

$$n\in(L\cup R\cup N)^{\mathsf{c}}\cup J_M=M\cup J_M \iff (\exists j,k)\ \langle 2,n,j,k\rangle\in\mathrm{Nbase}(z).$$

This verifies that $A\leq_e\mathrm{Nbase}(z)$. Conversely, we first recover $y(n)$ from an enumeration of $Y\oplus Y^{\mathsf{c}}$. Then consider the following.

(1) If $y(n)$ is a successor ordinal, then one finds it at finite stage. As it is isolated, we get the exact information of $y(n)$ at such a stage, so one can compute the unique $x(n)$ such that $y(n)\in I_{x(n)}$, which determines $z(n)=(x(n),y(n))$. Enumerate all neighborhoods of $z(n)$.
(2) Even if $y(n)$ is a limit ordinal, if $n\notin I_0\cup I_{\overline{0}}\cup I_{0_\zeta}$, we see $\omega^2k+\omega u+2j-1<y(n)\leq\omega^2k+\omega(u+1)$ at some finite stage. If u is even, $u=2\ell$ say, it is ensured that $j\leq x(n)\leq(-j)_\zeta$, and hence we can enumerate $\langle 3,n,j,k,\ell\rangle$ into $\mathrm{Nbase}(z)$. If u is odd, $u=2\ell+1$ say, it is ensured that $j_\zeta\leq x(n)\leq\overline{j}$, and hence we can enumerate $\langle 4,n,j,k,\ell\rangle$ into $\mathrm{Nbase}(z)$.
(3) Simultaneously, wait until n is enumerated into $(L\cup J_L)\oplus(R\cup J_R)\oplus(M\cup J_M)$.
   - If we see $n\in L\cup J_L$, enumerate $\langle 0,n,j\rangle$ for any j such that $y(n)>\omega^2 j$.
   - If we see $n\in R\cup J_R$, enumerate $\langle 1,n,j\rangle$ for any j such that $y(n)>\omega^2 j$.
   - If we see $n\in M\cup J_M$, enumerate $\langle 2,n,j,k\rangle$ for any j,k such that $\omega^2 j+\omega k<y(n)<\omega^2(j+1)$.

One can check that the above procedure witnesses that $\mathrm{Nbase}(z)\leq_e A$. Next, given such $L,R,N,J_L,J_R,J_M,Y\subseteq\omega$, we define a point $z\in\mathcal{QA}^\omega$. We first define $z(2n+1)=(1,1)$ if $n\in Y$; otherwise $z(2n+1)=(2,3)$. This clearly ensures that $Y\oplus Y^{\mathsf{c}}\leq_e\mathrm{Nbase}(z)$.

Fix Y-computable enumerations of N,J_L,J_R,J_M, and $(L\cup R)^{\mathsf{c}}$. If $n\in L$, define $z(2n)=(0,\omega^3)$. If $n\in R$, define $z(2n)=(\overline{0},\omega^3)$. Put $M=(L\cup R\cup N)^{\mathsf{c}}$. If $n\notin(L\cup R)$ happens, then let s be the first stage when we confirm that (w.r.t. a Y-computable enumeration of $(L\cup R)^{\mathsf{c}}$). If $n\in M$, we define $z(2n)=(0_\zeta,\omega^2\cdot(s+1))$. If $n\notin(L\cup R\cup M)$ happens, that is, if $n\in N$, then we see either $n\in H_L$ or $n\in H_R$ at some stage t. Put $j=1$ if $n\in H_L$, and $j=2$ if $n\in H_R$. If $n\in N\setminus(J_L\cup J_R\cup J_M)$, define $z(2n)=(\infty,\omega^2\cdot s+\omega(2t+j))$. If $n\in J_L\cup J_R\cup J_M$ happens, let u be the first stage when we confirm that. If $n \in J_L \cup J_R$, define the second coordinate of $z(2n)$ to be $\omega^2 \cdot s + \omega(2t + j - 1) + 2u + 1$; otherwise define it to be $\omega^2 \cdot s + \omega(2t + j - 1) + 2u$. The second coordinate uniquely determines the first coordinate.

Now, it is not hard to verify that the coded neighborhood basis of the second coordinate of z is e-reducible to $Y \oplus Y^{\mathsf{c}}$. Moreover, the sets L, R, N, J_L, J_R, J_M satisfy the equations mentioned in the first paragraph in this proof (where $z(2n) = (x_n, y_n)$). Hence, the above argument shows that $A \equiv_e \mathrm{Nbase}(z)$ as desired. □

3.6.2. *Roy's lattice space.* We next introduce another space which has a similar property as the Arens space $\mathcal{QA}$. The space is a quasi-Polish version of Roy's lattice space (see Steen-Seebach [74, II.126]).

To introduce the space, first recall that the ordinal ω^3 is order isomorphic to the lexicographical ordering on the set ω^3 of strings of length 3 by identifying an ordinal $\omega^2 \cdot j + \omega \cdot k + n < \omega^3$ with the string $\langle j, k, n\rangle \in \omega^3$. Now, consider the Kleene-Brouwer ordering $\leq_{\mathrm{KB}}$ on the well-founded tree $\mathcal{O}_{\omega^\omega} = \{\sigma \in \omega^{<\omega} : |\sigma| \leq \sigma(0) + 1\}$. Then, $(\mathcal{O}_{\omega^\omega}, \leq_{\mathrm{KB}})$ is order isomorphic to the ordinal $(\omega^\omega + 1, \leq)$. Note that $|\sigma|_{\mathrm{KB}}$ is a successor ordinal iff σ is a leaf (i.e., a terminal node), and that $|\langle\rangle|_{\mathrm{KB}} = \omega^\omega$. Hereafter, we use $\mathcal{O}^{\mathrm{leaf}}_{\omega^\omega}$ to denote the set of leaves of $\mathcal{O}_{\omega^\omega}$. Given $\sigma, \tau \in \mathcal{O}_{\omega^\omega}$, we define the interval $[\sigma, \tau]_{\mathrm{KB}} = \{\gamma \in \mathcal{O}_{\omega^\omega} : |\sigma|_{\mathrm{KB}} \leq |\gamma|_{\mathrm{KB}} \leq |\tau|_{\mathrm{KB}}\}$. We define the half-open interval $(\sigma, \tau]_{\mathrm{KB}}$ etc. in a similar manner. One can see that for $\sigma \in \mathcal{O}_{\omega^\omega} \setminus \mathcal{O}^{\mathrm{leaf}}_{\omega^\omega}$ and $j \in \omega$,

$$(\sigma j, \sigma]_{\mathrm{KB}} = \{\tau \in \mathcal{O}_{\omega^\omega} : \tau = \sigma \text{ or } (\exists k > j)\ \tau \succeq \sigma k\}.$$

We topologize $\mathcal{O}_{\omega^\omega}$ by declaring the following as an open basis:

$$\{\{\sigma\} : \sigma \in \mathcal{O}^{\mathrm{leaf}}_{\omega^\omega}\} \cup \{(\sigma j, \sigma]_{\mathrm{KB}} : \sigma \notin \mathcal{O}^{\mathrm{leaf}}_{\omega^\omega} \text{ and } j \in \omega\}.$$

One can see that $\mathcal{O}_{\omega^\omega}$ is homeomorphic to the order topology on the ordinal $\omega^\omega + 1$. As in Remark 3.28, one can see that the ordinal space $\omega^\omega + 1$ is zero-dimensional, compact, and metrizable. For $k > 0$, consider the following subsets of the space $\mathcal{O}_{\omega^\omega}$.

$$\begin{aligned}
I_0 = I_\infty &= \{\sigma \in \mathcal{O}_{\omega^\omega} : |\sigma| = 0\} = \{\langle\rangle\},\\
I_{2k} &= \{\sigma \in \mathcal{O}_{\omega^\omega} : |\sigma| = k \text{ and } \sigma \text{ is not a leaf}\}\\
I_1 &= \{\sigma \in \mathcal{O}^{\mathrm{leaf}}_{\omega^\omega} : |\sigma| \geq 2 \text{ and } (\forall j > 1)\ \sigma(j) = 0\},\\
I_{2k+1} &= \{\sigma \in \mathcal{O}^{\mathrm{leaf}}_{\omega^\omega} : |\sigma| \geq k + 2,\ \sigma(k+1) > 0 \text{ and } (\forall j > k+1)\ \sigma(j) = 0\}.
\end{aligned}$$

Note that $(I_k : k \in \omega)$ is a partition of the ordinal $\omega^\omega + 1$. Moreover, each set I_k is cofinal in ω^ω for $k \in \omega \setminus \{0\}$. We now introduce the *quasi-Polish Roy space*, whose underlying set is given as follows.

$$\mathcal{QR} = \{(x, y) : x \in \hat{\omega} \text{ and } y \in I_x\}.$$

Let τ be the topology on $\mathcal{O}_{\omega^\omega}$ defined as above. The set $\mathcal{QR}$ is topologized by declaring the following as an open basis.

$$\{\{2n+1\} \times U,\ \{0,1\} \times U,\ [2n+1, 2n+3] \times U,\ [2n+1, \infty] \times \mathcal{O}_{\omega^\omega} : n \in \omega,\ U \in \tau\}.$$

As in Remark 3.28, one can also ensure that the projection $\pi : \mathcal{QR} \to \omega^\omega + 1$ defined by $\pi(x, y) = y$ is (computably) continuous. In a similar way to $\mathcal{QA}$, one can show that $\mathcal{QR}$ is also quasi-Polish (Proposition 3.71).

Proposition 3.34. *The quasi-Polish Roy space $\mathcal{QR}$ is a second-countable $T_{2.5}$ space which is not completely Hausdorff.*

Proof. It is clear that the space $\mathcal{QR}$ is second-countable. To see that $\mathcal{QR}$ is $T_{2.5}$, let $(x_0, y_0), (x_1, y_1) \in \mathcal{QR}$ be given two distinct points. If $y_0 \neq y_1$, since the ordinal space $\omega^\omega + 1$ is metrizable; hence $T_{2.5}$, choose open sets $U, V \subseteq \mathcal{O}_{\omega^\omega}$ such that $y_0 \in U$, $y_1 \in V$, and $\overline{U} \cap \overline{V} = \emptyset$. Then $\hat{\omega} \times U$ and $\hat{\omega} \times V$ separate (x_0, y_0) and (x_1, y_1). If $y_0 = y_1$, then we must have $\{x_0, x_1\} = \{0, \infty\}$ and $y_0 = y_1$ are the empty string $\langle\rangle$, since $(I_x : x \in \omega \setminus \{0\})$ is a partition of the ordinal ω^ω. Then define $U = \{0,1\} \times \mathcal{O}_{\omega^\omega}$ and $V = [5, \infty] \times \mathcal{O}_{\omega^\omega}$. Clearly, we have $(0, \langle\rangle) \in U$, $(\infty, \langle\rangle) \in V$, $\overline{U} \subseteq [0,2] \times \mathcal{O}_{\omega^\omega}$ and $\overline{V} \subseteq [4, \infty] \times \mathcal{O}_{\omega^\omega}$. Hence, U and V separates (x_0, y_0) and (x_1, y_1).

To see that $\mathcal{QR}$ is not completely Hausdorff, suppose for the sake of contradiction that there is a continuous function $f : \mathcal{QR} \to [0,1]$ such that $f(0, \langle\rangle) = 0$ and $f(\infty, \langle\rangle) = 1$. Since every open neighborhood of $\langle \infty, \langle\rangle\rangle$ is of the form $[2n+1, \infty] \times \mathcal{O}_{\omega^\omega}$, there is $k \in \omega$ such that $f(n, \sigma) > 3/4$ for any $n > 2k$ and $\sigma \in \mathcal{O}_{\omega^\omega}$. Similarly, since the open sets of the form $\{0,1\} \times (\langle m \rangle, \langle\rangle]_{\mathrm{KB}}$ form a local basis at $\langle 0, \langle\rangle\rangle$, there is m such that $\{1\} \times (\langle m \rangle, \langle\rangle]_{\mathrm{KB}} \subseteq f^{-1}[0, \varepsilon)$. Put $\varepsilon = 4^{-1}$, and $\ell = \max\{m, k\} + 1$. Then, in particular, we have $\{1\} \times (\langle \ell - 1 \rangle, \langle \ell \rangle]_{\mathrm{KB}} \subseteq f^{-1}[0, 4^{-1})$.

Note that $(2, \langle \ell \rangle) \in \mathcal{QR}$ since the length of $\langle \ell \rangle$ is 1, and thus $\langle \ell \rangle \in I_2$. The closure of $\{1\} \times (\langle \ell - 1 \rangle, \langle \ell \rangle]_{\mathrm{KB}}$ contains the point $(2, \langle \ell \rangle)$ since every open neighborhood of $(2, \langle \ell \rangle)$ contains $\{1\} \times (\langle \ell, t \rangle, \langle \ell \rangle]_{\mathrm{KB}}$ for almost all $t \in \omega$. In particular, the closure of $f^{-1}[0, 4^{-1})$ contains $(2, \langle \ell \rangle)$. This shows that $f(2, \langle \ell \rangle) \leq 4^{-1}$. Since $f^{-1}[0, 4^{-1} + 4^{-2})$ is an open set containing the point $(2, \langle \ell \rangle)$, it also includes $\{3\} \times (\langle \ell, t \rangle, \langle \ell \rangle]_{\mathrm{KB}}$ for almost all $t \in \omega$. Then there is t_1 such that $\{3\} \times (\langle \ell, t_1 - 1 \rangle, \langle \ell, t_0 \rangle]_{\mathrm{KB}} \subseteq f^{-1}[0, 4^{-1} + 4^{-2})$. By the same argument as above, one can see that $(4, \langle \ell, t_1 \rangle)$ is contained in the closure of the above set, and thus $f(4, \langle \ell, t_1 \rangle) \leq 4^{-1} + 4^{-2}$. Continue this procedure. We eventually get a string $\sigma = \langle \ell, t_1, \dots, t_{\ell-1} \rangle$ such that $f(2\ell, \sigma) < 2^{-1}$. Since $2\ell > 2k$, by our choice of k, we get $3/4 < f(2\ell, \sigma) < 1/2$, a contradiction. □

We now begin to examine the degree structure of the product space $\mathcal{QR}^\omega$. The coded neighborhood basis of $z = (x_n, y_n) \in \mathcal{QR}^\omega$ is given as follows.

$$\begin{aligned}
&\{\langle 0, n, k, \sigma \rangle : x_n = 2k+1 \text{ and } y_n = \sigma, \text{ where } \sigma \text{ is a leaf}\} \\
\cup &\{\langle 1, n, k, \sigma j \rangle : |x_n - 2k| \leq 1 \text{ and } y_n \in (\sigma j, \sigma]_{\mathrm{KB}}\} \\
\cup &\{\langle 2, n, k \rangle : x_n > 2k\}.
\end{aligned}$$

From top to bottom, each set corresponds to basic open sets $\{2k+1\} \times U$, $[2k+1, 2k+3] \times U$, and $[2k+1, \infty] \times \mathcal{O}_{\omega^\omega}$, respectively, and one can see that $\{0,1\} \times U$ is absorbed into the second one.

Fix two symbols $\bot_0, \bot_1 \notin \omega$, and consider $\tilde{\omega} = \omega \cup \{\bot_0, \bot_1\}$. Then, consider the following *halfgraph* of a function $f : \omega \to \tilde{\omega}$:

$$\begin{aligned}
\mathrm{HalfGraph}(f) = &\{2\langle n, m \rangle : f(n) = 2m\} \\
&\cup \{2\langle n, m \rangle + 1 : f(n) \in \omega \text{ and } f(n) \geq 2m\}.
\end{aligned}$$

We say that a function $f : \omega \to \tilde{\omega}$ is *half-c.e.* if it has a c.e. halfgraph, that is, HalfGraph(f) is c.e. We also say that a function $f : \omega \to \tilde{\omega}$ is *computably dominated* if there is a partial computable function $g :\subseteq \omega \to \omega$ such that

$$(\forall n \in \omega)\ [f(n) \in \omega \implies n \in \mathrm{dom}(g) \text{ and } f(n) < g(n)].$$

Then, we consider the *extended halfgraph* of $f : \omega \to \tilde{\omega}$:

$$\begin{aligned}\mathrm{HalfGraph}^{+}(f) = &\{2\langle n,m\rangle : f(n) = \bot_0 \text{ or } f(n) \leq 2m\}\\ &\cup \{2\langle n,m\rangle + 1 : f(n) = \bot_1 \text{ or } f(n) \geq 2m\}.\end{aligned}$$

We say that $\mathbf{d}$ is *Roy halfgraph-above* if $\mathbf{d}$ contains a set S of the form

$$S = Y \oplus Y^{\mathsf{c}} \oplus \mathrm{HalfGraph}^{+}(f)$$

for some $Y \subseteq \omega$ and $f : \omega \to \tilde{\omega}$ such that f is half-c.e. and computably dominated relative to Y.

Proposition 3.35. *Every co-d-CEA e-degree is Roy halfgraph-above. Every Roy halfgraph-above e-degree is doubled co-d-CEA.*

Proof. Let $\mathbf{d}$ is a co-d-CEA degree. Then, $X \oplus X^{\mathsf{c}} \oplus (A \cup P) \in \mathbf{d}$ for some $X, A, P \subseteq \omega$ such that A^{c} and P are c.e. in X, and A and P are disjoint. Define $f(n) = \bot_0$ if $n \in A$; $f(n) = 0$ if $n \in P$; otherwise put $f(n) = 1$. One can check that f is half-c.e. in X since A^{c} and P are c.e. in X. Clearly, f is X-computably dominated. We claim that $X \oplus X^{\mathsf{c}} \oplus (A \cup P) \equiv_e X \oplus X^{\mathsf{c}} \oplus \mathrm{HalfGraph}^{+}(f)$. For the direction $\leq_e$, one can see that $n \in A \cup P$ iff $2\langle n,0\rangle \in \mathrm{HalfGraph}^{+}(f)$. For the converse direction, $2\langle n,0\rangle + 1 \in \mathrm{HalfGraph}^{+}(f)$ iff $n \notin A$, which is a X-c.e. condition. Moreover, whenever $m > 0$, we always have $2\langle n,m\rangle \in \mathrm{HalfGraph}^{+}(f)$ and $2\langle n,m\rangle + 1 \notin \mathrm{HalfGraph}^{+}(f)$. Hence, every co-$d$-CEA degree is Roy halfgraph-above.

For the second assertion, let $\mathbf{d}$ be a Roy halfgraph-above degree. Then, $\mathbf{d}$ contains a set of the form $Y \oplus Y^{\mathsf{c}} \oplus \mathrm{HalfGraph}^{+}(f)$, where f is Y-half-c.e. and Y-computably dominated. We define $A = \{n : f(n) = \bot_0\}$, $B = \{n : f(n) = \bot_1\}$, and given $I \subseteq \omega$, we also define $C_I = \{n : f(n) \in I\}$. We claim that $S = Y \oplus Y^{\mathsf{c}} \oplus \mathrm{HalfGraph}^{+}(f)$ is e-equivalent to the following set Q.

$$Q = Y \oplus Y^{\mathsf{c}} \oplus (A \cup C_{\{0\}}) \oplus \bigoplus_{k\in\omega}(B \cup C_{[2k,\infty)}) \oplus \bigoplus_{k\in\omega} C_{[2k,2k+2]}.$$

It is obvious that $n \in A \cup C_{\{0\}}$ iff $2\langle n,0\rangle \in \mathrm{HalfGraph}^{+}(f)$, and that $n \in B \cup C_{[2k,\infty)}$ iff $2\langle n,k\rangle \in \mathrm{HalfGraph}^{+}(f)$. To see $Q \leq_e S$, note that $n \in C_{[2k,2k+2]}$ iff $2\langle n,k\rangle + 1, 2\langle n,k+1\rangle \in \mathrm{HalfGraph}^{+}(f)$. Similarly, to see $S \leq_e Q$, one can see that $2\langle n,m\rangle \in \mathrm{HalfGraph}^{+}(f)$ iff $n \in A \cup C_{\{0\}}$ or $n \in C_{[2k,2k+2]}$ for some $k < m$. This verifies the claim.

We define $A_{2k} = A$, $B_{2k} = B$, $P_{2k} = C_{\{0\}} = \{n : f(n) = 0\}$, and $N_{2k} = C_{[2k,\infty)} = \{n : f(n) \geq 2k\}$. Note that $n \notin A_{2k} \cup B_{2k}$ iff there is m such that $2\langle n,m\rangle + 1 \in \mathrm{HalfGraph}(f)$. It is also easy to see that $n \in P_{2k}$ iff $2\langle n,0\rangle \in \mathrm{HalfGraph}(f)$, and that $n \in N_{2k}$ iff $2\langle n,k\rangle + 1 \in \mathrm{HalfGraph}(f)$. Since f is Y-half-c.e., the above shows that P_{2k}, N_{2k}, and $(A_{2k} \cup B_{2k})^{\mathsf{c}}$ are c.e. in Y. It is clear that A_{2k}, B_{2k}, P_{2k}, and N_{2k} are disjoint. Hence, $Z_{2k} = Y \oplus Y^{\mathsf{c}} \oplus (A_{2k} \cup P_{2k}) \oplus (B_{2k} \cup N_{2k})$ is doubled-co-d-CEA.

We now start to code $C_{[2k,2k+2]}$ in a set of a doubled co-d-CEA degree. Note that $C_{[2k,2k+2]}$ is 3-c.e. in Y, and hence it is e-equivalent to a set which is co-d-c.e. in Y (see Cooper [11]). For the sake of completeness (and to check uniformity of the proof) we explicitly write the coding procedure. Since f is Y-half-c.e., there is a uniform Y-computable enumeration of $(C_{[2k,\infty)})_{k\in\omega}$. We use the symbol $C_{[2k,\infty)}[s]$ to denote the set of elements enumerated into $C_{[2k,\infty)}$ by stage s. Note that $C_{[2k,\infty)}[s]$ is computable uniformly in k and s. Then, we define A_{2k+1} and P_{2k+1} as follows.

$$A_{2k+1} = \{\langle n,s\rangle : \text{either } n \notin C_{[2k,\infty)}[s] \text{ or } n \in A\cup B\cup C_{[0,2k+2)}\},$$
$$P_{2k+1} = \{\langle n,s\rangle : n \in C_{[2k,\infty)}[s] \text{ and } n \in C_{\{2k+2\}}\}.$$

We define $B_{2k+1} = N_{2k+1} = \emptyset$. Note that $n \in A_{2k+1}$ iff $n \in C_{[2k,\infty)}[s]$ or $2\langle n,k\rangle + 1 \notin \mathrm{HalfGraph}(f)$. Hence, A_{2k+1} is co-c.e. in Y. It is also easy to see that $n \in P_{2k+1}$ iff $n \notin C_{[2k,\infty)}[s]$ and $2\langle n,k+1\rangle \in \mathrm{HalfGraph}(f)$, which is a Y-c.e. condition. Therefore, $Z_{2k+1} = Y \oplus Y^{\mathsf{c}} \oplus (A_{2k+1} \cup P_{2k+1})$ is co-d-CEA.

We claim that Z_{2k+1} is e-equivalent to $Y \oplus Y^{\mathsf{c}} \oplus C_{[2k,2k+2]}$. Note that $n \in C_{[2k,\infty)}[s]$ ensures that $n \notin A\cup B$, and hence if $n \in C_{[2k,\infty)}[s]$, then the condition $n \in A_{2k+1}\cup P_{2k+1}$ is equivalent to that $n \in C_{[2k,2k+2]}$. Hence, $n \in C_{[2k,2k+2]}$ if and only if there is s such that $n \in C_{[2k,\infty)}[s]$ and $\langle n,s\rangle \in A_{2k+1}\cup P_{2k+1}$. For the converse direction, one can see that $\langle n,s\rangle \in A_{2k+1}\cup P_{2k+1}$ iff either $n \notin C_{[2k,\infty)}[s]$ or $n \in C_{[2k,2k+2]}$. Thus, we conclude that $\bigoplus_i Z_i$ is e-equivalent to Q.

We claim that $\bigoplus_i Z_i$ is also doubled co-d-CEA. To see this, consider $Z = Y \oplus Y^{\mathsf{c}} \oplus (\bigoplus_i A_i \cup \bigoplus_i P_i) \oplus (\bigoplus_i B_i \cup \bigoplus_i N_i)$. Note that $(\bigoplus_i A_i)\cup(\bigoplus_i B_i) = \omega\times(A_i\cup B_i)$ is co-c.e. in Y, and $\bigoplus_i A_i$, $\bigoplus_i B_i$, $\bigoplus_i P_i$, and $\bigoplus_i N_i$ are disjoint. Hence, Z is doubled co-d-CEA. It is easy to check that Z is e-equivalent to $\bigoplus_i Z_i$, which is also e-equivalent to S. Consequently, every Roy halfgraph-above degree is doubled-co-d-CEA. □

Theorem 3.36. *The $\mathcal{QR}^\omega$-degrees are exactly the Roy-halfgraph-above degrees, that is,*

$$\mathcal{D}_{\mathcal{QR}^\omega} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \textit{ is Roy-halfgraph-above}\}.$$

Proof. Assume that $z = (x_n,y_n)_{n\in\omega} \in \mathcal{QR}^\omega$ is given. To see that $\mathrm{Nbase}(z)$ has a Roy halfgraph-above degree, we define $f:\omega\to\tilde{\omega}$ as follows.

$$f(n) = \bot_0 \iff x_n = 0,$$
$$f(n) = \bot_1 \iff x_n = \infty,$$
$$f(n) = k \iff x_n = k+1 \iff y_n \in I_{k+1}.$$

Let Y be a set coding the second coordinate $(y_n)_{n\in\omega}$, which has a total degree. By definition, we have $f(n) = 2m$ iff $y_n \in I_{2m+1}$. This condition is c.e. in Y since I_{2m+1} is a computable set of isolated points (successor ordinals). We also note that $f(n) \in \omega$ and $f(n) \geq 2m$ iff $2m+1 \leq x_n < \omega$ iff, $|y_n| \geq m$, and, whenever y_n is a leaf, there is $\ell \geq m$ such that $y_n(\ell) > 0$. It is equivalent to saying that we see that, for some $\sigma \in \mathcal{O}_{\omega^\omega}$ with $|\sigma| \geq m$, the open set $(\sigma 0,\sigma]_{\mathrm{KB}}$ is a neighborhood of y_n. This shows that f is half-c.e. in Y. To see that f is Y-computably dominated, we define $g :\subseteq \omega\to\omega$ as follows.

$$g(n) = 2y_n(0) + 1.$$

Note that $y_n(0) = k$ iff $y_n \in (\langle k-1\rangle, \langle k\rangle]_{\mathrm{KB}}$. Hence, one can recover $y_n(0)$ from Y by a partial computable way. Obviously, $f(n) \in \omega$ if and only if $y_n(0)$ is defined. This verifies that g is Y-computable, and $\mathrm{dom}(g) = \{n : f(n) \in \omega\}$. One can see that $g(n) \leq 2k$ iff y_n extends $\langle j\rangle$ for some $j \leq k-1$. This implies that $|y_n| \leq k$. Note that $\sigma \in I_\ell$ for some $\ell > 2k$ implies that $|\sigma| > k$. Hence, $|y_n| \leq k$ implies $x_n \leq 2k$, that is, $f(n) \leq 2k-1$. Therefore, we have $f(n) < g(n)$ whenever $f(n) \in \omega$. Consequently, f is Y-computably dominated.

For $\mathrm{HalfGraph}^+(f) \leq_e \mathrm{Nbase}(z)$, it is easy to see the following.

$$\begin{aligned} 2\langle n,k\rangle \in \mathrm{HalfGraph}^+(f) &\iff x_n \leq 2k+1 \\ &\iff (\exists i \leq 1)(\exists \ell \leq k)(\exists \sigma)\ \langle i,n,\ell,\sigma\rangle \in \mathrm{Nbase}(z), \\ 2\langle n,k\rangle+1 \in \mathrm{HalfGraph}^+(f) &\iff x_n \geq 2k+1 \iff \langle 2,n,k\rangle \in \mathrm{Nbase}(z). \end{aligned}$$

To see that $Y \oplus \overline{Y} \leq_e \mathrm{Nbase}(z)$, if $x_n < \infty$ (that is, $\langle i,n,\ell,\sigma\rangle$ is enumerated into $\mathrm{Nbase}(z)$ for some i,n,ℓ,σ), then just follow the approximation of the second coordinate y_n. If we see $\langle 2,n,k\rangle \in \mathrm{Nbase}(z)$, then we know that $x_n > 2k$. As seen above, $x_n > 2k$ implies $|y_n| > k$, and thus $y_n \in (\langle k\rangle, \langle\rangle]_{\mathrm{KB}}$ is ensured. Hence, in any cases, we can recover a code $Y \oplus Y^{\mathsf{c}}$ of $(y_n)_{n\in\omega}$ from $\mathrm{Nbase}(z)$ in a uniform manner.

For $\mathrm{Nbase}(z) \leq_e S := Y \oplus Y^{\mathsf{c}} \oplus \mathrm{HalfGraph}^+(f)$, by monitoring an enumeration of Y, we can recover the second coordinate $(y_n)_{n\in\omega}$. Then, one can see the following.

$$\begin{aligned} \langle 0,n,k,\sigma\rangle \in \mathrm{Nbase}(z) &\iff y_n = \sigma \text{ and } (\forall i < 2)\ 2\langle n,k\rangle + i \in \mathrm{HalfGraph}^+(f), \\ \langle 1,n,k,\sigma j\rangle \in \mathrm{Nbase}(z) &\iff y_n \in (\sigma j, \sigma]_{\mathrm{KB}} \\ &\qquad \text{and } 2\langle n,k\rangle, 2\langle n,k-1\rangle+1 \in \mathrm{HalfGraph}^+(f), \\ \langle 2,n,k\rangle \in \mathrm{Nbase}(z) &\iff 2\langle n,k\rangle+1 \in \mathrm{HalfGraph}^+(f). \end{aligned}$$

Hence, we obtain $S \equiv_e \mathrm{Nbase}(z)$, and conclude that every $\mathcal{QR}^\omega$-degree is Roy-halfgraph-above.

Now, given such $S = Y \oplus Y^{\mathsf{c}} \oplus \mathrm{HalfGraph}^+(f)$, we define a point $z \in \mathcal{QR}^\omega$ such that $\mathrm{Nbase}(z) \equiv_e S$. We first define $z_{2n+1} = (1, \langle 1,1\rangle)$ if $n \in Y$; otherwise $z_{2n+1} = (1, \langle 1,0\rangle)$. It is clear that $Y \oplus Y^{\mathsf{c}}$ is e-equivalent to the coded neighborhood basis of $(z_{2n+1})_{n\in\omega}$ in the Roy space $\mathcal{QR}^\omega$.

We now describe how to define $z_{2n} = (x_n, y_n)$. Given $n \in \omega$, we begin with $y_n[0] = \langle\rangle$. We wait until we see $g(n)$ (witnessing computable domination) is defined, say $g(n) = 2k-1$. In this case, we have $f(n) < 2k-1$. Then, we declare that $y_n \in (\langle k-1\rangle, \langle k\rangle]_{\mathrm{KB}}$, that is, $y_n(0) = k$. If we see $2\langle n,1\rangle + 1 \in \mathrm{HalfGraph}(f)$, that is, $f(n) \geq 2$, at stage s_1, then we declare that $y_n \in (\langle k, s_1-1\rangle, \langle k, s_1\rangle]_{\mathrm{KB}}$, that is, $y_n(1) = s_1$. Continue this procedure. Assume that we have already seen $f(n) \geq 2m$, and thus $y_n[s_m] = \sigma = \langle k, s_1, \dots, s_m\rangle$. If we see $f(n) \geq 2m+2$ at stage s_{m+1}, then we declare $y_n \in (\sigma^\frown(s_{m+1}-1), \sigma^\frown s_{m+1}]_{\mathrm{KB}}$, that is, $y_n(m+1) = s_{m+1}$. If we see $f(n) = 2m$ at some stage t_m, then we declare $y_n = \sigma^\frown t_m 0\dots 0 \in \mathcal{O}^{\mathrm{leaf}}_{\omega^\omega}$, where we can assume that $t_m > 0$. If $f(n) \neq 2m$ for any m, then, since $f(n) < 2k-1$, our construction ensures that $|y_n| \leq k$, and hence $y_n \in \mathcal{O}_{\omega^\omega}$. Note that this procedure gives us an $\mathcal{O}_{\omega^\omega}$-name of y_n in a Y-computable manner. If y_n is nonempty, let x_n be the unique x such that $y_n \in I_x$. If y_n is empty and $f(n) = \bot_0$, define $x_n = 0$; otherwise put $x_n = \infty$.

We claim that if $f(n) \in \omega$, then $x_n = f(n)+1$. If $f(n) = 2m$, then we see $f(n) = 2m$ at some stage, and our construction ensures that y_n is of the form $\sigma^\frown t_m{}^\frown 0\dots 0 \in \mathcal{O}^{\mathrm{leaf}}_{\omega^\omega}$, where $|\sigma| = m+1$ and $t_m > 0$. Hence, $y_n(m+1) > 0$ and $y_n(k) = 0$ for any $k > m+1$. This means that $y_n \in I_{2m+1}$, and hence $x_n = 2m+1 = f(n)+1$. If $f(n) = 2m+1$, then we see $f(n) \geq 2m$ at some stage, and neither $f(n) \geq 2m+2$ nor $f(n) = 2m$ happens. Hence, our construction ensures that y_n is a string of length $m+1$. Also, we have $f(n) = 2m+1 < 2k-1$, so $m+1 < k = y_n(0) < |y_n|$; hence y_n is not a leaf. This means that $y_n \in I_{2m+2}$, and hence $x_n = 2m+2 = f(n)+1$.

We define $z_{2n} = (x_n, y_n)$ for each $n \in \omega$. As mentioned above, the coded neighborhood basis of the second coordinate of $(y_n)_{n\in\omega}$ in the product ordinal space $(\mathcal{O}_{\omega^\omega})^\omega$ (that is, the Y constructed from $(x_n, y_n)_{n\in\omega}$ as in the second paragraph of this proof) is e-reducible to $Y \oplus Y^{\mathsf{c}}$. Moreover, the function f satisfies the equations mentioned in the first paragraph in this proof (where $z_{2n} = (x_n, y_n)$). Hence, the above argument shows that $S \equiv_e \mathrm{Nbase}(z)$ as desired. □

3.7. **Degrees of points: submetrizable topology.** We say that a space is *submetrizable* if it admits a continuous metric, that is, either it is metrizable or it has a coarser metrizable topology (see [34]). Every submetrizable space is completely Hausdorff, and every completely Hausdorff space is $T_{2.5}$:

$$\text{submetrizable} \implies \text{completely Hausdorff} \implies T_{2.5}.$$

The Arens square and Roy's lattice space are not completely Hausdorff, and hence, not submetrizable. Extending the topology of a metrizable space always gives us a submetrizable space which is not necessarily metrizable.

3.7.1. *Extension topology.* One of the most basic constructions to obtain a non-metrizable, completely Hausdorff topology is extending a metrizable topology by adding new open sets. Concretely speaking, given a space $\mathcal{X}$ with a metrizable topology $\tau_{\mathcal{X}}$, choose a collection $\mathcal{U}$ of subsets of $\mathcal{X}$, and consider the topology generated by $\tau_{\mathcal{X}} \cup \mathcal{U}$. We denote the obtained space by $\mathcal{X}_{\mathcal{U}}$. By definition, $\mathcal{X}_{\mathcal{U}}$ is submetrizable (hence, completely Hausdorff, and $T_{2.5}$); however $\mathcal{X}_{\mathcal{U}}$ is not necessarily metrizable. In this article, since we are only interested in a second-countable topology, we always assume that $\mathcal{U}$ is countable.

Example 3.37. Let us begin with Cantor space 2^ω. Let $\mathcal{U} = (U_e)_{e\in\omega}$ be a countable collection of subsets of 2^ω. This induces a representation of the extension topology on 2^ω induced by $\mathcal{U}$, and for any point $x \in (2^\omega)_{\mathcal{U}}$, its coded neighborhood filter is given as:

$$\mathrm{Nbase}(x) = \{\langle 0, \sigma\rangle : \sigma \prec x\} \cup \{\langle 1, e\rangle : x \in U_e\}.$$

Let Γ be a countable collection of subsets of $\omega \times 2^\omega$. We say that a set $A \subseteq \omega$ is Γ *relative to* $X \in 2^\omega$ (written $A \in \Gamma^X$) if there is $G \in \Gamma$ such that $A = \{n : \langle n, X\rangle \in G\}$. We also say that an e-degree $\mathbf{d}$ is Γ-*above* if $A \oplus X \oplus X^{\mathsf{c}} \in \mathbf{d}$ for some $A, X \subseteq \omega$ such that A is Γ relative to X.

Let β be a representation of a space $\mathcal{X}$, and let γ be a countable collection of (not necessarily open) subsets of $\mathcal{X}$. We say that γ *computably extends* β if there is a c.e. set W such that $\beta_e = \bigcup\{\gamma_d : \langle d, e\rangle \in W\}$. Then, γ is a representation of $\mathcal{X}_\gamma$ (the result of adding sets from γ to the original space $\mathcal{X}$ as new open sets).

Observation 3.38. *If γ computably extends β, then $\mathrm{Nbase}_\beta(x) \leq_e \mathrm{Nbase}_\gamma(x)$ for any $x \in \mathcal{X}$.*

Proof. This is because $e \in \mathrm{Nbase}_\beta(x)$ if and only if there is d such that $\langle d, e\rangle \in W$ and $d \in \mathrm{Nbase}_\gamma(x)$. □

Let λ be the canonical representation of Cantor space, that is, λ_e is the set of all extensions of the e-th binary string. We characterize Γ-above e-degrees in terms of extension topology.

Proposition 3.39. *The following are equivalent for a collection $\mathcal{C}$ of e-degrees:*

(1) *There is* β *computably extending* λ *such that* $\mathcal{C} = \mathcal{D}_{(2^\omega)_\beta}$.
(2) *There is a countable collection* Γ *of subsets of* 2^ω *such that*

$$\mathcal{C} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \text{ is } \Gamma\text{-above}\}.$$

Proof. Assume that β computably extends λ. Then, define $G = \{\langle e, x\rangle : x \in \beta_e\}$ and $\Gamma = \{G\}$. Given x whose e-degree is in $\mathcal{C}$, define $A = \{n : \langle n, x\rangle \in G\}$, which is clearly Γ relative to x. Note that

$$e \in A \iff \langle e, x\rangle \in G \iff e \in \mathrm{Nbase}_\beta(x).$$

Thus, by the above equivalence and Observation 3.38, clearly $x \oplus x^{\mathsf{c}} \oplus A \equiv_e \mathrm{Nbase}_\beta(x)$, and thus, $\mathrm{Nbase}_\beta(x)$ is Γ-above. Conversely, if $\mathbf{d}$ is Γ-above, then there are $x \in 2^\omega$ and $A \in \Gamma^x$ such that $x \oplus x^{\mathsf{c}} \oplus A \in \mathbf{d}$. Since $\Gamma = \{G\}$, we must have $A = \{n : \langle n, x\rangle \in G\}$. As in the previous argument, one can see that $x \oplus x^{\mathsf{c}} \oplus A \equiv_e \mathrm{Nbase}_\beta(x)$.

To show the converse direction, fix a countable collection $\Gamma = (G_e)_{e\in\omega}$. Define β by $\beta_{\langle 0,e\rangle} = \lambda_e$ and $\beta_{\langle 1,e,n\rangle} = \{\langle e, y\rangle : \langle n, y\rangle \in G_e\}$. Clearly, β computably extends λ. For each $x \in (2^\omega)_\beta$, let e be the first entry of x, that is, $x = \langle e, y\rangle$. Define $A_e = \{n : \langle n, y\rangle \in G_e\}$. Then, A_e is Γ relative to y, and therefore $y \oplus y^{\mathsf{c}} \oplus A_e$ is Γ-above. Note that

$$n \in A_e \iff \langle n, y\rangle \in G_e \iff \langle 1, e, n\rangle \in \mathrm{Nbase}_\beta(\langle e, y\rangle) = \mathrm{Nbase}_\beta(x).$$

By the above equivalence and Observation 3.38, we have that $x \oplus x^{\mathsf{c}} \oplus A_e \leq_e \mathrm{Nbase}_\beta(x)$. Moreover, for any $d \neq e$ and $n \in \omega$, $\langle 1, d, n\rangle \notin \mathrm{Nbase}_\beta(x)$, and the 0-th section of $\mathrm{Nbase}_\beta(x)$ is obviously e-equivalent to $x \oplus x^{\mathsf{c}}$. Hence, $x \oplus x^{\mathsf{c}} \oplus A_e \equiv_e \mathrm{Nbase}_\beta(x)$. Since $x \oplus x^{\mathsf{c}} \oplus A_e$ is clearly e-equivalent to $y \oplus y^{\mathsf{c}} \oplus A_e$, this shows that $\mathrm{Nbase}_\beta(x)$ is Γ-above.

Conversely, if $\mathbf{d}$ is Γ-above, then there are $y \in 2^\omega$ and $A \in \Gamma^y$ such that $y \oplus y^{\mathsf{c}} \oplus A \in \mathbf{d}$. Then there is e such that $A = \{n : \langle n, y\rangle \in G_e\}$. Consider $x = \langle e, y\rangle$. As in the previous argument, one can see that $y \oplus y^{\mathsf{c}} \oplus A \equiv_e \mathrm{Nbase}_\beta(x)$. □

3.7.2. *A horrible behavior of extension topology.* In this section, we examine the notion of a $\mathcal{T}$-quasi-minimal e-degree for a collection $\mathcal{T}$ of represented cb_0 spaces, and try to construct such a degree.

