COMPUTABLY AND PUNCTUALLY UNIVERSAL SPACES

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Abstract. We prove that the standard computable presentation of the space $C[0,1]$ of continuous real-valued functions on the unit interval is computably and punctually (primitively recursively) universal. From the perspective of modern computability theory, this settles a problem raised by Sierpiński in the 1940s.

We prove that the original Urysohn's construction of the universal separable Polish space $\mathbb{U}$ is punctually universal. We also show that effectively compact, punctual Stone spaces are punctually homeomorphically embeddable into Cantor space 2^ω; note that we do not require effective compactness be primitive recursive. We also prove that effective compactness cannot be dropped from the premises by constructing a counterexample.

1. Introduction

This paper contributes to a fast-developing branch of computable analysis that uses methods of effective algebra and applies them to separable spaces. In this paper, we study the effective content of several classical results in topology that are concerned with universal spaces. In particular, we prove effective versions of universality for the space $C[0,1]$ of continuous functions on the unit interval, the Urysohn space, and Cantor space (among Stone spaces). We also continue the systematic development of primitive recursive (punctual) analysis which was initiated in [DMN21] and, in the context of ordered fields, proposed in [SS21]. We prove computable versions of the universality results, and then we also establish their primitive recursive analogs. These stronger results sometimes require new ideas, and these extensions are not necessarily straightforward. Before we state the results we need to give some background.

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1.1. **Universality and closure results in mathematics.** In topology and metric space theory there are plenty of results that establish universality of some natural space(s) in the respective class. For example, the Urysohn space is universal among all separable metric spaces under isometry, the space of continuous functions on the unit interval $C[0,1]$ is universal among all separable Banach spaces up to linear isometry and, more generally, among all separable metric spaces under isometry. Other universality results include the universality of the Hilbert cube and the Tikhonov cube, Baire space, the Menger curve, etc.

In classical (discrete) algebra there are various results that state that a given algebraic structure can be embedded into some natural existentially closed structure in the same class. For example, every ordered field is contained in its real closure, and every differential field can be isomorphically embedded into its differential closure. All these results can be interpreted as universality-type results. For example, it follows that every algebraic field (over $\mathbb{Q}$) can be embedded into the field of algebraic numbers $acl(\mathbb{Q})$ of $\mathbb{Q}$; and $acl(\mathbb{Q}(x_i)_{i\in\omega})$ is universal among all countable fields of characteristic zero. Similarly, the countable direct power of the Prüfer p-group is universal among all countable abelian p-groups, the ordered field of reals is universal among all Archimedean fields, etc.

1.2. **Computable mathematics.** While classical mathematics typically studies mathematical structures under isomorphism, computable mathematics investigates the algorithmic content of *computably presented* structures under *computable* isomorphism. For instance, in effective algebra [AK00, EG00], the algorithmic content of the closure results discussed above have been studied extensively and for many decades. Building on the earlier works of van der Waerden [vdW30] and Fröhlich–Shepherdson [FS56], Rabin [Rab60] proved that every computable field can be computably embedded into its computable algebraic closure. Similar effective algebraic theorems hold for other classes of algebraic structures including ordered fields [Ers68, Mad70], differential fields [Har74], abelian groups [Smi81], and difference-closed fields [HTMM17]. These results lay in the foundations of computable structure theory. Similarly to the respective classical results, these computability-theoretic results can often be viewed as *computable* universality results. For instance, every computable algebraic field (over $\mathbb{Q}$) can be computably embedded into a computable presentation of the algebraic closure $acl(\mathbb{Q})$ of $\mathbb{Q}$.

Recall also that there are many universality results that play a significant role in topology. However, in contrast with the situation in effective algebra, the computable versions of these fundamental results have been studied only relatively recently. We will discuss the little that is known in due course. For that, we need a bit more background.

To investigate the effective content of the classical universality results in topology we will use tools of computable analysis. The field of computable analysis originated in the work of Turing [Tur36] and then was studied by Specker [Spe49, Spe59], Rice [Ric54], Zaslavsky [Zas55, Zas62], Goodstein [Goo61], Kushner [Kus80], Aberth [Abe80], Pour-El and Richards [PER89], Weihrauch [Wei00] and others. Turing defined computability on real numbers using rational numbers and approximation by them. The same idea is used for the definition of a computable space by means of a dense sequence of points. For example, in the space of continuous functions, polynomials with rational coefficients can be taken as a dense sequence. This way one can develop computability in uncountable spaces provided that they are separable. We will give the formal details in the preliminaries.

We now discuss the very little that is known about computable universality results in topology. It is not too hard to prove that the Urysohn space is computably universal among computable Polish spaces under computable isometry ([Kam05, Kam06] and follows from [Mel13]). Also, it is known that the Hilbert cube is computably universal under computable homeomorphism among effectively compact Polish spaces (folklore). There is another related effective universality result that combines methods of effective algebra and rudiments of computable analysis: Goncharov, Lempp, and Solomon [GLS03] proved that every computable Archimedean ordered abelian group can be computably embedded into the ordered group of the reals $(\mathbb{R}, +, <)$ in the sense that each g is uniformly mapped to some computable real $f(g)$. (This is the natural effective analog of the well-known Hölder's theorem.) However, this is more of an effective algebraic result rather than an effective topological result. There is one more very recent result of this sort established in [DM23] that will be mentioned later. With the exception of a few more observations, these results are essentially all that is known (to the authors) about the effective content of universality results in topology and functional analysis.

Melnikov [Mel13] proposed that many aspects of computable metric space and Banach space theory are very similar to computable algebra and, therefore, it is often possible to apply methods of computable algebra to study these spaces. There have been many works that use methods of effective algebra in computable analysis as well as some new methods [SS21, McN17, FAK$^+$20, GMKT18]. As we discussed above, many classical results in effective algebra can be interpreted as universality-type results. The main aim of this article is to apply methods and ideas from computable algebra to establish computability-theoretic versions of classical universality results in topology.

1.3. **Primitive recursive mathematics.** In effective algebra, there has been a recent line of study into the primitive ('punctual') recursive content of effective algebraic results. This research program was proposed by Kalimullin, Melnikov, Ng in [KMN17]. (A related program was independently proposed by Alaev; e.g., [Ala16].)

Somewhat unexpectedly, the seemingly simple idea of attempting to eliminate an unbounded search in computable algebra resulted in a very rich and technically deep theory; see surveys [BDKM19, DMN21]. For instance, many effective algebraic results have been discovered to fail primitively recursively. Some other results actually hold but require completely different proofs. The class of primitive recursive functions can be viewed as an abstraction reflecting certain aspects of some natural complexity class (such as polynomial-time or logspace etc.) without necessarily specifying what this class exactly might be. This allows one to remove the extra layer of combinatorics specific to the complexity class and focus on the difficulties related to eliminating unbounded search in computable procedures. Also, there is a version of Church-Turing thesis for primitive recursive procedures which usually makes proofs less tedious. This approach has proved to be rather fruitful in effective algebra. Finally, primitive recursion is often viewed as a formalisation of 'finitism' in mathematics (see [Lei21]) which gives this line of study a clear philosophical motivation.

It is quite natural to test similar ideas in separable spaces. In fact, historically a lot of elementary computable analysis and computable calculus was in fact developed primitively recursively, see book [Goo61]. However, gradually, primitive recursiveness had been partially abandoned perhaps because of technical difficulties that arise while dealing with primitive recursive procedures. Beginning with the mid-eighties pretty much all computable analysis has been done using general Turing computability; see, e.g., [PER89], [Wei00]. Recently in [DMN21] and [SS21], it has been proposed to revive this program of applying primitive recursiveness in analysis using modern methods. For instance, very recently Selivanov and Selivanova [SS21] have established a primitive recursive ('punctual') version of the aforementioned Ershov–Madison theorem [Ers68, Mad70] establishing that every primitive recursive Archimedean field can be primitively recursively embedded into its primitive recursive real closure. Investigations into the primitive recursive content of closure results in algebra is still ongoing. For instance, Dorzhieva and Guo have recently announced that a similar result holds of primitive recursive fields and their algebraic closures, as well as for ordered fields that are not necessarily Archimedean. The second main aim of this article is to test these methods to establish primitive recursive versions of several universality results in the theory of separable spaces.

We are now ready to state and discuss our results.