For instance, on the one hand, in Section 5.1.3, we will see that if $\mathcal{T}$ is a countable collection of T_1 spaces, then there is a $\mathcal{T}$-quasi-minimal e-degree. On the other hand, we will show that if $\mathcal{T}$ is the collection of all decidable T_1 spaces, there is no $\mathcal{T}$-quasi-minimal e-degree (Theorem 3.40). At first glance, this looks paradoxical. — When we talk about an effective version of some notion $\mathcal{P}$, we often implicitly assume that there are only countably many objects which are effectively-$\mathcal{P}$. However, it is, strictly speaking, sometimes incorrect. There is a possibility of the existence of uncountably many effective objects.

Consider partial computable functions on Baire space. Any restriction $f \upharpoonright A$ of a partial computable function $f :\subseteq \omega^\omega \to \omega^\omega$ is also partial computable, and therefore, there are uncountably many partial computable functions. Fortunately, every partial computable function is merely a restriction of a partial computable function with a Π^0_2 domain, and there are only countably many such functions. Hence, essentially, we only need to deal with countably many partial computable functions.

Can we say the same thing about computable or decidable T_i-spaces? Is there a countable collection $(\mathcal{X}_n)_{n\in\omega}$ of represented T_i-spaces such that every computable cb_0 space embeds into $\mathcal{X}_n$ for some $n \in \omega$? Of course, it is true if $i = 0$ since there is a universal decidable cb_0 space, $\mathbb{S}^\omega$ say, or if $i = 3$ since there is a universal decidable metric space, $[0,1]^\omega$ say. Contrary to these cases, we will show that the answer is "no," for any $i \in \{1, 2, 2.5\}$.

In this section, we show that if $\mathcal{T}$ is a countable collection of T_1 spaces, then there is a $\mathcal{T}$-quasi-minimal e-degree, whereas if $\mathcal{T}$ is a collection of all decidable T_1 spaces, there is no $\mathcal{T}$-quasi-minimal e-degree.

Non-existence of universal spaces. Recall that every metrizable space is submetrizable, and every submetrizable space is $T_{2.5}$. By using an argument of extending topology, we show that every e-degree is realized as the degree of a point in a decidable submetrizable space. In particular, every e-degree is the degree of a point of a decidable $T_{2.5}$ space.

Theorem 3.40. *Every e-degree is an $\mathcal{X}$-degree for some decidable, submetrizable, cb_0 space $\mathcal{X}$, that is,*

$$\mathcal{D}_e = \bigcup\{\mathcal{D}_\mathcal{X} : \mathcal{X} \text{ is a decidable, submetrizable, } cb_0 \text{ space}\}.$$

Proof. Note that by Theorem 5.14, the enumeration degrees are not covered by countably many T_1-spaces. Therefore, the above equivalence must be realized by an *uncountable* union. As a corollary, there are uncountably many decidable submetrizable spaces in an essential sense, that is, for any countably many submetrizable (indeed T_1) spaces $(\mathcal{X}_i)_{i\in\omega}$, there is a decidable submetrizable space $\mathcal{Y}$ that cannot be embedded into $\mathcal{X}_i$ for any $i \in \omega$.

To prove Theorem 3.40, given an enumeration degree $\mathbf{d}$, we will construct an decidable submetrizable space $\mathcal{X}_\mathbf{d}$ and a point $x \in \mathcal{X}$ such that the enumeration degree of $\mathrm{Nbase}(x)$ is exactly $\mathbf{d}$. Moreover, if $\mathbf{d}$ is Δ^0_n with $n \geq 4$, $\mathcal{X}_\mathbf{d}$ can be strongly Π^0_n-named (see below for the definition).

Construction. Given a topological space $(\mathcal{X}, \tau_X)$ and a set $D \subseteq \mathcal{X}$ let $\mathcal{X}_D$ be the *extension topology* of $\mathcal{X}$ plus D, that is, the topological space with the underlying set $\mathcal{X}$ and the topology generated by $\tau_X \cup \{D\}$. If $\mathcal{X}$ is submetrizable, then so is the D-extension $\mathcal{X}_D$. However, the D-extension $\mathcal{X}_D$ is not necessarily metrizable even if $\mathcal{X}$ is.

Consider the ω-power $(\mathcal{B}_D)^\omega$ of the D-extension of Baire space $\mathcal{B} := \omega^\omega$. Note that an open subbasis of $(\mathcal{B}_D)^\omega$ is given by $B_{0,n,\sigma} = \{x \in (\mathcal{B}_D)^\omega : x(n) \succ \sigma\}$ and $B_{1,n} = \{x : x(n) \in D\}$. Therefore, the coded neighborhood filter of $x \in (\mathcal{B}_D)^\omega$ is given as follows:

$$\mathrm{Nbase}(x) = \{\langle 0, n, \sigma\rangle : \sigma \prec x(n)\} \cup \{\langle 1, n\rangle : x(n) \in D\}.$$

Observation 3.41. *If D is dense and co-dense, then $(\mathcal{B}_D)^\omega$ is a decidable, submetrizable, cb_0 space.*

Proof. If D is dense and co-dense, given $\sigma, \tau \in \omega^{<\omega}$, we have that $[\sigma] \not\subseteq D \cap [\tau]$, and that $[\sigma] \cap (D \cap [\tau]) = \emptyset$ if and only if $\sigma \perp \tau$. This gives a decidable basis of $(\mathcal{B}_D)^\omega$. □

We now describe how we extend a metric topology to code a given e-degree. Let $\mathbb{Q} \subseteq \omega^\omega$ be the set of all infinite binary strings x such that $x(n) = 0$ for almost all n. Define $\mathbb{Q}^- = \mathbb{Q} \setminus \{0^\omega\}$. Given $A \subseteq \omega$, define

$$D_A = \{n^\frown x \in \omega^\omega : [n \in A \text{ and } x \notin \mathbb{Q}^-] \text{ or } [n \notin A \text{ and } x \notin \mathbb{Q}]\}.$$

It is clear that D_A is dense and co-dense. Note that $\mathbb{Q} \cap D_A = \{n^\frown 0^\omega : n \in A\}$. We show that every e-degree is realized in the space of the form $(\mathcal{B}_{D_A})^\omega$.

Given $D \subseteq \omega^\omega$ and $x \in (\mathcal{B}_D)^\omega$, define $X = \{\langle k, n, m\rangle : x(k)(n) = m\}$. Then, it is not hard to see the following.

$$\mathrm{Nbase}(x) \equiv_e X \oplus X^{\mathsf{c}} \oplus \{n \in \omega : x(n) \in D\}.$$

Given an e-degree $\mathbf{d}$, choose $A \in \mathbf{d}$. By Observation 3.41, $(\mathcal{B}_{D_A})^\omega$ is a decidable, submetrizable, cb_0 space since D_A is dense and co-dense. Define $x(n) = n^\frown 0^\omega$. Then clearly, $x(n) \in D_A$ if and only if $n \in A$. Moreover, since X and X^{c} are c.e., we have the following.

$$\mathrm{Nbase}(x) \equiv_e X \oplus X^{\mathsf{c}} \oplus \{n \in \omega : x(n) \in D_A\} \equiv_e A.$$

Thus, by putting $\mathcal{X}_{\mathbf{d}} = (\mathcal{B}_{D_A})^\omega$, this verifies our claim. □

We will give a more detailed analysis of Theorem 3.40. Recall from Proposition 2.6 that a represented space is $\mathbf{\Pi}^0_2$-named if and only if it is quasi-Polish. We will also use a stronger naming condition. Consider the following sets:

$$\mathrm{Sup}(\mathcal{X}) = \{p \in \omega^\omega : (\exists x \in \mathcal{X})\ \mathrm{Nbase}(x) \subseteq \mathrm{rng}(p)\},$$
$$\mathrm{Sub}(\mathcal{X}) = \{p \in \omega^\omega : (\exists x \in \mathcal{X})\ \mathrm{rng}(p) \subseteq \mathrm{Nbase}(x)\}.$$

We always have $\mathrm{Name}(\mathcal{X}) \subseteq \mathrm{Sup}(\mathcal{X}) \cap \mathrm{Sub}(\mathcal{X})$. Moreover,

$$\mathcal{X} \text{ is } T_1 \implies \mathrm{Name}(\mathcal{X}) = \mathrm{Sup}(\mathcal{X}) \cap \mathrm{Sub}(\mathcal{X}).$$

We say that $\mathcal{X}$ is *strongly* Γ*-named* if there are Γ sets P, N such that

$$\mathrm{Sub}(\mathcal{X}) \subseteq N,\ \mathrm{Sup}(\mathcal{X}) \subseteq P, \text{ and } \mathrm{Name}(\mathcal{X}) = P \cap N.$$

For instance, one can easily see that Baire space ω^ω is strongly Π^0_2-named, and the telophase space $(\hat{\omega}_{TP})^\omega$ is strongly Π^0_3-named.

Proposition 3.42. *Let $n \geq 4$. If $\mathbf{d}$ is an e-degree of a Δ^0_n set, then there is a decidable, strongly Π^0_n-named, submetrizable, cb_0 space $\mathcal{X}_{\mathbf{d}}$ such that $\mathbf{d}$ is an $\mathcal{X}_{\mathbf{d}}$-degree.*

Proof. We introduce ad-hoc technical notions. A countable set $E \subseteq \omega^\omega$ is (Λ, Γ)-*enumerable* if there is a sequence $(r_e)_{e\in\omega}$ of reals such that

$$E \subseteq \{r_e : e \in \omega\},\ \{(e, \sigma) : r_e \succ \sigma\} \in \Lambda, \text{ and } \{e : r_e \in E\} \in \Gamma.$$

We also says that a countable set $E \subseteq \omega^\omega$ is *strongly* (Δ^0_m, Δ^0_n)*-enumerable* if it is (Δ^0_m, Δ^0_n)-enumerable, and moreover it satisfies the following condition:

$$(\forall S \subseteq \omega)\ [S \in \Pi^0_2 \implies \{n \in \omega : (\exists e)\ [r_e \in E \text{ and } \langle e, n\rangle \in S]\} \in \Delta^0_n].$$

For instance, $\mathbb{Q}$ is (Δ^0_1, Σ^0_2)-enumerable, and strongly (Δ^0_1, Δ^0_4)-enumerable. Here, recall that $\mathbb{Q} \subseteq \omega^\omega$ is the set of all infinite binary strings x such that $x(n) = 0$ for almost all n.

Lemma 3.43. *For any $k \geq 4$, if $A \in \Delta^0_k$ then $\omega^\omega \setminus D_A$ is strongly (Δ^0_1, Δ^0_k)-enumerable.*

Proof. Define $r_{n,e} = n^\frown \sigma_e{}^\frown 0^\omega$, where σ_e is the e-th finite string. We note that

$$\omega^\omega \setminus D_A = \mathbb{Q} \setminus \{n^\frown 0^\omega : n \in A\},$$

and therefore, we have $\omega^\omega \setminus D_A \subseteq \mathbb{Q} = \{r_{n,e} : n, e \in \omega\}$. Moreover, we have that

$$r_{n,e} \notin D_A \iff n \notin A \text{ or } (\exists s < |\sigma_e|)\ \sigma_e(s) \neq 0.$$

This condition is Δ^0_k, and thus, $\omega^\omega \setminus D_A$ is co-(Δ^0_1, Δ^0_k)-enumerable. Moreover, given a set $S \in \Pi^0_2$,

$$\begin{aligned}(\exists e)\ [r_{n,e} \notin D_A \text{ and } \langle e,n\rangle \in S] \iff & [n \in A \text{ and } (\exists e)[r_{n,e} \in \mathbb{Q}^- \text{ and } \langle e,n\rangle \in S] \\ & \text{and } [n \notin A \text{ and } (\exists e)[r_{n,e} \in \mathbb{Q} \text{ and } \langle e,n\rangle \in S]].\end{aligned}$$

Clearly, this condition is Δ^0_k since $A \in \Delta^0_k$ and $k \geq 4$. □

Lemma 3.44. *For any $n \geq 4$ and $m \leq n-2$, if D^{c} is strongly (Δ^0_m, Δ^0_n)-enumerable, then $(\mathcal{B}_D)^\omega$ is strongly Π^0_n-named.*

Proof. Let $(r_e)_{e\in\omega}$ witness that D^{c} is strongly (Δ^0_m, Δ^0_n)-enumerable. We define the predicate $p \in P$ as follows:

$$\begin{aligned}&(\forall n)(\forall \ell)(\exists \sigma \in \omega^\ell)\ \langle 0,n,\sigma\rangle \in \mathrm{rng}(p),\\ \text{and } &(\forall n)\ [\langle 1,n\rangle \notin \mathrm{rng}(p) \implies (\exists e)\ r_e \notin D \text{ and } (\forall \sigma \prec r_e)\ \langle 0,n,\sigma\rangle \in \mathrm{rng}(p)].\end{aligned}$$

The first line says that p extends the Baire name of a point $x \in \mathcal{B}^\omega$, and the second line says that if p does not enumerate $\langle 1,n\rangle$, then some of such x satisfies that $x(n) \notin D$. To see that $\mathrm{Sup}((\mathcal{B}_D)^\omega) \subseteq P$, fix $p \in \mathrm{Sup}((\mathcal{B}_D)^\omega)$, that is, p extends a $(\mathcal{B}_D)^\omega$-name q. We show that p satisfies the contrapositive of the second line in the definition of $p \in P$. Assume that any point x whose Baire name is extended by p satisfies $x(n) \in D$. Then the unique point x coded by q must satisfy $x(n) \in D$ since p extends q and therefore extends the Baire name of x. Then q must enumerate $\langle 1,n\rangle$, and so does p.

We next define the predicate $p \in N$ as follows:

$$\begin{aligned}&(\forall n)(\forall \sigma,\tau)\ [\sigma \perp \tau \implies \langle 0,n,\sigma\rangle \notin \mathrm{rng}(p) \text{ or } \langle 0,n,\tau\rangle \notin \mathrm{rng}(p)],\\ \text{and } &(\forall n)\ [(\langle 1,n\rangle \in \mathrm{rng}(p) \text{ and } (\exists^\infty \sigma)\ \langle 0,n,\sigma\rangle \in \mathrm{rng}(p))\\ &\qquad \implies (\forall e)[r_e \notin D \to (\exists \sigma \prec r_e)(\exists \tau)\ \tau \perp \sigma \text{ and } \langle 0,n,\tau\rangle \in \mathrm{rng}(p)].\end{aligned}$$

The first line says that p does not enumerate two incomparable strings for each coordinate. Note that, in this case, p generates a sequence $x^p = (x^p(n))_{n\in\omega} \in \omega^{\leq\omega}$. The second and third lines say that if p enumerates $\langle 1,n\rangle$ and such $x^p(n)$ is an infinite string, then $x^p(n) \in D$. Note that every $p \in \mathrm{Sub}((\mathcal{B}_D)^\omega)$ satisfies this condition. Otherwise, p enumerates $\langle 1,n\rangle$, $x^p(n) \in \omega^\omega$ is determined but $x^p(n) \notin D$. Thus, if q is a $(\mathcal{B}_D)^\omega$-name extending p, we must have $x^q(n) \notin D$, and then q never enumerates $\langle 1,n\rangle$, which is impossible. Hence, we get that $\mathrm{Sub}((\mathcal{B}_D)^\omega) \subseteq N$.

It is not hard to check that $P \in \Pi^0_n$ since D^{c} is strongly (Δ^0_m, Δ^0_n)-enumerable. It is also straightforward to see that $N \in \Pi^0_n$ since D^{c} is (Δ^0_m, Δ^0_n)-enumerable. Finally, we claim that $P \cap N \subseteq \mathrm{Name}((\mathcal{B}_D)^\omega)$. The first lines in the definitions of P and N say that p determines $x^p = (x^p(n))_{n\in\omega} \in (\omega^\omega)^\omega$. Then the second line of P ensures that if p does not enumerate $\langle 1,n\rangle$ then $x^p(n) \notin D$. Conversely, the second and third lines of N ensures that if p enumerates $\langle 1,n\rangle$ then $x^p(n) \in D$. This verifies our claim. □

Given $D \subseteq \omega^\omega$ and $x \in (\mathcal{B}_D)^\omega$, define $X = \{\langle k,m\rangle : x(k) = m\}$. Then, it is not hard to see the following.

$$\mathrm{Nbase}(x) \equiv_e X \oplus X^{\mathsf{c}} \oplus \{n \in \omega : x(n) \in D\}.$$

Let $\mathbf{d}$ be a Δ^0_n-enumeration degree for $n \geq 4$, and choose $A \in \mathbf{d}$. By Lemma 3.43, D_A is strongly co-(Δ^0_1, Δ^0_n)-enumerable. Therefore, by Lemma 3.44, $(\mathcal{B}_{D_A})^\omega$ is strongly

Π^0_n-named. Define $x_n = n^\frown 0^\omega$. Then clearly, $x_n \in D_A$ if and only if $n \in A$. Moreover, since X and X^{c} are c.e., we have the following.

$$\mathrm{Nbase}(x) \equiv_e X \oplus X^{\mathsf{c}} \oplus \{n \in \omega : x(n) \in D_A\} \equiv_e A.$$

Thus, by putting $\mathcal{X}_{\mathbf{d}} = (\mathcal{B}_{D_A})^\omega$, this verifies our claim. □

In particular, there is a quasi-minimal e-degree realized in a strongly Π^0_4-named submetrizable space since there is a 3-c.e. (hence Δ^0_2) quasi-minimal e-degree (see Proposition 3.50). Indeed, Proposition 3.50 shows that a quasi-minimal e-degree can be realized in a strongly Π^0_3-named submetrizable space.

3.7.3. *Gandy-Harrington topology.* We now consider a special kind of extension topology. Let GH_e be the e-th Σ^1_1 subset of Baire space ω^ω. The extension topology τ_{GH} on ω^ω generated by $GH = (GH_e)_{e\in\omega}$ is known as the *Gandy-Harrington topology.* This topology is known to have a number of applications in descriptive set theory and related areas (e.g. [47, 37]).

We always assume that the Gandy-Harrington space $(\omega^\omega)_{GH} = (\omega^\omega, \tau_{GH})$ is represented by GH, and then, for any point $x \in (\omega^\omega)_{GH}$, its coded neighborhood filter is given as:

$$\mathrm{Nbase}(x) = \{e \in \omega : x \in GH_e\}.$$

It is clear that every $(\omega^\omega)_{GH}$-degree is Σ^1_1-above, but be careful that the converse is not true. We now assume that λ is the canonical representation of Baire space ω^ω.

Proposition 3.45. *For every $x \in \omega^\omega$ and $\alpha < \omega_1^{\mathrm{CK},x}$ (that is, α is an x-computable ordinal), we have the following inequalities:*

$$\mathrm{Nbase}_\lambda(x^{(\alpha)}) \leq_e \mathrm{Nbase}_{GH}(x) \leq_e \mathrm{Nbase}_\lambda(x^{\mathrm{HJ}}),$$

where $x^{(\alpha)}$ denotes the α-th Turing jump of x, and x^{HJ} denotes the hyperjump of x.

Proof. First it is easy to see $\mathrm{Nbase}_{GH}(x) \leq_e \mathrm{Nbase}_\lambda(x^{\mathrm{HJ}})$ since the hyperjump of x determines whether the e-th Σ^1_1 set contains x or not. To see $\mathrm{Nbase}_\lambda(x^{(\alpha)}) \leq_e \mathrm{Nbase}_{GH}(x)$, given e, one can effectively find Σ^1_1 indices p_e and n_e of a $\Sigma^0_{1+\alpha}$ set $\{y \in \omega^\omega : y^{(\alpha)}(e) = 1\}$ and a $\Pi^0_{1+\alpha}$ set $\{y \in \omega^\omega : y^{(\alpha)}(e) = 0\}$. Then for each e either p_e or n_e is enumerated into $\{e : x \in GH_e\}$. By waiting for either one to occur, one can make an enumeration procedure witnessing $\mathrm{Nbase}_\lambda(x^{(\alpha)}) \leq_e \mathrm{Nbase}_{GH}(x)$. □

Recall that an e-degree is said to be *continuous* if it is an $\mathcal{H}$-degree, where $\mathcal{H}$ is Hilbert cube $[0,1]^{\mathbb{N}}$ with the canonical representation. Concretely speaking, the coded neighborhood filter of $x = (x(n))_{n\in\omega} \in \mathcal{H}$ is:

$$\mathrm{Nbase}_{\mathcal{H}}(x) = \{\langle n, s, p\rangle : |x(n) - p| < 2^{-s}\},$$

where n and s range over ω, and p ranges over $\mathbb{Q} \cap [0,1]$.

Theorem 3.46. *No $(\omega^\omega)_{GH}$-degree is continuous.*

Proof. Suppose that $\mathrm{Nbase}_{\mathcal{H}}(z) \leq_e \mathrm{Nbase}_{GH}(x)$ for $z \in \mathcal{H}$. Then there is a c.e. set Ψ such that $\langle n, s, p\rangle \in \mathrm{Nbase}_{\mathcal{H}}(z)$ if and only if $(n, s, p, D) \in \Psi$ for some finite set $D \subseteq \mathrm{Nbase}_{GH}(x)$. Let L_s be the set of all $\langle n, t, p\rangle$ such that $t \geq s$ and

$$(\forall (m, u, q, D) \in \Psi)\ [(m = n \text{ and } D \subseteq \mathrm{Nbase}_{GH}(x)) \ \to\ |q - p| < 2^{-t} + 2^{-u}].$$

In other words, the diameter of the ball $B_{t,p} = \{y \in [0,1] : |y - p| < 2^{-t}\}$ determined by $\langle n, t, p\rangle$ is less than 2^{-s}, and the ball $B_{t,p}$ must intersect with any ball enumerated by $\Psi^{\mathrm{Nbase}_{GH}(x)}$ at the n-th coordinate.

Note that $\mathrm{Nbase}_{GH}(x)$ is a $\Sigma^1_1(x)$ subset of ω. Therefore, L_s is a $\Pi^1_1(x)$ subset of ω uniformly in s, and clearly nonempty. We claim that $z(n) \in \overline{B_{t,p}}$ for any $\langle n, t, p\rangle \in L_s$. To see this, let V be an arbitrary open neighborhood of $z(n)$. Then, there is v and q such that $z(n) \in B_{v,q} \subseteq V$. Since $\langle n, v, q\rangle \in \mathrm{Nbase}_{\mathcal{H}}(z)$, $\Psi^{\mathrm{Nbase}_{GH}(x)}$ enumerates $\langle n, v, q\rangle$, and then, as mentioned above, $B_{t,p}$ intersects with such $B_{v,q}$. Therefore, $B_{t,p}$ intersects with any open neighborhood of $z(n)$, that is, $z(n) \in \overline{B_{t,p}}$; hence $z(n) \in B_{t-1,p}$.

Since L_s is $\Pi^1_1(x)$ uniformly in s, by uniformization (see [63, Theorem II.2.3]), there is a $\Pi^1_1(x)$ total function $\langle n, s\rangle \mapsto h(n, s)$ such that $\langle n, s, h(n, s)\rangle \in L_s$. By totality, h is $\Delta^1_1(x)$. Thus, we obtain a $\Delta^1_1(x)$-sequence $(B_{s-1,h(n,s)})_{s\in\omega}$ of open balls such that $z(n) \in \bigcap_s B_{s-1,h(n,s)}$ for all s. Indeed, we have $\{z(n)\} = \bigcap_s B_{s-1,h(n,s)}$ (that is, $(h(n, s))_{s\in\omega}$ is a Cauchy sequence rapidly converging to $z(n)$) since the diameter of $B_{s-1,h(s)}$ is at most 2^{-s+2}. Hence, one can enumerate $\mathrm{Nbase}_{\mathcal{H}}(z)$ using h. Since h is $\Delta^1_1(x)$, this shows that $\mathrm{Nbase}_{\mathcal{H}}(z) \leq_e \mathrm{Nbase}_{\lambda}(x^{(\alpha)})$ for some $\alpha < \omega_1^{\mathrm{CK},x}$. However, this implies that $\mathrm{Nbase}_{GH}(x) \not\leq_e \mathrm{Nbase}_{\mathcal{H}}(z)$ by Proposition 3.45. □

3.7.4. *Irregular Lattice Topology.* As another example of an extension topology we consider the irregular lattice topology. It indeed fails to be metrizable (and hence is irregular) – see below; and this is also a consequence of Proposition 3.50 below. Let $\hat{\omega} = \omega \cup \{\infty\}$ be the one-point compactification of ω. For a point $x \in \hat{\omega}^\omega$, its coded neighborhood filter is given as:

$$\mathrm{Nbase}_{\hat{\omega}^\omega}(x) = \{\langle 0, n, k\rangle : x(n) = k\} \cup \{\langle 1, n, k\rangle : x(n) \geq k\},$$

where n and k range over ω. Note that $\hat{\omega}^\omega$ is a zero-dimensional compact metrizable space. In the computability-theoretic context, this is rephrased as follows:

Observation 3.47. *The $\hat{\omega}^\omega$-degrees are exactly the total degrees.*

Proof. To see this (that is, to show totality of a point in $\hat{\omega}^\omega$), we identify $x \in \hat{\omega}^\omega$ with a partial function by interpreting $x(n) = \infty$ as that $x(n)$ is undefined. Then, it is not hard to check that $\mathrm{Nbase}(x)$ is e-equivalent to $\mathrm{Graph}(x) \oplus \mathrm{Graph}(x)^{\mathsf{c}}$. The latter set is clearly total. □

We consider an extension topology on (a subset of) $\hat{\omega}^\omega \times \hat{\omega}^\omega$. Let $\mathcal{L}$ be the space whose underlying set is $L = ((\omega \times \hat{\omega}) \cup \{(\infty, \infty)\})^\omega$ whose topology is generated by the (subspace) topology on $\hat{\omega}^\omega \times \hat{\omega}^\omega$. Then define $\mathcal{L}_{IL}$ as the space obtained by extending $\mathcal{L}$ by adding new open sets $IL = (V_{a,b})_{a,b\in\omega}$, where

$$V_{a,b} = \{(c, d) \in \omega^2 : c \geq a \text{ and } d \geq b\} \cup \{(\infty, \infty)\}.$$

We note that $(n, \infty) \notin V_{a,b}$ for any $n \in \omega$. This is called the *irregular lattice topology* and known to be non-metrizable (see Steen-Seebach [74, II.79]).

We introduce a countable open subbasis of $(\mathcal{L}_{IL})^\omega$ as follows. First put $P_{a,b} = \{(a, b)\}$, $U_{a,b} = \{(a, d) \in \omega \times \hat{\omega} : d \geq b\}$, and $V_{a,b}$ for $a, b \in \omega$. Given $Q \subseteq \mathcal{L}_{IL}$ and $n \in \omega$, we write $Q^n = \{x \in (\mathcal{L}_{IL})^\omega : x(n) \in Q\}$. Then, $(P^n_{a,b}, U^n_{a,b}, V^n_{a,b} : a, b \in \omega)$ forms a subbasis of $(\mathcal{L}_{IL})^\omega$. For instance, an open neighborhood $\{(c, d) \in L : c \geq a \text{ and } d \geq b\}$ of (∞, ∞) can be written as $V_{a,b} \cup \bigcup_{c\geq a} U_{c,b}$.

Each point $z \in (\mathcal{L}_{IL})^\omega$ can be thought of as the unique pair (x,y) satisfying $z(n) = (x(n), y(n))$ for all $n \in \omega$. Then, the coded neighborhood filter of $(x,y) \in (\mathcal{L}_{IL})^\omega$ is given as:

$$\begin{aligned}\mathrm{Nbase}_{(\mathcal{L}_{IL})^\omega}(x,y) =& \{\langle n,0,a,b\rangle : x(n)=a \text{ and } y(n)=b\}\\ &\cup \{\langle n,1,a,b\rangle : x(n)=a \text{ and } b \leq y(n)\}\\ &\cup \{\langle n,2,a,b\rangle : x(n)=y(n)=\infty \text{ or } [a\leq x(n) \text{ and } b\leq y(n)<\infty]\}.\end{aligned}$$

We will show that

$$\begin{aligned}\mathcal{D}_{(\mathcal{L}_{IL})^\omega} &= \{\mathbf{d}\in\mathcal{D}_e : \mathbf{d} \text{ is co-}d\text{-CEA}\}\\ &= \{\mathbf{d}\in\mathcal{D}_e : \mathbf{d} \text{ is } [*,\Pi^0_1,\Pi^0_1]\text{-separating-above}\}.\end{aligned}$$

Proposition 3.48. *The $(\mathcal{L}_{IL})^\omega$-degrees are exactly the co-d-CEA degrees.*

Proof. Given $(x,y)\in(\mathcal{L}_{IL})^\omega$, define X as the coded neighborhood filter of (x,y) in $\mathcal{L}^\omega$ (which is equivalent to $\mathrm{Nbase}_{\hat{\omega}^\omega}(x)\oplus\mathrm{Nbase}_{\hat{\omega}^\omega}(y)$). Note that X is total as mentioned above. Then, define A and P as follows:

$$\begin{aligned}A = A(x) &:= \{n\in\omega : x(n)=y(n)=\infty\},\\ P = P(y) &:= \{n\in\omega : y(n)\in\omega\}.\end{aligned}$$

Clearly, A^{c} and P are c.e. relative to X. Since $(\mathcal{L}_{IL})^\omega$ is finer than $\mathcal{L}^\omega$, we can recover the coded $\mathcal{L}^\omega$-neighborhood filter X of x from an enumeration of $\mathrm{Nbase}_{(\mathcal{L}_{IL})^\omega}(x)$. One can see that

$$n\in A\cup P \iff (\exists a,b\in\omega)(\exists i\in\{0,2\})\ \langle n,i,a,b\rangle\in\mathrm{Nbase}(x,y).$$

Thus, $X\oplus X^{\mathsf{c}}\oplus(A\cup P)$ is e-reducible to $\mathrm{Nbase}_{(\mathcal{L}_{IL})^\omega}(x,y)$.

Conversely, from X, we first decode $\mathrm{Nbase}_{\hat{\omega}^\omega}(x)$ and $\mathrm{Nbase}_{\hat{\omega}^\omega}(y)$. Then, it is easy to see that for any $i<2$ and $n,a,b\in\omega$,

$$\langle n,i,a,b\rangle\in\mathrm{Nbase}_{(\mathcal{L}_{IL})^\omega}(x,y) \iff \langle n,0,a\rangle\in\mathrm{Nbase}_{\hat{\omega}^\omega}(x) \text{ and } \langle n,i,b\rangle\in\mathrm{Nbase}_{\hat{\omega}^\omega}(y).$$

Moreover, one can see that

$$\langle n,2,a,b\rangle\in\mathrm{Nbase}_{(\mathcal{L}_{IL})^\omega}(x,y) \iff \langle n,1,a\rangle\in\mathrm{Nbase}_{\hat{\omega}^\omega}(x),\ \langle n,1,b\rangle\in\mathrm{Nbase}_{\hat{\omega}^\omega}(y) \\ \text{and } n\in A\cup P.$$

The above two equality given us an e-reduction from $\mathrm{Nbase}_{(\mathcal{L}_{IL})^\omega}(x,y)$ to $X\oplus X^{\mathsf{c}}\oplus(A\cup P)$. Consequently, every $(\mathcal{L}_{IL})^\omega$-degree is co-$d$-CEA.

Next, assume that a co-d-CEA set is given, i.e. sets X,A,P such that A^{c} and P are X-c.e. are given, and consider $A\cup P$. Without loss of generality, we can assume that $A\cap P=\emptyset$ since replacing P with the new X-c.e. set $P\setminus A$ does not affect on the set $A\cup P$. Then, we construct $(x,y)\in(\mathcal{L}_{IL})^\omega$ as follows. Fix X-computable enumerations of A^{c} and P. First we use $(x(2n),y(2n))_{n\in\omega}$ to code X. Then define $x(2n+1)$ and

$y(2n+1)$ as follows:

$$x(2n+1) = \begin{cases} \infty & \text{if } n \in A, \\ s & \text{if we see } n \in A^{\mathsf{c}} \text{ at stage } s, \end{cases}$$

$$y(2n+1) = \begin{cases} \infty & \text{if } n \notin P, \\ t & \text{if we see } n \in P \text{ at stage } t, \end{cases}$$

Then, A and P are recovered from (x, y) as above, i.e., $A = A(x)$ and $P = P(y)$. The above argument shows that for any $(x, y) \in (\mathcal{L}_{IL})^\omega$, $\mathrm{Nbase}_{(\mathcal{L}_{IL})^\omega}(x, y)$ is e-equivalent to $X \oplus X^{\mathsf{c}} \oplus (A(x) \cup P(y))$. This concludes the proof. □

Proposition 3.49. *The $(\mathcal{L}_{IL})^\omega$-degrees (hence the co-d-CEA degrees) are exactly the $[*, \Pi^0_1, \Pi^0_1]$-separating-above e-degrees. In other words, a nonempty set $E \subseteq \omega$ is co-d-CEA if and only if there are $X, A, B \subseteq \omega$ such that B and $A \cup B$ are X-co-c.e., $A \cap B = \emptyset$, and*

$$\mathrm{Enum}(E) \equiv_M \{X\} \times \mathrm{Sep}(A, B).$$

Proof. Fix $X, A, B \subseteq \omega$ such that B and $A \cup B$ are X-co-c.e., and A and B are disjoint. Note that $A^{\mathsf{c}} = B \cup (A \cup B)^{\mathsf{c}}$, that is, it is the union of an X-co-c.e. set and an X-c.e. set, and thus, A^{c} is co-d-c.e. relative to X. We claim that $\mathrm{Enum}(X \oplus X^{\mathsf{c}} \oplus A^{\mathsf{c}})$ is Medvedev equivalent to $\{X\} \times \mathrm{Sep}(A, B)$.

We first show that there is a X-computable function that, given enumeration of A^{c}, returns a set C separating A from B. Fix an enumeration of B^{c} relative to X. Then, given an enumeration of A^{c}, wait until we see either $n \in A^{\mathsf{c}}$ or $n \in B^{\mathsf{c}}$ (by using an enumeration relative to X). Since A and B are disjoint, this happens at some stage. If we see $n \in A^{\mathsf{c}}$ (before seeing $n \in B^{\mathsf{c}}$), we enumerate n into C^{c}. If we see $n \in B^{\mathsf{c}}$ (before seeing $n \in A^{\mathsf{c}}$), we enumerate n into C. Clearly, C separates A from B.

Conversely, assume that a set C separating A from B is given. Then, note that $A^{\mathsf{c}} = C^{\mathsf{c}} \cup (A \cup B)^{\mathsf{c}}$. Thus, wait until we see either $n \in C^{\mathsf{c}}$ or $n \in (A \cup B)^{\mathsf{c}}$ (by using an enumeration relative to X). If we see this, enumerate n into A^{c}. This procedure gives us a correct enumeration of A^{c}.

Next, assume that a co-d-CEA set is given, that is, disjoint sets $B, P \subseteq \omega$ such that B is X-co-c.e. and P is X-c.e. are given. Define $A = (B \cup P)^{\mathsf{c}}$. Note that A and B are disjoint, and that B and $A \cup B = P^{\mathsf{c}}$ are X-co-c.e. Since $B \cup P = A^{\mathsf{c}}$, by the same argument as above, we can show that $\mathrm{Enum}(X \oplus X^{\mathsf{c}} \oplus (B \cup P))$ is Medvedev equivalent to $\{X\} \times \mathrm{Sep}(A, B)$. □

It is known that there is a quasi-minimal co-d-c.e. e-degree; see e.g. a few paragraphs after [11, Theorem 3.3]. As a consequence:

Proposition 3.50. *The product irregular lattice space $(\mathcal{L}_{IL})^\omega$ contains a Δ^0_2 point of quasi-minimal degree.*

By relativizing Proposition 3.50, one may also conclude that $\mathcal{L}_{IL}$ is far from being metrizable. This is because the degree structure of a metrizable space is the continuous degrees relative to some oracle [43], and there is no quasi-minimal continuous degree [54]. Indeed, by [43], the former property (that the degree structure is the continuous degrees relative to some oracle) is equivalent to saying that such a space is σ-*metrizable*, that is, it is written as the union of countably many metrizable subspaces (see also [34]).

It is natural to ask if the product irregular lattice space is strictly smaller than the product double-origin space in the sense of degrees. We leave this question open. However we shall show that the set of doubled co-d-c.e. degrees is strictly larger than the set of co-d-c.e. degrees:

Proposition 3.51. *There is a doubled co-d-c.e. e-degree which is not co-d-c.e.*

Proof. We construct $Z = (A \cup P) \oplus (B \cup N)$. Let E_e be the e-th co-d-c.e. set. Begin with $n \in B$. Wait until $2n+1$ is enumerated into $\Psi \circ \Phi(Z_s)$ with Φ-use $\varphi_s(n)$. If we see this, remove n from B, and enumerate n into A. That is, define $Z_{s+1} = (Z_s \setminus \{2n+1\}) \cup \{2n\}$. Restrain $Z_{s+1} \upharpoonright \varphi_s(n)$. Wait until $2n$ is enumerated into $\Psi \circ \Phi(Z_t)$ with Φ-use $\varphi_t(n)$. Restrain $Z_t \upharpoonright \varphi_t(n)$. Given S, we write $S^0 = (S \setminus \{2n\}) \cup \{2n+1\}$ and $S^1 = (S \setminus \{2n+1\}) \cup \{2n\}$. For any stage u after t, either both $2n$ and $2n+1$ are enumerated into $\Psi \circ \Phi(Z^i_u)$ for some $i < 2$ or there is $m < \max\{\varphi_s(n), \varphi_t(n)\}$ such that $m \in \Phi(Z^i_u)$, but $m \notin \Phi(Z^{1-i}_u)$ by monotonicity of an enumeration operator. In the former case, put $Z_{u+1} = Z^i_u$, and restrain the Φ-use. In the latter case, search for such m, and choose i such that the current guess of $\Phi(Z^i_u; m)$ is unequal to the current approximation of $E_e(m)$. Then put $Z_{u+1} = Z^i_u$. Note that, at some later stage $v > u$, we may see that $\Phi(Z^i_v; m) = E(m)$. In this case, we search for new m and i, and continue the similar procedure. This procedure converges at some stage, and therefore, this is finite injury. $\square$

Note that each quasi-minimal e-degree that is co-d-CEA is co-d-c.e. Therefore to show that the product irregular lattice space is strictly smaller than the product double-origin space in the sense of degrees, it is enough to make the degree constructed in Proposition 3.51 quasi-minimal. We also leave this question open.

3.8. Degrees of points: G_δ-topology.

The next stop in our investigation are the degrees of points in G_δ-spaces and see that these are just the cototal degrees. We briefly recall the definition of G_δ-spaces:

Definition 3.52. A G_δ-*space* or a *perfect space* is a topological space in which every closed set is G_δ (i.e. an intersection of countably many open sets).

In the classical Borel hierarchy, a set is (classically) $\boldsymbol{\Pi}^0_{n+1}$ if it can be obtained as a countable intersection of $\boldsymbol{\Sigma}^0_n$ sets. For the modified Borel hierarchy, recall from Section 2.5 that a set is (modified) $\boldsymbol{\Pi}^0_2$ if it can be obtained as a countable intersection of constructible sets (Boolean combinations of open sets). In general, a set is (modified) $\boldsymbol{\Pi}^0_{n+1}$ if it can be obtained as a countable intersection of Boolean combinations of $\boldsymbol{\Sigma}^0_n$ sets (see e.g. [15]). Every modified $\boldsymbol{\Pi}^0_n$ set is always modified $\boldsymbol{\Pi}^0_{n+1}$, but a classical $\boldsymbol{\Pi}^0_n$ set is not necessarily classical $\boldsymbol{\Pi}^0_{n+1}$.

Note that a G_δ space is a space in which the classical Borel hierarchy is well-behaved (without considering the modified Borel hierarchy); indeed, the classical Borel hierarchy coincides with the modified Borel hierarchy. Note that we always have $G_\delta \subseteq \boldsymbol{\Pi}^0_2$.

Observation 3.53. *A space is* G_δ *if and only if* $G_\delta = \boldsymbol{\Pi}^0_2$.

Proof. Clearly, if $G_\delta = \mathbf{\Pi}^0_2$ in a space $\mathcal{X}$, then $\mathcal{X}$ is a G_δ space since every closed set is constructible, and hence $\mathbf{\Pi}^0_2$. To see the converse, note that the class of G_δ sets is closed under finite union and countable intersection. If $\mathcal{X}$ is a G_δ space, then any open or closed set is G_δ. Hence, every constructible set is G_δ, and therefore, any $\mathbf{\Pi}^0_2$ set is G_δ. □

3.8.1. *Closed networks and G_δ-spaces.* One of the key notions in this article is a network of a topological space introduced by Arhangel'skii in 1959, which has become one of the most fundamental notions in modern general topology.

Definition 3.54 (Arhangel'skii)**.** A *network* $\mathcal{N}$ for a topological space $\mathcal{X}$ is a collection of subsets (not necessarily open) of $\mathcal{X}$ such that, for any open neighborhood $U \subseteq \mathcal{X}$ of a point $x \in \mathcal{X}$, there is $N \in \mathcal{N}$ such that $x \in N \subseteq U$.

If Γ is a collection of subsets of $\mathcal{X}$, by a Γ *network*, we mean a network consisting of Γ sets. For instance, an open network for $\mathcal{X}$ is precisely an open basis of $\mathcal{X}$. A closed network is a network consisting of closed sets. Later, we will see that the notion of a *closed network* plays an important role in our work. We first see the following characterization of T_1-spaces.

Observation 3.55. *A T_0 space $\mathcal{X}$ is T_1 if and only if $\mathcal{X}$ has a closed network.*

Proof. If $\mathcal{X}$ is T_1, every point is closed. Thus, $\mathcal{N} = \{\{x\} : x \in \mathcal{X}\}$ forms a closed network. We show the converse direction. Fix $x \neq y$. Since $\mathcal{X}$ is T_0, there is an open set U such that either $x \in U \not\ni y$ or $x \notin U \ni y$. Without loss of generality, we may assume that $x \in U$ and $y \notin U$. Since $\mathcal{X}$ has a closed network, there is a closed set F such that $x \in F \subseteq U$. Then, $V = \mathcal{X} \setminus F$ is open, and we have $x \notin V$ and $y \in V$. This shows that $\mathcal{X}$ is T_1. □

Proposition 3.56. *A second-countable space $\mathcal{X}$ is a G_δ-space if and only if $\mathcal{X}$ has a countable closed network.*

Proof. Let $\mathcal{X}$ be a T_0 space with a countable basis $(\beta_e)_{e\in\omega}$. If $\mathcal{X}$ is a G_δ-space, then every open set is F_σ, and therefore, for any $e \in \omega$, there is a countable collection $(F^e_n)_{n\in\omega}$ of closed sets such that $\beta_e = \bigcup_{n\in\omega} F^e_n$. Since $(\beta_e)_{e\in\omega}$ is a basis, $(F^e_n)_{e,n\in\omega}$ forms a countable closed network for $\mathcal{X}$.