1.4. **The space of continuous functions.** We first discuss the well-known result of Banach which states that every Polish metric space (and, thus, every separable Banach space) can be isometrically embedded into the space of continuous functions on the unit interval under the supremum metric. (This result is sometimes refereed to as the Banach–Mazur theorem.) Banach proved this result in his famous monograph [Ban32] using non-constructive methods. For instance, he used the dual space and the Hahn–Banach theorem to establish the universality. However, it is known that the Hahn–Banach theorem is not computable in general [MNS85, Bra08]. The highly non-algorithmic nature of Banach's proof of the universality of $C[0,1]$ was noticed by Sierpiński who proposed the problem of discovering a more direct and more effective way of embedding every separable space into $C[0,1]$. For a detailed discussion, see [Hol08]. Sierpiński [Sie45] came up with an elegant and more direct way of embedding the Urysohn space (thus, any Polish and any separable Banach space) into $C[0,1]$. He calls his result 'effective' and mentions 'effectivity' a few times throughout his paper. However, upon a closer examination we discovered that what he thought was effective actually requires an application of the halting problem. It is well-known that the halting problem is algorithmically undecidable. Formal recursion theory was at its infancy in the 1940s, and the subtle difference between computable and computably enumerable processes was not evident even to such eminent mathematicians as Sierpiński. Indeed, even Turing's seminal paper [Tur36] contained a significant number of errors some of which were related to this subtle difference; Turing corrected several of these flaws in [Tur37]. Even these days, the exact issue with Sierpiński's proof is not easy to spot. We will present and discuss his original proof (that was published in French [Sie45]) later in the paper.

In our first main result we eliminate the use of undecidability techniques from Sierpiński's proof. To state the result formally, we use the standard terminology of computable analysis. Recall that a *computable Polish space* is a complete separable metric space M together with a dense sequence $(x_i)_{i\in\omega}$ such that, for every i, j, n, we can uniformly compute a rational $r = \frac{m}{n}$ such that $|d(x_i, x_j) - r| < 2^{-n}$. (In other words, $d(x_i, x_j)$ are uniformly computable reals.) Fix the standard computable presentation $\mathcal{C}$ of $(C[0,1], d_{\mathrm{sup}})$ given by the dense sequence of piecewise linear functions with finitely many rational breaking points. (We remark that this presentation is *not* computably unique up to computable isometry [MN16] even if we require the Banach space operations to be computable.) To handle computable Polish spaces, we had to design a *new proof* even more explicit than the proof of Sierpiński.

Theorem 1.1. *Given any computable Polish space $\mathcal{M}$, there is a computable isometry from $\mathcal{M}$ onto a subset of $\mathcal{C}$.*

In our proof, we use careful dynamic approximations to build an embedding in stages. As far as we know, this proof has no analogs in the literature. Even though it does share some features with Sierpiński's proof, it seems that, even classically, it gives a new (and highly explicit) method of illustrating universality of $C[0,1]$.

In fact, our technique enables us to prove the following, even stronger result. We define a *punctual Polish space* by replacing *computable* by *primitive recursive* throughout the definition of a computable Polish space[1].

Theorem 1.2. *Given any punctual Polish space $\mathcal{M}$, there is a primitive recursive isometry from $\mathcal{M}$ onto a subset of $\mathcal{C}$.*

The result is stronger in the sense that it actually provides us with a uniform primitive recursive operator. We suspect that the result can perhaps be further refined to obtain a polynomial-time embedding, but this does not look straightforward (if true). We leave the verification (or refutation) of this as an open problem.

From the perspective of modern computability theory, Theorem 1.2 settles the problem raised and attacked by Sierpiński in the 1940s. We refer to [Hol08] for a detailed discussion of this problem and several interesting results related to universality of $C[0,1]$. Indeed, we settle the problem in a rather strong sense: recall that primitive recursion is often viewed as a formalisation of 'finitistic' mathematics; e.g., [Lei21].

1.5. **The Urysohn space.** In his proof, Sierpiński [Sie45] used universality of the Urysohn space $\mathbb{U}$, the universal separable metric space. In our direct proof, we do not use the computable universality of the Urysohn space among all separable spaces. In our particular proof, $\mathbb{U}$ does not seem to help[2]. Nonetheless, the computable universality of the Urysohn space $\mathbb{U}$ is evidently a problem that has independent interest. It is natural to ask for the Urysohn space, which is a computable space by design, if it is universal among all computable Polish spaces. It is not too difficult to prove that the Urysohn space is computably universal [Kam05, Kam06]. A much simpler proof of this fact can be obtained using methods from [Mel13]. It

[1] We also require that points x_i and x_j in the dense sequence are unequal whenever $i \neq j$. In the case of a computable Polish space this restriction would not make any difference because one can always eliminate repetitions using unbounded search.