Conversely, if $\mathcal{N}$ is a countable closed network for $\mathcal{X}$, for any open set U, consider the F_σ set $N(U) = \bigcup\{N \in \mathcal{N} : N \subseteq U\}$. We claim that $U = N(U)$. The inclusion $N(U) \subseteq U$ is clear. For the inclusion $U \subseteq N(U)$, given $x \in U$, since $\mathcal{N}$ is a network, there is $N \in \mathcal{N}$ such that $x \in N \subseteq U$. This means that $x \in N(U)$, and therefore, $U = N(U)$, that is, U is F_σ. This concludes that $\mathcal{X}$ is a G_δ-space. □

Note that even if $\mathcal{X}$ is not second-countable, the proof of Proposition 3.56 shows the following implications:

$$\mathcal{X} \text{ has a countable closed network } \implies \mathcal{X} \text{ is } G_\delta \implies \mathcal{X} \text{ has a closed network.}$$

Thus, the property being a G_δ-space can be thought of as a strengthening of being T_1 in the category of T_0 spaces. One can see further implications as follows:

Observation 3.57. *For a second-countable T_0 space, we have the following implications:*

$$\begin{array}{rcl} \textit{compact and } T_1 \implies & & \\ & G_\delta \implies T_1. & \\ \textit{metrizable} \implies & & \end{array}$$

Proof. Let $(B_i)_{i\in\omega}$ be a countable basis for $\mathcal{X}$. Assume that $\mathcal{X}$ is compact and T_1. For each finite set D, let N_D be the complement of $\bigcup_{i\in D} B_i$. We claim that $(N_D)_{D\subseteq_{\rm fin}\omega}$ forms a countable closed network for $\mathcal{X}$. To see this, fix a point $x \in \mathcal{X}$ and open neighborhood U of x. Since $\mathcal{X}$ is T_1, the complement of $\{x\}$ is open, and thus it is written as $\bigcup_{i\in I} B_i$. Since $\mathcal{X} \setminus \{x\}$ covers the closed subset $\mathcal{X} \setminus U$ of the compact space $\mathcal{X}$, there is a finite set $D \subseteq I$ such that $\mathcal{X} \setminus U$ is covered by $\bigcup_{i\in D} B_i$, which means that $N_D \subseteq U$. Moreover, we have that $\bigcup_{i\in D} B_i \subseteq \mathcal{X} \setminus \{x\}$, and therefore $x \in N_D$. This shows that $(N_D)_{D\subseteq_{\rm fin}\omega}$ forms a countable closed network for $\mathcal{X}$. Thus, $\mathcal{X}$ is G_δ by Proposition 3.56.

Next, if $\mathcal{X}$ is metrizable, it is easy to see that every open ball is a countable union of closed balls. Hence, every metrizable space is G_δ. The implication from being G_δ to being T_1 follows from Observation 3.55 and Proposition 3.56. □

We will also see that none of the above implications can be reversed.

Observation 3.58. *There is a non-compact non-metrizable second-countable G_δ-space. For instance, the double origin space $(\mathcal{Q}_{DO})^\omega$ is non-compact, non-metrizable, but G_δ.*

Proof. See Steen-Seebach [74, II.74]. □

Proposition 3.59. *There exist a second-countable submetrizable space which is not G_δ. For instance, the indiscrete irrational extension of $\mathbb{R}$ is second-countable, submetrizable, but not G_δ.*

Proof. To simplify our argument, we consider the indiscrete irrational extension of $\mathcal{C} = 2^\omega$ rather than $\mathbb{R}$. More formally, let J be the set of all infinite binary sequences containing infinitely many 0's, and then consider the J-extension $\mathcal{C}_J$ of the Cantor topology $\tau_\mathcal{C}$, i.e., the topology generated by $\tau_\mathcal{C} \cup \{J\}$.

Suppose for the sake of contradiction that $\mathcal{C}_J$ is G_δ. Let $(B_i)_{i\in\omega}$ and $(N_i)_{i\in\omega}$ be a countable basis and a countable closed network for $\mathcal{C}_J$ (by Proposition 3.56). Note that every N_i is closed in $\mathcal{C}_J$, and therefore N_i can be written as the complement of $V_i \cap J$ or V_i for some $\tau_\mathcal{C}$-open set V_i. Then, there is an oracle Z such that V_i is Z-c.e. open for any $i \in \omega$. Let x be a 1-generic real relative to Z, i.e. let x be a real that meets all dense Z-c.e. open sets. Clearly, $x \in J$. Therefore, there is i such that $x \in B_i \subseteq J$. Then, since $(N_i)_{i\in\omega}$ is a closed network for $\mathcal{C}_J$, there is j such that $x \in N_j \subseteq B_i$. Since $x \in J$, we have $x \notin V_j$. Note that the complement of B_i is dense, and thus V_j is dense. However, since x is 1-generic relative to Z, and V_j is a dense Z-c.e. open set, we must have $x \in V_j$, a contradiction. □

As seen in Section 3.7.1, extending topology has an undesirable degree-theoretic behavior, that is, even if we assume decidability of bases, any e-degree can be realized by extending a metrizable topology. Therefore, extension topologies must avoid any nontrivial degree-theoretic characterizations. An important observation obtained from

Proposition 3.59 is that extending topology destroys the property being G_δ. We will explain the reason of this phenomenon in terms of degree theory: In contrast to extension topologies, G_δ-spaces are, degree-theoretically, extremely well-behaved.

3.8.2. *Cototal enumeration degrees.* In this section, we describe how the notion of a G_δ-space is useful in computability theory.

Recall that a set $A \subseteq \omega$ is *cototal* if $A \leq_e A^{\mathsf{c}}$ holds. As one of the most important results in this article, we will see that the notion of *cototality* is captured by G_δ-spaces. A represented space is computably G_δ if given a code of a closed set, one can effectively find its G_δ code (see Definition 3.67 for the precise definition).

Theorem 3.60. *An e-degree is cototal if and only if it is an $\mathcal{X}$-degree of a computably G_δ, cb_0 space $\mathcal{X}$, that is,*

$$\{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \textit{ is cototal}\} = \bigcup \{\mathcal{D}_{\mathcal{X}} : \mathcal{X} \textit{ is a computably } G_\delta,\ cb_0 \textit{ space}\}$$

Moreover, we will construct a decidable G_δ-space $A^{\mathrm{co}}_{\max}$ which is universal in the sense that $A^{\mathrm{co}}_{\max}$ captures exactly the cototal e-degrees:

Theorem 3.61. *There exists a decidable, computably G_δ, cb_0 space $\mathcal{X} = A^{\mathrm{co}}_{\max}$ such that*

$$\mathcal{D}_{\mathcal{X}} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \textit{ is cototal}\}.$$

The proofs of Theorems 3.60 and Theorems 3.61 will be given later.

Uniform cototality. We first introduce notions of cototality for a space.

Definition 3.62. We say that a space $\mathcal{X}$ is *relatively cototal* if there are an oracle C and an enumeration operator Ψ such that

$$(\forall x \in \mathcal{X})\ \mathrm{Nbase}(x) = \Psi(\mathrm{Nbase}(x)^{\mathsf{c}} \oplus C \oplus C^{\mathsf{c}}).$$

If C can be empty, we say that $\mathcal{X}$ is *uniformly cototal.*

It is clear that if x is a point in a uniformly cototal space $\mathcal{X}$, then $\mathrm{Nbase}_{\mathcal{X}}(x)$ is cototal. Baire space, the Hilbert cube, the double origin space, the telophase space, the cocylinder space, etc. are all uniformly cototal.

Example 3.63 (Jeandel [40])**.** In universal algebra, a *quasi-variety* is a class of structures satisfying a Horn clause. Jeandel [40] coded this notion as follows. Let $S \subseteq \omega^{<\omega}$ be a set. We say that $X \subseteq \omega$ satisfies S if for any $\sigma \in S$, if $\sigma(n) \in X$ for all $0 < n < |\sigma|$, then $\sigma(0) \in X$. A quasi-variety V defined by a set S is the class of all $X \subseteq \omega$ satisfying S. If S is c.e. we call V a c.e. quasi-variety. For instance, the set of all (forbidden languages of) subshifts Sf_Σ over a finite alphabet Σ, and the set of all (words of) groups Gr_n with n generators are c.e. quasi-varieties.

For each $Y \subseteq \omega$, we define $[Y] = \{X \subseteq \omega : Y \subseteq X\}$. A set $Y \subseteq \omega$ is a presentation of $X \in V$ if $V \cap [Y] = V \cap [X]$. A point $X \in V$ is *finitely presented* if X has a finite presentation. A point $X \in V$ is *maximal* if $X \neq \omega$ and $V \cap [X] = \{X, \omega\}$. Maximal points in Sf_Σ and Gr_n are minimal subshifts and simple groups, respectively.

Given a quasi-variety V, consider the following space (as a subspace of $\mathcal{P}(\omega) \simeq \mathbb{S}^\omega$):

$$V^{\mathrm{co}}_{\max} = \{X \subseteq \omega : X^{\mathsf{c}} \in V, \text{ and } X^{\mathsf{c}} \text{ is maximal in } V\}.$$

Jeandel [40] showed that if V is a c.e. quasi-variety such that ω is finitely presented, then $V^{\rm co}_{\max}$ is uniformly cototal. In particular, the spaces $(\mathsf{Sf}_\Sigma)^{\rm co}_{\max}$ and $(\mathsf{Gr}_n)^{\rm co}_{\max}$ are uniformly cototal.

Remark 3.64. For a quasi-variety V, the space $V^{\rm co}_{\max}$ is computably homeomorphic to the space of all maximal elements in V equipped with the basis $([D]_{\rm co})$, where $[D]_{\rm co} = \{X : X \cap D = \emptyset\}$, and D ranges over all finite subsets of ω.

Example 3.65 (McCarthy [48])**.** We topologize $\mathcal{P}(\omega^{<\omega})$ by putting $[D] = \{X \subseteq \omega^{<\omega} : D \subseteq X\}$ as a basic open set for any finite set $D \subseteq \omega^{<\omega}$. Consider the following subspace of $\mathcal{P}(\omega^{<\omega})$:

$$A^{\rm co}_{\max} = \{X \subseteq \omega^{<\omega} : X^{\mathsf{c}} \text{ is a maximal antichain}\}.$$

Then, $A^{\rm co}_{\max}$ is uniformly cototal. Moreover, McCarthy [48] showed that the $A^{\rm co}_{\max}$-degrees and the $(\mathsf{Sf}_\Sigma)^{\rm co}_{\max}$-degrees are exactly the cototal e-degrees:

$$\mathcal{D}_{A^{\rm co}_{\max}} = \mathcal{D}_{(\mathsf{Sf}_\Sigma)^{\rm co}_{\max}} = \{\mathbf{d} \in \mathcal{D}_e : \mathbf{d} \text{ is cototal}\}.$$

One can check that a represented cb$_0$ space (X, β) is *decidable* (see Section 2.4.2) if $\{\langle \sigma, \tau\rangle : \beta^+_\sigma \subseteq \beta^+_\tau\}$ is computable, where we assume that $\emptyset$ and X appear in β.

Observation 3.66. *$A^{\rm co}_{\max}$ is a decidable cb$_0$ space.*

Proof. For finite sets $D, E \subseteq \omega^{<\omega}$, we claim that $D \subseteq E$ if and only if $A^{\rm co}_{\max} \cap [E] \subseteq A^{\rm co}_{\max} \cap [D]$. It suffices to show that $D \not\subseteq E$ implies $A^{\rm co}_{\max} \cap [E] \not\subseteq A^{\rm co}_{\max} \cap [D]$. Choose $\sigma \in D \setminus E$. Then, it is easy to construct a maximal antichain $X \subseteq \omega^{<\omega}$ such that $X \cap E = \emptyset$ and $\sigma \in X$. For instance, consider $X = \{\sigma\} \cup \{\tau \in \omega^\ell : \sigma \not\preceq \tau\}$ for a sufficiently large ℓ. Then, $E \subseteq X^{\mathsf{c}}$, but $D \not\subseteq X^{\mathsf{c}}$. This shows that $X^{\mathsf{c}} \in [E] \setminus [D]$; therefore $A^{\rm co}_{\max} \cap [E] \not\subseteq A^{\rm co}_{\max} \cap [D]$. □

Then, what kind of topological property is shared by these spaces? To answer this question, we effectivize the notion of a G_δ-space. It is obvious that a space is G_δ if and only if every open set is F_σ. We introduce an effective version of this property.

Definition 3.67. We say that $\mathcal{X}$ is *computably G_δ* if there is a computable procedure that, given a code of open set, returns its F_σ-code, that is, there is a computable function f such that

$$(\forall e \in \omega)\ \beta_e = \bigcup_{n\in\omega} P_{f(e,n)}, \text{ where } P_d = \mathcal{X} \setminus \bigcup\{\beta_c : c \in W_d\}.$$

Note that a second-countable T_0 space is G_δ if and only if it is computably G_δ relative to some oracle, or equivalently, it is computably G_δ w.r.t. some representation β.

We now show an effective topological characterization of uniform cototality.

Theorem 3.68. *Let $\mathcal{X} = (X, \beta)$ be a represented cb$_0$ space. Then, $\mathcal{X}$ is computably G_δ if and only if there is a representation $\gamma \equiv \delta$ of X such that (X, γ) is uniformly cototal.*

Proof. To prove Theorem 3.68, we will see that one can assume that, in a computably G_δ space, every open set can be written as a computable union of finitary closed sets, where a set is finitary closed (w.r.t. β) if it is the complement of finitely many open sets in the basis generated by β. Be careful that the notion of being finitary closed depends on the choice of the representation β; hence it is not a topological property.

Recall from Section 2.4.2 the notion of reducibility of representations; for instance, by $\gamma \equiv \delta$ we mean that γ is bi-reducible to δ.

Observation 3.69. *Let $\mathcal{X} = (X, \beta)$ be a represented cb_0 space which is computably G_δ. Then, there is a representation $\gamma \equiv \beta$ of X such that, given $e \in \omega$, one can effectively find a computable sequence $(Q^e_n)_{n\in\omega}$ of γ-finitary closed sets with $\gamma_e = \bigcup_n Q^e_n$.*

Proof. Let f be a computable function witnessing that $\mathcal{X}$ is computably G_δ. Define $\gamma_{2e} = \beta_e$ and $\gamma_{2\langle e,n\rangle+1} = X \setminus P_{f(e,n)} = \bigcup\{\beta_d : d \in W_{f(e,n)}\}$. Then, $\gamma_{2e} = \beta_e = \bigcup_n P_{f(e,n)} = \bigcup_n (X \setminus \gamma_{2\langle e,n\rangle+1})$, which is a computable union of γ-finitary closed sets. Moreover, $\gamma_{2\langle e,n\rangle+1}$ is a computable union of sets of the form β_d, where β_d can be written as a computable union of γ-finitary closed sets. This concludes the proof. $\square$

Observation 3.70. *Let (X, β) be a represented cb_0 space, and let γ be a representation of X such that $\beta \equiv \gamma$. Then, if (X, β) is a computably G_δ space, so is (X, γ).*

Proof. Note that $\beta \leq \gamma$ iff, given a β-code of an open set in $\mathcal{X}$, one can effectively find its γ-code. Given a γ-basic open set U, one can find its β-code since $\gamma \leq \beta$. Since (X, β) is computably G_δ, one can find a β-computable sequence of closed sets whose union is U. Since $\beta \leq \gamma$, it is also computable w.r.t. γ. Hence, (X, γ) is computably G_δ. $\square$

We now assume that $\mathcal{X} = (X, \beta)$ is computably G_δ. By Observation 3.69, there is $\gamma \equiv \delta$ such that every basic open set can be written as a computable union of finitary closed sets w.r.t. γ in an effective manner. We will construct an enumeration operator Ψ. Let W be a c.e. set such that $\gamma_e = \bigcup_{\langle e,D\rangle\in W} N_D$, where $N_D = \mathcal{X} \setminus \bigcup_{j\in D} \gamma_j$. Then we claim that W witnesses uniform cototality of (X, γ), that is,

$$e \in \mathrm{Nbase}_\gamma(x) \iff (\exists D)\,[D \subseteq \mathrm{Nbase}_\gamma(x)^{\mathsf{c}} \text{ and } \langle e, D\rangle \in W].$$

To see the implication "$\Leftarrow$", we first note that $D \subseteq \mathrm{Nbase}_\gamma(x)^{\mathsf{c}}$ if and only if $x \in N_D$ by definition of N_D. Moreover, if $\langle e, D\rangle \in W$, then $N_D \subseteq \gamma_e$, and therefore, the right formula implies $e \in \mathrm{Nbase}(x)$. For the implication "$\Rightarrow$", if $e \in \mathrm{Nbase}(x)$, since $\gamma_e = \bigcup_{\langle e,D\rangle\in W} N_D$, there is a finite set D such that $\langle e, D\rangle \in W$ and $x \in N_D \subseteq \gamma_e$. Then we have $D \subseteq \mathrm{Nbase}(x)^{\mathsf{c}}$, and $\langle e, D\rangle \in W$ as desired. Consequently, (X, γ) is uniformly cototal.

Conversely, we assume that (X, γ) is uniformly cototal via an enumeration operator Ψ. Then for any finite set D, define $N_D = \mathcal{X} \setminus \bigcup_{n\in D} \gamma_n$. We claim that $\gamma_n = \bigcup\{N_D : \langle n, D\rangle \in \Psi\}$. For the inclusion "$\subseteq$," if $x \in \gamma_n$, then since $\Psi(\mathrm{Nbase}(x)^{\mathsf{c}}) = \mathrm{Nbase}(x)$, there is a finite set $D \subseteq \mathrm{Nbase}(x)^{\mathsf{c}}$ (i.e., $x \notin \bigcup_{j\in D} \gamma_j$, and therefore $x \in N_D$) such that $\langle n, D\rangle \in \Psi$. For the inclusion "$\supseteq$," we show that if $\langle n, D\rangle \in \Psi$, then $\gamma_n \cup \bigcup_{j\in D} \gamma_j = \mathcal{X}$ (i.e., $N_D \subseteq \gamma_n$). Otherwise, there is $y \in \mathcal{X}$ such that $y \notin \gamma_n \cup \bigcup_{j\in D} \gamma_j$. However, we then have $D \subseteq \mathrm{Nbase}(y)^{\mathsf{c}}$, which implies $n \in \Psi(\mathrm{Nbase}(y)^{\mathsf{c}})$, while $n \notin \mathrm{Nbase}(y)$. Then, we get $\Psi(\mathrm{Nbase}(y)^{\mathsf{c}}) \neq \mathrm{Nbase}(y)$, which contradicts our choice of Ψ. This shows that given n, one can effectively find a computable sequence $(N_D : \langle n, D\rangle \in \Psi)$ of finitary closed sets whose union is γ_n, that is, (X, γ) is computably G_δ. Hence, (X, β) is also computably G_δ by Observation 3.70. $\square$

Theorem 3.60 clearly follows from Theorem 3.68. This gives us an (effective) topological explanation of why continuous degrees, double origin degrees, graph cototal degrees, etc. are cototal.

Proof of Theorem 3.61. As mentioned in Example 3.65, McCarthy [48] showed that the space $A^{\rm co}_{\max}$ is uniformly cototal, and moreover, the $A^{\rm co}_{\max}$-degrees are exactly the cototal e-degrees. Hence, by Theorem 3.68, $A^{\rm co}_{\max}$ is computably G_δ. Moreover, as seen in Observation 3.66, $A^{\rm co}_{\max}$ is a decidable ${\rm cb}_0$ space. Consequently, $A^{\rm co}_{\max}$ is a decidable G_δ-space which captures the cototal e-degrees. □

Theorems 3.60 and Theorems 3.61 conclude that the cototal e-degrees are characterized by the degrees of difficulty of enumerating (neighborhood bases of) points in (decidable) computably G_δ spaces.

3.9. **Quasi-Polish topology.**

Contrary to the fact that we always have the completion of a metric, there is no hope of getting the notion of "quasi-completion" which preserves T_i-separation axioms for $i \neq 0, 3$ (see Section 3.7.2). Instead of considering the notion of quasi-completion[3], we will directly show the existence of a quasi-Polish space at every separation level. Remember the spaces $(\hat{\omega}_{TP})^\omega$, $(\mathcal{P}_{DO})^\omega$, and $\mathcal{QA}^\omega$ introduced in Examples 3.17 and 3.24 and in Section 3.6.1, respectively.

Proposition 3.71. *For any $i \in \{0, 1, 2, 2.5\}$, there is a quasi-Polish T_i space which is not T_j for any $j > i$. Indeed,*

(1) *The telophase space $(\hat{\omega}_{TP})^\omega$ is a quasi-Polish T_1-space which is not T_2.*
(2) *The double origin space $(\mathcal{P}_{DO})^\omega$ is a quasi-Polish T_2-space which is not $T_{2.5}$.*
(3) *The Arens space $\mathcal{QA}^\omega$ and the Roy space $\mathcal{QR}^\omega$ are quasi-Polish $T_{2.5}$-spaces which are not submetrizable.*
(4) *The irregular lattice space $(\mathcal{L}_{IL})^\omega$ is a quasi-Polish submetrizable space which is not metrizable.*
(5) *The quasi-completion $\overline{\mathbb{R}_<} = \mathbb{R}_< \cup \{\infty\}$ of the lower real line is a quasi-Polish T_0-space which is not T_1.*

Proof. It is easy to see that if $\mathcal{X}$ is $\mathbf{\Pi}^0_2$-named, then so is $\mathcal{X}^\omega$. For (1), by Fact 2.8, it suffices to show that $\hat{\omega}_{TP}$ is an open continuous image of a Polish space. Define a function δ by $\delta(j0^n10^\omega) = n$ for each $j < 2$, $\delta(0^\omega) = \infty$, and $\delta(10^\omega) = \infty_\star$. It is clear that the domain of δ is a closed subset of 2^ω; hence Polish. For continuity, the preimages of basic open sets $\{n\}$, $[n, \infty]$, and $[n, \infty_\star]$ are the clopen sets $[00^n1] \cup [10^n1]$, $[00^n]$, and $[10^n]$. For openness, the images of basic open sets $[j0^n1\tau]$, $[00^n]$, and $[10^n]$ are the open sets $\{n\}$, $[n, \infty]$, and $[n, \infty_\star]$. Hence, δ is open and continuous.

For (2), define a partial surjection $\delta :\subseteq \omega^\omega \to \mathcal{P}_{DO}$ as follows.

$$\begin{aligned}
&\delta(0n0^m10^\omega) = (n, m),\ \delta(0n0^m20^\omega) = (n, \overline{m}),\ \delta(0n0^\omega) = (n, *),\\
&\delta(10^\omega) = \mathbf{0}_\star,\ \delta(20^\omega) = \mathbf{0},\\
&\delta(10^s1m0^n10^\omega) = (n + s, m + s),\ \delta(10^s1m0^\omega) = (\infty, m + s),\\
&\delta(20^s1m0^n10^\omega) = (n + s, \overline{m + s}),\ \delta(20^s1m0^\omega) = (\infty, \overline{m + s}).
\end{aligned}$$

[3]Some positive results about the possibility of having a quasi-completion were recently obtained by de Brecht [17].

It is clear that $\mathrm{dom}(\delta)$ is a closed subset of ω^ω; hence Polish. For continuity, the preimages of some basic open sets are:

$$\delta^{-1}[(n,\overline{m})] = [0n0^m2] \cup \bigcup_{s\leq m}[20^s1(m-s)0^{n-s}],$$
$$\delta^{-1}[[n,\infty]\times\{m\}] = \bigcup_{s\leq m}[10^s1(m-s)0^{n-s}] \cup \bigcup_{k\geq n}\delta^{-1}[(k,m)],$$
$$\delta^{-1}[\{n\}\times[m,\overline{m}]] = [0n0^m] \cup \bigcup_{k\in\omega\cup\omega^*}\delta^{-1}[(n,k)],$$
$$\delta^{-1}[([n,\infty]\times(*,\overline{n}])\cup\{\mathbf{0}\}] = [20^n] \cup \bigcup_{m\geq n}\delta^{-1}[[n,\infty]\times\{\overline{m}\}].$$

These sets are open, and therefore, δ is continuous. For openness, the images of some basic open sets $[0n0^m]$, $[0n0^m2\tau]$, $[10^s]$, $[20^s1m0^n]$, $[10^s1m0^n1\tau]$ are the open sets $\{n\}\times[m,\overline{m}]$, $\{(n,\overline{m})\}$, $([s,\infty]\times[s,*))\cup\{\mathbf{0}_\star\}$, $[n+s,\infty]\times\{\overline{m+s}\}$, and $\{(n+s,m+s)\}$. Consequently, δ is open and continuous.

For (3), we define a partial surjection $\delta :\subseteq \omega^\omega \to \mathcal{QA}$ as follows.

$$\begin{aligned}
\delta(0^\omega) &= (0,\omega^3),\ \delta(10^\omega) = (\overline{0},\omega^3),\\
\delta(00^j1k\ell\tau) &= (\ell+1,\omega^2\cdot j+\omega\cdot(2k)+2\ell+1),\\
\delta(10^j1k\ell\tau) &= (\overline{\ell+1},\omega^2\cdot j+\omega\cdot(2k+1)+2\ell+1),\\
\delta(2k0^\omega) &= (0_\zeta,\omega^2\cdot k+1),\\
\delta(2k0^{2j}1\ell\tau) &= ((-\ell-1)_\zeta,\omega^2\cdot k+\omega\cdot(2j)+2\ell+2),\\
\delta(2k0^{2j+1}1\ell\tau) &= ((\ell+1)_\zeta,\omega^2\cdot k+\omega\cdot(2j+1)+2\ell+2),\\
\delta(3k\ell0^\omega) &= (\infty,\omega^2\cdot k+\omega\cdot(2\ell+1)),\\
\delta(3k\ell0^{2j}1\tau) &= (j+1,\omega^2\cdot k+\omega\cdot(2\ell)+2j+1),\\
\delta(3k\ell0^{2j+1}1\tau) &= ((-j-1)_\zeta,\omega^2\cdot k+\omega\cdot(2\ell)+2j+2),\\
\delta(4k\ell0^\omega) &= (\overline{\infty},\omega^2\cdot k+\omega\cdot(2\ell+2)),\\
\delta(4k\ell0^{2j}1\tau) &= (\overline{j+1},\omega^2\cdot k+\omega\cdot(2\ell+1)+2j+1),\\
\delta(4k\ell0^{2j+1}1\tau) &= ((j+1)_\zeta,\omega^2\cdot k+\omega\cdot(2\ell+1)+2j+2).
\end{aligned}$$

where $j,k,\ell\in\omega$, and τ is an arbitrary finite string. It is not hard to check that δ is open continuous. A similar argument applies to $\mathcal{QR}$.

For (4), we define a partial surjection $\delta :\subseteq \omega^\omega \to \mathcal{L}_{IL}$ as follows.

$$\begin{aligned}
\delta(0^\omega) &= (\infty,\infty), & \delta(00^jab\tau) &= (j+a,j+b),\\
\delta(1n0^\omega) &= (n,\infty), & \delta(1n0^j1\tau) &= (n,j).
\end{aligned}$$

It is clear that the domain of δ is a closed subset of ω^ω. It is also easy to check that δ is open continuous.

For (5), a neighborhood basis of $x\in\overline{\mathbb{R}_<}$ is given by $\{q\in\mathbb{Q} : q<x\}$. Being unbounded and downward closed is a Π^0_2 condition. Thus, Proposition 2.6 ensures that $\overline{\mathbb{R}_<}$ is quasi-Polish. □

We also show that several natural spaces are not quasi-Polish.

Proposition 3.72.

(1) *(De Brecht) The Gandy-Harrington space* $(\omega^\omega)_{GH}$ *is not quasi-Polish.*
(2) *The Golomb space* $\mathbb{N}_{\mathrm{rp}}$ *(see Section 5.3) is not quasi-Polish.*
(3) *The maximal antichain space* $\mathcal{A}^{\mathrm{co}}_{\max}$ *is not quasi-Polish.*

Proof. (1) Let A be a (light-face) strictly co-analytic subset of Baire space, and let A' be the same subset inside $(\omega^\omega)_{GH}$. Then A' is closed in $(\omega^\omega)_{GH}$, so if $(\omega^\omega)_{GH}$ were quasi-Polish, then A' as a subspace of $(\omega^\omega)_{GH}$ would be quasi-Polish, too. Thus, there would exist a continuous function $g : (\omega^\omega) \to (\omega^\omega)_{GH}$ such that $g(\omega^\omega) = A'$. Note that $\mathrm{id} : (\omega^\omega)_{GH} \to (\omega^\omega)$ is trivially continuous, and that $A = (\mathrm{id} \circ g)(\omega^\omega)$, i.e. A is a continuous image of Baire space, hence analytic, which contradicts the choice of A as a strictly co-analytic set.

(2) As mentioned before 2.6, every $\undertilde{\mathbf{\Pi}}^0_2$ subspace of a quasi-Polish space is quasi-Polish ([15, Theorem 24]). Also, it is well-known that $\mathbb{Q}$ is not (quasi-)Polish. Thus, it suffices to show that $\mathbb{Q}$ embeds into $\mathbb{N}_{\mathrm{rp}}$ as a Π^0_2-subspace. Inductively choose $n(s)$ as a number satisfying $1 + \sum_{i<s} \prod_{k \leq n(i)} p_k < p_{n(s)}$. Then, given $b = b_0 b_1 b_2 \dots$ define

$$h(b) = 1 + \sum_i b_i \prod_{k \leq n(i)} p_k.$$

Claim. h is a computable embedding of the dyadic rationals into $\mathbb{N}_{\mathrm{rp}}$.

For computability of h, to check whether $h(b) \equiv u \bmod v$, let p_k be the largest prime factor of v. Then there is s such that $k \leq n(s)$. Then, to compute the value of $h(b)$ mod v, we only need to check the first s terms of $h(b)$.

For computability of h^{-1}, given $h(b)$, inductively assume that we have already computed $b_0, b_1, \dots, b_{s-1}$. To compute b_s, let $r_i = \prod_{k \leq n(i)} p_k$. We have already computed $h(b)[s] := 1 + \sum_{i<s} b_i r_i$. By our choice of $n(s+1)$ we have $h(b)[s] + r(s) < p_{n(s+1)}$. Hence, since $p_{n(s+1)}$ is prime, we have $\gcd(h(b)[s], p_{n(s+1)}) = 1$, and $\gcd(h(b)[s] + r(s), p_{n(s+1)}) = 1$. Thus, to the relatively prime integer topology, we can ask whether $h(b) \equiv h(b)[s] \bmod p_{n(s+1)}$, or $h(b) \equiv h(b)[s] + r(s) \bmod p_{n(s+1)}$. If the former holds, then $b_s = 0$, and if the latter holds, then $b_s = 1$.

(3) By Theorem 2.9, ω_{cof} is not quasi-Polish. Thus, as before, it suffices to show that ω_{cof} embeds into the maximal antichain space $\mathcal{A}^{\mathrm{co}}_{\max}$ as a Π^0_2-subspace. For that we map $n \in \mathbb{N}$ to the complement of the set of all strings of length n. It is not hard to check that this gives indeed an embedding. To see that the range is Π^0_2, note that we can define it as *for any pair of words of distinct length, at least one of them belongs to the complement of the antichain.* □

For (1), [57, Theorem 6.1] by Mummert and Stephan implies that the Gandy-Harrington space can be represented as the maximal elements of an ω-algebraic domain. Therefore, the Gandy-Harrington space embeds as a (necessarily strict) co-analytic subset of a quasi-Polish space.

4. Cs-networks and non-second-countability

In this section, we develop techniques which will be used in the next section. As we mentioned repeatedly, one can develop computability theory on some non-second-countable spaces (without using notions from higher computability theory such as

α-recursion, E-recursion, infinite time Turing machines, etc.) To explain this idea, we consider the following basic notion in general topology (originally introduced by Arhangel'skii in 1959).

Definition 4.1. Let $\mathcal{X}$ be a topological space, and $\mathcal{N}$ be a collection of subsets of $\mathcal{X}$. We say that $\mathcal{N}$ is a *network at a point* $x \in \mathcal{X}$ if for any open neighborhood U of x, there is $N \in \mathcal{N}$ such that $x \in N \subseteq U$. Moreover, if $\mathcal{N}$ is a network at x, and if $x \in N$ holds for all $N \in \mathcal{N}$, then we also say that $\mathcal{N}$ is a *strict network at* x.

We now consider a space $\mathcal{X}$ which has no countable basis, but has a countable network $\mathcal{N} = (N_e)_{e\in\omega}$. Recall that by the "*degree of a point* x" in a (represented) cb_0 space, we meant the *degree of difficulty of enumerating a neighborhood basis of* x. However, if a space is non-second-countable, there may be no ω-step enumeration of a neighborhood basis of a point. Instead, we consider the *degree of difficulty of enumerating a strict subnetwork of* $\mathcal{N}$ *at* $x \in \mathcal{X}$. That is, we consider the following representation:

$$p \text{ is a name of } x \iff \{N_{p(n)} : n \in \omega\} \text{ is a strict network at } x.$$

Now, the induced computability theory on $\mathcal{X}$ heavily depends on the choice of a network $\mathcal{N}$. Of course, the same was true for a basis representation. But the situation regarding a network is worse than the case of a basis. On the one hand, one can always recover the topology on $\mathcal{X}$ from a basis, and thus, any representations yield the same computability natures relative to some oracle. On the other hand, a network does not memorize information on topology, and thus, the computability structure induced from a network can be almost arbitrary. In summary, the notion of a network is too weak, and therefore, we need a more restrictive notion.

A number of variants of a network have been extensively studied in general topology (see [33, 49, 46]). Schröder [66, 65] clarified that the following variant captures the territory of computability theory.

Definition 4.2 (Guthrie [35])**.** A *cs-network* $\mathcal{N}$ for a topological space $\mathcal{X}$ is a collection of subsets of $\mathcal{X}$ such that, for any open set $U \subseteq \mathcal{X}$, if a sequence $(x_n)_{n\in\omega}$ converges to $x \in U$, then there are $N \in \mathcal{N}$ and $n_0 \in \omega$ such that

$$\{x\} \cup \{x_n : n \geq n_0\} \subseteq N \subseteq U.$$

Here, "cs" stands for "convergent sequence". The following implications are clear:

$$\text{basis} \implies \text{cs-network} \implies \text{network}.$$

For notational simplicity, in this article, we assume that a *cs-network* $\mathcal{N}$ *always contains the whole space*, that is, $\mathcal{X} \in \mathcal{N}$. Schröder [66, 65] showed that a topological space $\mathcal{X}$ has an admissible representation if and only if $\mathcal{X}$ is a T_0 space with a countable cs-network. Since then, the notion of a countable cs-network have become a key notion in the context of *a convenient category of domains* [25, 6, 7][4].

More explicitly, if $\mathcal{N} = (N_e)_{e\in\omega}$ is a countable cs-network for a T_0 space $\mathcal{X}$, recall from Section 2.4.4 that the induced ω^ω-representation of $\mathcal{X}$ from $\mathcal{N}$ is given as follows:

$$\delta_{\mathcal{N}}(p) = x \iff \{N_{p(n)} : n \in \omega\} \text{ is a strict network at } x.$$

[4]In [66, 65, 25, 7], a cs-network is called a *pseudobase* or a *sequential pseudobase*.

This map $\delta_{\mathcal{N}}$ always gives an admissible representation of $\mathcal{X}$ ([66, Theorem 12]). Note that the convention $\mathcal{X} \in \mathcal{N}$ makes it possible for p to output no information at each stage. Recall from Section 2.4.4 the definition of reducibility $\leq_{\mathbf{T}}$. By $y\colon \mathcal{Y} \leq_{\mathbf{T}} x\colon \mathcal{X}$ we mean that there is a partial computable function which, given an $\mathcal{X}$-name of x, returns a $\mathcal{Y}$-name of y.

Observation 4.3. *Let $\mathcal{X} = (X, \beta)$ be a represented cb_0 space, and $\mathcal{Y} = (Y, \mathcal{N})$ be a topological space with a countable cs-network. Then $y\colon \mathcal{Y} \leq_{\mathbf{T}} x\colon \mathcal{X}$ if and only if there is $J \leq_e \mathrm{Nbase}_{\mathcal{X}}(x)$ such that $\{N_e : e \in J\}$ is a strict network at y.*

Proof. The "if" direction is obvious. Assume that $y\colon \mathcal{Y} \leq_{\mathbf{T}} x\colon \mathcal{X}$. Then, there is a computable function Φ such that if p enumerates $\mathrm{Nbase}_{\mathcal{X}}(x)$, then $\Phi(p)$ enumerates a strict subnetwork of $\mathcal{N}$ at y. We define Ψ as follows:

$$\langle e, \mathrm{rng}(\tau)\rangle \in \Psi \iff (\exists n)\ \Phi(\tau)(n)\downarrow= e.$$

Clearly, Ψ is c.e. Then, we define $J = \Psi(\mathrm{Nbase}_{\mathcal{X}}(x))$. We claim that $\{N_e : e \in J\}$ is a strict network at y. For any open neighborhood U of y, if p enumerates $\mathrm{Nbase}_{\mathcal{X}}(x)$, then $\Phi(p \upharpoonright s)(n)$ must output e such that $y \in N_e \subseteq U$ for some $e, n, s \in \omega$. Since $\mathrm{rng}(p \upharpoonright s) \subseteq \mathrm{Nbase}_{\mathcal{X}}(x)$, we have $e \in J$. Therefore, $\{N_e : e \in J\}$ is a network at y. For strictness, suppose that $y \notin N_e$ for some $e \in J$. Then, there are τ, n such that $\mathrm{rng}(\tau) \subseteq \mathrm{Nbase}_{\mathcal{X}}(x)$ and $\Phi(\tau)(n)\downarrow= e$. Clearly, τ can be extended to an $\mathcal{X}$-name p of x; however, we have $y \notin N_{\Phi(p)(n)}$, and thus $\Phi(p)$ is not an enumeration of a strict network at y, which is a contradiction. $\square$

4.1. **Regular-like networks and closure representation.** For a topological space $\mathcal{X}$ with a countable network $\mathcal{N}$, we introduce a new represented space with the underlying space $\mathcal{X}$ names of whose points are given by a sequence of closures of network elements whose intersection captures the point. Formally, we define that p is a $\overline{\delta_{\mathcal{N}}}$-name of x if and only if

$$\{N_{p(n)} : n \in \omega\} \text{ is a network at } x, \text{ and } x \in \overline{N_{p(n)}} \text{ for all } n \in \omega.$$

Here, we do not require $\{N_{p(n)} : n \in \omega\}$ to be strict, that is, $x \notin N_{p(n)}$ can happen, while we always have $x \in \overline{N_{p(n)}}$. At first glance this definition may look very strange; however, we will later see that this is a very useful technical notion. In this section, we investigate how $(\mathcal{X}, \delta_{\mathcal{N}})$ and $(\mathcal{X}, \overline{\delta_{\mathcal{N}}})$ are related, and we will show the following.

Theorem 4.4. *Assume that $\mathcal{X}$ is a topological space with a countable cs-network.*

1. *If $\mathcal{X}$ is regular and Hausdorff, then $\mathcal{X}$ has a countable cs-network $\mathcal{N}$ such that $(\mathcal{X}, \delta_{\mathcal{N}})$ is isomorphic to $(\mathcal{X}, \overline{\delta_{\mathcal{N}}})$.*
2. *There is a non-regular Hausdorff space $\mathcal{X}$ which has a countable cs-network $\mathcal{N}$ such that $(\mathcal{X}, \delta_{\mathcal{N}})$ is isomorphic to $(\mathcal{X}, \overline{\delta_{\mathcal{N}}})$.*

Before proving this theorem, we have to warn the reader that $\overline{\delta_{\mathcal{N}}}$ may be a multi-representation in general (as studied e.g. by Weihrauch in [76]), that is, a single p can be a name of many points. For instance, if $\mathcal{N}$ is an open network (i.e. basis) of $(\omega^\omega)_{\mathrm{co}}$, then $\mathrm{id} : \omega \to \omega$ is an $\overline{\delta_{\mathcal{N}}}$-name of any point $x \in (\omega^\omega)_{\mathrm{co}}$. Then, when does p determine a single point x? It is only if $\{\bigcap_n \overline{N_{p(n)}}\}$ is a strict network at x. We first check that it is always true if $\mathcal{X}$ is Hausdorff.

Observation 4.5. *If a Hausdorff space $\mathcal{X}$ has a countable network $\mathcal{N}$, then $\overline{\delta_{\mathcal{N}}}$ is a (single-valued) representation of $\mathcal{X}$.*

Proof. Note that $\mathcal{X}$ is Hausdorff if and only if every point in $\mathcal{X}$ can be written as the intersection of all its closed neighborhoods. For a collection $\mathcal{M}$ of subsets of $\mathcal{X}$, we write $\overline{\mathcal{M}} = \{\overline{M} : M \in \mathcal{M}\}$. Thus, $\{x\} = \bigcap \overline{\mathcal{O}_x}$, where $\mathcal{O}_x$ is the set of all open neighborhoods of x. Now, assume that p is a $\overline{\delta_{\mathcal{N}}}$-name of $x_0, x_1 \in \mathcal{X}$. Then, for each $i < 2$, $\mathcal{N}_p = \{N_{p(n)} : n \in \omega\}$ forms a network at x_i, and therefore $x_i \in \bigcap \overline{\mathcal{N}_p} \subseteq \bigcap \overline{\mathcal{O}_{x_i}} = \{x_i\}$. Hence, $\bigcap \overline{\mathcal{N}_p} = \{x_0\} = \{x_1\}$, which implies $x_0 = x_1$. $\square$

We should be careful that, even if $\mathcal{X}$ is Hausdorff, $\{\overline{N_{p(n)}} : n \in \omega\}$ is not necessarily a network at x, that is, there may exist an open neighborhood U of x such that $\overline{N_{p(n)}} \not\subseteq U$ for all $n \in \omega$, while we eventually have $\bigcap_n \overline{N_{p(n)}} \subseteq U$.

Now, it is clear that the identity map $\mathrm{id} : (\mathcal{X}, \delta_{\mathcal{N}}) \to (\mathcal{X}, \overline{\delta_{\mathcal{N}}})$ is computable (in the sense of Section 2.4), that is, there is a partial computable function F such that $\overline{\delta_{\mathcal{N}}}(F(p)) = \delta_{\mathcal{N}}(p)$ for any $\delta_{\mathcal{N}}$-name p of a point in $\mathcal{X}$. Under a certain assumption on a network, we also have computability of its inverse.

Observation 4.6. *If a topological space $\mathcal{X}$ has a countable closed cs-network $\mathcal{N}$, then the identity map* $\mathrm{id} : (\mathcal{X}, \overline{\delta_{\mathcal{N}}}) \to (\mathcal{X}, \delta_{\mathcal{N}})$ *is computable.*

Proof. Assume that $\mathcal{X}$ has a countable closed cs-network $\mathcal{N}$. Let p be a $\overline{\delta_{\mathcal{N}}}$-name of a point $x \in \mathcal{X}$. Since $\mathcal{N}$ is closed, $\overline{\mathcal{N}_{p(n)}} = \mathcal{N}_{p(n)}$, and therefore, $\{\mathcal{N}_{p(n)} : n \in \omega\}$ forms a strict network at x. Thus, p is also a $\delta_{\mathcal{N}}$-name of x. Hence, the identity map $\mathrm{id} : (\mathcal{X}, \overline{\delta_{\mathcal{N}}}) \to (\mathcal{X}, \delta_{\mathcal{N}})$ is computable. $\square$

Recall that a topological space $\mathcal{X}$ is *regular* if for any open neighborhood U of a point $x \in \mathcal{X}$, there is an open neighborhood V of x such that $x \in V \subseteq \overline{V} \subseteq U$. It is equivalent to saying that if $\mathcal{M}$ is a neighborhood basis at x, then so is $\overline{\mathcal{M}} = \{\overline{N} : N \in \mathcal{M}\}$. This holds for $\overline{\mathcal{M}}^+$ as well, where $\overline{\mathcal{M}}^+ = \{\overline{N}_1 \cap \cdots \cap \overline{N}_\ell : N_1, \dots, N_\ell \in \mathcal{M}\}$. We say that a network $\mathcal{N}$ of $\mathcal{X}$ is *regular-like* if for any $\mathcal{M} \subseteq \mathcal{N}$ and $x \in \mathcal{X}$, if $\mathcal{M}$ is a network at x, then so is $\overline{\mathcal{M}}^+$. The definition does not change if we consider only the case where $\mathcal{M}$ is strict.

It is clear that every closed network is regular-like. The converse is not always true, but it is easy to see that a space has a closed network of cardinality κ iff it has a regular-like network of cardinality κ. In particular, if a space has a countable regular-like network, it is a G_δ-space by Proposition 3.56. However, it is unclear if every space with a regular-like cs-network always has a closed cs-network.

Observation 4.7. *Every network of a regular space is regular-like.*

Proof. Let $\mathcal{N}$ be a network for a regular space $\mathcal{X}$, and let $\mathcal{M} \subseteq \mathcal{N}$ be a network at $x \in \mathcal{X}$. Given an open neighborhood U of x, by regularity of $\mathcal{X}$, there is an open neighborhood V of x such that $x \in V$ and $\overline{V} \subseteq U$. Since $\mathcal{M}$ is a network for $\mathcal{X}$, there is $N \in \mathcal{M}$ such that $x \in N \subseteq V$. Therefore, $x \in \overline{N} \subseteq \overline{V} \subseteq U$. This shows that $\overline{M} = \{\overline{N} : N \in \mathcal{M}\}$ is a network at x as well. Consequently, $\mathcal{N}$ is regular-like. $\square$

A regular space which has a countable network is known as a *cosmic space*, and a regular space which has a countable cs-network is known as an $\aleph_0$-*space* (see [52, 35]).

Theorem 4.8. *A topological space $\mathcal{X}$ has a countable regular-like cs-network if and only if $\mathcal{X}$ has a countable cs-network $\mathcal{N}$ such that the identity map* $\mathrm{id} : (\mathcal{X}, \overline{\delta_\mathcal{N}}) \to (\mathcal{X}, \delta_\mathcal{N})$ *is continuous.*

Proof. Assume that $\mathcal{X}$ has a countable regular-like cs-network $\mathcal{N}$. Then, $\mathcal{M} = \mathcal{N} \cup \overline{\mathcal{N}}^+$ is also a countable regular-like cs-network, and enumerate $\mathcal{M} = (M_e)_{e\in\omega}$. Given a $\overline{\delta_\mathcal{M}}$-name p, let $h(p)$ be an enumeration of all $e \in \omega$ such that $\overline{M}_{p\restriction s} \subseteq M_e$ for some s, where $\overline{M}_{p\restriction s} = \bigcap_{n<s} \overline{M_{p(n)}}$. Clearly, h is continuous. We claim that h realizes $\mathrm{id} : (\mathcal{X}, \overline{\delta_\mathcal{M}}) \to (\mathcal{X}, \delta_\mathcal{M})$, that is, if p is a $\overline{\delta_\mathcal{M}}$-name of a point $x \in \mathcal{X}$, then $h(p)$ is a $\delta_\mathcal{M}$-name of the same point x. For any n, it is clear that $x \in M_{h(p)(n)}$ since there is s such that $\overline{M}_{p\restriction s} \subseteq M_{h(p)(n)}$, and $x \in \overline{M}_{p\restriction s}$ for any s. It remains to show that $\{M_{h(p)(n)} : n \in \omega\}$ is a network at x. Let U be an open neighborhood of x. Since p is a $\overline{\delta_\mathcal{M}}$-name of x, $\{M_{p(n)} : n \in \omega\} \subseteq \mathcal{M}$ is a network at x. Since $\mathcal{M}$ is regular-like, $\{\overline{M}_{p\restriction s} : s \in \omega\}$ is also a network at x. Therefore, there is $n \in \omega$ such that $x \in \overline{M}_{p\restriction s} \subseteq U$. Clearly, $\overline{M}_{p\restriction s} \in \mathcal{M}$, and in particular, $\overline{M}_{p\restriction s} \subseteq M_e \subseteq U$ for some e. This means that $h(p)$ enumerates such e at some stage, that is, $M_{h(p)(t)} \subseteq U$ for some t. Hence, $h(p)$ is a $\delta_\mathcal{M}$-name of x. This verifies the claim.

Conversely, assume that $\mathcal{X}$ has a countable cs-network $\mathcal{N}$ such that the identity map $\mathrm{id} : (\mathcal{X}, \overline{\delta_\mathcal{N}}) \to (\mathcal{X}, \delta_\mathcal{N})$ is continuous. Assume that $\mathcal{M} \subseteq \mathcal{N}$ is a strict network at $x \in \mathcal{X}$. Then, any enumeration p of $\mathcal{M}$ is a $\overline{\delta_\mathcal{N}}$-name of a point $x \in \mathcal{X}$. Suppose for the sake of contradiction that $\overline{\mathcal{M}} = \{\overline{N}_{p\restriction n} : n \in \omega\}$ is not a network at x. Then, there is an open neighborhood U of x such that $\overline{N}_{p\restriction n} \not\subseteq U$ for any $n \in \omega$. Let h be a realizer of $\mathrm{id} : (\mathcal{X}, \overline{\delta_\mathcal{N}}) \to (\mathcal{X}, \delta_\mathcal{N})$. Then, $h(p)$ is a $\delta_\mathcal{N}$-name of x. Therefore, there is $s \in \omega$ such that $N_{h(p\restriction s)} \subseteq U$. Choose any $\overline{\delta_\mathcal{N}}$-name q of $y \in \overline{N}_{p\restriction s} \setminus U$. Then, $(p \restriction s)^\frown q$ is also a $\overline{\delta_\mathcal{N}}$-name of y. However, since $y \notin U \supseteq N_{h(p\restriction s)}$, $h((p \restriction s)^\frown q)$ cannot be a $\delta_\mathcal{N}$-name of y, which is a contradiction. Hence, $\mathcal{N}$ is regular-like. $\square$

In particular, if $\mathcal{N}$ is a countable cs-network of an $\aleph_0$-space $\mathcal{X}$, then the identity map $\mathrm{id} : (\mathcal{X}, \overline{\delta_\mathcal{N}}) \to (\mathcal{X}, \delta_\mathcal{N})$ is continuous.

We now turn look to second-countable spaces. Recall that a basis can be thought of as a cs-network, and thus every countable basis β also induces the representation $\overline{\beta} := \overline{\delta_\beta}$. Recall that the identity map $\mathrm{id} : (\mathcal{X}, \beta) \to (\mathcal{X}, \overline{\beta})$ is always computable. The continuity of the inverse requires regularity (and thus, metrizability if the space is T_0, since a second-countable T_0 space is regular if and only if it is metrizable).

Proposition 4.9. *Let $(\mathcal{X}, \beta)$ be a second-countable space. Then $\mathcal{X}$ is regular if and only if the identity map* $\mathrm{id}\colon (\mathcal{X}, \overline{\beta}) \to (\mathcal{X}, \beta)$ *is continuous.*

Proof. Assume that $\mathcal{X}$ is regular. We construct h witnessing that $\mathrm{id}\colon (\mathcal{X}, \overline{\beta}) \to (\mathcal{X}, \beta)$ is continuous. Let p be a $\overline{\beta}$-name of a point $x \in \mathcal{X}$. If p enumerates e, then $h(p)$ enumerates all d such that $\overline{\beta_e} \subseteq \beta_d$. We claim that $h(p)$ is a β-name of x. Note that $x \in \beta_{h(p)(n)}$ for any n. This is because $h(p)(n) = d$ only if there are $s \in \omega$ such that $\overline{\beta_{p(s)}} \subseteq \beta_e$, and moreover $x \in \overline{\beta_{p(s)}}$ since p is a $\overline{\beta}$-name of x. Now, to see that $\{\beta_{h(p)(n)} : n \in \omega\}$ is equal to $\mathrm{Nbase}(x)$, assume that $x \in \beta_e$. By regularity of $\mathcal{X}$, there is d such that $x \in \beta_d \subseteq \overline{\beta_d} \subseteq \beta_e$. Since p is a $\overline{\beta}$-name of x, there is n such that $x \in \beta_{p(n)} \subseteq \beta_d$. Since $\overline{\beta_{p(n)}} \subseteq \overline{\beta_d}$, by definition, $h(p)$ enumerates e. This verifies the claim.

Conversely, assume that the identity map $\mathrm{id}\colon (\mathcal{X},\overline{\beta}) \to (\mathcal{X},\beta)$ is continuous. Then, the proof of Observation 4.6 shows that β is regular-like. However, one can easily check that if a space has a regular-like basis, it is actually regular. □

It is worth noting that there is a *non-regular* Hausdorff space which has a countable regular-like cs-network.

Example 4.10. The Kleene-Kreisel space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is a non-regular Hausdorff space which has a countable closed cs-network (hence, has a countable regular-like cs-network). Here, the Kleene-Kreisel space consists of (total) continuous functions from Baire space ω^ω to the natural numbers ω endowed with the quotient topology given by the following map:

$$\delta(e^\frown z) = f \iff \Phi_e^z = f,$$

where Φ_e^z is the e-th partial z-computable function from ω^ω to ω, and the domain of δ is the set of all $e^\frown z$ such that Φ_e^z is total, i.e., $\mathrm{dom}(\Phi_e^z) = \omega^\omega$.

Define $N_{\sigma,n} = \{f : (\forall x \succ \sigma)\ f(x) = n\}$ for each $\sigma \in \omega^{<\omega}$ and $n \in \omega$, and then $\mathcal{N} = (N_{\sigma,n})$ forms a countable cs-network for the Kleene-Kreisel space. We claim that $N_{\sigma,n}$ is closed. One can see that $\delta(e^\frown z) \notin N_{\sigma,n}$ if and only if there are s, k, and τ such that $\Phi_e^{z\upharpoonright s}(\tau)\downarrow = k \neq n$, and τ is comparable with σ. Therefore, $\delta^{-1}[N_{\sigma,n}]$ is closed in ω^ω. Since the topology on $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is given by the quotient map δ, we conclude that $N_{\sigma,n}$ is closed in $\mathcal{K}$. That is, $\mathcal{N}$ is a closed network for $\mathcal{K}$. Finally, Schröder [68] has shown that the Kleene-Kreisel space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is not regular.

Hence, by Theorem 4.8, $\mathrm{id} : (\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}, \overline{\delta_{\mathcal{N}}}) \to (\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}, \delta_{\mathcal{N}})$ is continuous. Indeed, one can easily see that it is actually computable.

4.2. **Near quasi-minimality.** We now start to study computability w.r.t. closure representations.

Definition 4.11. Let $\mathcal{X} = (X, \mathcal{N})$ be a topological space X with a countable cs-network $\mathcal{N}$. We say that a point $x \in \mathcal{X}$ is *nearly computable* if $x\colon \overline{\delta_{\mathcal{N}}}$ is computable.

Clearly, every computable point is nearly computable. If $\mathcal{X} = [0,1]^{\mathbb{N}}$ or $\mathcal{X} = \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, then the converse is also true. By Observation 4.5, if $\mathcal{X}$ is Hausdorff, there are only countably many nearly computable points. If $\mathrm{id} : (\mathcal{X}, \overline{\delta_{\mathcal{N}}}) \to (\mathcal{X}, \delta_{\mathcal{N}})$ is computable, then near computability and computability coincide. For instance, $f \in \mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$ is nearly computable if and only if f is computable.

Definition 4.12. Let $\mathcal{X} = (X, \mathcal{N})$ and $\mathcal{Y} = (Y, \mathcal{M})$ be topological spaces with countable cs-networks. Then, we say that a point $x \in \mathcal{X}$ is *nearly $\mathcal{Y}$-quasi-minimal* if

$$(\forall y \in \mathcal{Y})\ [y\colon \mathcal{Y} \leq_{\mathbf{T}} x\colon \mathcal{X} \implies y \text{ is nearly computable}].$$

If computability and near computability are equivalent in a space $\mathcal{Y}$, then so are $\mathcal{Y}$-quasi-minimality and near $\mathcal{Y}$-quasi-minimality. For instance, near $[0,1]^{\mathbb{N}}$-quasi-minimality is equivalent to quasi-minimality, and near $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$-quasi-minimality is equivalent to $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$-quasi-minimality.

This notion and its variant will play important roles in Sections 5.2 and 5.3. But before that, we will show a few results on the closure representation.

4.3. **Borel extension topology.** Recall from Section 3.7.3 that the Gandy-Harrington topology is generated by Σ^1_1 sets. Then, it is natural to study topologies generated by lightface Borel pointclasses. Indeed, the topology generated by $\Sigma^1_1 \cap \mathbf{\Pi}^0_\xi$ sets has played a prominent role in the proof of Louveau's separation theorem [47]. The topology generated by Π^0_1 sets has also been used, for instance, by Miller [53] and Monin [56, Section 3.1].

In this section, we discuss topologies generated by collections of Σ^0_α sets. Our first motivation was, for instance, to understand the degree-theoretic behavior of 2^ω equipped with the standard Cantor topology plus Martin-Löf conull sets. Here, a Martin-Löf test is a computable sequence $(U_n)_{n\in\omega}$ of c.e. open sets such that $\mu(U_n) \leq 2^{-n}$ for any $n \in \omega$. Abusing the terminology, for such a sequence, we sometimes say that $\bigcap_{n\in\omega} U_n$ is a Martin-Löf test. We say that a set $A \subseteq 2^\omega$ is Martin-Löf null if there is a Martin-Löf test $(U_n)_{n\in\omega}$ such that $A \subseteq U_n$ for any n, and a set is Martin-Löf conull if its complement is Martin-Löf null.

However, in contrast to the Gandy-Harrington topology, we will see that such a space is uninteresting from the perspective of enumeration degrees because the degree structure is exactly the same as the total degrees relative to some oracle. In particular, it is σ-metrizable (recall the paragraph below Proposition 3.50). Note that the Gandy-Harrington space $(\omega^\omega)_{GH}$ is not σ-metrizable by relativizing Theorem 3.46 (see also Theorem 5.53) and by Kihara-Pauly [43].

By Proposition 4.9, we know that a represented cb$_0$ space $(\mathcal{X}, \beta)$ is metrizable if and only if the identity map is an isomorphism between $(\mathcal{X}, \beta)$ and $(\mathcal{X}, \overline{\beta})$. However, for nonmetrizable $(\mathcal{X}, \beta)$, this proposition does not ensure that $(\mathcal{X}, \beta)$ and $(\mathcal{X}, \overline{\beta})$ are not isomorphic. By using a Borel extension topology, we will see the following.

Proposition 4.13. *There is a represented, submetrizable, σ-metrizable, cb$_0$ space $(\mathcal{X}, \beta)$ such that $(\mathcal{X}, \overline{\beta})$ is not isomorphic to $(\mathcal{X}, \beta)$.*

A *filter $\mathcal{F}$ on $\mathcal{X}$* is a nonempty collection of nonempty subsets of $\mathcal{X}$ such that $A \in \mathcal{F}$ and $A \subseteq B$ implies $B \in \mathcal{F}$ and that $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$. A *generator* of $\mathcal{F}$ is a subcollection G of $\mathcal{F}$ such that $\mathcal{X} \in G$ and for every $A \in \mathcal{F}$ there is $B \in G$ such that $B \subseteq A$. If $(\mathcal{X}, \tau)$ is a topological space, we say that a filter $\mathcal{F}$ on $\mathcal{X}$ is *τ-consistent* if every $A \in \mathcal{F}$ is τ-dense, and $\mathcal{F}$ is *τ-nontrivial* if there is $A \in \mathcal{F}$ whose complement is τ-dense in some τ-open set. For instance, the collection of all Martin-Löf conull subsets of 2^ω forms a non-trivial filter, and the complements $(2^\omega \setminus \bigcap_n U_n)$ of Martin-Löf tests $(U_n)_{n\in\omega}$ form a countable generator of this filter.

Let G be a countable generator of a filter $\mathcal{F}$ on a second-countable space $(\mathcal{X}, \tau)$. Then consider the new topology τ_G on $\mathcal{X}$ generated by the following set:

$$\{A \cap U : A \in G \text{ and } U \in \tau\}.$$

Note that the new topology τ_G is second-countable since G is countable, and that τ_G is finer than τ since $\mathcal{X} \in G$. The latter implies that $(\mathcal{X}, \tau_G)$ is submetrizable whenever $(\mathcal{X}, \tau)$ is submetrizable. If $(\mathcal{X}, \tau)$ is represented by β and if G is enumerated as $(G_n)_{n\in\omega}$, then $(\mathcal{X}, \tau_G)$ is represented by $\beta^G_{m,n} = G_m \cap \beta_n$.

Example 4.14. All of the following examples are countable generators of a τ-consistent τ-nontrivial filter on a separable metrizable space.

(1) The generator $G = \{\mathbb{Q}, \mathbb{R}\}$ of a principal filter on the Euclidean line $\mathbb{R}$ yields the so-called indiscrete rational extension of $\mathbb{R}$ (see [74, II.66]). The generator $G = \{\mathbb{R} \setminus \mathbb{Q}, \mathbb{R}\}$ of a principal filter on the Euclidean line $\mathbb{R}$ yields the so-called indiscrete irrational extension of $\mathbb{R}$ (see [74, II.67]).
(2) Let MLR be the complements of (the intersections of) Martin-Löf tests on 2^ω. Then, MLR is a countable generator of the filter consisting of all measure 1 subsets of 2^ω. This yields the space $\mathcal{X}_{\mathsf{MLR}} := (2^\omega, \tau_{\mathsf{MLR}})$ where τ is the usual Cantor topology.

Lemma 4.15. *Let G be a countable generator of a τ-consistent filter $\mathcal{F}$ on $\mathcal{X}$. Then, the identity map yields a homeomorphism between $(\mathcal{X}, \overline{\beta^G})$ and $(\mathcal{X}, \overline{\beta})$.*

Proof. Let $A \cap U$ and $B \cap V$ be τ_G-open sets. The assumption $A, B \in G$ implies $A \cap B \in \mathcal{F}$, and therefore $A \cap B$ is τ-dense since $\mathcal{F}$ is τ-consistent. Therefore, if $A \cap U$ and $B \cap V$ are disjoint, then U and V must be disjoint. Consequently, the τ_G-closure of $A \cap U$ is exactly the τ-closure of U. □

Lemma 4.16. *Let G be a countable generator of a τ-consistent τ-nontrivial filter $\mathcal{F}$ on $\mathcal{X}$. Then, $(\mathcal{X}, \tau_G)$ is not metrizable.*

Proof. It suffices to show that $(\mathcal{X}, \tau_G)$ is not regular. Since $\mathcal{F}$ is τ-nontrivial, there are $D \in \mathcal{F}$ and $T \in \tau$ such that $T \setminus D$ is τ-dense in T. Then, there exists $x \in D \cap T$ since D is τ-dense. We claim that there is no disjoint pair of open sets separating x and D^{c}. Suppose that $x \in A \cap U$ and $D^{\mathsf{c}} \subseteq B \cap V$. As in the proof of Lemma 4.15, if $A \cap U$ and $B \cap V$ are disjoint, so are U and V. Since $T \setminus D$ is τ-dense in T, V must include T. However, $x \in T$ and therefore $U \cap V \neq \emptyset$. □

For a computable ordinal α, we say that a generator G is Σ^0_α-*generated* if every element of G is Σ^0_α in τ, and the union of a τ_G-open set and a Σ^0_α-in-τ set is τ_G-open. For instance, the complements of Martin-Löf tests is Σ^0_2-generated.

If $(\mathcal{X}, \tau)$ is Polish, the topology τ_α generated by Σ^0_α sets in τ yields a zero-dimensional metrizable topology whenever $\alpha > 0$. This is because the collection of all Π^0_β-sets in τ for $\beta < \alpha$ forms a basis of the topology τ_α, and each Π^0_β set is τ_α-clopen. Therefore, τ_α has a basis consisting of clopen sets $(C_n)_{n\in\omega}$ and then the metric d_α on $(\mathcal{X}, \tau_\alpha)$ is given by $d_\alpha(x, y) = 2^{-n}$ where n is the least index n such that C_n separates x and y.

Lemma 4.17. *Let $(\mathcal{X}, \tau)$ be a Polish space. If G is Σ^0_α-generated, then $(\mathcal{X}, \tau_G)$ is σ-metrizable.*

Proof. We say that x is G-*quasi-generic* if $x \in A$ for any $A \in G$. If x is G-quasi-generic, $x \in A \cap U$ if and only if $x \in U$. Therefore, $(\mathcal{X} \upharpoonright \mathcal{R}_G, \tau_G)$ is homeomorphic to $(\mathcal{X} \upharpoonright \mathcal{R}_G, \tau)$ where $\mathcal{R}_G$ is the set of all G-quasi-generic points.

If x is not G-quasi-generic, there is $A \in G$ such that $x \notin A$. Let $\mathcal{S}_e$ be the set of all $x \in \mathcal{X}$ such that $x \notin A_e$ where A_e is the eth element of G. We claim that $(\mathcal{X} \upharpoonright \mathcal{S}_e, \tau_G)$ is homeomorphic to $(\mathcal{X} \upharpoonright \mathcal{S}_e, \tau_\alpha)$ for any e. It is clear that the identity map $\mathrm{id} : (\mathcal{X}, \tau_\alpha) \to (\mathcal{X}, \tau_G)$ is continuous, since G is Σ^0_α-generated, which implies that τ_α is finer than τ_G. For any $x \in \mathcal{S}_e$, given Σ^0_α set S, $x \in A_e$ if and only if $x \in A_e \cup S$. Since G is Σ^0_α-generated, we always have $A_e \cup S \in G$ and hence $A_e \cup S$ is τ_G-open. Therefore, the identity map $(\mathcal{X} \upharpoonright \mathcal{S}_e, \tau_G) \to (\mathcal{X} \upharpoonright \mathcal{S}_e, \tau_\alpha)$ is also continuous. □

Proof of Proposition 4.13. Let $(\mathcal{X},\tau)$ be a separable metrizable space and G be a countable Σ^0_α-generated generator of a τ-consistent τ-nontrivial filter on $\mathcal{X}$. For instance, let $(\mathcal{X},\tau)$ be Cantor space, and G be the collection of complements of Martin-Löf tests. Then, $(\mathcal{X},\tau_G)$ is submetrizable, and σ-metrizable by Lemma 4.17. Given a representation of $\mathcal{X}$, consider $\mathcal{X}_G=(\mathcal{X},\beta^G)$. Then $(\mathcal{X},\overline{\beta^G})$ is isomorphic to $(\mathcal{X},\overline{\beta})$ by Lemma 4.15, and this is isomorphic to $(\mathcal{X},\beta)$ since β is metrizable. Then, since $\mathcal{X}_G$ is non-metrizable by Lemma 4.16, we know that $(\mathcal{X},\beta^G)$ is not isomorphic to $(\mathcal{X},\overline{\beta^G})$. □

5. Further degree theoretic results

In the light of Theorem 3.40 showing that any enumeration degree can be realized in a decidable, submetrizable, cb_0 space, it may seem that separating separation axioms via degree-theoretic properties is not possible. One may overcome this difficulty by restricting our attention to countably many collections of spaces. We adopt this approach, and we often show strong separating results by considering the notion of quasi-minimality. A large part of computability theory is concerned with measuring the relative computational content between two sets or objects; this has led to the introduction of *reducibilities* such as the enumeration reducibility considered in this paper, and the degree structures arising thereof. In order to show that a degree $\boldsymbol{a}$ does not have property P, one only has to show that A is not *equivalent* to any set with property associated with property P, where A is a representative of the degree $\boldsymbol{a}$. However, such a result merely states that A does not have the same relative computational strength as any set having property P. A much more compelling way to argue that A is disassociated with P, is to prove that A does not even reduce any set with property P. For example, a quasi-minimal e-degree is much stronger than being a non-total e-degree (even the proof of both are not that different). In this spirit, we will introduce the notion of being quasi-minimal with respect to a (collection of) represented spaces in order to strongly separate different collection of spaces. Some of these quasi-mimimal separation of spaces will turn out to be much harder than a degree-theoretic separation.

Definition 5.1. Let $\mathcal{T}$ be a collection of represented spaces. We say that $x\colon \mathcal{X}$ is *$\mathcal{T}$-quasi-minimal in $y\colon \mathcal{Y}$* if

$$(\forall \mathcal{Z}\in\mathcal{T})(\forall z\in\mathcal{Z})\ z\colon\mathcal{Z}\leq_{\mathrm{T}} x\colon\mathcal{X} \implies z\colon\mathcal{Z}\leq_{\mathrm{T}} y\colon\mathcal{Y}.$$

i.e. if x cannot compute any more points in spaces from $\mathcal{T}$ than y can. If in addition it holds that $y\colon\mathcal{Y}<_{\mathrm{T}} x\colon\mathcal{X}$, we say that $x\colon\mathcal{X}$ a *strong $\mathcal{T}$-quasi-minimal cover of $y\colon\mathcal{Y}$*.

We just say that $x\colon\mathcal{X}$ is $\mathcal{T}$-quasiminimal, if $x\colon\mathcal{X}$ is a strong $\mathcal{T}$-quasi-minimal cover of $0^\omega\colon 2^\omega$, where $0^\omega\in 2^\omega$ is the computable infinite sequence consisting only of 0's (of course, one may replace 0^ω with any computable sequence).

Recall that an e-degree $\mathbf{a}$ is *quasi-minimal* if for every total degree $\mathbf{b}\leq_e\mathbf{a}$, we have $\mathbf{b}=\mathbf{0}$. Medvedev [51] first constructed a quasi-minimal e-degree by showing that every enumeration 1-generic is quasi-minimal. Our definition is a generalization, in as far as an enumeration degree is quasi-minimal iff it is $\{2^\omega\}$-quasi-minimal.

We use $\mathcal{D}_{\mathcal{T}}$ to denote the union $\bigcup\{\mathcal{D}_{\mathcal{Z}}:\mathcal{Z}\in\mathcal{T}\}$.

5.1. T_0-degrees which are not T_1.

5.1.1. *Quasi-minimality.* We return to the degree spectrum of $\mathbb{R}_<$ considered in Subsection 3.2, and recall that these degrees are just the semirecursive enumeration degrees. Arslanov-Cooper-Kalimullin [3] showed that if a point $x \in \mathbb{R}_<$ is neither left- nor right-c.e., then it has a quasi-minimal enumeration degree. We can relativize this property in the following sense.

Lemma 5.2 (see also Kihara-Pauly [43])**.** *Let $\mathcal{X}$ be any represented cb_0 space, $x \in 2^\omega$, $y \in \mathbb{R}_<$, and $z \in \mathcal{X}$. If $\mathrm{Nbase}_{2^\omega}(x) \leq_e \mathrm{Nbase}_<(y) \oplus \mathrm{Nbase}_{\mathcal{X}}(z)$, then either $\mathrm{Nbase}_{2^\omega}(x) \leq_e \mathrm{Nbase}_{\mathcal{X}}(z)$ or $\mathrm{Nbase}_<(-y) \leq_e \mathrm{Nbase}_{\mathcal{X}}(z)$ holds.*

Proof. Assume that $\mathrm{Nbase}_{2^\omega}(x) \leq_e \mathrm{Nbase}_<(y) \oplus \mathrm{Nbase}_{\mathcal{X}}(z)$. Then, there is a c.e. set $\Phi \subseteq \omega \times 2 \times \mathbb{Q} \times \omega$ such that for any $n \in \omega$ and $i < 2$,

$$x(n) = i \iff (\exists p \in \mathbb{Q})(\exists e \in \omega)\ [p < y,\ e \in \mathrm{Nbase}(z), \text{ and } \langle n, i, p, e\rangle \in \Phi].$$

Suppose that there is $\varepsilon > 0$ such that for any $p < y+\varepsilon$ and $e \in \mathrm{Nbase}(z)$, $\langle n, i, p, e\rangle \in \Phi$ implies $x(n) = i$. In this case, fix a rational $q \in \mathbb{Q}$ such that $y < q < y + \varepsilon$. Then,

$$x(n) = i \iff (\exists p < q)(\exists e \in \omega)\ [e \in \mathrm{Nbase}(z), \text{ and } \langle n, i, q, e\rangle \in \Phi].$$

This shows $\mathrm{Nbase}_{2^\omega}(x) \leq_e \mathrm{Nbase}_{\mathcal{X}}(z)$. Hereafter we assume that y is irrational; otherwise y is computable and thus $\mathrm{Nbase}_<(-y) \leq_e \mathrm{Nbase}_{\mathcal{X}}(z)$ trivially holds.

Otherwise, for all $\varepsilon > 0$, there are $p < y+\varepsilon$ and $e \in \mathrm{Nbase}(z)$ such that $\langle n, i, p, e\rangle \in \Phi$ but $x(n) \neq i$. We claim that $y < p$ if and only if there are n, i, q, d, e such that

$$q < p \ \ \&\ \ d, e \in \mathrm{Nbase}(z) \ \ \&\ \ \langle n, 1-i, q, d\rangle \in \Phi \ \ \&\ \ \langle n, i, p, e\rangle \in \Phi.$$

If $y < p$ is not true, we have $p < y$ since y is irrational, and then $q < p$ implies $q < p < y$. Since $\mathrm{Nbase}_{2^\omega}(x) \leq_e \mathrm{Nbase}_<(y) \oplus \mathrm{Nbase}_{\mathcal{X}}(z)$, for any $d, e \in \mathrm{Nbase}(z)$, whenever $\langle n, i, q, d\rangle$ and $\langle n, j, p, e\rangle$ are enumerated into Φ, we must have $i = j$. If $y < p$, then by our assumption, there are $\hat{p} < p$ and $e \in \mathrm{Nbase}(z)$ such that $\langle n, i, \hat{p}, e\rangle \in \Phi$ but $x(n) \neq i$. By monotonicity, one can assume that $\hat{p} = p$. Since $\mathrm{Nbase}_{2^\omega}(x) \leq_e \mathrm{Nbase}_<(y) \oplus \mathrm{Nbase}_{\mathcal{X}}(z)$, there also exist $q < y$ and $d \in \mathrm{Nbase}(z)$ such that $\langle n, j, q, d\rangle \in \Phi$ and $x(n) = j$, i.e., $j = 1 - i$. This verifies the claim.

Now note that $y < p$ if and only if $-p \in \mathrm{Nbase}_<(-y)$. Thus, by the above claim, we conclude $\mathrm{Nbase}_<(-y) \leq_e \mathrm{Nbase}(z)$. ☐

The total degrees in $\mathbb{R}_<$ can be characterized as follows.

Proposition 5.3. *Let* **a** *be an $\mathbb{R}_<$-degree. Then, the following are equivalent.*

(1) **a** *is a total e-degree.*
(2) **a** *is a Π^0_1 e-degree.*
(3) **a** *is the e-degree of* $\mathrm{Nbase}(x)$ *of a left- or right-c.e. real $x \in \mathbb{R}_<$.*
(4) **a** *is not quasi-minimal.*

Proof. Fix $x \in \mathbb{R}_<$. If x is rational, then x satisfies all conditions (1)–(4). We now assume that x is irrational. Clearly, the conditions (2) and (3) are equivalent. We show the equivalence of (1) and (3). If x is right-c.e., $\mathrm{Nbase}_{\mathbb{R}_<}(-x)$ is c.e., and thus $\mathrm{Nbase}_<(x)^{\mathsf{c}}$ is c.e. Hence, $\mathrm{Nbase}_<(x) \oplus \mathrm{Nbase}_<(x)^{\mathsf{c}} \equiv_e \mathrm{Nbase}_<(x)$. Thus, $\mathrm{Nbase}_<(x)$ is total. Conversely, if $A \oplus A^{\mathsf{c}} \equiv_e \mathrm{Nbase}_<(x)$ for a set $A \subseteq \mathbb{N}$, by Lemma 5.2, we have $A \oplus A^{\mathsf{c}}$ is c.e. (thus $\mathrm{Nbase}_<(x)$ is c.e.) or $\mathrm{Nbase}_<(-x)$ is c.e. In other words, x is either left- or right-c.e. The equivalence of (3) and (4) follows from Lemma 5.2. ☐

In particular, if $\mathrm{Nbase}_<(x) \leq_e \mathrm{Nbase}_<(y)$ and x is right-c.e., but not left-c.e., then y is right-c.e. Note that the structure $\mathcal{D}_{\mathbb{R}_<}$ has a maximal element, i.e., the e-degree of K^{c}. One can also see that the structure of the c.e. degrees $(\mathcal{R}, \leq_T)$ is isomorphic to the substructure of $\mathcal{D}_{\mathbb{R}_<}$ consisting of the right-c.e. reals (which are exactly the Π^0_1 e-degrees by Proposition 5.3).

Here, we review the definition of function n-genericity as introduced by Copestake [13]. Fix a new symbol $\perp \notin \omega$, and assume that $\omega \cup \{\perp\}$ is endowed with the discrete topology, that is, fix an effective bijection between ω and $\omega \cup \{\perp\}$. Then, $(\omega \cup \{\perp\})^\omega$ is effectively homeomorphic to Baire space ω^ω. Thus, the $(\omega \cup \{\perp\})^\omega$-degrees are the total degrees. Indeed, for any $g \in (\omega \cup \{\perp\})^\omega$,

$$\mathrm{Graph}(g) \equiv_e \{\langle n, m+1\rangle : g(n) = m\} \cup \{\langle n, 0\rangle : g(n) = \perp\}.$$

From $g \in (\omega \cup \{\perp\})^\omega$ we get a partial function $\hat{g} :\subseteq \omega \to \omega$ by interpreting $\perp$ as "undefined". Then,

$$\mathrm{Graph}(\hat{g}) = \{\langle n, m\rangle \in \omega^2 : g(n) \neq \perp, \text{ and } g(n) = m\}.$$

Definition 5.4. We say that $G \subseteq \omega$ is *function n-generic* if it is of the form $\mathrm{Graph}(\hat{g})$ for some n-generic point g in the Baire space $(\omega \cup \{\perp\})^\omega$.

As shown by Copestake, every function n-generic computes an enumeration n-generic, but the notions are distinct. Note that $\mathrm{Graph}(\hat{g}) \leq_e \mathrm{Graph}(g)$ is always true, but $\mathrm{Graph}(g) \leq_e \mathrm{Graph}(\hat{g})$ is not necessarily true.

As mentioned above, it is known that each enumeration 1-generic e-degree is quasi-minimal, and the same holds for function 1-generic [13, Proposition 2.6]. Therefore, we know at least two ways of constructing a quasi-minimal enumeration degree; choosing a point $x \in \mathbb{R}_<$ which is neither left- nor right-c.e., or choosing an enumeration/function generic point. Here we show that these two constructions are incomparable in the following sense.

Proposition 5.5. *No $\mathbb{R}_<$-degree computes a function 3-generic. I.e. if x is a real and $G \subseteq \omega$ be a function 3-generic, then $G \not\leq_e \mathrm{Nbase}_<(x)$.*

Proof. Suppose that $\mathrm{Graph}(\hat{g}) \leq_e \mathrm{Nbase}_<(x)$ for a partial function $\hat{g} :\subseteq \omega \to \omega$. Then, there is a c.e. set Φ such that $\hat{g}(n)\downarrow$ if and only if $p < x$ for some $\langle n, p\rangle \in \Phi$. For each n we let $\theta(n) = \inf\{q \in \mathbb{Q} : (n, q) \in \Phi\}$. There is an increasing sequence $\{n_k\}_{k\in\omega}$ such that $\{\theta(n_k)\}_{k\in\omega}$ is a monotonic sequence of reals. Note that the relation $\theta(n) \leq \theta(m)$ is Π^0_2. Therefore, such a sequence $\{n_k\}_{k\in\omega}$ can be found computably in $\emptyset''$. Now it is easy to check that $\hat{g}(n_k)$ is either defined for finitely many or for co-finitely many $k \in \omega$. Consider $S^0_\ell = \{f \in (\omega \cup \{\perp\})^\omega : (\exists k > \ell)\ f(n_k) = \perp\}$ and $S^1_\ell = \{f \in (\omega \cup \{\perp\})^\omega : (\exists k > \ell)\ f(n_k) \in \omega\}$. Note that g is not contained in S^0_ℓ or S^1_ℓ for any sufficiently large ℓ. However, S^0_ℓ and S^1_ℓ are dense $\emptyset''$-open sets with respect to the Baire topology on $(\omega \cup \{\perp\})^\omega$. Consequently, $\hat{g}$ is not function 3-generic. $\square$

Note that we can see that every nonzero $\mathbb{R}_<$-degree bounds a quasi-minimal e-degree.

Proposition 5.6. *For every $x \in \mathbb{R}_<$, either $\mathrm{Nbase}_<(x)$ is c.e. or there is quasi-minimal $S \subseteq \omega$ such that $S \leq_e \mathrm{Nbase}_<(x)$.*

Proof. To prove Proposition 5.6, we need the following lemma:

Lemma 5.7. *Given non-computable c.e. sets $A, B \subseteq \omega$, there are left-c.e. reals $z \leq_T A$ and $y \leq_T B$ such that $y - z$ is neither left- nor right-c.e.*

Proof. Given non-computable c.e. sets $A_0, A_1 \subseteq \omega$, we construct a c.e. reals $z^i = \Gamma_i^{A_i}$ with A_i-use $\gamma_i(n) = n$. The (e,i)-th strategy $\mathcal{R}_{e,i}$ tries to diagonalize $W_e = \mathrm{Left}(z^{1-i} - z^i)$, where $\mathrm{Left}(z) = \{q \in \mathbb{Q} : q \leq z\}$. We describe the action of $\mathcal{R}_{e,i}$ at stage s. If this is the first action of $\mathcal{R}_{e,i}$ after its initialization, choose a large number $n_{e,i}$ which is bigger than all numbers $+2$ mentioned in previous stages $< s$. Then, put $z^{1-i}_{s+1} = z^{1-i}_s + 2^{-n_{e,i}}$. Wait for $\mathrm{Left}(z^{1-i} - z^i) \upharpoonright n_{e,i} + 1 \subseteq W_e$. Here, $\mathrm{Left}(z^{1-i} - z^i) \upharpoonright n$ is defined as $\mathrm{Left}(z)$ for a dyadic rational $z = 0.(\sigma \upharpoonright n)000\dots$, where σ is a unique binary string satisfying $z^{1-i} - z^i = 0.\sigma000\dots$ (note that such σ exists since our strategy ensures that z^i and z^{1-i} are dyadic rationals). If it happens, wait for the change of $A_i \upharpoonright n_{e,i} + 1$, choose a fresh large number $n'_{e,i}$, injure all lower priority strategies, and go back to the first step with $n'_{e,i}$. If we see the change of $A_i \upharpoonright n_{e,i} + 1$ at stage $t \geq s$, the strategy $\mathcal{R}_{e,i}$ acts by putting $z^i_{t+1} = z^i_t + 2^{-n_{e,i}}$, and stop the action of $\mathcal{R}_{e,i}$.