[2] Downey and Melnikov [DM23] have recently suggested a proof of computable universality of $\mathcal{C}$ that (among many other tools) uses $\mathbb{U}$. However, it is not clear at all whether their proof gives primitive recursive universality of $\mathcal{C}$; we strongly suspect that it does not.

is not clear whether the technical proof contained in [Kam05, Kam06] is suitable for primitive recursive analysis of the Urysohn space universality, while methods in [Mel13] heavily rely on unbounded search. In Section 3, we will give a new proof of computable universality of $\mathbb{U}$ that is based on the original construction of Urysohn.

It is known that the Urysohn space is computably categorical [Mel13], and thus it is computably universal in the strongest sense possible. However, we discovered that the Urysohn space is *not punctually categorical* in the sense that there are at least two (actually, infinitely many) punctual presentations of the Urysohn space that are not punctually isometric.

Theorem 1.3. *Urysohn space is not punctually categorical.*

Therefore, the best we can prove is the following:

Theorem 1.4. *The presentation $\mathcal{U}$ of the Urysohn space constructed by Urysohn is punctually universal, in the sense that every punctual Polish space can be punctually isometrically embedded into $\mathcal{U}$.*

In fact, we will give two proofs of the theorem above. The first proof is a relatively straightforward application of punctual universality of $C[0,1]$, Theorem 1.2. Our proof of Theorem 1.2 is relatively involved, so perhaps using it to establish the punctual universality of $\mathbb{U}$ is an overkill. We thus also give another, much more explicit proof of Theorem 1.4. Interestingly, our second proof follows the original proof of Urysohn very closely.

1.6. **Cantor space and Stone spaces.** Finally, we finish this paper with an effective version of a folklore result saying that every Stone space can be homeomorphically embedded into Cantor space. Recall that a computable Polish space is effectively compact if it is possible to computably enumerate all its finite covers by basic open balls [MTY96]. It is not too difficult to prove that every computable, effectively compact Stone space is computably homeomorphically embeddable into Cantor space; we refer to Section 5 for more detail. It has recently been proven that Cantor space has a unique effectively compact presentation up to computable homeomorphism [BHTM23], and thus it is computably universal among effectively compact Stone spaces in the strongest possible sense[3].

In contrast with the somewhat routine and not particularly surprising computable universality of Cantor space among Stone spaces, the punctual version for Cantor space turned out to be much more interesting. We discovered that in the punctual case the necessary and sufficient conditions are not what one would perhaps expect. Here and in the next theorem, by Cantor space we mean its natural presentation by strings with the usual shortest common prefix ultrametric.[4] We prove:

Theorem 1.5. *Every punctual, effectively compact (in the Turing sense) Stone space X is primitively recursively homeomorphically embeddable into Cantor space.*

[3]We remark that there is another, more commonly known universality of Cantor space, the Alexandroff–Hausdorff Theorem asserting that every compact metric space is a homeomorphic image of 2^ω. The result was first established in [BdP12]; alternative proofs can be found in [DM23]. The punctual content of Alexandroff–Hausdorff Theorem has not been explored.

[4]It is not too difficult to show that Cantor space possesses punctual presentations that are not punctually homeomorphic; we shall not elaborate on this in this paper. A complete proof of a more general result will appear in Dorzhieva's Ph.D. thesis.

Note that effective compactness remains effective in the usual Turing computability sense. So we do *not* require effective compactness to be primitive recursive in any way. As far as we know, this is the first result of this 'partial punctual' sort in punctual analysis and, even more generally, in punctual structure theory.

It is natural to ask if effective compactness is necessary to prove the results above. In Theorem 1.6 we provide a counter-example.

Theorem 1.6. *There is a compact punctual Stone space which is not computably homeomorphically embeddable into Cantor space.*

The paper is arranged as follows. In Section 2, we give all the necessary definitions. In Section 3, we consider the space $C[0,1]$ and prove Theorem 1.2. In Section 4, we prove Theorems 1.3 and 1.4. The last Section 5 proves Theorems 1.5 and 1.6.

2. Preliminaries

Fix a Gödel numbering of the rational numbers. Under this numbering, we can speak about computable sets and functions on $\mathbb{Q}$. A real number x is called *computable* if there is a computable sequence of rational numbers $(c_n)_{n\in\omega}$ converging to x quickly, i.e. $|c_n - x| < 2^{-n}$ for all $n \in \omega$.