If $\mathrm{Left}(z^{1-i} - z^i) \upharpoonright n_{e,i} + 1 \subseteq W_e$ does not happen, then the requirement is clearly fulfilled. If it happens with an infinite increasing sequence $(n^k_{e,i})_{k \in \omega}$ since $A_i \upharpoonright n^k_{e,i} + 1$ never changes for all $k \in \omega$, then A_i is computable. It contradicts the choice of A_i. Hence, $\mathcal{R}_{e,i}$ acts with $n^k_{e,i}$ for some k. If it is never injured after the action, then the requirement is fulfilled sine the sum of weights added to z^{1-i} by all lower priority strategies is less than $2^{-n_{e,i}}$. Consequently, $z^{1-i} - z_i$ is neither left- nor right-c.e. $\square$

It suffices to show that if $r \in \mathbb{R}_<$ is right-c.e., but not left-c.e., then r bounds a quasi-minimal e-degree. As mentioned in the paragraph below Proposition 5.3, the $\mathbb{R}_<$-degrees of right-c.e. reals are exactly the c.e. Turing degrees. Therefore, by Lemma 5.7, for any right-c.e. real $r \in \mathbb{R}$ if r is not left-c.e., we have right-c.e. reals $y, z \in \mathbb{R}$ such that $y - z$ is neither left- nor right-c.e., and $\mathrm{Nbase}(y), \mathrm{Nbase}(z) \leq_e \mathrm{Nbase}_<(r)$. Put $x = y - z$. Then, $\mathrm{Nbase}(x) \leq_e \mathrm{Nbase}_<(r)$. By Lemma 5.2, $\mathrm{Nbase}_<(x)$ is quasi-minimal. $\square$

5.1.2. *Degree Structure.* It is immediate that every recursively presented perfect[5] Polish space $\mathcal{X}$ has a computable homeomorphic copy of Cantor space. Hence, the jump inversion theorem holds in $\mathcal{X}$. However, the lower reals $\mathbb{R}_<$ obviously contain no copy of Cantor space. Nevertheless, we can have the jump inversion theorem for $\mathbb{R}_<$, though the jump and join operations are not in the same space. Indeed, we here give a short proof of McEvoy's quasi-minimal jump inversion theorem [50] inside $\mathbb{R}_<$.

The enumeration jump operator is introduced by Cooper [10]. Given $A \subseteq \omega$, define $K^A = \{e \in \omega : e \in \Psi_e(A)\}$, where Ψ_e is the e-th enumeration operator. Then, the *enumeration jump of* A is the set $\mathrm{EJ}(A) = K^A \oplus (K^A)^{\mathsf{c}}$. Gregoriades-Kihara-Ng [32] introduced the jump of a point in a represented cb_0 space $\mathcal{X} = (X, \beta)$. The *jump of* $x \in \mathcal{X}$ is the point $J_{\mathcal{X}}(x) \in 2^\omega$ defined by $J_{\mathcal{X}}(x) = \{e \in \omega : x \in U^{\mathcal{X}}_e\}$, where $U^{\mathcal{X}}_e$ is the e-th c.e. open set in $\mathcal{X}$. One can see that for any $x \in \mathcal{X}$,

$$\mathrm{Nbase}_{2^\omega}(J_{\mathcal{X}}(x)) \equiv_e \mathrm{EJ}(\mathrm{Nbase}_{\mathcal{X}}(x)).$$

The notation such as $J_{\mathcal{X}}(x) \equiv_T C$ also makes sense. This is equivalent to saying that $\mathrm{Nbase}_{2^\omega}(J_{\mathcal{X}}(x)) \equiv_e C \oplus C^{\mathsf{c}}$.

[5]Unlike the classical case, here *perfect* cannot be replaced by *uncountable* as shown by Gregoriades [31].

We now show the *Semirecursive Jump Inversion Theorem*, which generalizes McEvoy's quasi-minimal jump inversion theorem; nevertheless our proof is far simpler than McEvoy's one:

Proposition 5.8. *For any $C \geq_T \emptyset'$ there is a semirecursive set $A \subseteq \omega$ such that A is quasi-minimal and* $\mathrm{EJ}(A) \equiv_e C \oplus C^{\mathsf{c}}$.

Proof. It suffices to find a real $x \in \mathbb{R}_<$ such that $\mathrm{Nbase}_<(x)$ is quasi-minimal and $J_{\mathcal{X}}(x) \equiv_T C$. First note that every c.e. open set in $\mathbb{R}_<$ is of the form $\bigcup_{q\in W_e}\{y : y > q\}$ for a c.e. set $W_e \subseteq \mathbb{Q}$. Clearly, $r_e = \inf W_e$ is right-c.e. Then, $J_{\mathbb{R}_<}(x) = \{e \in \omega : r_e < x\}$. Assume that x is not right-c.e. Then, either $x < r_e$ or $x > r_e$ holds. Since $r_e \leq_T \emptyset'$ in 2^ω, we have $J_{\mathbb{R}_<}(x) \equiv_T x \oplus \emptyset'$ (i.e., $\mathrm{Nbase}_{2^\omega}(J_{\mathbb{R}_<}(x)) \equiv_e \mathrm{Nbase}_{\mathbb{R}}(x) \oplus K \oplus K^{\mathsf{c}}$). Now, by the Friedberg jump inversion theorem in 2^ω, there is a 1-generic real $z \in \mathbb{R}$ such that $z' \equiv_T z \oplus \emptyset' \equiv_T C$. By 1-genericity, z is neither left-c.e. nor right-c.e. By Lemma 5.2, $\mathrm{Nbase}_<(z)$ is semirecursive and quasi-minimal. Moreover, we have $J_{\mathbb{R}_<}(z) \equiv_T z \oplus \emptyset' \equiv_T C$ since z is not right-c.e. Note that $J_{\mathbb{R}_<}(z)$ is e-equivalent to $\mathrm{EJ}(\mathrm{Nbase}_<(z))$ as mentioned above. Consequently, $A = \mathrm{Nbase}_<(z)$ satisfies the desired property. □

We now focus on the first order degree structure of the lower reals. One of the most fundamental questions in degree theory is whether a given degree structure forms an upper (lower) semilattice or not. It is not hard to see that $\mathcal{D}_{\mathbb{R}_<}$ has no supremum operation. A somewhat more involved argument also establishes that $\mathcal{D}_{\mathbb{R}_<}$ has no infimum operation.

Proposition 5.9. *The structure $\mathcal{D}_{\mathbb{R}_<}$ is not an upper semilattice.*

Indeed, if x is not Δ^0_2 (as a point in $\mathbb{R}$), then the pair $\mathrm{Nbase}_<(x)$ *and* $\mathrm{Nbase}_<(-x)$ *has no common upper bound in $\mathcal{D}_{\mathbb{R}_<}$.*

Proof. Assume that x is not Δ^0_2, and that $\mathrm{Nbase}_<(x) \oplus \mathrm{Nbase}_<(-x) \leq_e \mathrm{Nbase}_<(y)$. Then on the one hand, $\mathrm{Nbase}_<(y)$ is not quasi-minimal. On the other hand, y is not Δ^0_2 and hence neither left- nor right-c.e. But this contradicts Lemma 5.2. □

Proposition 5.10. *The structure $\mathcal{D}_{\mathbb{R}_<}$ is not a lower semilattice.*

Indeed, there are right-c.e. reals $x, y \in \mathbb{R}$ such that the pair $\mathrm{Nbase}_<(x)$ *and* $\mathrm{Nbase}_<(Y)$ *has no greatest lower bound in $\mathcal{D}_{\mathbb{R}_<}$.*

Lemma 5.11. *Let $A, B \subseteq \omega$ be c.e. sets, and $y \in \mathbb{R}_<$. Suppose that* $\mathrm{Nbase}_<(y) \leq_e A \oplus A^{\mathsf{c}}$ *and* $\mathrm{Nbase}_<(y) \leq_e B \oplus B^{\mathsf{c}}$. *Then there exists a total function $h \in \omega^\omega$ such that $h \leq_T A, B$ and y is left-c.e. relative to h.*

Proof. Suppose that $\mathrm{Nbase}_<(y) \leq_e A \oplus A^{\mathsf{c}}$ and $\mathrm{Nbase}_<(y) \leq_e B \oplus B^{\mathsf{c}}$. Then, there are computable functions $W, V : 2^\omega \to \mathbb{Q}^\omega$ such that $y = \sup W^A = \sup V^B$. Let $\alpha_s = \sup W^A[s]$ and $\beta_s = \sup V^B[s]$, where $W^A[s] = (W(A[s])(n))_{n<s}$ and $V^B[s] = (V(B[s])(n))_{n<s}$. Define s_0 to be the least stage such that for every $t \geq s_0$ we have $\alpha_t \geq q_0$ or $\beta_t \geq q_0$, where $q_0 = \min\{\alpha_{s_0}, \beta_{s_0}\}$. Similarly define $s_{n+1} > s_n$ and $q_{n+1} > q_n$. Let $h(n) = s_n$. Clearly h is total. We claim that h is computable from both A and B. For each stage $u > s_n$, compute $q[u] = \min\{\alpha_u, \beta_u\}$, and use A to compute stage $v \geq u$ such that $\alpha_t \geq q[u]$ for all $t \geq v$. Then, check whether $\beta_t \geq q[u]$ for each t such that $u \leq t \leq v$. Clearly, for the least such $u > s_n$ with $q[u] > q_n$, we have $u = s_{n+1}$. By the same argument, B can also compute h. Now given h we can recover the increasing sequence q_n with limit $y = \sup W^A = \sup V^B$. □

Proof of Proposition 5.10. Lachlan and Yates (see [72, Corollary IX.3.3]) proved the existence of c.e. sets A, B such that for any (not necessarily c.e.) set $H \leq_T A, B$ there exists a c.e. set C such that $H <_T C \leq_T A, B$. We claim that for $x = \sum_{n \in A^{\mathsf{c}}} 2^{-n}$ and $y = \sum_{n \in B^{\mathsf{c}}} 2^{-n}$, $\mathrm{Nbase}_<(x)$ and $\mathrm{Nbase}_<(y)$ have no greatest lower bound in $\mathcal{D}_{\mathbb{R}_<}$. Let $z \in \mathbb{R}_<$ be any point such that $\mathrm{Nbase}_<(z) \leq_e \mathrm{Nbase}_<(x), \mathrm{Nbase}_<(y)$. Since A and B are c.e., $\mathrm{Nbase}_<(x) \equiv_e A \oplus A^{\mathsf{c}}$ and $\mathrm{Nbase}_<(y) \equiv_e B \oplus B^{\mathsf{c}}$. Thus, we can apply Lemma 5.11 to obtain some total function $h \leq_T A, B$ such that $\mathrm{Nbase}_<(z) \leq_e \mathrm{Nbase}_{\omega^\omega}(h)$. Since A and B have no greatest lower-bound in the c.e. Turing degrees $\mathcal{R}_T$ and h is total, there is a c.e. set C such that $h <_T C <_T A, B$. Therefore, if we set $w = \sum_{n \in C^{\mathsf{c}}} 2^{-n}$, then $\mathrm{Nbase}_<(w) \equiv_e C \oplus C^{\mathsf{c}}$; hence,

$$\mathrm{Nbase}_<(z) <_e \mathrm{Nbase}_<(w) <_e \mathrm{Nbase}_<(x), \mathrm{Nbase}_<(y).$$

Consequently, there is no z such that $\mathrm{Nbase}_<(z)$ can be a greatest lower bound of $\mathrm{Nbase}_<(x)$ and $\mathrm{Nbase}_<(y)$. □

We have already noted that the degree structure of $\mathbb{R}_<$ has a maximal element. Now, it is natural to ask whether this degree structure has a minimal element. Recall that the degree structures of 2^ω and $[0,1]^\omega$ each have continuum many minimal elements, whereas Gutteridge [36] showed that there is no minimal degree in $\mathbb{S}^\omega$. We can see that $\mathbb{R}_<$ has no minimal degree. We will prove that every noncomputable point $X \in \mathbb{S}^\omega$ computes a noncomputable point $y \in \mathbb{R}_<$ (i.e. that every nonzero e-degree bounds a nonzero semirecursive e-degree; i.e. that there are no $\mathbb{R}_<$-quasi-minimal e-degrees). This allows us to transfer Gutteridge's result that there are no minimal e-degrees to the $\mathbb{R}_<$-degrees.

Theorem 5.12. *There is no $\mathbb{R}_<$-quasi-minimal e-degree.*

Proof. We show that given any $X \subseteq \omega$ not c.e., there is some $y \in \mathbb{R}_<$ such that y is not left-c.e., and $\mathrm{Nbase}_<(y) \leq_e X$.

If $0.X = \sum_{n \in X} 2^{-n}$ is not left-c.e., consider the real $y = 0.X$. Otherwise assume that $0.X$ is left-c.e. In this case, X is Δ^0_2. Furthermore we may fix an approximation X_s of X such that if n leaves X_s then there is some $m < n$ which is enumerated in X_s at the same time.

Let $(\alpha_e)_{e \in \omega}$ be an effective enumeration of all left-c.e. reals. We wish to construct a some left c.e. real y relative to $X \in \mathbb{S}^\omega$ such that $y \neq \alpha_e$ for all $e \in \omega$. In other words, we will construct a c.e. set Φ of pairs of rationals and finite subsets of ω such that $y = \sup\{q : (\exists D \subseteq X)\ (q, D) \in \Phi\}$. At stage s, the e-th strategy for $e \leq s$ is eligible to act with parameters $y_{e,s}$, $m_{e,s}$, $s^-_{e,s}$, $s_{e,s}$ and $p_{e,s}$. Here, $y_{-1,s} = m_{-1,s} = t_{-1,s} = 0$ for all s.

At substage $e \leq s$, the e-th strategy acts as follows:

(1) If this is the first action of the e-th strategy after its initialization, set

$$m_{e,s+1} = m_{e-1,s+1} + 2^{p_{e-1,s+1}} + 1,$$
$$y_{e,s+1} = y_{e-1,s+1} + 2^{-m_{e,s}-1}.$$

Put $p_{e,s+1} = p_{e-1,s+1}$, and $s_{e,s+1} = s + 1$. Initialize all lower priority strategies, and go to stage $s + 1$. Otherwise, go to step (2).

(2) If the e-th strategy is active, go to step (3). If the e-th strategy is inactive because of the previous action in (3b), go to step (5).

(3) Check whether $X_{s_{e,s}} \cap [p_{e-1,s}, p_{e,s}) \subseteq X_s$.
   (a) If yes, and go to step (4).
   (b) If no with $X_{s^-_{e,s}} \cap [p_{e-1,s}, p_{e,s} - 1) \not\subseteq X_s$, define $y_{e,s+1}$ as the current value of $\Phi(X_s)$ at stage s, and put $s^-_{e,s+1} = s^-_{e,s}$, $s_{e,s+1} = s_{e,s}$ and $p_{e,s+1} = p_{e,s}$. This strategy is shifted into the inactive state. Initialize all lower priority strategies, and go to stage $s + 1$.
   (c) Otherwise, define

$$y_{e,s+1} = y_{e,s} + 2^{-m_{e,s}-1},$$

   and maintain the computation by enumerating $(y_{e,s+1}, X_{s_{e,s+1}} \upharpoonright p_{e,s})$ into Φ. Put $s^-_{e,s+1} = s^-_{e,s}$, $s_{e,s+1} = s$ and $p_{e,s+1} = p_{e,s}$. Initialize all lower priority strategies, and go to stage $s + 1$.

(4) Check whether $\alpha_{e,s} \geq y_{e,s} - 2^{-m_{e,s}-2}$.
   (a) If yes, put $p_{e,s+1} = p_{e,s} + 1$, and

$$m_{e,s+1} = m_{e-1,s+1} + 2^{p_{e,s+1}} + 1,$$
$$y_{e,s+1} = y_{e,s} + 2^{-m_{e,s+1}-1}.$$

   Moreover, put $s^-_{e,s+1} = s_{e,s}$, $s_{e,s+1} = s$ and $p_{e,s+1} = p_{e,s}+1$. Then, enumerate $(y_{e,s+1}, X_{s_{e,s+1}} \upharpoonright p_{e,s+1})$ into Φ. Initialize all lower priority strategies, and go to stage $s + 1$.
   (b) If no, go to substrategy $e + 1$ unless $e = s$. If $e = s$, then go to stage $s + 1$.

(5) Check whether $X_{s-1} \cap [p_{e-1,s}, p_{e,s}) \subseteq X_s$.
   (a) If yes, do the action described in (4b).
   (b) If no, check whether $\Phi(X_s) < y_s$. If yes, the e-th strategy keeps on being in the inactive state, and do the action described in (3b). If no, we must have stage $t < s$ such that $X_t \subseteq X_s$ and the e-th strategy is active at stage t. We recover the parameters for the last such stage $t < s$, and the e-th strategy is shifted into the active state.

Suppose that the e-th strategy is never injured after stage t. In this case, any parameter for $e' < e$ never changes after stage t. If (4a) happens infinitely often, then (3b) never happens after t. Hence, we have a computable $\subseteq$-increasing sequence $(X_{s^-_{e,s}} \cap [p_{e-1,s}, p_{e,s} - 1))_{s \geq t}$ converging to $X \cap \{n : n \geq p_{e-1,s}\}$ since $p_{e-1,s} = p_{e-1,t}$ for all $s \geq t$, and moreover, $s^-_{e,s}$ and $p_{e,s}$ are nondecreasing, and tend to infinity. Therefore, X is c.e.; however, it is impossible by our assumption. Thus, (4a) never happens after some stage $t' \geq t$. Then, $p_{e,s} = p_{e,t'}$, $m_{e,s} = m_{e,t'}$, $s^-_{e,s} = s^-_{e,t'}$ and $\alpha_{e,s} \leq y_{e,s} - 2^{-m_{e,s}-2}$ for all $s \geq t'$. Suppose that (3b) never happens. First assume that $X_{s_{e,s}} \cap [p_{e-1,s}, p_{e,s}) \subseteq X_s$. If $X_{s_{e,s}} \upharpoonright p_{e-1,s} \not\subseteq X_s$, then a higher priority strategy acts with (5b), which is impossible by our assumption. Thus, $X_{s_{e,s}} \upharpoonright p_{e,s} \subseteq X_s$. Hence, our computation is maintained, that is, $\Phi(X_{s_{e,s}} \upharpoonright p_{e,s}) \geq y_{e,s}$. If $X_{s_{e,s}} \cap [p_{e-1,s}, p_{e,s}) \not\subseteq X_s$, the e-th strategy maintains the computation at (3c) since (3b) never happens. By monotonicity of our approximation of X, (3c) can happen only finitely often after stage t'. Fix stage $t'' \geq t'$ such that (3c) never happens after stage t''. Then, $y_{e,s} = y_{e,t''}$ for all $s \geq t''$. Consequently, $\Phi(X) \geq \Phi(X_{s_{e,t''}} \upharpoonright p_{e,t''}) \geq y_{e,t''} > y_{e,t''} - 2^{-m_{e,t''}-2} \geq \alpha_e$.

If (3b) happens, $X_{s^-_{e,s}} \cap [p_{e-1,s}, p_{e,s} - 1) \not\subseteq X_s$ at some stage $s \geq t$. Let $u \geq t$ be the least such stage, and let $u' < u$ be the last stage when (4a) happened. Then, $\alpha_e \geq y_{e,u'} - 2^{-m_{e,u'}-2}$. Note that $s^-_{e,u} < u'$ is the last stage when either (3c) or (4a) happens before u'. Thus, $X_{s^-_{e,u}} \cap [p_{e-1,s}, p_{e,s}) \subseteq X_s$ for $s^-_{e,u} < s \leq u'$ since neither (3c) nor (3c) happens between $s^-_{e,u}$ and u', and $s_{e,s} = s^-_{e,u}$ for such s. Moreover, since $s^-_{e,s} = s^-_{e,u}$ for all s with $u' < s < u$, we have $X_{s^-_{e,u}} \cap [p_{e-1,s}, p_{e,s} - 1) \subseteq X_s$ for such s by our choice of u. Consequently, $X_s \cap [p_{e-1,s}, p_{e,u'}) \not\subseteq X_u$ for all s with $s^-_{e,u} < s < u$. In particular, if $s^-_{e,u} < s < u$ and $e' \geq e$, then $X_s \upharpoonright p_{e',s+1} \not\subseteq X_u$ since $p_{e,s} \leq p_{e',s} \leq p_{e',s+1}$. Now, note that every computation enumerated into Φ at stage $s \geq t$ is of the form $(y_{e',s+1}, X_s \upharpoonright p_{e',s+1})$ for some $e' \geq e$. Therefore, every computation enumerated into Φ at stage s with $s^-_{e,u} < s < u$ is destroyed at stage u. We also note that $(y_{e,s})$ is monotone in the sense that $e \leq e'$ implies $y_{e,s} \leq y_{e',s}$ and that $t \leq s \leq s' \leq u$ implies $y_{e,s} \leq y_{e,s'}$. Moreover, $y_{e,u''} + 2^{-m_{e,u''+1}-1} = y_{e,u''+1} \leq y_{e,u'}$, where $u'' = s^-_{e,u}$. Therefore, $y_{e,u''} + 2^{-m_{e,u''+1}-2} \leq \alpha_u$. By the previous argument, the e-th strategy only enumerate the value $y_{e,u''}$ under the oracle X_u, and therefore, the value enumerated by lower priority is at most $y_{e,u''} + 2^{-m_{e,u''+1}-2}$. Consequently, we have $\Phi(X) < \alpha_e$. □

The result of Theorem 5.12 was independently obtained by Downey, Greenberg, Harrison-Trainor, Patey and Turetsky [23].

Corollary 5.13. *There is no minimal semirecursive degree.*

Proof. If there is some nonzero minimal element $\mathrm{Nbase}_<(x)$ in $\mathcal{D}_{\mathbb{R}_<}$, then we claim that $\mathrm{Nbase}_<(x)$ has minimal e-degree (which is a contradiction to Gutteridge [36]). Clearly $\mathrm{Nbase}_<(x)$ is not c.e. If there is some non-c.e. set $Y <_e \mathrm{Nbase}_<(x)$ then by Theorem 5.12 we have some y where $\emptyset <_e \mathrm{Nbase}_<(y) \leq_e Y <_e \mathrm{Nbase}_<(x)$, contradicting minimality of $\mathrm{Nbase}_<(x)$ in $\mathcal{D}_{\mathbb{R}_<}$. □

5.1.3. *T_1-quasi-minimal degrees.* We extend our idea of the proof of Lemma 5.2 to show the existence of a T_1-quasi-minimal degree.

Theorem 5.14. *Let $\mathcal{T}$ be a countable collection of second-countable T_1 spaces. Then, there is a $\mathcal{T}$-quasi-minimal semirecursive e-degree.*

Proof. A *lightface pointclass* Γ is a countable collection of subsets of ω^ω which is closed under computable substitution, that is, if $S \in \Gamma$, then $\Phi^{-1}[S] \in \Gamma$ for any computable function $\Phi : \omega^\omega \to \omega^\omega$. By $\check{\Gamma}$ we denote the dual of Γ, that is, $\check{\Gamma} = \{A : A^{\mathsf{c}} \in \Gamma\}$.

We say that a set $A \subseteq \omega$ is Γ if there is a Γ set S such that for any $n \in \omega$, $n \in A$ if and only if $n0^\omega \in S$. We say that a real $x \in \mathbb{R}$ is *left*-Γ if $\{n \in \omega : p_n < x\}$ is in Γ, where p_n is the n-th rational. Similarly, we say that a real $x \in \mathbb{R}$ is *right*-Γ if $\{n \in \omega : x < p_n\}$ is in Γ. Then we say that $x \colon \mathbb{R}$ is Δ if it is both left-Γ and right-Γ.

Lemma 5.15. *Let Γ be a lightface pointclass. For any $x \in \mathbb{R}$, if $x \colon \mathbb{R}$ is not Δ, then $x \colon \mathbb{R}_<$ is quasi-minimal w.r.t. strongly $\check{\Gamma}$-named T_1 spaces.*

Proof. We show a relativized version, which improves Lemma 5.2. For any $r \in \omega^\omega$, we say that $A \subseteq \omega^\omega$ is Γ *relative to* r, or simply $\Gamma(r)$ if there is a Γ set $G \subseteq \omega^\omega$ such that

$A = \{y : \langle r, y\rangle \in G\}$. Let $\mathcal{X}$ be any represented cb_0 space. Given $o \in \mathcal{X}$, we say that $A \subseteq \omega^\omega$ is Γ *relative to* o, or simply $\Gamma(o)$, if A is Γ relative to any $\mathcal{X}$-name of o in a uniform manner, that is, there is a Γ set $G \subseteq \omega^\omega$ such that $A = \{y : \langle r, y\rangle \in G\}$ for any $\mathcal{X}$-name r of o. Then, we define the notion of a $\Delta(o)$ real in a straightforward manner.

Let $\mathcal{X}$ be a represented cb_0 space, and $\mathcal{Z}$ be a strongly $\check{\Gamma}$-named T_1 space. We will show that for any $x \in \mathbb{R}$, $o \in \mathcal{X}$, and $z \in \mathcal{Z}$,

$$\mathrm{Nbase}_{\mathcal{Z}}(z) \leq_e \mathrm{Nbase}_<(x) \oplus \mathrm{Nbase}_{\mathcal{X}}(o) \implies \mathrm{Nbase}_{\mathcal{Z}}(z) \leq_e \mathrm{Nbase}_{\mathcal{X}}(o) \text{ or } x\colon \mathbb{R} \text{ is } \Delta(o).$$

Since the specialization order on $\mathbb{R}_<$ forms a chain while the specialization order on a T_1 space forms an antichain, for any continuous function Φ from $\mathcal{X} \times \mathbb{R}_<$ to a T_1 space $\mathcal{Z}$ and any $o \in \mathcal{X}$, for $\Phi_o : y \mapsto \Phi(o, y)$, the cardinality of $\mathrm{rng}(\Phi_o)$ is at most one. In the context of an enumeration operator, this is because, for any enumeration operator Φ, any point $o \in \mathcal{X}$, and any reals $x, y \in \mathbb{R}$,

$$x < y \implies \Phi_o(\mathrm{Nbase}_<(x)) \subseteq \Phi_o(\mathrm{Nbase}_<(y)),$$

where we define $\Phi_o(A) = \Phi(\mathrm{Nbase}_{\mathcal{X}}(o) \oplus A)$, and recall that if $\mathcal{Z}$ is T_1, for any $z_0, z_1 \in \mathcal{Z}$, $z_0 \neq z_1$ implies $\mathrm{Nbase}_{\mathcal{Z}}(z_i) \not\subseteq \mathrm{Nbase}_{\mathcal{Z}}(z_{1-i})$ for any $i < 2$. Here, by symbols $\mathrm{dom}(\Phi_o)$ and $\mathrm{rng}(\Phi_o)$ we mean the domain and the range of the function $\tilde{\Phi}_o$ from $\mathbb{R}_<$ to a T_1 space $\mathcal{Z}$ induced from the operator Φ_o, that is,

$$\mathrm{dom}(\Phi_o) = \{x \in \mathbb{R}_< : (\exists z \in \mathcal{Z})\ \Phi_o(\mathrm{Nbase}_<(x)) = \mathrm{Nbase}_{\mathcal{Z}}(z)\},$$
$$\mathrm{rng}(\Phi_o) = \{z \in \mathcal{Z} : (\exists x \in \mathbb{R}_<)\ \Phi_o(\mathrm{Nbase}_<(x)) = \mathrm{Nbase}_{\mathcal{Z}}(z)\}.$$

Assume that $\mathrm{rng}(\Phi_o)$ is nonempty in a T_1 space $\mathcal{Z}$, that is, there are $x \in \mathbb{R}$ and $z \in \mathcal{Z}$ such that $\Phi_o(\mathrm{Nbase}_<(x)) = \mathrm{Nbase}_{\mathcal{Z}}(z)$. If $\mathrm{dom}(\Phi_o)$ is not a singleton, $x, y \in \mathrm{dom}(\Phi_o)$ with $x < y$ say, it contains a non-degenerate interval $[x, y]$, and therefore contains a rational $q \in [x, y] \subseteq \mathrm{dom}(\Phi_o)$. Thus, $\Phi_o(\mathrm{Nbase}_<(q))$ gives us a unique element z of $\mathrm{rng}(\Phi_o)$, and then we must have $\mathrm{Nbase}_{\mathcal{Z}}(z) \leq_e \mathrm{Nbase}_{\mathcal{X}}(o)$ since q is computable.

Therefore, if $\mathrm{rng}(\Phi_o)$ contains a point z such that $\mathrm{Nbase}_{\mathcal{Z}}(z) \not\leq_e \mathrm{Nbase}_{\mathcal{X}}(o)$, then $\mathrm{dom}(\Phi_o)$ has to be a singleton $\{x\}$. Let P and N witness that $\mathcal{Z}$ is strongly $\check{\Gamma}$-named. Let F be a computable realizer of $\tilde{\Phi}$, that is, given $\alpha, \beta \in \omega^\omega$, if s is the first stage such that we see $D \subseteq \mathrm{rng}(\alpha \upharpoonright s)$ and $E \subseteq \mathrm{rng}(\beta \upharpoonright s)$ and $\langle k, D, E\rangle \in \Phi$ by stage s, then put $F(\alpha, \beta)(s) = k$. One can assume that F is a total computable function on $\omega^\omega \times \omega^\omega$ since it is generated from an enumeration operator. If we fix an $\mathcal{X}$-name α of $o \in \mathcal{X}$, then $F(\alpha, \beta) \in \mathrm{Sub}(\mathcal{Z}) \cup \mathrm{Sup}(\mathcal{Z})$ for any $\beta \in \mathrm{Name}(\mathcal{R}_<)$. Thus, we have the following:

$$F(\alpha, \beta) \notin P \iff F(\alpha, \beta) \in \mathrm{Sub}(\mathcal{Z}) \setminus \mathrm{Name}(\mathcal{Z}),$$
$$F(\alpha, \beta) \notin N \iff F(\alpha, \beta) \in \mathrm{Sup}(\mathcal{Z}) \setminus \mathrm{Name}(\mathcal{Z}).$$

By our assumption, both $L = F^{-1}[P^{\mathsf{c}}]$ and $R = F^{-1}[N^{\mathsf{c}}]$ are Γ subsets of ω^ω. If α be an $\mathcal{X}$-name of o, and β be an $\mathbb{R}_<$-name of z, one can see that $(\alpha, \beta) \in L$ implies $z < x$, and similarly, $(\alpha, \beta) \in R$ implies $z > x$. For each rational p_n, choose an $\mathbb{R}_<$-name β_n of p_n in an effective manner. One can ensure that $g : (\alpha, n0^\omega) \mapsto (\alpha, \beta_n)$ is computable. Therefore, $g^{-1}[L]$ and $g^{-1}[R]$ are in Γ since Γ is closed under computable substitution. If α is an $\mathcal{X}$-name of o, we get that

$$p_n < x \iff (\alpha, \beta_n) \in L \iff (\alpha, n0^\omega) \in g^{-1}[L],$$
$$x < p_n \iff (\alpha, \beta_n) \in R \iff (\alpha, n0^\omega) \in g^{-1}[R].$$

Thus, these give left- and right-Γ approximations of x uniformly relative to any name of o. This concludes that x is $\Delta(o)$, which verifies our claim. If $o = \emptyset$ in 2^ω, this means that x is quasi-minimal w.r.t. strongly $\check{\Gamma}$-named T_1 spaces. $\square$

For any countable collection $\mathcal{T}$ of T_1-spaces, clearly there is a lightface pointclass Γ such that $\mathcal{X}$ is strongly $\check{\Gamma}$-named for any $\mathcal{X} \in \mathcal{T}$. Thus, by Lemma 5.15, if $x\colon \mathbb{R}$ is not Δ, then $x\colon \mathbb{R}_<$ is $\mathcal{T}$-quasi-minimal. Recall that every $\mathbb{R}_<$-degree is semirecursive. This concludes the proof. $\square$

Note that Lemma 5.15 essentially show that all but countably many semirecursive e-degree satisfy the above property; see also Proposition 5.17 below. In particular, almost no point $x \in \mathbb{R}_<$ computes a nontrivial point in a T_1 space.

5.1.4. *Products of lower topology.* We say that an e-degree $\mathbf{d}$ is *n-semirecursive* if there are semirecursive e-degrees $\mathbf{c}_1, \dots, \mathbf{c}_n$ such that

$$\mathbf{d} = \mathbf{c}_1 \oplus \cdots \oplus \mathbf{c}_n.$$

Clearly, the $\mathbb{R}^n_<$-degrees are exactly the n-semirecursive e-degrees. These degrees have also been studied by Kihara-Pauly [43]. In this section, using the idea developed in Sections 3.2 and 5.1.3, we provide a more detailed analysis of n-semirecursive e-degrees.

We first see degree-theoretic differences among $\mathbb{R}$, $\mathbb{R}_<$, $\mathbb{R} \times \mathbb{R}_<$, and $\mathbb{R}^2_<$. We say that $y \in \mathbb{R}$ is *left-c.e. in $x \in \mathcal{X}$* if $y\colon \mathbb{R}$ is $\Sigma^0_1(x)$, that is, $\mathrm{Nbase}_<(y) \leq_e \mathrm{Nbase}_{\mathcal{X}}(x)$. Similarly, $y \in \mathbb{R}$ is *right-c.e. in $x \in \mathcal{X}$* if $\mathrm{Nbase}_<(-y) \leq_e \mathrm{Nbase}_{\mathcal{X}}(x)$.

The latter result shows that $\mathbb{R}_<$-points are useless to compute a 2^ω-point.

Proposition 5.16. *Let $\mathcal{X}$ be a represented cb_0 space. For any $x \in \mathcal{X}$ and $y \in \mathbb{R}$, if y is neither left- nor right-c.e. in x, then $(x, y)\colon \mathcal{X} \times \mathbb{R}_<$ is a strong quasi-minimal cover of $x\colon \mathcal{X}$. In particular, we have the following.*

1. *Every $\mathcal{X} \times \mathbb{R}_<$-degree is either an $(\mathcal{X} \times \mathbb{R})$-degree or a strong quasi-minimal cover of an $\mathcal{X}$-degree.*
2. *For any $\mathcal{X}$-degree $\mathbf{d}$, there is an $(\mathcal{X} \times \mathbb{R}_<)$-degree which is a strong quasi-minimal cover of $\mathbf{d}$.*

Proof. Clearly $\mathrm{Nbase}_{\mathcal{X}}(x) <_e \mathrm{Nbase}_{\mathcal{X}\times\mathbb{R}_<}(x, y) \equiv_e \mathrm{Nbase}_{\mathcal{X}}(x) \oplus \mathrm{Nbase}_<(y)$ since y is not left-c.e. in x. Moreover, if $\mathrm{Nbase}_{2^\omega}(z) \leq_e \mathrm{Nbase}_{\mathcal{X}}(x) \oplus \mathrm{Nbase}_<(y)$ for some $z \in 2^\omega$, by Lemma 5.2, we have $\mathrm{Nbase}_{2^\omega}(z) \leq_e \mathrm{Nbase}_{\mathcal{X}}(x)$ since y is not right-c.e. in x, that is, $\mathrm{Nbase}_<(-y) \not\leq_e \mathrm{Nbase}_{\mathcal{X}}(x)$. Therefore, $(x, y)\colon \mathcal{X} \times \mathbb{R}_<$ is a strong quasi-minimal cover of $x\colon \mathcal{X}$. For (1), if y is left-c.e. in x, then $\mathrm{Nbase}(y) \leq_e \mathrm{Nbase}(x)$, and therefore, $\mathrm{Nbase}_{\mathcal{X}}(x) \equiv_e \mathrm{Nbase}_{\mathcal{X}}(x) \oplus \mathrm{Nbase}_<(y)$. This means that $\mathrm{Nbase}_{\mathcal{X}\times\mathbb{R}_<}(x, y)$ has an $\mathcal{X}$-degree, and in particular, has an $\mathcal{X}\times\mathbb{R}$-degree. If y is right-c.e. in x, then $\mathrm{Nbase}_<(-y) \leq_e \mathrm{Nbase}_{\mathcal{X}}(x)$, and therefore, $\mathrm{Nbase}_{\mathcal{X}}(x) \oplus \mathrm{Nbase}(y) \equiv_e \mathrm{Nbase}_{\mathcal{X}}(x) \oplus \mathrm{Nbase}_<(y)$. This means that $\mathrm{Nbase}_{\mathcal{X}\times\mathbb{R}_<}(x, y)$ has an $\mathcal{X} \times \mathbb{R}$-degree. If y is neither left- nor right-c.e. in x, then $\mathrm{Nbase}_{\mathcal{X}\times\mathbb{R}_<}(x, y)$ is a strong quasi-minimal cover of x as shown above. For (2), let $\mathbf{d}$ be an $\mathcal{X}$-degree. Then given a point $x \in \mathcal{X}$ of degree $\mathbf{d}$, choose a real y which is neither left- nor right-c.e. in x. Such a y must exist. Then $\mathrm{Nbase}_{\mathcal{X}\times\mathbb{R}_<}(x, y)$ is a strong quasi-minimal cover of x as shown above. $\square$

Indeed, we can show the following:

Proposition 5.17. *Let $\mathcal{X}$ be a represented cb_0 space, and let $\mathcal{T}$ be a countable collection of second-countable T_1-spaces. For any $x \in \mathcal{X}$ for all but countably many $y \in \mathbb{R}$ it holds that $(x, y)\colon \mathcal{X} \times \mathbb{R}_<$ is a strong $\mathcal{T}$-quasi-minimal cover of $x\colon \mathcal{X}$.*

Proof. Let Γ be some countably pointclass such that every space from $\mathcal{T}$ is strongly Γ-named. We adapt the proof of Proposition 5.16 by using Lemma 5.15 instead of Lemma 5.2. We conclude that for any $x \in \mathcal{X}$ and $y \in \mathbb{R}$, if $y\colon \mathbb{R}$ is not $\Delta(x)$, then $(x, y)\colon \mathcal{X} \times \mathbb{R}_<$ is a strong $T_1[\Gamma]$-quasi-minimal cover of $x\colon \mathcal{X}$, where $T_1[\Gamma]$ is the collection of strongly Γ-named, second-countable, T_1 spaces. $\square$

We now turn our attention to the lower real plane $\mathbb{R}^2_<$. The space $\mathbb{R}^2_<$ also has a point which is a strong quasi-minimal cover of $\emptyset'$. This is because any right-c.e. real has a total degree by Proposition 5.3; indeed, $\mathcal{X} = \mathbb{R}_<$ has a point of degree $\emptyset'$. Apply Proposition 5.16 (2) to this. However, we will show that $\mathbb{R}^2_<$ has no point which is a strong quasi-minimal cover of $\emptyset''$, which implies $\mathcal{D}_{\mathbb{R}\times\mathbb{R}_<} \not\subseteq \mathcal{D}_{\mathbb{R}^2_<}$.

Proposition 5.18. *Every 2-semirecursive e-degree is either total or quasi-minimal in $\emptyset'$.*

Proof. Let $x \in \mathbb{R}$, and $y, z \in \mathbb{R}_<$. By iterating Lemma 5.2, if $\mathrm{Nbase}(x) \leq_e \mathrm{Nbase}_<(y) \oplus \mathrm{Nbase}_<(z)$, then either (1) $\mathrm{Nbase}(x) \leq_e \emptyset$ (i.e., $x \leq_T \emptyset$), (2) $\mathrm{Nbase}(x) \leq_e \mathrm{Nbase}_<(z)$ and $\mathrm{Nbase}_<(-z) \leq_e \emptyset$, or (3) $\mathrm{Nbase}_<(-y) \leq_e \mathrm{Nbase}_<(z)$. If (1) or (2) holds, then $x \leq_T \emptyset'$. Thus, assume that (3) holds. We claim that either $\mathrm{Nbase}_<(-y) \leq_e \emptyset$ or $\mathrm{Nbase}_<(-z) \leq_e \mathrm{Nbase}_<(y)$ holds. Let Φ be a witness of our assumption (3), that is, for any rational $p \in \mathbb{Q}$, $y < p$ if and only if there is $q < z$ such that $\langle p, q\rangle \in \Phi$. If there is $\varepsilon > 0$ such that $q < z + \varepsilon$ and $\langle p, q\rangle \in \Phi$ implies $y \leq p$, then y is right-c.e. Otherwise, for all $\varepsilon > 0$, there are $p, q \in \mathbb{Q}$ such that $q < z + \varepsilon$, $\langle p, q\rangle \in \Phi$, and $p < y$. Now, by using a left-approximation of y, search for q such that $\langle p, q\rangle \in \Phi$ and $p < y$. Such a q must satisfy $q \geq z$ by our choice of Φ, and for any $\varepsilon > 0$ there is such $q < z + \varepsilon$ by our assumption. Therefore, by enumerating all such q's, we obtain a right-approximation of z. This shows that $\mathrm{Nbase}_<(-z) \leq_e \mathrm{Nbase}_<(y)$. Consequently, $\mathrm{Nbase}_<(-y) \oplus \mathrm{Nbase}_<(-z) \leq_e \mathrm{Nbase}_<(y) \oplus \mathrm{Nbase}_<(z)$, which implies that $(y, z)\colon \mathbb{R}^2_<$ is total. $\square$

Next we will discuss degree-theoretic properties of n-semirecursive e-degrees. We first construct an $(n + 1)$-semirecursive e-degree which cannot be obtained from an n-semirecursive e-degree by joining a T_1-degree.

Theorem 5.19. *Let $\mathcal{T}$ be a countable collection of second-countable T_1 spaces. Then, there is an $(n + 1)$-semirecursive e-degree which cannot be written as the join of an n-semirecursive e-degree and a $\mathcal{T}$-degree. That is,*

$$\mathcal{D}_{\mathbb{R}^{n+1}_<} \not\subseteq \left\{\mathbf{c} \oplus \mathbf{d} : \mathbf{c} \in \mathcal{D}_{\mathbb{R}^n_<} \text{ and } (\exists \mathcal{X} \in \mathcal{T})\ \mathbf{d} \in \mathcal{D}_{\mathcal{X}}\right\}.$$

Proof. One can assume that $\mathcal{T} = T_1[\Gamma]$, i.e. the strongly Γ-named, second countable, T_1-spaces, for some lightface pointclass Γ. To prove Theorem 5.19, we need the notion of an independent point. We say that a point $(x_i)_{i\leq n} \in \mathbb{R}^{n+1}_<$ is *independent* if neither $\mathrm{Nbase}_<(x_i)$ nor $\mathrm{Nbase}_<(-x_i)$ is e-reducible to $\mathrm{Nbase}((x_j)_{j\neq i})$ for any $i \leq n$. Here, note that the latter is the coded neighborhood basis of $(x_j)_{j\neq i}\colon \mathbb{R}^n$; hence, $\mathrm{Nbase}((x_j)_{j\neq i}) \equiv_e \mathrm{Nbase}_<((x_j)_{j\neq i}) \oplus \mathrm{Nbase}_<((-x_j)_{j\neq i})$. It is not hard to see that if $(x_i)_{i\leq n}$ is 1-generic as a point in $\mathbb{R}^{n+1}$, then $(x_i)_{i\leq n}$ is independent. One can also see that if $(x_i)_{i\leq n}$ is a Martin-Löf random point in $\mathbb{R}^{n+1}$ (w.r.t. the product measure), then $(x_i)_{i\leq n}$ is independent. Given a lightface pointclass Γ, we say that a point $(x_i)_{i\leq n}\in\mathbb{R}^{n+1}_<$ is *Δ-independent* if it is independent, and if for any $i\leq n$, neither $x_i\colon\mathbb{R}_<$ nor $-x_i\colon\mathbb{R}_<$ is Δ in $(x_j)_{j\neq i}\colon\mathbb{R}^n$. If $(x_i)_{i\leq n}$ is sufficiently generic or random, then $(x_i)_{i\leq n}$ is Δ-independent. Therefore, the set of Δ-independent points is comeager and conull in $\mathbb{R}^{n+1}$.

Lemma 5.20. *If $\boldsymbol{x}\in\mathbb{R}^{n+1}_<$ is independent, then* $\mathrm{Nbase}(\boldsymbol{x})$ *does not have an $\mathbb{R}\times\mathbb{R}^n_<$-degree. If $\boldsymbol{x}\in\mathbb{R}^{n+1}_<$ is Δ-independent, then* $\mathrm{Nbase}(\boldsymbol{x})$ *does not have an $\mathcal{X}\times\mathbb{R}^n_<$-degree for any strongly Γ-named T_1-space $\mathcal{X}$.*

Proof. Suppose not. Then, there are an independent point $\boldsymbol{x}=(x_i)_{i\leq n}\in\mathbb{R}^{n+1}_<$ and points $z\in\mathcal{X}$ and $\boldsymbol{y}=(y_j)_{j<n}\in\mathbb{R}^n_<$ such that $\mathrm{Nbase}(z,\boldsymbol{y})\equiv_e\mathrm{Nbase}(\boldsymbol{x})$. In particular, $\mathrm{Nbase}(z)\leq_e\mathrm{Nbase}(\boldsymbol{x})$. If $\boldsymbol{x}$ is independent and $\mathcal{X}=2^\omega$, then $\boldsymbol{x}$ is quasi-minimal by iterating Lemma 5.2. If $\boldsymbol{x}$ is Δ-independent, then $\boldsymbol{x}$ is $T_1[\Gamma]$-quasi-minimal by iterating (the general claim in the proof of) Lemma 5.15. Therefore, $\mathrm{Nbase}(z)\leq_e\mathrm{Nbase}(\boldsymbol{x})$ implies $\mathrm{Nbase}(z)\leq_e\emptyset$. Thus, we now have $\mathrm{Nbase}(\boldsymbol{y})\equiv_e\mathrm{Nbase}(\boldsymbol{x})$. Let Φ and Ψ be enumeration operators witnessing this e-equivalence, that is,

$$\Phi(\mathrm{Nbase}(\boldsymbol{x}))=\mathrm{Nbase}(\boldsymbol{y}),\text{ and }\Psi(\mathrm{Nbase}(\boldsymbol{y}))=\mathrm{Nbase}(\boldsymbol{x}).$$

Here, the coded neighborhood basis is given as $\mathrm{Nbase}((x_i)_{i\leq n})=\{\langle i,p\rangle:p<x_i\}$. First consider the case that there are $k<n$ and $\varepsilon>0$ such that for any $p_k<y_k+\varepsilon$,

$$(\forall i\leq n)(\forall q\in\mathbb{Q})\;[\langle i,q\rangle\in\Psi(\mathrm{Nbase}(y_0,\dots,y_{k-1},p_k,y_{k+1},\dots,y_{n-1}))\implies q<x_i].$$

Then, we get that $\mathrm{Nbase}((x_i)_{i\leq n})\leq_e\mathrm{Nbase}((y_j)_{j\neq k})$ as in the proof of Lemma 5.2. By induction, we can ensure that it is impossible.

Thus, for any $k<n$ and $\varepsilon>0$, there has to be $q_k<y_k+\varepsilon$ such that

$$(\exists i_k\leq n)(\exists q\in\mathbb{Q})\;[\langle i_k,q\rangle\in\Psi(\mathrm{Nbase}(y_0,\dots,y_{k-1},q_k,y_{k+1},\dots,y_{n-1}))\text{ and }x_{i_k}<q].$$

By pigeon hole principle, one can assume that i_k only depends on k, and does not depend on ε. Without loss of generality, we may assume that $n\notin\{i_k:k<n\}$. Now, let $g_\Phi:\mathbb{R}^{n+1}_<\to\mathbb{R}^n_<$ be the computable function induced from the enumeration operator Φ, that is, $g_\Phi(\boldsymbol{r})=\boldsymbol{s}$ if and only if $\Phi(\mathrm{Nbase}(\boldsymbol{r}))=\mathrm{Nbase}(\boldsymbol{s})$. Then, given $p\in\mathbb{Q}$, we define $\boldsymbol{z}^p=(z^p_j)_{j<n}$ as follows:

$$\boldsymbol{z}^p=g_\Phi(x_0,\dots,x_{n-1},p),$$

It is easy to see that for any $j<n$, if $p\leq x_n$ then $z^p_j\leq y_j$, and if $p\geq x_n$ then $y_j\leq z^p_j$. If there is $p>x_n$ such that $z^p_j=y_j$ for any $j<n$, then clearly,

$$\mathrm{Nbase}(\boldsymbol{y})=\mathrm{Nbase}(\boldsymbol{z}^p)\leq_e\mathrm{Nbase}((x_i)_{i<n})<_e\mathrm{Nbase}((x_i)_{i\leq n})=\mathrm{Nbase}(\boldsymbol{x}),$$

which contradicts our assumption. Therefore, for any $p>x_n$, there is $k<n$ such that $y_k<z^p_k$. By pigeon hole principle, one can fix such a k. Let q_k be a rational such that $y_k<q_k<z^p_k$. Then,

$$\mathrm{Nbase}(y_0,\dots,y_{k-1},q_k,y_{k+1},\dots,y_{n-1})\subseteq\mathrm{Nbase}(\boldsymbol{z}^p).$$

Hence, there is $q\in\mathbb{Q}$ such that $\langle i_k,q\rangle\in\Psi(\mathrm{Nbase}(\boldsymbol{z}^p))$ and $x_{i_k}<q$. By combining these observations, we have

$$x_n<p\iff(\exists q\in\mathbb{Q})\;[\langle i_k,q\rangle\in\Psi(\mathrm{Nbase}(\boldsymbol{z}^p))\text{ and }x_{i_k}<q].$$

This gives us a right-approximation of x_n from a left-approximation of $\boldsymbol{z}^p$ and a right-approximation of x_{i_k}. This concludes that

$$\mathrm{Nbase}(-x_n) \leq_e \mathrm{Nbase}(\boldsymbol{z}^p) \oplus \mathrm{Nbase}(-x_{i_k}) \leq_e \mathrm{Nbase}((x_j)_{j<n}) \oplus \mathrm{Nbase}(-x_{i_k}).$$

This contradicts our assumption on independence of $\boldsymbol{x}$ as $i_k \neq n$. Consequently, we obtain that $\mathrm{Nbase}(z, \boldsymbol{y}) \not\equiv_e \mathrm{Nbase}(\boldsymbol{x})$. □

By Lemma 5.20, if $\mathbf{a}$ is an e-degree of an independent point in $\mathbb{R}^{n+1}_<$, then $\mathbf{a}$ is not the join of an $\mathbb{R}^n_<$-degree $\mathbf{c}$ and a $T_1[\Gamma]$-degree $\mathbf{d}$. □

In particular, we have $\mathcal{D}_{\mathbb{R}^{n+1}_<} \not\subseteq \mathcal{D}_{\mathbb{R}\times\mathbb{R}^n_<}$. We next see a degree-theoretic behavior of the join of a n-semirecursive e-degree and a total e-degree.

Theorem 5.21. *There are an n-semirecursive e-degree $\mathbf{c} \leq \mathbf{0}''$ and a total e-degree $\mathbf{d} \leq \mathbf{0}''$ such that the join $\mathbf{c} \oplus \mathbf{d}$ is not $(n+1)$-semirecursive. In particular,*

$$\mathcal{D}_{\mathbb{R}\times\mathbb{R}^n_<} \not\subseteq \mathcal{D}_{\mathbb{R}^{n+1}_<}.$$

Proof. For $z \in \mathbb{R}$, we say that a point $(x_i)_{i<n} \in \mathbb{R}^n_<$ is *independent relative to z* if neither $\mathrm{Nbase}_<(x_i)$ nor $\mathrm{Nbase}_<(-x_i)$ is e-reducible to $\mathrm{Nbase}(z, (x_j)_{j\neq i})$ for any $i \leq n$. One can easily see that $(z, \boldsymbol{x})$ is 1-generic or Martin-Löf random as a point in $\mathbb{R}^{n+1}$, then $\boldsymbol{x}$ is independent relative to z.

Lemma 5.22. *If $\boldsymbol{x} \in \mathbb{R}^n_<$ is independent relative to $z \in \mathbb{R}$, and $z \not\leq_T \emptyset'$, then $\mathrm{Nbase}(z, \boldsymbol{x})$ is not $(n+1)$-semirecursive.*

Proof. Let $z \in \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^n_<$ be such that $\boldsymbol{x}$ is independent relative to z, and that $\mathrm{Nbase}(z) \not\leq_e K \oplus K^{\mathsf{c}}$. Suppose that there is $\boldsymbol{y} = (y_j)_{j\leq n} \in \mathbb{R}^{n+1}_<$ such that $\mathrm{Nbase}(\boldsymbol{y}) \equiv_e \mathrm{Nbase}(z, \boldsymbol{x})$. Consider any permutation σ on $n+1$. First assume that $\mathrm{Nbase}(z) \leq_e \mathrm{Nbase}(y_{\sigma(0)})$. Then, by Lemma 5.2, we have $\mathrm{Nbase}(z) \leq_e \emptyset$ or $\mathrm{Nbase}(-y_{\sigma(0)}) \leq_e \emptyset$. The latter inequality implies that $\mathrm{Nbase}(z) \leq_e \mathrm{Nbase}(y_{\sigma(0)}) \leq_e K \oplus K^{\mathsf{c}}$, which is impossible by our assumption.

Assume that for some $k \leq n$, there is a permutation σ on $n+1$ such that $\mathrm{Nbase}(z) \leq_e \mathrm{Nbase}((y_{\sigma(j)})_{j\leq k})$, but there is no permutation τ on $k+1$ such that $\mathrm{Nbase}(z) \leq_e \mathrm{Nbase}((y_{\sigma\circ\tau(j)})_{j<k})$. By Lemma 5.2, the inequality $\mathrm{Nbase}(z) \leq_e \mathrm{Nbase}((y_{\sigma(j)})_{j\leq k})$ implies that for any $\ell \leq k$, either $\mathrm{Nbase}(z) \leq_e \mathrm{Nbase}((y_{\sigma(j)})_{j\neq\ell, j\leq k})$ or $\mathrm{Nbase}(-y_{\sigma(\ell)}) \leq_e \mathrm{Nbase}((y_{\sigma(j)})_{j\neq\ell, j\leq k})$. The former is impossible by our choice of k. Hence, we get that $\mathrm{Nbase}((-y_{\sigma(j)})_{j\leq k}) \leq_e \mathrm{Nbase}((y_{\sigma(j)})_{j\leq k})$. Note that by induction, there must exist such a k, and we have $k \geq 1$.

Eventually, we must have $\mathrm{Nbase}(\boldsymbol{y}, -y_s, -y_t) \equiv_e \mathrm{Nbase}(\boldsymbol{y})$ for some $s \neq t \leq n$. Hence, one can identify $\boldsymbol{y}$ with an element of $\mathbb{R} \times \mathbb{R}^{n-1}_<$. However, by Lemma 5.20 relative to z and by independence of $\boldsymbol{x}$ relative to z, it is impossible to have $\mathrm{Nbase}(\boldsymbol{y}) \equiv_e \mathrm{Nbase}(z, \boldsymbol{x})$. □

By choosing a 2-generic or a 2-random, it is easy to construct an independent sequence $(z, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^n_<$ of Δ^0_3 reals such that $z \not\leq_T \emptyset'$. Let $\mathbf{c}$ be the e-degree of $\mathrm{Nbase}(\boldsymbol{x})$, and $\mathbf{d}$ be that of $\mathrm{Nbase}(z)$. Since these are Δ^0_3, we have $\mathbf{c}, \mathbf{d} \leq \mathbf{0}''$. By Lemma 5.22, the join $\mathbf{c} \oplus \mathbf{d}$ is not $(n+1)$-semirecursive. □

The next theorem reveal a strong quasi-minimal behavior of a finite join of semirecursive e-degrees.

Theorem 5.23. *For any $n \in \omega$, an n-semirecursive e-degree is either total or a strong quasi-minimal cover of a total e-degree.*

Proof. Fix $\boldsymbol{x} = (x_i)_{i \leq n} \in \mathbb{R}^n_<$. Suppose that $\boldsymbol{x}$ bounds a total degree $z \in 2^\omega$. As in the proof of Lemma 5.22, by iterating Lemma 5.2, for any permutation σ on $n+1$, either $\mathrm{Nbase}(z) \leq_e \emptyset$ or $\mathrm{Nbase}(-x_{\sigma(k)}) \leq_e \mathrm{Nbase}((x_{\sigma(i)})_{i<k})$ holds for some $k \leq n$. We assume $\mathrm{Nbase}(z) \not\leq_e \emptyset$. Therefore, by induction, we can find a permutation σ on $n+1$ and a number $k \leq n$ such that $\mathrm{Nbase}(-x_{\sigma(\ell)}) \not\leq_e \mathrm{Nbase}((x_{\sigma(i)})_{i<\ell})$ for any $k < \ell \leq n$ and $\mathrm{Nbase}(-x_{\sigma\circ\tau(k)}) \leq_e \mathrm{Nbase}((x_{\sigma\circ\tau(i)})_{i<k})$ for any permutation τ on $k+1$. The latter condition implies that $\mathrm{Nbase}((x_{\sigma(i)}, -x_{\sigma(i)})_{i\leq k}) \equiv_e \mathrm{Nbase}((x_{\sigma(i)})_{i\leq k})$. This shows that $(x_{\sigma(i)})_{i\leq k}$ has a total degree. We claim that $(x_{\sigma(i)})_{i\leq k}$ has the greatest total degree below $\mathrm{Nbase}(\boldsymbol{x})$. Suppose that Y is a subset of ω such that $Y \oplus Y^{\mathsf{c}} \leq_e \mathrm{Nbase}(\boldsymbol{x})$. Note that $\mathrm{Nbase}(\boldsymbol{x}) \equiv_e \mathrm{Nbase}((x_{\sigma(i)})_{i\leq n})$. Therefore, by iterating Lemma 5.2, either $Y \oplus Y^{\mathsf{c}} \leq_e \mathrm{Nbase}((x_{\sigma(i)})_{i\leq k})$ or $\mathrm{Nbase}(-x_{\sigma(\ell)}) \leq_e \mathrm{Nbase}((x_{\sigma(i)})_{i<\ell})$ holds for some $k < \ell \leq n$. However, the latter condition cannot hold by our choice of σ and k. Consequently, $\boldsymbol{x}$ is either total or a strong quasi-minimal cover of $(x_{\sigma(i)})_{i\leq k}$. □

5.1.5. *Quasi-minimality w.r.t. telograph-cototal degrees.* In Section 5.1.3, we developed techniques constructing T_1-quasi-minimal degrees. In contrast to non-T_1 cb_0 spaces, there are a number of interesting T_1-space, and therefore T_1-degrees, e.g. graph-cototal degrees, cylinder-cototal degrees, telograph-cototal degrees, etc. We examine the behavior of T_1-degrees using quasi-minimality arguments.

First recall from Theorem 5.14 that every semirecursive, non-Δ^0_2 e-degree is quasi-minimal w.r.t. strongly Π^0_2-named T_1-spaces (see Lemma 5.15). However, we essentially know just one proven example of a Π^0_2-named T_1 space, namely ω^ω (and its variants). We do conjecture that the spaces $(\omega_{\mathrm{cof}})^\omega$, $(\omega^\omega)_{\mathrm{co}}$ and $(\hat{\omega}_{\mathrm{TP}})^\omega$ are not strongly Π^0_2-named. Then we introduce a different technique to prove that, if $x \in \mathbb{R}$ is not Δ^0_2, then $\mathrm{Nbase}_<(x)$ is $(\hat{\omega}_{\mathrm{TP}})^\omega$-quasi-minimal.

Theorem 5.24. *Every semirecursive, non-Δ^0_2 e-degree is quasi-minimal w.r.t. telograph-cototal e-degrees.*

We will also show that quasi-minimality w.r.t. telograph-cototal e-degrees is strictly stronger than quasi-minimality (w.r.t. total e-degrees).

Theorem 5.25. *There is a semirecursive set $A \subseteq \omega$ which is quasi-minimal, but not quasi-minimal w.r.t. telograph-cototal e-degrees.*

Total-like properties. To prove Theorems 5.24 and 5.36, we introduce some variants of totality, and show that every telograph-cototal e-degree is close to being total in this sense.

The power set of ω can be topologized as the countable product $\mathbb{S}^\omega$ of the Sierpinskí space $\mathbb{S} = \{0, 1\}$, where open sets in $\mathbb{S}$ are $\emptyset$, $\{1\}$, and $\mathbb{S}$. Every element $S \in \mathbb{S}^\omega$ is identified with $S^{-1}\{1\} \subseteq \omega$. We consider the effective Borel hierarchy on $\mathbb{S}^\omega$. Here, we note that Selivanov and his collaborators (see e.g. [15, 71]) have studied a modified version of the Borel hierarchy. As seen in Observation 3.53, the underlying space is a G_δ-space if and only if the classical Borel hierarchy and the modified Borel hierarchy coincide. In particular, a $\boldsymbol{\Pi}^0_2$ set is not necessarily G_δ in the space $\mathbb{S}^\omega$.

We say that $\mathcal{P} \subseteq \mathbb{S}^\omega$ is *C-computably e-closed* if there is a C-computable sequence $(D_n)_{n\in\omega}$ of finite subsets of ω such that

$$A \in \mathcal{P} \iff (\forall n)\ D_n \not\subseteq A.$$

We also say that $\mathcal{P} \subseteq \mathbb{S}^\omega$ is *C-computably e-G_δ* if there is a C-computable sequence $(D_{n,i})_{n,i\in\omega}$ of finite subsets of ω such that

$$A \in \mathcal{P} \iff (\forall n)(\exists i)\ D_{n,i} \subseteq A.$$

As mentioned above, a computably e-closed set is not necessarily computably e-G_δ. Therefore we avoid use of the terminology such as "e-Π^0_2". Note that every computably e-closed set is downward closed, and every computably e-G_δ set is upward closed.

Definition 5.26. A set $A \subseteq \omega$ is said to be *G_δ-left-total* if there are a set $C \subseteq \omega$, and an computably e-G_δ set $\mathcal{P}$ such that

$$C \oplus C^{\mathsf{c}} \leq_e A, \text{ and } A \in \mathcal{P},$$
$$(\forall X)\ [X \in \mathcal{P} \text{ and } X \subseteq A \implies A \leq_e X \oplus C \oplus C^{\mathsf{c}}].$$

A set $A \subseteq \omega$ is said to be *jump-right-total* if there are a set $C \subseteq \omega$, and a C'-computably e-closed set $\mathcal{N}$ such that

$$C \oplus C^{\mathsf{c}} \leq_e A, \text{ and } A \in \mathcal{N},$$
$$(\forall X)\ [X \in \mathcal{N} \text{ and } A \subseteq X \implies A \leq_e X \oplus C \oplus C^{\mathsf{c}}].$$

Example 5.27. Every telograph-cototal e-degree is G_δ-left-total and jump-right-total.

Proof. We first show that every telograph-cototal e-degree is G_δ-left-total. Let A be $\mathrm{Nbase}(x)$ for a point x in the product telophase space $(\hat{\omega}_{TP})^\omega$. Let C be the total part of x, that is, $C = \{2\langle n,m\rangle : x(n) = m\} \cup \{2\langle n,m\rangle + 1 : x(n) \neq m\}$, and define $R = \{n \in \omega : x(n) = \infty\}$ and $S = \{n \in \omega : x(n) = \infty_\star\}$. The proof of Theorem 3.22 shows that $A = \mathrm{Nbase}(x)$ is Medvedev equivalent to $\{C\} \times \mathrm{Sep}(R,S)$. In particular $C \oplus C^{\mathsf{c}} \leq_e A$. Define $D_{n,i} = \{\langle n,j,m\rangle\}$, where $i = \langle j,m\rangle$. This sequence generates a computably e-G_δ-set:

$$\mathcal{P} = \{X : (\forall n)(\exists i)\ D_{n,i} \subseteq X\}.$$

Recall the definition of $\mathrm{Nbase}(x)$ in Example 3.17. Clearly, for any n, there is $i = \langle j,\sigma\rangle$ such that $D_{n,i} = \{\langle n,j,\sigma\rangle\} \subseteq A = \mathrm{Nbase}(x)$. Fix $X \in \mathcal{P}$ such that $X \subseteq A$. Then, for any n, $D_{n,i_n} \subseteq X$ for some i_n. Given an enumeration of X, one can find a sequence $(i_n)_{n\in\omega}$ of such witnesses. The sequence $(i_n)_{n\in\omega}$ may depend on how to enumerate X. As in the proof of Theorem 3.22, we can construct $B \subseteq \omega$ according to $(i_n)_{n\in\omega}$ as follows. Define $n \notin B$ iff i_n is of the form $\langle 2,m\rangle$, that is, i_n is an index of the interval $[m,\infty_\star]$. We claim that $B \in \mathrm{Sep}(R,S)$. If $n \in R$, then $x(n) = \infty$, and therefore, $x(n) \notin [m,\infty]$ for any m, and thus $\langle n,2,m\rangle \notin A$. Since $X \subseteq A$, we also have $\langle n,2,m\rangle \notin X$. Therefore, $n \in B$, that is, $R \subseteq B$. If $n \in S$, then $x(n) = \infty_\star$, and thus, $\langle n,j,m\rangle \in A$ implies $j = 2$. Since $X \subseteq A$, i_n must be of the form $\langle 2,m\rangle$. This implies that $n \notin B$, that is, $B \cap S = \emptyset$. Consequently, we have $B \in \mathrm{Sep}(R,S)$. This procedure gives us an enumeration operator Ψ (independent of X) such that $\Psi(X \oplus C \oplus C^{\mathsf{c}}) = A$. Hence, A is G_δ-left-total.

We next show that every telograph-cototal e-degree is jump-right-total. Let A be $\mathrm{Nbase}(x)$ for a point x in the product telophase space $(\hat{\omega}_{TP})^\omega$. We define C, R, and S as above. Consider the following C'-computably e-closed set:

$$\mathcal{N} = \{X : (\forall n \in R \cup S)(\forall i, j, s, t)\ [\langle n, i, s\rangle, \langle n, j, t\rangle \in X \implies i = j \neq 0]\}.$$

It is clear that $A \in \mathcal{N}$. Note that $R \cup S$ is C-co-c.e., and in particular, C'-computable. To see that $\mathcal{N}$ is C'-computably e-closed, let I be the set of all $\langle n, i, j, s, t\rangle$ such that $n \in R \cup S$ and either $i \neq j$ or $i = 0$. Define $D_k = \{\langle n_k, i_k, s_k\rangle, \langle n_k, j_k, t_k\}$, where $\langle n_k, i_k, j_k, s_k, t_k\rangle$ is the k-th element of I. Clearly, $(D_k)_{k\in\omega}$ is a C'-computable sequence of finite sets. It is easy to see that $X \in \mathcal{N}$ iff $D_k \not\subseteq X$ for all $k \in \omega$.

Fix $X \in \mathcal{N}$ such that $A \subseteq X$. Consider the procedure Ψ which enumerates all $\langle n, i, m\rangle \in X$ which agree with C, that is, Ψ enumerates $\langle n, 0, m\rangle$ if $\langle n, 0, m\rangle \in X$ and $2\langle n, m\rangle \in C$, and for $i \neq 0$, Ψ enumerates $\langle n, i, m\rangle$ if $\langle n, i, m\rangle \in X$ and $2\langle n, s\rangle + 1 \in C$ for any $s < m$.

We claim that this procedure $\Psi(X \oplus C \oplus C^{\mathsf{c}})$ precisely enumerates A. To see this, for $n, m \in \omega$, first note that $x(n) = m$ (that is, $\langle n, 0, m\rangle \in A$) if and only if $2\langle n, m\rangle \in C$. Thus, $x(n) = m$ if and only if Ψ enumerates $\langle n, 0, m\rangle$ since $\langle n, 0, m\rangle \in A \subseteq X$. Next, $x(n) \geq m$ if and only if $2\langle n, s\rangle + 1 \in C$ for any $s < m$. If $x(n) \in \omega$, then for each $i \neq 0$ and $m \leq x(n)$, we have $\langle n, i, m\rangle \in A \subseteq X$, and therefore Ψ enumerates $\langle n, i, m\rangle$. If $x(n) \notin \omega$, then this means that $n \in R \cup S$. Assume that $x(n) = \infty$. Then $\langle n, 1, m\rangle \in A$ for any m. Since $A \subseteq X \in \mathcal{N}$, $\langle n, i, m\rangle \in X$ if and only if $i = 1$. Thus, Ψ enumerates $\langle n, i, m\rangle \in X$ if and only if $i = 1$. Similarly, when $x(n) = \infty_\star$, Ψ enumerates $\langle n, i, m\rangle \in X$ if and only if $i = 2$. Consequently, we have $\Psi(X \oplus C \oplus C^{\mathsf{c}}) = A$. Hence, A is jump-right-total. $\square$

5.1.6. *Quasi-minimality w.r.t. left-/right-totality.* We now describe how to use these total-like properties introduced in Section 5.1.5 to show some results on quasi-minimality mentioned in Section 5.1.5.

Lemma 5.28. *Let x be a real. If x is not left-c.e. in $\emptyset'$, then* $\mathrm{Nbase}_<(x)$ *is quasi-minimal w.r.t. G_δ-left-total degrees. If x is not right-c.e. in $\emptyset'$, then* $\mathrm{Nbase}_<(x)$ *is quasi-minimal w.r.t. jump-right-total degrees.*

Proof. Let x be a real which is not left-c.e. in $\emptyset'$. In particular, x is neither left-c.e. nor right-c.e. For the first assertion, let A be a G_δ-left-total set, witnessed by $(D_{n,i})_{n,i\in\omega}$, and C. That is, let $\mathcal{P}$ be the computable e-G_δ set defined by $X \in \mathcal{P}$ iff for each n, there is i such that $D_{n,i} \subseteq X$. Suppose that $\Phi(\mathrm{Nbase}_<(x)) = A \in \mathcal{P}$. By Lemma 5.2, if x is neither left- nor right-c.e., then $\mathrm{Nbase}_<(x)$ is quasi-minimal, and therefore C is computable since $C \oplus C^{\mathsf{c}} \leq_e A \leq_e \mathrm{Nbase}_<(x)$. Hence, $(D_{n,i})_{n,i\in\omega}$ is a computable sequence.

As mentioned above, $\mathcal{P}$ is upward closed, and so is $\Phi^{-1}[\mathcal{P}]$ by monotonicity of the enumeration operator Φ. We define $\Phi^*[\mathcal{P}] = \{z \in \mathbb{R} : \Phi(\mathrm{Nbase}_<(z)) \in \mathcal{P}\}$, which is upward closed w.r.t. the standard ordering on $\mathbb{R}$. Consider $q = \inf \Phi^*[\mathcal{P}]$. Then, $q \leq x$ since $x \in \Phi^*[\mathcal{P}]$. Note that $q \leq p$ iff $p + \varepsilon \in \Phi^*[\mathcal{P}]$ for any $\varepsilon > 0$, and therefore, for any rational p,

$$p < q \iff \neg(\forall \varepsilon > 0)(\forall n)(\exists i)\ D_{n,i} \subseteq \Phi(\mathrm{Nbase}_<(p + \varepsilon)).$$

This shows that $\mathrm{Nbase}_<(q)$ is Σ^0_2, and thus, q is left-c.e. in $\emptyset'$. Since x is not $\emptyset'$-left-c.e., we have $q < x$. Therefore, there is a rational p such that $q < p < x$.

Since $\mathrm{Nbase}_<(p) \subseteq \mathrm{Nbase}_<(x)$, by monotonicity of an enumeration operator, we have $\Phi(\mathrm{Nbase}_<(p)) \subseteq \Phi(\mathrm{Nbase}_<(x)) = A$. Moreover, we have $p \in \Phi^*[\mathcal{P}]$, and therefore, $\Phi(\mathrm{Nbase}(p)) \in \mathcal{P}$. Since p is rational, C is computable, and A is G_δ-left-total (via $\mathcal{P}$), we have $A \leq_e \Phi(\mathrm{Nbase}_<(p)) \leq_e \emptyset$. Consequently, $\mathrm{Nbase}_<(x)$ is quasi-minimal w.r.t. G_δ-left-total degrees.

For the second assertion, let A be a jump-right-total set, witnessed by $(D_n)_{n\in\omega}$, and C. That is, $(D_n)_{n\in\omega}$ is a C'-computable sequence, and let $\mathcal{N}$ be the C'-computable closed set defined by $X \in \mathcal{N}$ iff for all $n \in \omega$, $D_n \not\subseteq X$. Suppose that $\Phi(\mathrm{Nbase}_<(x)) = A \in \mathcal{N}$. By the same argument as before, C has to be computable, and thus $(D_n)_{n\in\omega}$ is a $\emptyset'$-computable sequence.

As mentioned above, $\mathcal{N}$ is downward closed, and so is $\Phi^{-1}[\mathcal{N}]$ by monotonicity of the enumeration operator Φ. We define $\Phi^*[\mathcal{N}] = \{z \in \mathbb{R} : \Phi(\mathrm{Nbase}_<(z)) \in \mathcal{N}\}$, which is downward closed w.r.t. the standard ordering on $\mathbb{R}$. Consider $q = \sup \Phi^*[\mathcal{N}]$. Then, $x \leq q$ since $x \in \Phi^*[\mathcal{N}]$. Moreover, by a similar argument as above, for any rational p,

$$q < p \iff \neg(\forall \varepsilon > 0)(\forall n)\ D_n \not\subseteq \Phi(\mathrm{Nbase}_<(p+\varepsilon)).$$

This shows that $\mathrm{Nbase}_<(-q)$ is Σ^0_2, and thus, q is right-c.e. in $\emptyset'$. Therefore, by our assumption on x, we have $x < q$. Thus, there is a rational p such that $x < p < q$. Since $\mathrm{Nbase}_<(x) \subseteq \mathrm{Nbase}_<(p)$, by monotonicity of an enumeration operator, we have $A = \Phi(\mathrm{Nbase}_<(x)) \subseteq \Phi(\mathrm{Nbase}_<(p))$. Moreover, we have $p \in \Phi^*[\mathcal{N}]$, and therefore, $\Phi(\mathrm{Nbase}(p)) \in \mathcal{N}$. Since p is rational, C is computable, and A is jump-right-total (via $\mathcal{N}$), we have $A \leq_e \Phi(\mathrm{Nbase}_<(p)) \leq_e \emptyset$. Consequently, $\mathrm{Nbase}_<(x)$ is quasi-minimal w.r.t. jump-right-total degrees. □

Proof of Theorem 5.24. Let $\mathbf{d}$ be a semirecursive, non-Δ^0_2 e-degree. As mentioned in Section 3.2, the semirecursive degrees are characterized as the $\mathbb{R}_<$-degrees. Hence, there is a real x such that $\mathrm{Nbase}_<(x) \in \mathbf{d}$ and $\mathrm{Nbase}_<(x)$ is not Δ^0_2. In particular, x is not $\emptyset'$-left-c.e. or x is not $\emptyset'$-right-c.e. As seen in Example 5.27, every telograph-cototal e-degree is G_δ-left-total and jump-right-total. Therefore, by Lemma 5.28, $\mathrm{Nbase}_<(x)$ is quasi-minimal w.r.t. telograph-cototal e-degree. □

Theorem 5.24 implies that there is a Σ^0_2 e-degree which is quasi-minimal w.r.t. telograph-cototal e-degrees. However, the proof of Theorem 5.25 indeed implies that a semirecursive, Δ^0_2, e-degree is not necessarily quasi-minimal w.r.t. telograph-cototal e-degrees (see Theorem 5.12).

Lemma 5.29. *There is a Δ^0_2 real x such that x is neither left- nor right-c.e., but $\mathrm{Nbase}_<(x)$ is not quasi-minimal w.r.t. telograph-cototal e-degrees.*

Proof. We construct a Δ^0_2 real x and a computably inseparable pair (A, B) such that x is not right-c.e., $A \cup B$ is co-c.e., and that any enumeration of $\mathrm{Nbase}_<(x)$ computes a separator of (A, B), where recall from Theorem 3.21 that the telograph-cototal e-degrees are characterized in terms of separating sets. Let r_e be the e-th right-c.e. real. Consider the following requirements:

$$\begin{aligned}
P_e &\colon \Phi_e \text{ is total} \implies \Phi_e \notin \mathrm{Sep}(A, B),\\
N_e &\colon \mathrm{Nbase}_<(x) \neq r_e,\\
G &\colon (\exists \Gamma)(\forall p)\ \mathrm{rng}(p) = \mathrm{Nbase}_<(x) \implies \Gamma^p \in \mathrm{Sep}(A, B).
\end{aligned}$$

Begin with $x_0 = 0$, $A_0 = \emptyset$, $B_0 = \omega$, and $\Gamma_0 = \emptyset$. The global strategy G constructs Γ_s as follows: The operator Γ_s is a collection of tuples of the form $\langle n, i, p\rangle$, which indicates that $\Gamma(\mathrm{Nbase}_<(x))(n)\downarrow= i$ for any $x > p$. For any s, we ensure that only finitely many tuples of the form $\langle n, i, p\rangle$ is enumerated into Γ_s. At the beginning of each stage s, the global strategy tries to recover the destroyed computations as follows. Given $n < s$, the G-strategy check if there is a rational $p < x_s$ such that $\langle n, i, p\rangle$ is enumerated into Γ_s for some $i < 2$. If there is no such p, enumerate $\langle n, i, x_s - 2^{-2n}\rangle$ into Γ_{s+1}, where $i = 1$ iff $n \in A_s$.

A priority ordering is given by $P_e < N_e < P_{e+1}$, where $S < T$ means that S is a higher priority strategy than T. The P_e-action may decrease the value of x_s, and the N_e-action may increase the value of x_s. At stage s, a P_e-strategy acts as follows:

(P1) Choose a large $n_e \in B_s$.
(P2) Wait for $\Phi_{e,s}(n_e)\downarrow= 0$.
(P3) Move n_e from B to A, that is, define $B_{s+1} = B_s \setminus \{n_e\}$ and $A_{s+1} = A_s \cup \{n_e\}$.
(P4) Try to destroy the computation of $\Gamma^x(n_e)$ by putting $x_{s+1} = x_s - 2^{-2n_e+1}$.
(P5) Injure all lower priority strategies by resetting all parameters.

An N_e-strategy acts as follows:

(N1) Choose a large m_e.
(N2) Wait for $x_s < r_{e,s} < x_s + 2^{-2m_e}$.
(N3) Put $x_{s+1} = x_s + 2^{-2m_e}$, and for a sufficiently large u such that $r_{e,s} < x_{s+1} - 2^{-2u}$, we hereafter require that a large number should be bigger than u.
(N4) The above action may cause inconsistent computations on $\Gamma^x(n)$, that is, it is possible to have $p < q < x_{s+1}$ such that both $\langle n, i, p\rangle$ and $\langle n, 1-i, q\rangle$ is contained in Γ_{s+1}. For all such n, remove n from $A \cup B$.
(N5) Injure all lower priority strategies by resetting all parameters.

If a strategy reaches (P5) or (N5), then the strategy never acts unless it is later initialized. Thus, every strategy acts once with the same n_e or m_e. The P-requirements can only be injured by (N3), and the N-requirements can only be injured by (P4).

Claim. The action (P4) always forces $\Gamma_{s+1}(x_{s+1}; n_e)$ to be undefined, that is, there is no $p < x_{s+1}$ and i such that $\langle n_e, i, p\rangle \in \Gamma_{s+1}$.

Proof. An action of a higher priority strategy injures P_e at some stage $t < s$, which forces P_e to redefine n_e as a large number $> t$. At some stage v, $\langle n_e, i, x_v - 2^{2n_e}\rangle$ is enumerated into Γ_v. The G-strategy ensures $v > n_e > t$. After n_e is settled after stage t, only a lower priority strategy can act. Only N-strategies can increase the value of x. Therefore, for $m = \sum_{d\geq e} 2^{-2m_d}$, and we have $x_s < x_v + m$. One can see that $m < 2^{-2n_e-1}$, and hence

$$x_{s+1} = x_s - 2^{-2n_e+1} < x_v + m - 2^{-2n_e+1} < x_v - 3 \cdot 2^{-2n_e-1} < x_v - 2^{2n_e}.$$

This forces $\Gamma_{s+1}(\mathrm{Nbase}_<(x_{s+1}); n_e)$ to be undefined. $\square$

Thus, the G-strategy can recover the correct value for a separating set. We would like to make sure that Γ^x is total. Of course, enumerating inconsistent computations makes Γ^x be partial. Hence, instead of dealing with Γ^x, we consider Γ^p for any enumeration p of $\mathrm{Nbase}_<(x)$. The computation Γ^p is given as follows. Let p be an enumeration of $\mathrm{Nbase}_<(x)$. For each n, wait for s such that $q < p(s)$ and $\langle\langle n, i\rangle, q\rangle \in \Gamma_s$ for some $i < 2$, and then for the first such, we define $\Gamma^p(n) = i$. By the action of the G-strategy, Γ^p becomes total.

Claim. The G-requirement is fulfilled.

Proof. Straightforward. □

Claim. The action (N3) only cause inconsistent $\Gamma^x(n)$ for $n \geq m_e$. Hence, if the P_e-strategy acts, and is never injured, then P_e is fulfilled.

Proof. Otherwise, $\langle n, 1, p\rangle$ is already enumerated. In this case, we must have $n_d = n$ for some d. Assume that P acts with $n_d < m_e$ at some stage $t < s$. As in the above argument, at some stage v, $\langle n_d, i, x_v - 2^{2n_d}\rangle$ is enumerated into Γ_v. As seen above, $x_{t+1} < x_v - 3 \cdot 2^{-2n_d-1}$. Thus, to make $\Gamma^x(n_d)$ inconsistent, we need to increase x by 2^{-2n_d-1}. As before, we have $m < 2^{-2n_d-1}$. □

Claim. If the N_e-strategy acts, and is never injured, then the property $r_e < x$ is preserved forever.

Proof. To see this, note that the P_e-action (P3) at stage s only injure lower priority strategies. This is because, for $d < e$, if N_d has already acted at some stage $t < s$, and not injured until s, then n_e must be bigger than u chosen by N_d. Therefore, $r_e \leq r_{e,t} < x_{t+1} - 2^{-u} \leq x_s$. □

These claims conclude the proof. □

Proof of Theorem 5.25. By Lemma 5.29. □

5.1.7. *An alternate approach via Kalimullin pairs.* Mariya Soskova (personal communication) has pointed out to us that some of the results in Subsection 5.1 can alternatively be obtained via Kalimullin pairs:

Definition 5.30 (Kalimullin [42])**.** A $\mathcal{K}$-pair is a pair (A, B) where $A, B \subseteq \omega$ and there is a c.e. set $W \subseteq \omega \times \omega$ such that $A \times B \subseteq W$ and $A^c \times B^c \subseteq W^c$. The pair is trivial, if A is c.e., and non-trivial otherwise.

As observed by Kalimullin, $\mathcal{K}$-pairs generalize the notion of semirecursive set in way that preserves many of their good properties. Namely, whenever A is semirecursive, then (A, A^c) is a $\mathcal{K}$-pair. Conversely, as shown in [8] if (A, B) is a non-trivial $\mathcal{K}$-pair, then there is a semirecursive C with $A \leq_e C$ and $B \leq_e C^c$. Further relevant properties of $\mathcal{K}$-pairs were provided by Ganchev, Soskova and others in [42, 8, 9, 27, 30].

Soskova in particular gave alternative proofs of Lemma 5.2, Proposition 5.3, Proposition 5.5, Proposition 5.6, Proposition 5.8 and Proposition 5.18 by reasoning about $\mathcal{K}$-pairs. By invoking some highly non-trivial results from [28], she also provided an alternative proof of Theorem 5.14. Given the usefulness of $\mathcal{K}$-pairs and the context of this work, the question is raised whether we can characterise the halves of non-trivial $\mathcal{K}$-pairs as the degrees of points in some natural topological space (see Question 6.11).

5.2. **T_1-degrees which are not T_2.** We consider two cb-representations of ω^ω. When we simply write ω^ω, we assume that ω^ω is endowed with the usual Baire representation. We again use $(\omega^\omega)_{\rm co}$ to denote the represented space whose underlying set is ω^ω whose cb-representation is given by $B_\sigma := \{f \in \omega^\omega : \sigma \not\prec f\}$. In this section, we will show the following:

Theorem 5.31. *For any represented Hausdorff space $\mathcal{X}$, there is a cylinder-cototal e-degree which is not an $\mathcal{X}$-degree, that is,*

$$\mathcal{D}_{(\omega^\omega)_{\rm co}} \not\subseteq \mathcal{D}_{\mathcal{X}}.$$

We say that a function $f : \omega \to \omega$ is *computably dominated* if there is a computable function $h : \omega \to \omega$ such that $f(n) \leq h(n)$ for almost all $n \in \omega$. For an oracle C, we say that $f \in \omega^\omega$ is *C-computably dominated* if h above can be chosen to be C-computable. We also say that a function $f : \omega \to \omega$ is *computably dominating* if for any computable function $h : \omega \to \omega$, $h(n) \leq f(n)$ for almost all $n \in \omega$.

Observation 5.32.

(1) *For any $f : \omega \to \omega$,* $\mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f) \leq_e \mathrm{Nbase}_{\omega^\omega}(f)$.
(2) *If $f : \omega \to \omega$ is computably dominated, then* $\mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f) \equiv_e \mathrm{Nbase}_{\omega^\omega}(f)$. *Hence,* $\mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$ *has a total degree.*
(3) *There is a computably dominating function f such that* $\mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$ *is c.e.*

Proof. For (1), if we see $\sigma \in \mathrm{Nbase}_{\omega^\omega}(f)$ (that is, $\sigma \preceq f$), we know that $\tau \not\preceq f$ whenever τ is incomparable with σ, and therefore enumerate all such τ into $\mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$. For (2), it suffices to show that $\mathrm{Nbase}_{\omega^\omega}(f) \leq_e \mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$. Assume that f is bounded by a computable function g. We may assume $f(n) \leq g(n)$ for any n. Then, for any ℓ, there are only finitely many strings σ of length ℓ such that $\sigma(n) < g(n)$ for any $n < \ell$. Let $I(g, \ell)$ be the set of all such strings. Then, all but one string in $I(g, \ell)$ is enumerated into $\mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$. By finiteness of $I(g, \ell)$, all such strings are enumerated by some finite stage, and then one can know what the unique string $\sigma \in I(g, \ell) \setminus \mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$ is. Then, enumerate σ into $\mathrm{Nbase}_{\omega^\omega}(f)$. This procedure clearly witnesses that $\mathrm{Nbase}_{\omega^\omega}(f) \leq_e \mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$. For (3), let $f(n)$ be the least stage s such that for each $e \leq n$, if $\Phi_e(n)$ halts then $\Phi_e(n)$ halts by stage s. Clearly f is computably dominating. Moreover, $\sigma \prec f$ iff for every $n < |\sigma|$ and $e \leq n$, $\Phi_e(n)$ does not halt or $\Phi_e(n)$ halts by stage $\sigma(n)$. Clearly, this condition is Π^0_1. Therefore, $\mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$ is c.e. $\square$

Lemma 5.33. *Suppose that $f : \omega \to \omega$ is a function which is not $\emptyset'$-computably dominated. For every computable sequence $(D_s)_{s\in\omega}$ of finite sets of finite strings, if $f \in \bigcap_s [D_s]$ then there is $\ell \in \omega$ such that $[f \upharpoonright \ell] \subseteq \bigcap_s [D_s]$, i.e., for any s there is $\sigma \preceq f \upharpoonright \ell$ such that $\sigma \in D_s$.*

Proof. For a given computable sequence $(D_s)_{s\in\omega}$ we will define a finite-branching c.e. subtree T of $\omega^{<\omega}$. Let T_0 be the tree only having the root, that is, $T_0 = \{\langle\rangle\}$. Given T_s, if σ is a leaf of T_s and there is no $\tau \preceq \sigma$ such that $\tau \in D_s$, then enumerate all strings α such that $\sigma \prec \alpha \preceq \tau$ for some $\tau \in D_s$ into T_{s+1}. Define $T = \bigcup_s T_s$. Note that T is finite-branching. This is because we only enumerate finitely many elements into T_{s+1} extending a leaf $\sigma \in T_s$ since D_s is finite, and then σ never become a leaf of T_t for any $t > s$. Since T is a finite-branching c.e. tree, there is a $\emptyset'$-computable function dominating all infinite paths through T.

If $f \in \bigcap_s [D_s]$ but $[f \upharpoonright \ell] \not\subseteq \bigcap_s [D_s]$ for all $\ell \in \omega$, then we claim that f is an infinite path through T. Assume that $f \upharpoonright n$ is a leaf of T_s. Then there is $t \geq s$ such that $f \upharpoonright m \notin D_t$ for any $m \leq n$ since $[f \upharpoonright n] \not\subseteq \bigcap_s [D_s]$. Let t be the first such stage. Since $f \in \bigcap_s [D_s]$, there is $k > n$ such that $f \upharpoonright k \in D_t$. By our construction, all initial segments of $f \upharpoonright k$ are enumerated into T_{t+1}. Consequently, f is an infinite path through T; however this implies that f is $\emptyset'$-computably dominated. □

By using the above lemma, we will show that if f is not $\emptyset'$-computably dominated then $\mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$ is quasi-minimal. Indeed, the following abstract lemma implies more concrete results. Let $\mathcal{N} = (N_e)_{e\in\omega}$ be a network. Then, define we define the *disjointness diagram* of $\mathcal{N}$ as $\mathrm{Disj}_{\mathcal{N}} = \{\langle d, e\rangle : N_d \cap N_e = \emptyset\}$.

Lemma 5.34. *Let $f\colon \omega \to \omega$ be a function which is not C'-computably dominated. For any space $\mathcal{Z} = (Z, \mathcal{N})$ where $\mathcal{N}$ is a countable cs-network with a C-c.e. disjointness diagram, $f\colon (\omega^\omega)_{\rm co}$ is nearly $\mathcal{Z}$-quasi-minimal.*

Proof. Now suppose that $z\colon \mathcal{Z} \leq_{\mathbf{T}} f\colon (\omega^\omega)_{\rm co}$ holds where $\mathcal{Z}$ is a given space with a countable cs-network $\mathcal{N} = (N_e)_{e\in\omega}$. By Observation 4.3, there is $J \leq_e \mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$ such that $\{N_e : e \in J\}$ forms a strict network at z. Let Ψ witness that $J \leq_e \mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$. We first note that if there are d, e such that (d, D) and (e, E) are enumerated in Ψ while N_d and N_e are disjoint, then we must have $D \cup E \not\subseteq \mathrm{Nbase}_{(\omega^\omega)_{\rm co}}(f)$, that is, there is $\sigma \in D \cup E$ such that $\sigma \prec f$. Enumerate all such tuples $(d_s, e_s, D_s, E_s)_{s\in\omega}$. Such a C-computable enumeration exists since the disjointness diagram of $\mathcal{N}$ is C-c.e.

Then, either this gives a finite sequence, or else $(D_s \cup E_s)_{s\in\omega}$ is a C-computable sequence such that $f \in \bigcap_s [D_s \cup E_s]$. In any case, by relativizing Lemma 5.33, there is p such that $[f \upharpoonright p] \subseteq \bigcap_s [D_s \cup E_s]$, that is, for every s there is $\sigma \in D_s \cup E_s$ such that $\sigma \preceq f \upharpoonright p$. We then consider

$$L = \{e \in \omega : (\exists E)\ [(e, E) \in \Psi \text{ and } (\forall \sigma \in E)\ \sigma \not\preceq f \upharpoonright p]\}.$$

Clearly, $J \subseteq L$, and therefore, $\{N_e : e \in L\}$ forms a network at z. We claim that $z \in \overline{N_e}$ for any $e \in L$. By our choice of p, if $e \in L$, then there is no $d \in L$ such that $N_d \cap N_e = \emptyset$. If $z \notin \overline{N_e}$, there is an open neighborhood U of z such that $U \cap N_e = \emptyset$. Then, there is $d \in L$ such that $z \in N_d \subseteq U$. In particular, $N_d \cap N_e = \emptyset$, which implies a contradiction. Consequently, any enumeration of L gives a $\overline{\delta_{\mathcal{N}}}$-name of z. Since L is c.e., we conclude that z is $\overline{\delta_{\mathcal{N}}}$-computable. □

Proof of Theorem 5.31. Let $\mathcal{X}$ be a topological space with a countable cs-network $\mathcal{N}$. Let C be an oracle such that the disjointness diagram of $\mathcal{N}$ is C-c.e. By relativizing Lemma 5.34, if f is not C'-computably dominated, then $f\colon (\omega^\omega)_{\rm co}$ is nearly $\mathcal{X}$-quasi-minimal, that is, if $x\colon \mathcal{X} \leq_{\mathbf{T}} f\colon (\omega^\omega)_{\rm co}$ then $x\colon \mathcal{X}$ is nearly computable. If $\mathcal{X}$ is a Hausdorff space, then only countably many points in $\mathcal{X}$ can be nearly computable by Observation 4.5. However, there are uncountably many functions which is not C'-computably dominated. Thus, one can choose such a function which is not $\mathbf{T}$-equivalent to any nearly computable points in $\mathcal{X}$. □

Our proof of Theorem 5.31 is applicable to show the existence of a quasi-minimal degree w.r.t. some non-second-countable space. We will later introduce the notion of degrees of points of certain non-second-countable (but still separable) spaces. In particular, we will deal with the degree structure of the Kleene-Kreisel space $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$, which has been studied by Hinman [38], Normann [58], and others. Then we will show the following:

Theorem 5.35. *There is a cylinder-cototal e-degree which is $\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}$-quasi-minimal.*

Proof. The canonical network $\mathcal{N}$ of $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$ has a computable disjointness diagram. Moreover, $\mathrm{id}\colon (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \overline{\delta_\mathcal{N}}) \to (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \delta_\mathcal{N})$ is computable as seen in Example 4.10. Therefore, near $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimality is equivalent to $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimality by definition. Now, by Lemma 5.34, if f is not $\emptyset'$-computably dominated, then $f\colon (\omega^\omega)_{\mathrm{co}}$ is $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimal. □

Applying the above lemmas, we eventually show the following:

Proof of Proposition 3.13. Let A be a co-d-c.e. set relative to $K = \emptyset'$ such that $K \oplus K^\mathsf{c} \leq_T A$ and the e-degree of A is non-total. Suppose that $A \equiv_e \mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(f)$ for any f. If f is $\emptyset'$-computably dominated, then by Observation 5.32 relative to $\emptyset'$, we have that $\mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(f) \oplus K \oplus K^\mathsf{c}$ is total. Since $K \oplus K^\mathsf{c} \leq_e \mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(f)$ by our assumption, $\mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(f) \equiv_e \mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(f) \oplus K \oplus K^\mathsf{c}$ is total, which is impossible since A is nontotal. If f is not $\emptyset'$-computably dominated, then by Lemma 5.34, $\mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(f)$ is quasi-minimal, which is impossible since A is not quasi-minimal. □

We next see that the property of jump-right-totality is not shared by (strongly arithmetically named) decidable T_1 cb_0 spaces. Indeed, the cocylinder space $(\omega^\omega)_{\mathrm{co}}$ (see Section 3.4.1), one of the simplest decidable T_1 cb_0 space, is not jump-right-total. By using a similar idea as in Section 5.1.5, we will show the following:

Theorem 5.36. *There is a cylinder-cototal e-degree which is quasi-minimal w.r.t. telograph-cototal e-degrees.*

Proof. We begin with a lemma.

Lemma 5.37. *If $x \in \omega^\omega$ is not $\emptyset''$-computably dominated, then* $\mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(x)$ *is quasi-minimal w.r.t. jump-right-total e-degrees.*

Proof. Assume that $x \in \omega^\omega$ is not $\emptyset''$-computably dominated. By Lemma 5.34, such x is quasi-minimal. Let A be a jump-right-total set, witnessed by C and $(D_n)_{n\in\omega}$ (generating $\mathcal{N}$), and assume that $A \leq_e \mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(x)$ via an enumeration operator Ψ. Note that, by quasi-minimality of x, the total part C has to be computable. Thus, $(D_n)_{n\in\omega}$ is a $\emptyset'$-computable sequence.

Let (E_s) be a $\emptyset'$-computable enumeration of finite sets E such that $D_n \subseteq \Psi(E)$ for some n. Then, $A \in \mathcal{N}$ implies that $\Psi(E_s) \not\subseteq A$ for any s. Since $\Psi(\mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(x)) = A$, we have $E_s \not\subseteq \mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(x)$, that is, there is $\sigma \prec x$ such that $\sigma \in E_s$. Since x is not $\emptyset''$-computably dominated, by Lemma 5.33 (relative to $\emptyset'$), there is ℓ such that for any s, there is $\tau \preceq x \upharpoonright \ell$ such that $\tau \in E_s$. Consider the following c.e. set:

$$L = \{n : (\exists H)\ [\langle n, H\rangle \in \Psi \text{ and } (\forall \sigma \prec x \upharpoonright \ell)\ \sigma \notin H]\}.$$

It is easy to see that $A \subseteq L$ since $\Psi(\mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(x)) \subseteq L$. We claim that $L \in \mathcal{N}$. Otherwise, there is n such that $D_n \subseteq L$. For each $m \in D_n$, there is H_m such that $\langle m, H_m\rangle \in \Psi$ and $\sigma \notin H_m$ for all $\sigma \prec x \upharpoonright \ell$. Thus, for $E = \bigcup_{m\in D_n} H_m$ we have $D_n \subseteq \Psi(E)$, and hence, $E = E_s$ for some $s \in \omega$. By our choice of ℓ, there is $\sigma \prec x \upharpoonright \ell$ such that $\sigma \in E = E_s$, that is, $\sigma \in H_m$ for some $m \in D_n$. This contradicts our choice of H_m.

We thus obtain $A \subseteq L \in \mathcal{N}$. Since A is jump-right-total, C is computable and L is c.e., we conclude that $A \leq_e L \leq_e \emptyset$. □

Choose $x \in \omega^\omega$ which is not $\emptyset''$-computably dominated. Then, $\mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(x)$ is cylinder-cototal as seen in Section 3.4.1, and quasi-minimal w.r.t. jump-right-total e-degrees by Lemma 5.37. In particular, $\mathrm{Nbase}_{(\omega^\omega)_{\mathrm{co}}}(x)$ is quasi-minimal w.r.t. telograph-cototal e-degree since every telograph-cototal e-degree is jump-right-total as seen in Example 5.27. □

5.2.1. *T_1-degrees which are T_2-quasi-minimal.* In this section, we show that there is a T_1-degree which is T_2-quasi-minimal. This is one of the most nontrivial theorems in this article. Indeed, the product telophase space $(\hat{\omega}_{TP})^\omega$ contains a point of T_2-quasi-minimal degree in the following sense.

Theorem 5.38. *Given any countable collection $\{S_i\}_{i\in\omega}$ of computable Hausdorff spaces, there is a telograph-cototal e-degree which is S_i-quasi-minimal for any $i \in \omega$.*

Proof. By Theorem 3.22, it suffices to construct sets $A, B \subseteq \omega$ such that $A \cup B$ is co-c.e. and $\mathrm{Sep}(A, B)$ is quasi-minimal with respect to $\{S_i\}_{i\in\omega}$.

The requirements. We construct disjoint sets $A, B \subseteq \omega$ and a c.e. set X such that $A \cup B = \omega - X$, (A, B) are computably inseparable, and we have to satisfy the following requirements:

$$R_e: \quad \text{If } \forall D_0, D_1 \in \mathrm{Sep}(A, B)\ W_e^{D_0} = W_e^{D_1} \text{ and } W_e^{D_0} = \mathrm{Nbase}(y) \text{ for some point } y \in S_e, \text{ then } \exists \text{ c.e. set } V = W_e^A.$$

Here W_e^X is the e^{th} c.e. set relative to $X \oplus (\omega - X)$. We will construct a path $Y \in \{a, b, 0\}^\omega$. Given any path $Z \in \{a, b, 0\}^\omega$ we define $Z^+, Z^- \in 2^\omega$ by $Z^+(n) = 1$ iff $Z(n) = a$, and $Z^-(n) = 1$ iff $Z(n) = b$. Clearly Z^+ and Z^- are the characteristic functions of disjoint sets. (At the end we will take $A = Y^+$ and $B = Y^-$).

We also assume that all approximations of a Π^0_1-class Q are slowed down such that for every s, the complement of Q_s is presented by a finite set of strings of length $< s$.

The definition of X. We first describe how to construct the c.e. set X. We consider requirements $S_{e,\sigma}$ indexed by $e \in \omega$ and $\sigma \in \{a, b, 0\}^{<\omega}$, and arrange the requirements in some order of priority. The c.e. set X is constructed by a straightforward finite injury construction, and whenever a requirement $S_{e,\sigma}$ performs any action, all lower priority requirements are initialized and starts afresh with a value of m much larger than before.

The basic activity of requirement $S_{e,\sigma}$ is as follows. First pick a very large number $m_{e,\sigma}$, and for each node $\tau \in \{a, b, 0\}^{m_{e,\sigma}}$ do the following. Search for a pair $\eta_0, \eta_1 \supset \tau$ and some $i, j \in \omega$ such that $B_i^{S_e} \cap B_j^{S_e} = \emptyset$ and $i \in W_e^{\eta_0^+}$, $j \in W_e^{\eta_1^+}$. Here $B_i^{S_e}$ refers to the i-th (sub)basic open set in S_e. For the first pair η_0, η_1 found at some stage s, where $|\eta_0|, |\eta_1| < s$, enumerate the interval $\{m_{e,\sigma}, \cdots, s\}$ into X_{s+1}. Note that $S_{e,\sigma}$ will change X up to $3^{m_{e,\sigma}}$ many times (until it is initialized).

It is easy to see that X is c.e. and co-infinite. Now define the closed set $C \subset \{a, b, 0\}^\omega$ to consist of precisely the paths $Z \in \{a, b, 0\}^\omega$ such that $Z^+ \cup Z^- = \mathbb{N} - X$. Also for each s define C_s to be the set of paths Z such that $(Z^+ \cup Z^-) \upharpoonright s = (\mathbb{N} - X_s) \upharpoonright s$. Note that C is a Π^0_1-class relative to $\emptyset'$ and each C_s is a Π^0_1-class. However, $\{C_s\}$ is not monotone. This non-computability feature of C will cause various difficulties in our construction. First, clearly we will need to take $Y \in C$, and second we will need to enumerate V without oracles; however, C_s are guesses about the true C which will help us in the construction.

Given a string $Z \in \{a,b,0\}^\omega$ and $n, s \in \omega$, we say that $Z(n)$ *is compatible with* C_s if $n \in X_s \Leftrightarrow Z(n) = 0$. Similarly we say that $Z(n)$ *is compatible with* C if $n \in X \Leftrightarrow Z(n) = 0$. Thus the set of all Z such that $Z(n)$ is compatible with C_s for all $n < s$ is the set C_s itself.

The class $\mathtt{Ex}(Q,\rho)$. Our construction will define a sequence of approximations to $P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$ of Π^0_1-classes and a sequence of nodes $\rho_0 \subset \rho_1 \subset \cdots$ with a unique Y such that $\{Y\} = \bigcap_{k\in\omega} P_k$. In fact, we will ensure a little more. Given any s and a node $\rho \in \{a,b,0\}^{<\omega}$, we define $\rho \boxplus C_s$ to contain all strings $Z \in \{a,b,0\}^\omega$ such that $Z \supset \rho$ and for every $n \geq |\rho|$, $Z(n)$ is compatible with C_s. We also define for each (approximation to) Q and ρ, the class $\mathtt{Ex}(Q,\rho)$ by the following.

We define the c.e. set of strings $E = \cup_s E_s$ by the following. First enumerate into E_0 all σ incompatible with ρ. Then at stage $s+1$, enumerate all strings σ, where $|\sigma| < s$ and $\sigma \supseteq \rho$ into E_{s+1} if $(\sigma \boxplus C_{s+1}) \subseteq [E_s] \cup (\{a,b,0\}^\omega - Q[s])$.

Now take $\mathtt{Ex}(Q,\rho) = \{a,b,0\}^\omega - [E]$. Clearly $\mathtt{Ex}(Q,\rho) \subseteq Q$ is still a Π^0_1-class. Note that $\mathtt{Ex}(Q,\rho)$ depends on the approximation $\{Q[s]\}$ of Q; different approximations of Q may give rise to different versions of $\mathtt{Ex}(Q,\rho)$. Our aim in the construction is to define approximations to P_k and ensure that $\{Y\} = \bigcap_{k\in\omega} P_k = \bigcap_{k\in\omega} \mathtt{Ex}(P_k,\rho_k) \cap C$. For convenience, we will denote $[E_s]$ by $\mathtt{Ex}^c(Q,\rho)[s]$ and $\mathtt{Ex}(Q,\rho)[s] = \{a,b,0\}^\omega - \mathtt{Ex}^c(Q,\rho)[s]$.

We now fix some conventions about the approximations to Π^0_1-classes. Given an approximation $Q[s]$ of Q and a string α, define the natural approximation $(Q \cap [\alpha])[s]$ of $Q \cap [\alpha]$ by taking $\{a,b,0\}^\omega - (Q\cap[\alpha])[s] = (\{a,b,0\}^\omega - Q[s]) \cup (\{a,b,0\}^\omega - [\alpha])$ for all s. If $P[s]$ and $Q[s]$ are approximations to P and Q, then the natural approximation to $P \cap Q$ is given by $\{a,b,0\}^\omega - (P\cap Q)[s] = (\{a,b,0\}^\omega - P[s]) \cup (\{a,b,0\}^\omega - Q[s])$ for all s.

The initial condition P_0, ρ_0. It is easy to see that there is a Π^0_1-class $P_0 \subset \{a,b,0\}^\omega$ such that for every $Z \in P_0 \cap C$, Z^+ and Z^- are computably inseparable. P_0 can be constructed from an effective approximation to X and the knowledge that X is coinfinite. For each s, let $x_{0,s} < x_{1,s} < \cdots$ be the elements of $\omega - X_s$ listed in increasing order. Notice that as X is constructed by dumping an entire interval into X at each stage, the approximation above has the property that if $x_{i,s+1} \neq x_{i,s}$ then $x_{i,s+1} > s$.

Now at stage s, for each $i < s$ such that $\varphi_i(x_{i,s})\downarrow = j$, enumerate $[\sigma[a]$ into $\{a,b,0\}^\omega - P_0$ if $j = 1$ and enumerate $[\sigma^\frown b]$ into $\{a,b,0\}^\omega - P_0$ if $j = 0$ for each $\sigma \in 3^{x_{i,s}}$. It is easy to see that the definition of P_0 above ensures that for any $Z \in P_0 \cap C$, there is no computable set R such that $Z^+ \subseteq R$ and $R \cap Z^- = \emptyset$. Furthermore, by the observation above that if $x_{i,s+1} \neq x_{i,s}$ then $x_{i,s+1} > s$, we see that each level only has nodes removed from P_0 at most once. Therefore $P_0 \cap C$ and $P_0 \cap C_s$ are nonempty for any s. Note that P_0 and C_s are both homogeneous, that is, given any string α which is extendible to an infinite path in P_0, and any s, there is always an infinite extension $\alpha^\frown Y$ of α such that $\alpha^\frown Y \in P_0 \cap (\alpha \boxplus C_s)$.

We now check that $\mathtt{Ex}(P_0,\langle\rangle) = P_0$: Suppose there is some $Z \in P_0 \cap \mathtt{Ex}^c(P_0,\langle\rangle)$. Let s be the least stage such that there is some $Z \in P_0 \cap \mathtt{Ex}^c(P_0,\langle\rangle)[s]$. Suppose $Z \upharpoonright k \in \mathtt{Ex}^c(P_0,\langle\rangle)[s]$ (here we make the obvious identification between finite strings and the open sets they generate). Now apply homogeneity of P_0 above to $\alpha = Z \upharpoonright k$, and we get a contradiction to the fact that $Z \upharpoonright k \in \mathtt{Ex}^c(P_0, \langle\rangle)[s]$ (and the minimality of s). This shows that $\mathtt{Ex}(P_0, \langle\rangle) = P_0$, and in particular,

$$\mathtt{Ex}(P_0, \langle\rangle) \cap C \neq \emptyset.$$

Obviously we shall take $\rho_0 = \langle\rangle$.

Forcing R_e and the condition P_e, ρ_e. Now assume that at step e we are given (approximations to) a sequence $P_0 \supset P_1 \supset \cdots \supset P_{e-1}$ of Π^0_1-classes such that $\mathtt{Ex}(P_i, \rho_i) \cap C \neq \emptyset$, and a sequence $\rho_0 \subset \rho_1 \subset \cdots \subset \rho_{e-1}$ such that for every $i < e$, we have $P_i \subseteq [\rho_i]$. We assume $P_i \subset \{a, b, 0\}^\omega$ and $\rho_i \in \{a, b, 0\}^{<\omega}$ for every $i < e$. Our aim is to define P_e and ρ_e such that the following condition $(\star)$ holds:

$$(\star)\ P_e[s] \subset P_{e-1}[s] \text{ for every } s,\ \mathtt{Ex}(P_e, \rho_e) \cap C \neq \emptyset,$$
$$\mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap [\rho_e] \neq \emptyset, \text{ and } \rho_e \supset \rho_{e-1}. \text{ Furthermore}$$
$$\text{if } Z \in \mathtt{Ex}(P_e, \rho_e) \cap C \text{ then } R_e \text{ is satisfied along } Z.$$

We begin with a technical lemma.

Lemma 5.39. *Let Q_0 and Q_1 be Π^0_1-classes with approximations such that $Q_0[s] \cap [\alpha_1] \subseteq Q_1[s] \cap [\alpha_1]$ for every s, and where $\alpha_0 \subseteq \alpha_1$. Then*

$$\mathtt{Ex}(Q_0, \alpha_0) \cap [\alpha_1] \subseteq \mathtt{Ex}(Q_1, \alpha_1).$$

Proof. We prove by induction on s that

$$\mathtt{Ex}^c(Q_1, \alpha_1)[s] \subseteq \mathtt{Ex}^c(Q_0, \alpha_0)[s] \cup (\{a, b, 0\}^\omega - [\alpha_1]).$$

For $s = 0$, this is obviously true. Now suppose that σ is enumerated in $\mathtt{Ex}^c(Q_1, \alpha_1)[s+1]$ at that stage. This means that $\sigma \supseteq \alpha_1$, and

$$\begin{aligned}(\sigma \boxplus C_{s+1}) &\subseteq \mathtt{Ex}^c(Q_1, \alpha_1)[s] \cup (\{a, b, 0\}^\omega - Q_1[s]) \\ &\subseteq \mathtt{Ex}^c(Q_0, \alpha_0)[s] \cup (\{a, b, 0\}^\omega - Q_0[s]) \cup (\{a, b, 0\}^\omega - [\alpha_1]).\end{aligned}$$

As $\sigma \supseteq \alpha_1$, this implies that

$$(\sigma \boxplus C_{s+1}) \subseteq \mathtt{Ex}^c(Q_0, \alpha_0)[s] \cup (\{a, b, 0\}^\omega - Q_0[s]).$$

Thus $\sigma \in \mathtt{Ex}^c(Q_0, \alpha_0)[s+1]$. $\square$

Let $\hat{m}$ be the final parameter used by $S_{e,\rho_{e-1}}$ during the construction of the c.e. set X. Since $\mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap C \neq \emptyset$, there must be some $\tau \in \{a, b, 0\}^{\hat{m}}$ such that $\mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap C \cap [\tau] \neq \emptyset$ and $\tau \supset \rho_{e-1}$. Fix any such τ. We will now meet requirement R_e in this cone. For each $n \in \omega$ define

$$T_n = \left\{ Z \in \{a, b, 0\}^\omega \mid n \notin W_e^{Z^+} \right\}.$$

Then T_n is a Π^0_1-class for each $n \in \omega$, and we fix an approximation $T_n[s]$ of T_n. There are two cases.

Case 1: Assume that there exists some $n \in \omega$ such that $\mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau) \cap C \neq \emptyset$, and there exists some $\alpha \supseteq \tau$ such that $[\alpha] \cap T_n = \emptyset$ and $\mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap [\alpha] \cap C \neq \emptyset$. Fix n and α as in the case assumption. There are now three subcases.

Subcase 1.1: For every $Z \in \mathtt{Ex}([\alpha] \cap P_{e-1}, \alpha) \cap C$, $W_e^{Z^+} \neq \mathrm{Nbase}(y)$ for any point $y \in S_e$. In this case we take $P_e = [\alpha] \cap P_{e-1}$ and $\rho_e = \alpha$. Now note that by Lemma 5.39, we have $\emptyset \neq \mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap [\alpha] \cap C \subseteq \mathtt{Ex}([\alpha] \cap P_{e-1}, \alpha) \cap C$.

Then we clearly have condition $(\star)$ holds. Note that we have $Z^+ \in \mathrm{Sep}(Z^+, Z^-)$.

Subcase 1.2: For every $Z \in \mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau) \cap C$, $W_e^{Z^+} \neq \mathrm{Nbase}(y)$ for any point $y \in S_e$. In this case we take $P_e = T_n \cap P_{e-1} \cap [\tau]$ and take $\rho_e = \tau$. Then we clearly also have $(\star)$. Note that in this case $\mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap [\tau] \neq \emptyset$ by the choice of τ.

Subcase 1.3: Otherwise. This means that there exists $Z_1 \in [\alpha]$ and $Z_2 \in T_n \cap [\tau]$ such that $W_e^{Z_1^+} = \mathrm{Nbase}(y_1)$ and $W_e^{Z_2^+} = \mathrm{Nbase}(y_2)$ for some points y_1, y_2 in S_e. Since $n \in W_e^{Z_1^+} - W_e^{Z_2^+}$, hence $y_1 \neq y_2$. Since S_e is a T_2 space, this means that there are disjoint balls B_k and B_l such that $k \in W_e^{Z_1^+}$ and $l \in W_e^{Z_2^+}$.

This means that during the construction of X, the requirement $S_{e,\rho_{e-1}}$ must have found a pair $\eta_0, \eta_1 \supset \tau$ successfully where η_0^+ and η_1^+ enumerate disjoint S_e-balls. Hence we must have the entire interval $\{\hat{m}, \cdots, \max\{|\eta_0|, |\eta_1|\}\} \subset X$. Note that the pair η_0, η_1 found in the construction for X might not be along Z_1 and Z_2, and in fact, they might not even be extendible in P_{e-1} or C, but this will not matter, as we will soon explain.

In this subcase we take $P_e = P_{e-1} \cap [\tau]$ and $\rho_e = \tau$. Note that as $\mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap C \cap [\tau] \neq \emptyset$ by the choice of τ and by Lemma 5.39, $\mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap [\tau] \subseteq \mathtt{Ex}(P_{e-1} \cap [\tau], \tau)$, we have that $\mathtt{Ex}(P_e, \rho_e) \cap C \neq \emptyset$.

Take any $Z \in \mathtt{Ex}(P_{e-1} \cap [\tau], \tau) \cap C$. Since $Z \supset \tau$ and $Z \in C$, this means that

$$\eta_0^{+\frown} Z^+(|\eta_0|) Z^+(|\eta_0|+1) Z^+(|\eta_0|+2) \cdots \in \mathrm{Sep}(Z^+, Z^-) \text{ and}$$
$$\eta_1^{+\frown} Z^+(|\eta_1|) Z^+(|\eta_1|+1) Z^+(|\eta_1|+2) \cdots \in \mathrm{Sep}(Z^+, Z^-),$$

which means that R_e is met along any such Z. Hence condition $(\star)$ holds.

Case 2: No such n in Case 1 exists. This means that for every $n \in \omega$, one of the following holds:

(I): $\mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau) \cap C = \emptyset$, or

(II): For every $\alpha \supseteq \tau$ such that $[\alpha] \cap T_n = \emptyset$, we have $\mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap [\alpha] \cap C = \emptyset$.

In this case we take $P_e = P_{e-1} \cap [\tau]$ and $\rho_e = \tau$. Again by Lemma 5.39, $\mathtt{Ex}(P_e, \rho_e) \cap C \neq \emptyset$. It only remains to check that R_e is met along all $Z \in \mathtt{Ex}(P_e, \rho_e) \cap C$.

Define the c.e. set V as follows:

$$V = \{n \in \omega : (\exists s)\ \mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau)[s] = \emptyset\}.$$

We may assume that at stage s, $X_s \upharpoonright |\tau| = X \upharpoonright |\tau|$.

Lemma 5.40. *For each n, if (I) holds, then $n \in V$.*

Proof. Fix n such that $\mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau) \cap C = \emptyset$. By compactness, fix $l > |\tau|$ such that $\mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau)[l-1] \cap C_l = \emptyset$. We want to verify that $\mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau)[l] = \emptyset$. Suppose for a contradiction that there is some $Z \in \mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau)[l]$.

We have $Z \notin C_l$, and we let k be the least number such that $Z(k)$ is not compatible with C_l. We know that $k < l$ since the complement of C_l can be presented by strings of length less than l. Furthermore we also know that $k \geq |\tau|$: Since $Z \in \mathtt{Ex}(T_n \cap P_{e-1} \cap$

$[\tau],\tau)[l]$, we have $Z \supset \tau$, but since τ is compatible with C (as $[\tau]\cap C \neq \emptyset$) and we can assume l is large enough so that $X_l \restriction |\tau| = X \restriction |\tau|$, we have $k \geq |\tau|$.

Now applying the definition of $Z \in \mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau)[l]$, we get that

$$\Big((Z \restriction k \boxplus C_l) \cap T_{n,l} \cap P_{e-1,l} \cap [\tau] \Big) - \mathtt{Ex}^c(T_n \cap P_{e-1} \cap [\tau], \tau)[l-1] \neq \emptyset.$$

Fix $\hat{Z}$ in the set above. By the definition of k, we know that $\hat{Z} \in C_l$. At the same time, $\hat{Z} \notin \mathtt{Ex}^c(T_n \cap P_{e-1} \cap [\tau], \tau)[l-1]$. This is a contradiction to the assumption that $\mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau)[l-1] \cap C_l = \emptyset$. $\square$

Now fix $Z \in \mathtt{Ex}(P_{e-1} \cap [\tau], \tau) \cap C$, and we want to argue that requirement R_e is met along Z. Obviously we begin by assuming that $\forall D_0, D_1 \in \mathrm{Sep}(Z^+, Z^-)$, $W_e^{D_0} = W_e^{D_1}$ and $W_e^{D_0} = \mathrm{Nbase}(y)$ for some point $y \in S_e$. We wish to now verify that $V = W_e^{Z^+}$.

Lemma 5.41. *$V \subseteq W_e^{Z^+}$.*

Proof. Suppose that $n \in V$. This means that there is a stage s such that $\mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau)[s] = \emptyset$, and so $Z \notin \mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau)[s]$. This means there is a $k \geq |\tau|$ and some $t \leq s$ such that $Z \restriction k$ is enumerated in $\mathtt{Ex}^c(T_n \cap P_{e-1} \cap [\tau], \tau)[t]$ at stage t. We fix t to be the smallest stage which enumerates some initial segment of Z this way.

Let $x_m = \min(X - X_t) \restriction t$. First of all, if x_m does not exist then $X \restriction t = X_t \restriction t$, and so $Z(l)$ is compatible with C_t for all $l < t$. This means that $[Z \restriction t] \cap T_{n,t} \cap P_{e-1,t} \cap [\tau] \cap \mathtt{Ex}(T_n \cap P_{e-1} \cap [\tau], \tau)[t-1] = \emptyset$ and we apply the fact that $[Z \restriction t] \cap P_{e-1,t} \neq \emptyset$ and the minimality of t to conclude that $[Z \restriction t] \cap T_{n,t} = \emptyset$. Thus, $Z \notin T_n$ and hence $n \in W_e^{Z^+}$.

So we will assume that $x_m < t$ exists. By the construction of X, since x_m is enumerated after stage $t+1$, we observe that $\{x_m, \cdots, t\} \subseteq X$. Since $Z(l)$ is compatible with C_t for all $l < x_m$, we may assume that $k \geq x_m$, because otherwise we have $Z \restriction x_m \boxplus C_t \subseteq Z \restriction k \boxplus C_t$ and we can use x_m in place of k. So we assume the order $x_m \leq k < t$.

Now suppose that there is some $\alpha \in \{a, b, 0\}^{t-k}$ such that $\big[(Z \restriction k)^\frown \alpha\big] \cap T_n = \emptyset$. Take $D = \Big(Z \restriction k^\frown \alpha^\frown Z(t)Z(t+1)Z(t+2)\cdots\Big)^+$, and as $x_m \leq k < t$ and $\{x_m, \ldots, t\} \subseteq X$, we see that $D \in \mathrm{Sep}(Z^+, Z^-)$. So $n \in W_e^D = W_e^{Z^+}$. So we suppose that no such α exists.

Now we prove by induction on $v \leq t$ that for every σ enumerated into $\mathtt{Ex}^c(T_n \cap P_{e-1} \cap [\tau], \tau)[v]$ such that $\sigma \supseteq Z \restriction k$, we have that σ is also enumerated into $\mathtt{Ex}^c(P_{e-1} \cap [\tau], \tau)[v]$. At $v = 0$ this is trivially true. Now suppose that σ is enumerated into $\mathtt{Ex}^c(T_n \cap P_{e-1} \cap [\tau], \tau)[v]$ at stage v, and that $\sigma \supseteq Z \restriction k$. This means that $\sigma \boxplus C_v$ is covered by $\mathtt{Ex}^c(T_n \cap P_{e-1} \cap [\tau], \tau)[v-1]$ and the complements of $T_{n,v}$ and $P_{e-1,v}$. But since α above is assumed not to exist, and $v \leq t$, this means that $\sigma \boxplus C_v$ is covered by $\mathtt{Ex}^c(T_n \cap P_{e-1} \cap [\tau], \tau)[v-1]$ and the complement of $P_{e-1,v}$. But any σ' enumerated in $\mathtt{Ex}^c(T_n \cap P_{e-1} \cap [\tau], \tau)[v-1]$ cannot have $\sigma' \subseteq Z \restriction k$ by the minimality of t. Thus by induction hypothesis, we see that $\sigma \boxplus C_v$ is in fact covered by $\mathtt{Ex}^c(P_{e-1} \cap [\tau], \tau)[v-1]$ and the complement of $P_{e-1,v}$. Thus, σ is in $\mathtt{Ex}^c(P_{e-1} \cap [\tau], \tau)[v]$. This concludes the induction.

By our choice of t and k, we have that $Z \restriction k$ is enumerated in $\mathtt{Ex}^c(T_n \cap P_{e-1} \cap [\tau], \tau)[t]$. By our induction above, we see that $Z \restriction k$ is also enumerated in $\mathtt{Ex}^c(P_{e-1} \cap [\tau], \tau)[t]$. However, recall that we had assumed that $Z \in \mathtt{Ex}(P_{e-1} \cap [\tau], \tau)$, and thus we have a contradiction. $\square$

Lemma 5.42. *Let Q_0 and Q_1 be Π^0_1-classes with approximations such that $Q_0[s] \subseteq Q_1[s]$ for every s, and $\alpha_0 \supseteq \alpha_1$ such that $[\alpha_0] \cap \mathtt{Ex}(Q_1, \alpha_1) \neq \emptyset$. Then*

$$\mathtt{Ex}(Q_0, \alpha_0) \subseteq \mathtt{Ex}(Q_1, \alpha_1).$$

Proof. We proceed by induction on s, the statement

$$\mathtt{Ex}^c(Q_1, \alpha_1)[s] \subseteq \mathtt{Ex}^c(Q_0, \alpha_0)[s].$$

For $s = 0$ it is surely trivial as $\alpha_1 \subseteq \alpha_0$. At stage $s+1$, suppose that σ is enumerated in $\mathtt{Ex}^c(Q_1, \alpha_1)[s+1]$. If $\sigma \supseteq \alpha_0$ then we apply the induction hypothesis to get $\sigma \in \mathtt{Ex}^c(Q_0, \alpha_0)[s+1]$. If σ is incomparable with α_0 then $[\sigma] \subseteq \mathtt{Ex}^c(Q_0, \alpha_0)[0]$. If $\sigma \subset \alpha_0$ then we get $[\alpha_0] \cap \mathtt{Ex}(Q_1, \alpha_1) = \emptyset$, contrary to the assumption. $\square$

Finally we check that $W_e^{Z+} \subseteq V$. Fix any n and suppose that (II) holds for n. If $n \in W_e^{Z+}$, i.e., $Z \notin T_n$, then by taking $\alpha = Z \upharpoonright k$ for some appropriate k in (II), we see that $\mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap [Z \upharpoonright k] \cap C = \emptyset$. By Lemma 5.42 we see that

$$\mathtt{Ex}(P_{e-1} \cap [\tau], \tau) \subseteq \mathtt{Ex}(P_{e-1}, \rho_{e-1}).$$

As $Z \in \mathtt{Ex}(P_{e-1} \cap [\tau], \tau) \cap C$, we conclude that $Z \in \mathtt{Ex}(P_{e-1}, \rho_{e-1}) \cap [Z \upharpoonright k] \cap C$, a contradiction.

Thus if $n \in W_e^{Z+}$ then (II) does not hold for n, which means that (I) has to hold, and by Lemma 5.40 we see that $n \in V$. Thus $W_e^{Z+} \subseteq V$. This shows that once again condition $(\star)$ is met in Case 2.

We produce a sequence $P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$ of Π^0_1-classes and a sequence of nodes $\rho_0 \subset \rho_1 \subset \cdots$. At the end we take $Y = \bigcup_k \rho_k$, and $A = Y^+$ and $B = Y^-$. By condition $(\star)$ and Lemma 5.42 we have $\mathtt{Ex}(P_k, \rho_k) \supseteq \mathtt{Ex}(P_{k+1}, \rho_{k+1})$ for every k. We also see that $Y \in \mathtt{Ex}(P_k, \rho_k) \cap C$ for every k. Thus requirement R_k is met along Y for every k. $\square$

5.2.2. *Continuous degrees.* Recall that a continuous degree is a degree of a point in a computable metric space. It is known that there is no quasi-minimal continuous degree (see Miller [54]). Therefore, there is a co-d-c.e. e-degree which is not continuous (see Proposition 3.50). Conversely, by using the notion of cospectrum, we can show the following:

Proposition 5.43. *There is a continuous degree which is neither telograph-cototal nor cylinder-cototal.*

Proof. The *cospectrum* of a point $x \in X$ is the set of all $z \in 2^\omega$ such that $z \leq_{\mathbf{T}} x$ (cf. Kihara-Pauly [43]). Equivalently, the cospectrum of $x \in X$ is the following set:

$$\{Z \subseteq \omega : Z \oplus Z^{\mathrm{c}} \leq_e \mathrm{Nbase}_X(x)\}.$$

If the cospectrum is closed under the Turing jump, it is called a jump ideal.

Lemma 5.44. *There is no telograph-cototal e-degree whose cospectrum is a jump ideal.*

Proof. Given $x \in (\hat{\omega}_{TP})^\omega$, let $c(x) \in \hat{\omega}^\omega$ be its total information, that is, $c(x)(n) = x(n)$ if $x(n) \in \omega$; otherwise $c(x)(n) = \infty$. Since $c(x)$ is an element of the one-point compactification of ω, by Observation 3.47, $c(x)$ is total. As seen in the proof of Proposition 3.18, we have $c(x) \leq_{\mathbf{T}} x$. Given X, let X_d denote the space whose underlying space is the same as X, but its topology is endowed by the discrete topology. By asking to the jump of $c(x)$, for each n, whether $c(x)(n)$ converges to ∞ or not, one can easily see that $c(x)'$

computes $c(x)\colon (\hat{\omega}_d)^\omega$. Then it is not hard to see that the pair $(x, c(x))\colon (\hat{\omega}_{TP})^\omega \times (\hat{\omega}_d)^\omega$ computes $x\colon ((\hat{\omega}_{TP})_d)^\omega$, which is total. It is also obvious that $x\colon ((\hat{\omega}_{TP})_d)^\omega$ computes $x\colon (\hat{\omega}_{TP})^\omega$; hence $x\colon (\hat{\omega}_{TP})^\omega \leq_{\mathbf{T}} x\colon ((\hat{\omega}_{TP})_d)^\omega \leq_{\mathbf{T}} (x, c(x)')\colon (\hat{\omega}_{TP})^\omega \times 2^\omega$. It the cospectrum of x is a jump ideal, then $c(x) \leq_{\mathbf{T}} x$ implies $c(x)' \leq_{\mathbf{T}} x$, so we obtain $x\colon (\hat{\omega}_{TP})^\omega \equiv_{\mathbf{T}} x\colon ((\hat{\omega}_{TP})_d)^\omega$; hence x has a total degree. However, if x is total, then its cospectrum must be a principal Turing ideal, and thus it cannot be a jump ideal. □

Lemma 5.45. *There is no cylinder-cototal* e*-degree whose cospectrum is a jump ideal.*

Proof. Let $f \in \omega^\omega_{\mathrm{co}}$ be given. If f is not $\emptyset'$-computably dominated, then by Lemma 5.34, f is quasi-minimal (see also the proof of Theorem 5.35). If f is $\emptyset'$-dominated, by relativizing Observation 5.32, $f \oplus \emptyset'$ computes $f\colon \omega^\omega$. Hence, either f is total or the cospectrum of f does not contains $\emptyset'$. In any case, the cospectrum of f cannot be a jump ideal. □

Miller [54] showed that every countable Scott ideal is realized as a cospectrum of a point in the Hilbert cube. Thus, take a countable jump ideal $\mathcal{I}$, and choose $x \in [0,1]^\omega$ whose cospectrum is $\mathcal{I}$. Then, by Lemmas 5.44 and 5.45, the e-degree of $\mathrm{Nbase}(x)$ is continuous, but neither telograph-cototal nor cylinder-cototal. □

5.3. **T_2-degrees which are not $T_{2.5}$.** Let $\mathbb{Z}_+$ be the set of all positive integers. The *relatively prime integer topology* τ_{rp} on $\mathbb{Z}_+$ is generated by $\{a + b\mathbb{Z} : \gcd(a,b) = 1\}$, where $a + b\mathbb{Z} = \{a + bn \in \mathbb{Z}_+ : n \in \mathbb{Z}\}$. This space is also known as the *Golomb space*. We write $\mathbb{N}_{\mathrm{rp}} := (\mathbb{Z}_+, \tau_{\mathrm{rp}})$. It is known that $\mathbb{N}_{\mathrm{rp}}$ is second-countable, Hausdorff, but not $T_{2.5}$ (see Steen-Seebach [74, II.60]). Its countable product $\mathbb{N}_{\mathrm{rp}}^\omega$ also has the same properties. Note that the coded neighborhood basis of $x \in \mathbb{N}_{\mathrm{rp}}^\omega$ is described as follows.

$$\mathrm{Nbase}(x) = \{\langle n, a, b\rangle : x(n) \equiv a \bmod b, \text{ and } a \text{ and } b \text{ are relatively prime}\}.$$

Theorem 5.46. *For any represented $T_{2.5}$-space $\mathcal{X}$, there is an $(\mathbb{N}_{\mathrm{rp}})^\omega$-degree which is not an $\mathcal{X}$-degree, that is,*

$$\mathcal{D}_{(\mathbb{N}_{\mathrm{rp}})^\omega} \not\subseteq \mathcal{D}_{\mathcal{X}}.$$

As before, our proof of Theorem 5.46 is applicable to show the existence of a quasi-minimal degree w.r.t. some non-second-countable space.

Theorem 5.47. *There is an $(\mathbb{N}_{\mathrm{rp}})^\omega$-degree which is $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimal.*

To prove Theorems 5.46 and 5.47, we need a special property of the relatively prime integer topology. We say that a space $\mathcal{X}$ is *nowhere $T_{2.5}$* if for any open sets $U, V \subseteq \mathcal{X}$, $\overline{U} \cap \overline{V}$ is nonempty.

Fact 5.48 (see Steen-Seebach [74, II.60])**.** *$b\mathbb{Z} \subseteq \overline{a + b\mathbb{Z}}$, and therefore $\mathrm{lcm}(b,d)\mathbb{Z} \subseteq \overline{a + b\mathbb{Z}} \cap \overline{c + d\mathbb{Z}}$ in the relatively prime integer topology. In particular, the relatively prime integer topology is nowhere $T_{2.5}$.*

Instead of dealing with $\mathbb{N}_{\mathrm{rp}}$, we consider any countable, second-countable, nowhere $T_{2.5}$ space $\mathcal{H}$, and conclude that Theorems 5.46 and 5.47 hold true for $\mathcal{H}^\omega$ relative to some oracle. Combining the argument in [43], this, in particular, implies that the ω-power $\mathcal{H}^\omega$ of a countable, second-countable, nowhere $T_{2.5}$ space cannot be written as a countable union of $T_{2.5}$ subspaces.

Now we modify the closure representation argument. Given a network $\mathcal{N}$ of a space $\mathcal{X}$, we consider the following representation $\widetilde{\delta_{\mathcal{N}}}$ defined as follows. We say that p is a $\widetilde{\delta_{\mathcal{N}}}$-name of x if and only if

$$\{N_{p(n)} : n \in \omega\} \text{ is a network at } x, \text{ and } (\forall m, n)\ \overline{N_{p(m)}} \cap \overline{N_{p(n)}} \neq \emptyset.$$

Here recall that an element of a network at x does not need to contain x.

Observation 5.49. *Let $\mathcal{N}$ be a network of $\mathcal{X}$.*

1. *The identity map* $\mathrm{id} : (\mathcal{X}, \overline{\delta_{\mathcal{N}}}) \to (\mathcal{X}, \widetilde{\delta_{\mathcal{N}}})$ *is always computable.*
2. *If $\mathcal{N}$ is a regular-like network, then* $\mathrm{id} : (\mathcal{X}, \widetilde{\delta_{\mathcal{N}}}) \to (\mathcal{X}, \overline{\delta_{\mathcal{N}}})$ *is computable.*
3. *If $\mathcal{X}$ is $T_{2.5}$, then $\widetilde{\delta_{\mathcal{N}}}$ is single-valued.*
4. *If $\mathcal{X}$ is Hausdorff, and $\mathcal{N}$ is regular-like, then $\widetilde{\delta_{\mathcal{N}}}$ is single-valued.*

Proof. For (2), let p is a $\widetilde{\delta_{\mathcal{N}}}$-name of x. We show that p is also a $\overline{\delta_{\mathcal{N}}}$-name of x. Suppose for the sake of contradiction that $x \notin \overline{N_{p(k)}}$ for some $k \in \omega$. Since p is a $\widetilde{\delta_{\mathcal{N}}}$-name of x, $\overline{N_{p(k)}}$ intersects with $\overline{N_{p(n)}}$ for all n. Moreover, since $\mathcal{N}$ is regular-like, $\{\overline{N_{p(n)}} : n \in \omega\}$ is a network at x. Therefore, $\overline{N_{p(k)}}$ must intersect with all open neighborhoods of x. However, $\mathcal{X} \setminus \overline{N_{p(k)}}$ is an open neighborhood of x since $x \notin \overline{N_{p(k)}}$. Hence, p is a $\overline{\delta_{\mathcal{N}}}$-name of x.

For (3), assume that $\mathcal{X}$ is $T_{2.5}$, and p is a $\widetilde{\delta_{\mathcal{N}}}$-name of x and y. If $x \neq y$, there are open sets $U, V \subseteq \mathcal{X}$ such that $x \in U$, $y \in V$, and $\overline{U} \cap \overline{V} = \emptyset$. Since $\{N_{p(n)} : n \in \omega\}$ is a network at x and y, there are $d, e \in \omega$ such that $x \in N_{p(d)} \subseteq U$ and $y \in N_{p(e)} \subseteq V$. However, this implies that $\overline{N_{p(d)}} \cap \overline{N_{p(e)}} = \emptyset$. Then, p cannot be a $\widetilde{\delta_{\mathcal{N}}}$-name.

For (4), assume that $\mathcal{X}$ is Hausdorff and regular-like, and p is a $\widetilde{\delta_{\mathcal{N}}}$-name of x and y. If $x \neq y$, there are open sets $U, V \subseteq \mathcal{X}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. Since $\mathcal{N}$ is regular-like, and $\{N_{p(n)} : n \in \omega\}$ is a network at x and y, $\{\overline{N_{p(n)}} : n \in \omega\}$ is also a network at x and y. Therefore, there are $d, e \in \omega$ such that $x \in \overline{N_{p(d)}} \subseteq U$ and $y \in \overline{N_{p(e)}} \subseteq V$. However, this implies that $\overline{N_{p(d)}} \cap \overline{N_{p(e)}} = \emptyset$. Then, p cannot be a $\widetilde{\delta_{\mathcal{N}}}$-name. $\square$

By Observation 5.49, if either $\mathcal{X}$ is $T_{2.5}$ or $\mathcal{X}$ is Hausdorff and $\mathcal{N}$ is regular-like, then there are only countably many points x such that $x\colon \widetilde{\delta_{\mathcal{N}}}$ is computable.

Definition 5.50. We say that a point $x \in \mathcal{X}$ is *$\widetilde{*}$-nearly computable* if x is $\widetilde{\delta_{\mathcal{M}}}$-computable, that is, there is a computable $p \in \omega^\omega$ such that $\widetilde{\delta_{\mathcal{M}}}(p) = x$.

Definition 5.51. Let $\mathcal{X} = (X, \mathcal{N})$ and $\mathcal{Y} = (Y, \mathcal{M})$ be topological spaces with countable cs-networks. Then, we say that a point $x \in \mathcal{X}$ is *$\widetilde{*}$-nearly $\mathcal{Y}$-quasi-minimal* if

$$(\forall y \in \mathcal{Y})\ [y\colon \mathcal{Y} \leq_{\mathbf{T}} x\colon \mathcal{X} \implies y \text{ is } \widetilde{*}\text{-nearly computable}].$$

Every nearly computable point is $\widetilde{*}$-nearly computable. Similarly, if a point is nearly $\mathcal{Z}$-quasi-minimal, then it is $\widetilde{*}$-nearly $\mathcal{Z}$-quasi-minimal. Let $\mathcal{H} = (\omega, (H_e)_{e \in \omega})$ be a represented second-countable space. A *witness for being nowhere $T_{2.5}$* is a set $\Lambda \subseteq \omega^3$ such that for any $d, e \in \omega$, if both H_d and H_e are nonempty, then $\Lambda_{d,e} = \{n : (d, e, n) \in \Lambda\}$ is nonempty, and $\Lambda_{\langle a,b\rangle,\langle c,d\rangle} \subseteq \overline{H_d} \cap \overline{H_e}$. For instance, Fact 5.48 shows that $\mathbb{N}_{\mathrm{rp}}$ has a computable witness for being nowhere $T_{2.5}$, that is, $\Lambda_{d,e} = \mathrm{lcm}(b,d)\mathbb{Z}$. For a network $\mathcal{N}$, we define the *strong disjointness diagram* as $\mathrm{Disj}^-_{\mathcal{N}} = \mathrm{Disj}_{\mathcal{N}} \oplus \{\langle d,e\rangle : \overline{N_d} \cap \overline{N_e} = \emptyset\}$.

Recall that $x \in \omega^\omega$ is 1-*generic* if it meets or avoids every c.e. open set. For an oracle C, a point $x \in \omega^\omega$ is 1-C-*generic* if it meets or avoids every C-c.e. open set.

Lemma 5.52. *Let $\mathcal{H}$ be a represented, countable, second-countable space with a C-c.e. witness for being nowhere $T_{2.5}$, and let $x \in \omega^\omega$ be 1-C-generic. For any topological space $\mathcal{Y}$ with a cs-network with a C-c.e. strong disjointness diagram, $x\colon \mathcal{H}^\omega$ is $\widetilde{*}$-nearly $\mathcal{Y}$-quasi-minimal.*

Proof. Since $\mathcal{H}$ is countable, we can assume that $\mathcal{H}$ is of the form $(\omega, (H_e)_{e\in\omega})$, where $(H_e)_{e\in\omega}$ is an enumeration of countable basis of the space $\mathcal{H}$. We code a basic open set in $\mathcal{H}^\omega$ by a finite sequence α, that is, α codes the open set

$$U_\alpha = \{x \in \mathcal{H}^\omega : (\forall n < |\alpha|)\ x(n) \in H_{\alpha(n)}\}.$$

Note that $(U_\alpha)_{\alpha<\omega^{<\omega}}$ forms a basis of $\mathcal{H}^\omega$. Hereafter we use $\mathrm{Nbase}(x)$ to denote $\{\alpha : x \in U_\alpha\}$.

Now, assume that $y\colon \mathcal{Y} \leq_{\mathbf{T}} x\colon \mathcal{H}^\omega$. We will show that $y\colon \mathcal{Y}$ is $\widetilde{*}$-nearly computable. By Observation 4.3, there is $J \leq_e \mathrm{Nbase}(x)$ such that $\{N_e : e \in J\}$ forms a strict network at y. Let Ψ witness that $J \leq_e \mathrm{Nbase}(x)$. Since (U_α) forms a basis, one can assume that Ψ is a c.e. set of pairs of indices and singletons, that is,

$$e \in J \iff (\exists \alpha)\ [\alpha \in \mathrm{Nbase}(x) \text{ and } \langle e,\alpha\rangle \in \Psi].$$

Consider the following three cases:

Case 1. There is $\ell \in \omega$ such that for any d, e, α, and β,

$$U_\alpha \cap [x \upharpoonright \ell] \neq \emptyset,\ U_\beta \cap [x \upharpoonright \ell] \neq \emptyset, \text{ and } \langle d,\alpha\rangle, \langle e,\beta\rangle \in \Psi \ \Rightarrow\ \overline{N_d} \cap \overline{N_e} \neq \emptyset.$$

Then let p be a computable sequence such that $p(n) = e+1$ for some n if and only if there is α such that $\langle e,\alpha\rangle \in \Psi$ and $U_\alpha \cap [x \upharpoonright \ell] \neq \emptyset$. Then, it is easy to check that p is a $\widetilde{\delta_{\mathcal{N}}}$-name of y.

Case 2. For any $\ell \in \omega$, there are d, e, α, and β such that

$$U_\alpha \cap U_\beta \cap [x \upharpoonright \ell] \neq \emptyset,\ \langle d,\alpha\rangle, \langle e,\beta\rangle \in \Psi, \text{ and } N_d \cap N_e = \emptyset.$$

In this case, inconsistent Ψ-computations are dense along x, that is, consider

$$E = \{\langle \alpha,\beta\rangle : (\exists d,e)\ [\langle d,\alpha\rangle, \langle e,\beta\rangle \in \Psi, \text{ and } N_d \cap N_e = \emptyset]\},$$

and then define $V_E = \bigcup\{U_\alpha \cap U_\beta : \langle \alpha,\beta\rangle \in E\}$. Then, Ψ is undefined on V_E, that is, for any $z \in V_E$, $\Psi(\mathrm{Nbase}(z))$ is undefined. Note that each U_α is clopen with respect to the standard Baire topology on ω^ω. Therefore, since the disjointness diagram of $\mathcal{N}$ is C-c.e., V_E is C-c.e. open and dense along x with respect to the standard Baire topology on ω^ω. Since x is 1-C-generic, we have $x \in V_E$. Therefore, $\Psi(\mathrm{Nbase}(x))$ is undefined.

Case 3. Otherwise, let ℓ be a witness of the failure of Case 2. Then, since Case 1 fails, there are d, e, α, and β such that

$$U_\alpha \cap [x \upharpoonright \ell] \neq \emptyset,\ U_\beta \cap [x \upharpoonright \ell] \neq \emptyset,\ \langle d,\alpha\rangle, \langle e,\beta\rangle \in \Psi, \text{ and } \overline{N_d} \cap \overline{N_e} = \emptyset.$$

That is, there are splitting Ψ-computations above $x \upharpoonright \ell$ in a strong sense. Consider

$$D = \{\langle \alpha,\beta\rangle : (\exists d,e)\ [\langle d,\alpha\rangle, \langle e,\beta\rangle \in \Psi, \text{ and } \overline{N_d} \cap \overline{N_e} = \emptyset]\}.$$

Then, the set D is C-c.e., since the strong disjointness diagram is C-c.e. by our assumption. Let Λ be a C-c.e. witness for being nowhere $T_{2.5}$. Consider the following ω^ω-clopen set:

$$Q_{\alpha,\beta} = \{z \in \omega^\omega : (\exists \sigma \prec z)\ \ell \leq |\sigma| \leq \max\{\ell, |\alpha|, |\beta|\},\ U_\alpha \cap [\sigma] \neq \emptyset,\ U_\beta \cap [\sigma] \neq \emptyset,$$
$$\text{and } (\forall n)[|\sigma| \leq n < \max\{|\alpha|, |\beta|\})\ z(n) \in \Lambda_{\alpha(n),\beta(n)}\},$$

where let $H_{\alpha(n)}$ be an index of the whole space ω whenever $\alpha(n)$ is undefined. Note that $Q_{\alpha,\beta} \subseteq \overline{U_\alpha} \cap \overline{U_\beta}$ since $\Lambda_{\alpha(n),\beta(n)} \subseteq \overline{H_{\alpha(n)}} \cap \overline{H_{\beta(n)}}$. Define $Q_D = \bigcup\{Q_{\alpha,\beta} : \langle \alpha, \beta\rangle \in D\}$. Then, Q_D is C-c.e. open, and dense along x with respect to the standard Baire topology on ω^ω. To see this, for any m, since Case 1 fails, there is $\langle \alpha, \beta\rangle \in D$ such that $U_\alpha \cap [x \upharpoonright m] \neq \emptyset$ and $U_\beta \cap [x \upharpoonright m] \neq \emptyset$. By our assumption, we can always choose $z(k) \in H_{\alpha(k)} \cap H_{\beta(k)}$ for any $k \geq m$, and thus we can get some $z \in Q_{\alpha,\beta}$ extending $x \upharpoonright m$. Therefore, by 1-C-genericity of x, we have $x \in Q_D$.

We claim that Ψ is undefined on $Q_D \cap [x \upharpoonright \ell]$. Otherwise, $\Psi(\mathrm{Nbase}(z))$ is defined for some $z \in Q_D \cap [x \upharpoonright \ell]$. Since $z \in Q_D \cap [x \upharpoonright \ell]$, there is $\langle \alpha, \beta\rangle \in D$ such that $z \in Q_{\alpha,\beta} \subseteq \overline{U_\alpha} \cap \overline{U_\beta}$. Let $\langle d, e\rangle$ be a pair witnessing $\langle \alpha, \beta\rangle \in D$, that is, $\langle d, \alpha\rangle, \langle e, \beta\rangle \in \Psi$, and $\overline{N_d} \cap \overline{N_e} = \emptyset$. Clearly $\Psi(\mathrm{Nbase}(z)) \in \mathcal{Y} \setminus \overline{N_d}$ or $\Psi(\mathrm{Nbase}(z)) \in \mathcal{Y} \setminus \overline{N_e}$. Without loss of generality, we may assume that $\Psi(\mathrm{Nbase}(z)) \in \mathcal{Y} \setminus \overline{N_d}$. Then, there is $\langle c, \gamma\rangle \in \Psi$ such that

$$z \in U_\gamma, \text{ and } \Psi(\mathrm{Nbase}(z)) \in N_c \subseteq \mathcal{Y} \setminus \overline{N_d}.$$

In particular, we have $N_c \cap N_d = \emptyset$. Since $U_\gamma \cap [z \upharpoonright \ell]$ is an open neighborhood of z and $z \in \overline{U_\alpha}$, $U_\alpha \cap U_\gamma \cap [z \upharpoonright \ell]$ is nonempty. Since $x \upharpoonright \ell = z \upharpoonright \ell$, we conclude that

$$U_\alpha \cap U_\gamma \cap [x \upharpoonright \ell] \neq \emptyset,\ \langle d, \alpha\rangle, \langle c, \gamma\rangle \in \Psi, \text{ and } N_d \cap N_c = \emptyset.$$

This contradicts our choice of ℓ. Consequently, $\Psi(\mathrm{Nbase}(x))$ is undefined. □

Proof of Theorem 5.46. Let $\mathcal{X}$ be a topological space with a countable cs-network $\mathcal{N}$. Let C be an oracle such that the strong disjointness diagram is C-c.e. By Lemma 5.52, for any 1-C-generic point $x \in \omega^\omega$, $x\colon (\mathbb{N}_{\mathrm{rp}})^\omega$ is $\widetilde{*}$-nearly $\mathcal{X}$-quasi-minimal, that is, if $z\colon \mathcal{X} \leq_{\mathbf{T}} x\colon (\mathbb{N}_{\mathrm{rp}})^\omega$ then $z\colon \mathcal{X}$ is $\widetilde{*}$-nearly computable. By Observation 5.49, if $\mathcal{X}$ is a $T_{2.5}$-space, then only countably many points in $\mathcal{X}$ can be $\widetilde{*}$-nearly computable. However, there are uncountably many points which are 1-C-generic. Thus, one can choose such a point which is not $\equiv_{\mathbf{T}}$-equivalent to any $\widetilde{*}$-nearly computable points in $\mathcal{X}$. □

Proof of Theorem 5.47. The canonical network $\mathcal{N}$ of $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$ has a computable strong disjointness diagram. Since $\mathcal{N}$ is regular-like (see Example 4.10), by Observation 5.49, $\mathrm{id}\colon (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \widetilde{\delta_\mathcal{N}}) \to (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \overline{\delta_\mathcal{N}})$ is computable. Moreover, as seen in Example 4.10, $\mathrm{id}\colon (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \overline{\delta_\mathcal{N}}) \to (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \delta_\mathcal{N})$ is computable, and so is $\mathrm{id}\colon (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \widetilde{\delta_\mathcal{N}}) \to (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \delta_\mathcal{N})$ is computable. Hence, $\widetilde{*}$-near $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimality is equivalent to $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimality by definition. Therefore, by Lemma 5.52, for any 1-C-generic point x, $x\colon (\mathbb{N}_{\mathrm{rp}})^\omega$ is $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimal. □

5.4. **$T_{2.5}$-degrees which are not T_3.** Recall that $(\omega^\omega)_{GH}$ is the set ω^ω endowed with the Gandy-Harrington topology. Recall from Theorem 3.46 that no $(\omega^\omega)_{GH}$-degree is continuous. In this section, we will prove much stronger results.

Theorem 5.53. *Let $\mathcal{X} = (X, \mathcal{N})$ be a regular Hausdorff space with a countable cs-network. Then there is an $(\omega^\omega)_{GH}$-degree which is not an $\mathcal{X}$-degree, that is,*

$$\mathcal{D}_{(\omega^\omega)_{GH}} \not\subseteq \mathcal{D}_{\mathcal{X}}.$$

Theorem 5.54. *The Gandy-Harrington space has no point of $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-degree, that is,*

$$\mathcal{D}_{(\omega^\omega)_{GH}} \cap \mathcal{D}_{\mathbb{N}^{\mathbb{N}^\mathbb{N}}} = \emptyset.$$

To prove Theorems 5.53 and 5.54, we need the following lemma.

Lemma 5.55. *Let $\mathcal{X}$ be a topological space with a countable cs-network $\mathcal{N}$. If $\mathcal{N}$ has a Σ^1_1 disjointness diagram, then for any $x \in (\omega^\omega)_{GH}$ and $z \in \mathcal{X}$,*

$$z\colon \delta_{\mathcal{N}} \leq_{\mathbf{T}} x\colon (\omega^\omega)_{GH} \implies x\colon (\omega^\omega)_{GH} \not\leq_{\mathbf{T}} z\colon \overline{\delta_{\mathcal{N}}}.$$

Proof. For $x \in \omega^\omega$, consider $G_x := \mathrm{Nbase}_{GH}(x) = \{e : x \in GH_e\}$, where recall that GH_e is the e-th Σ^1_1 set in ω^ω. Clearly, G_x is a $\Sigma^1_1(x)$ subset of ω. Suppose that $z\colon \delta_{\mathcal{N}} \leq_T x\colon (\omega^\omega)_{GH}$ for $z \in \mathcal{X}$, and $\mathcal{N}$ is a countable cs-network for $\mathcal{X}$ such that $\mathrm{Disj}_{\mathcal{N}}$ is Σ^1_1. By Observation 4.3, there is $J \leq_e G_x$ such that $\{N_e : e \in J\}$ forms a strict network at z. Let Ψ witness $J \leq_e G_x$, that is, $e \in J$ iff there is a finite set $D \subseteq G_x$ such that $\langle e, D\rangle \in \Psi$. Then consider

$$L = \{n \in \omega : (\forall \langle m, D\rangle \in \Psi)\ D \subseteq G_x \ \to\ N_m \cap N_n \neq \emptyset\}.$$

Note that L is a $\Pi^1_1(x)$ subset of ω since $\mathrm{Disj}_{\mathcal{N}}$ is Σ^1_1. One can also see that $J \subseteq L$, since $n \in J$ implies that $z \in N_n$, and moreover, if $\langle m, D\rangle \in \Psi$ and $D \subseteq G_x$, then $m \in J$, and therefore, $z \in N_m \cap N_n$. This implies that $\{N_e : e \in L\}$ forms a network at z. We claim that $z \in \overline{N}_n$ for any $n \in L$. This is because, if $z \notin \overline{N}_n$ then there is an open set $U \subseteq \mathcal{X}$ such that $z \in U$ and $U \cap N_n = \emptyset$. By our choice of Ψ, there is $\langle e, D\rangle \in \Psi$ such that $D \subseteq G_x$ and $z \in N_e \subseteq U$. Since $N_e \cap N_n = \emptyset$, we have $n \notin L$. This verifies the claim, and in particular, every enumeration of L gives an $\overline{\delta_{\mathcal{N}}}$-name of z.

Suppose that $x\colon (\omega^\omega)_{GH} \leq_{\mathbf{T}} z\colon \overline{\delta_{\mathcal{N}}}$. Then, in particular, G_x is enumeration reducible to L, that is, there is a c.e. set Γ such that

$$e \in G_x \iff (\exists D \text{ finite})\ [(e, D) \in \Gamma \text{ and } D \subseteq L].$$

Since L is $\Pi^1_1(x)$, this gives a $\Pi^1_1(x)$ definition of G_x. However, G_x is clearly a complete $\Sigma^1_1(x)$ subset of ω, which implies a contradiction. Consequently, $x\colon (\omega^\omega)_{GH} \not\leq_{\mathbf{T}} z\colon \overline{\delta_{\mathcal{N}}}$. □

Proof of Theorem 5.53. Let $\mathcal{X} = (X, \mathcal{N})$ be a regular Hausdorff space with a countable cs-network. By Observation 4.7, $\mathcal{N}$ is regular-like. By Theorem 4.8, $\mathcal{X}$ has a countable cs-network $\mathcal{M}$ such that $\mathrm{id} : (\mathcal{X}, \overline{\delta_{\mathcal{M}}}) \to (\mathcal{X}, \delta_{\mathcal{M}})$ is continuous; hence, computable relative to some oracle C_0. As mentioned in Section 2.4.4, cs-networks induce admissible representations, that is, $\delta_{\mathcal{M}}$ and $\delta_{\mathcal{N}}$ are both $\leq$-maximal among continuous representations of X, and thus $\mathcal{M}$ and $\mathcal{N}$ are equivalent; hence, computably equivalent relative to some oracle C_1. Moreover, $\mathrm{Disj}_{\mathcal{M}}$ is Σ^1_1 relative to some oracle C_2. We now put $C = C_0 \oplus C_1 \oplus C_2$.

Choose $x \in \omega^\omega$ such that $C \leq_T x$. Then, we have $C\colon 2^\omega \leq_{\mathbf{T}} x\colon (\omega^\omega)_{GH}$ by Proposition 3.45. Thus, the condition $z\colon \delta_{\mathcal{N}} \leq_{\mathbf{T}} x\colon (\omega^\omega)_{GH}$ is equivalent to saying that $z\colon \delta_{\mathcal{M}} \leq_{\mathbf{T}}$

$x\colon (\omega^\omega)_{GH}$ since $C_1 \leq_T C$. By relativizing the proof of Lemma 5.55, since $C_2 \leq_T C$ and we now have $\Sigma^1_1(x \oplus C) = \Sigma^1_1(x)$ and $\Pi^1_1(x \oplus C) = \Pi^1_1(x)$, we get the following.

$$z\colon \delta_{\mathcal{N}} \leq_{\mathbf{T}} x\colon (\omega^\omega)_{GH} \implies x\colon (\omega^\omega)_{GH} \not\leq_{\mathbf{T}} (z\colon \overline{\delta_{\mathcal{M}}}) \oplus C.$$

We now assume that $z\colon \delta_{\mathcal{N}} \leq_{\mathbf{T}} x\colon (\omega^\omega)_{GH}$. Since $C_0 \leq_T C$, we have $z\colon \delta_{\mathcal{M}} \leq_{\mathbf{T}} (z\colon \overline{\delta_{\mathcal{M}}}) \oplus C$. Combining this with the above implication, we get $x\colon (\omega^\omega)_{GH} \not\leq_{\mathbf{T}} (z\colon \delta_{\mathcal{M}}) \oplus C$. Since $C_1 \leq_T C$, we have $z\colon \delta_{\mathcal{N}} \leq_{\mathbf{T}} (z\colon \delta_{\mathcal{M}}) \oplus C$, and thus $x\colon (\omega^\omega)_{GH} \not\leq_{\mathbf{T}} z\colon \delta_{\mathcal{N}}$. Hence, there is no $z \in \mathcal{X}$ such that $x\colon (\omega^\omega)_{GH} \equiv_M z\colon \delta_{\mathcal{N}}$. $\square$

The above proof indeed shows that if $C\colon 2^\omega \leq_{\mathbf{T}} x\colon \omega^\omega$ then $x\colon (\omega^\omega)_{GH}$ does not have an $\mathbf{X}$-degree. In particular, if $\mathcal{X}$ is a sufficiently effective regular Hausdorff space, then we have $\mathcal{D}_{(\omega^\omega)_{GH}} \cap \mathcal{D}_{\mathcal{X}} = \emptyset$.

Proof of Theorem 5.54. The canonical network $\mathcal{N}$ of $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$ has a computable disjointness diagram. Moreover, as seen in Example 4.10, $\mathrm{id}\colon (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \overline{\delta_{\mathcal{N}}}) \to (\mathbb{N}^{\mathbb{N}^\mathbb{N}}, \delta_{\mathcal{N}})$ is computable. Therefore, by Lemma 5.55, for any $x \in (\omega^\omega)_{GH}$ and $z \in \mathbb{N}^{\mathbb{N}^\mathbb{N}}$,

$$z\colon \mathbb{N}^{\mathbb{N}^\mathbb{N}} \leq_{\mathbf{T}} x\colon (\omega^\omega)_{GH} \implies x\colon (\omega^\omega)_{GH} \not\leq_{\mathbf{T}} z\colon \mathbb{N}^{\mathbb{N}^\mathbb{N}}.$$

This shows that there are no $x \in (\omega^\omega)_{GH}$ and $z \in \mathbb{N}^{\mathbb{N}^\mathbb{N}}$ such that $x\colon (\omega^\omega)_{GH} \equiv_M z\colon \mathbb{N}^{\mathbb{N}^\mathbb{N}}$. Hence, the Gandy-Harrington degrees and the $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-degrees have no common element. $\square$

For an ω-parametrized pointclass Γ, the Γ-*Gandy-Harrington topology* is the topology τ_Γ on ω^ω generated by the subbasis consisting of all Γ subsets of ω^ω. By $(\omega^\omega)_{GH(n)}$, we denote ω^ω endowed with the Σ^1_n-Gandy-Harrington topology. We show that there is a hierarchy of degree structures of Gandy-Harrington topologies.

Theorem 5.56. *For any distinct numbers $n, m \in \omega$, there is no e-degree which is both an $(\omega^\omega)_{GH(n)}$-degree and an $(\omega^\omega)_{GH(m)}$-degree, that is,*

$$n \neq m \implies \mathcal{D}_{(\omega^\omega)_{GH(n)}} \cap \mathcal{D}_{(\omega^\omega)_{GH(m)}} = \emptyset.$$

Proof. Suppose for the sake of contradiction that $y\colon (\omega^\omega)_{GH(n)} \leq_{\mathbf{T}} x\colon (\omega^\omega)_{GH(m)}$ for $n < m$; that is, $\{e : y \in S^n_e\} \leq_e \{e : x \in S^m_e\}$, where S^n_e is the e-th Σ^1_n set in ω^ω. As in Proposition 3.45, one can see that $y\colon \omega^\omega \leq_{\mathbf{T}} y\colon (\omega^\omega)_{GH(n)}$. This implies that $\mathrm{Graph}(y) \oplus \mathrm{Graph}(y)^{\mathsf{c}} \leq_e \{e : x \in S^m_e\}$. Note that $\mathrm{Graph}(y) \leq_e \{e : x \in S^m_e\}$ implies that y is Σ^1_m relative to x. Hence, we get that y is Δ^1_m relative to x. Note also that $\{e : y \in S^n_e\}$ is Σ^1_n-complete relative to y, so Δ^1_m-complete relative to y as $n < m$. In particular, $\{e : y \in S^n_e\}$ is Δ^1_m relative to x since Δ^1_m-reducibility is transitive (see Rogers [62, Theorem 16.XXXIV]). Thus, we get $\{e : y \in S^n_e\} \oplus \{e : y \notin S^n_e\} \leq_e \{e : x \in S^m_e\}$. By our assumption, this shows that $\{e : x \in S^m_e\}$ is e-equivalent to $\{e : y \in S^n_e\} \oplus \{e : y \notin S^n_e\}$, which is total. However, as in Theorem 3.46, observe that $x\colon (\omega^\omega)_{GH(m)}$ cannot have a total degree. $\square$

6. Open Questions

Here we list the current open problems.

Major Questions. We have shown that, in a certain sense, there are a T_1-quasi-minimal e-degree (Theorem 5.14), and a T_2-quasi-minimal T_1-degree (Theorem 5.38). Thus, whether there exist a $T_{2.5}$-quasi-minimal T_2-degree is the one of the most important open problems:

Question 6.1. *Does there exist a represented Hausdorff space $\mathcal{X}$ such that given $T_{2.5}$ space $\mathcal{Y}$, there is $x \in \mathcal{X}$ which is $\mathcal{Y}$-quasi-minimal?*

Currently we do not know if we can separate $T_{2.5}$ degrees and submetrizable degrees. Hence, the following problem is also important:

Question 6.2. *Does there exist a represented $T_{2.5}$-space $\mathcal{X}$ such that, given a submetrizable space $\mathcal{Y}$, there is $x \in \mathcal{X}$ which is not of $\mathcal{Y}$-degree?*

We are also interested in whether we can show separation results in the category of effective quasi-Polish spaces. For instance, co-d-CEA, chained (Arens) co-d-CEA, doubled co-d-CEA, telophase, and semirecursive e-degrees are realized as the degrees of points in effective quasi-Polish spaces.

Question 6.3. *Given a submetrizable space $\mathcal{Y}$, does there exist a Arens co-d-CEA (or Roy halfgraph-above) degree which is not a $\mathcal{Y}$-degree?*

Note that the affirmative answer to the above question gives a quasi-Polish solution to Question 6.2. Similarly, we have found a non-$T_{2.5}$-degree in a T_2-space, namely, the product Golomb space $\mathbb{N}_{\rm rp}^{\omega}$ (Theorem 5.46); however this space is not quasi-Polish. We know that there is a quasi-Polish space Hausdorff space which is not $T_{2.5}$, e.g. the double origin space. Therefore, one can ask the following:

Question 6.4. *Given a $T_{2.5}$ space $\mathcal{Y}$, does there exist a doubled co-d-CEA degree which is not a $\mathcal{Y}$-degree?*

Another big open problem is concerning graph-cototal degrees was raised by Joseph Miller:

Question 6.5. *Does there exist a continuous degree which is not graph-cototal?*

With our framework, this is equivalent to asking whether there is a σ-embedding of the Hilbert cube $[0,1]^\omega$ into the product cofinite space $(\omega_{\rm cof})^\omega$. Note that the continuous functions into $\omega_{\rm cof}$ correspond to countable partitions into closed sets. A classic result by Sierpiński shows that connected compact Polish space do not admit non-trivial countable partitions into closed sets (cf [24, Theorem 6.1.27]. In particular, there is no *embedding* of $[0,1]^\omega$ into $(\omega_{\rm cof})^\omega$. There exist, however, infinite dimensional spaces without any connected compact Polish subspaces (these are called punctiform), so an answer to the question is not immediate. A piece of the puzzle could that the result that the Hilbert cube cannot be decomposed into countably-many hereditarily disconnected spaces [5].

To solve a question, one may examine the behavior of the co-spectrum of a space. For instance, by an argument using the notion of co-spectrum, we have shown that there is a continuous degree which is neither telograph-cototal nor cylinder-cototal (by Proposition 5.43). However, we do not know even the following:

Question 6.6. *Does there exist a continuous degree which is not 2-cylinder-cototal?*

In general, we are also interested in analyzing the behavior of the cospectrum of a given space. For instance, it is important to ask the following:

Question 6.7. *Is every countable Turing ideal realized as the cospectrum of a graph-cototal e-degree?*

We next consider cototal e-degrees. Recall from Theorem 3.68 that a space is cototal if and only if it is computably G_δ. Since every computably G_δ space is effectively T_1 (i.e., every singleton is Π^0_1 relative to it) by Observation 3.57, in particular, every point in a cototal space has a T_1-degree. Then, can we separate cototal degrees and T_1-degrees?

Question 6.8. *Does there exist a point in an effective quasi-Polish T_1-space which has no cototal e-degree?*

Recall that our universal (in the degree-theoretic sense) computably G_δ space $\mathcal{A}^{\rm co}_{\rm max}$, the maximal antichain space, is not quasi-Polish (Proposition 3.72). One of the most important questions on cototal degrees is whether a universal computably G_δ quasi-Polish space exists:

Question 6.9. *Does there exist a computably G_δ, quasi-Polish, space which contains all cototal e-degrees?*

In this article, we have also discussed $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimality. However, currently we do not know whether quasi-minimality is different from $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimality.

Question 6.10. *Does there exist a quasi-minimal e-degree which is not $\mathbb{N}^{\mathbb{N}^\mathbb{N}}$-quasi-minimal?*

Mariya Soskova raised the following question (see Subsection 5.1.7 for context):

Question 6.11. *Is there a topological characterization of the halves of non-trivial $\mathcal{K}$-pairs?*

Minor Questions. We also list some minor questions. Recall that every telograph-cototal (double-origin) e-degree is graph-cototal (Propositions 3.21 and 3.26). There is a graph-cototal (indeed cylinder-cototal) e-degree which is neither telograph-cototal nor doubled co-d-CEA (by Theorem 5.36). Can every telograph-cototal (doubled co-d-CEA) e-degree be embedded into some level of the hierarchy of graph-cototal e-degrees?

Question 6.12. *Is every telograph-cototal e-degree n-cylinder-cototal for some $n \in \omega$?*

Recall from Proposition 3.13 that there is a co-d-CEA e-degree which is not cylinder-cototal. The following question is also open.

Question 6.13. *Does there exist a co-d-CEA e-degree which is not 2-cylinder-cototal?*

We also do not know the relationship among variations of co-d-CEA degrees.

Question 6.14. *What is the relationship among doubled co-d-CEA degrees, Arens co-d-CEA degrees, and Roy halfgraph-above degrees?*

Recall that every 3-c.e. e-degree is telograph-cototal while there is a Σ^0_2 e-degree which is not telograph-cototal (by Theorem 5.24).

Question 6.15. *For any n, is every n-c.e. e-degree telograph-cototal?*

There is also a problem related to left-totality.

Question 6.16. *Is there a cylinder-cototal e-degree which is not G_δ-left-total?*

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