A Polish space $(X, d, (\alpha_i)_{i\in\omega})$ with a distinguished dense subspace $(\alpha_i)_{i\in\omega}$ is called *computable* if the distances $d(\alpha_i, \alpha_j)$ are computable real numbers uniformly in i, j, i.e., there is a computable function $\varphi\colon \omega^3 \to \mathbb{Q}$ such that $|\varphi(i, j, n) - d(\alpha_i, \alpha_j)| < 2^{-n}$ for all i, j, n. Points α_i are called *special points* of X.

Given a computable Polish space $(X, d, (\alpha_i)_{i\in\omega})$, a *Cauchy name* for a point $x \in X$ is a function $f\colon \omega \to \omega$ such that $d(\alpha_{f(n)}, x) < 2^{-n}$ for all n. A point $x \in X$ is *computable* if it has a computable Cauchy name.

Let $(X, d, (\alpha_i)_{i\in\omega})$ and $(Y, d', (\beta_i)_{i\in\omega})$ be computable Polish spaces. A mapping $F\colon X \to Y$ is called *computable* if there is a Turing functional Φ_e such that Φ_e^f is a Cauchy name for $F(x)$ for every $x \in X$ and every Cauchy name f for x.

The above definitions can be transferred to the primitive recursive setting as follows.

Definition 2.1 (Selivanov and Selivanova [SS21])**.** A Polish space $(X, d, (\alpha_i)_{i\in\omega})$ is called *primitive recursive*, or *punctual*, if the distance $d(\alpha_i, \alpha_j)$ between the points of the dense set is uniformly primitive recursive, i.e., there exists a primitive recursive function $f(i, j, k)$ such that for all i, j, k, we have $|d(\alpha_i, \alpha_j) - q_{f(i,j,k)}| \leqslant 2^{-k}$.

For the space to be 'truly' punctual, we can additionally require the distance between α_i and α_j to be non-zero whenever $i \neq j$. Our universality results will not rely on this assumption, however, the counterexample constructed in the last section will have this property.

Let $f, g \in \omega^\omega$. We say that f is *primitively recursively reducible* to g ($f \leq_{pr} g$) if there is a *primitive recursive scheme* (consisting of the basic functions $o(x) = 0$, $I_n^m(x_1, \ldots, x_n) = x_m$, $s(x) = x+1$, the function g and the operators of composition and primitive recursion) that outputs the function f. A computable numbering of all correct primitive recursive schemata yields a computable numbering of *primitive recursive operators* $\Psi\colon \omega^\omega \to \omega^\omega$.

Let $(X, d, (\alpha_i)_{i\in\omega})$ and $(Y, d', (\beta_i)_{i\in\omega})$ be punctual Polish spaces. A mapping $F\colon X \to Y$ is called *primitive recursive* if there is a primitive recursive operator Ψ such that $\Psi(f)$ is a Cauchy name for $F(x)$ whenever f is a Cauchy name for $x \in X$.

It is not hard to see that an isometric mapping $F\colon X \to Y$ is primitive recursive if and only if there is a primitive recursive function h of two variables such that $d'(F(\alpha_i), \beta_{h(i,j)}) \leqslant 2^{-j}$ for all i, j (i.e., we can uniformly primitively recursively compute images $F(\alpha_i)$ of the elements of the dense subspace $(\alpha_i)_{i\in\omega}$). Indeed, a primitive recursive operator Ψ realizing the mapping F gives rise to a uniformly primitive recursive procedure of computation of $F(\alpha_i)$. On the other hand, if there exists a primitive recursive function h as above, then for every Cauchy name f for an element $x \in X$ it holds

$$\begin{aligned} d'(\beta_{h(f(n+1),n+1)}, F(x)) &\leqslant d'(\beta_{h(f(n+1),n+1)}, F(\alpha_{f(n+1)})) + d'(F(\alpha_{f(n+1)}), F(x)) \\ &= d'(\beta_{h(f(n+1),n+1)}, F(\alpha_{f(n+1)})) + d(\alpha_{f(n+1)}, x) \\ &\leqslant 2^{-n-1} + 2^{-n-1} = 2^{-n}, \end{aligned}$$

and $\Psi(f)(n) = h(f(n+1), n+1)$ defines a primitive recursive operator realizing F.

An isometry F of punctual metric spaces X and Y is called *punctual* if F and F^{-1} are primitive recursive. A punctual metric space $(X, d, (\alpha_i)_{i\in\omega})$ is called *punctually categorical* if for any punctual copy $(Y, d', (\beta_i)_{i\in\omega})$ of X there exists a punctual surjective isometry $F\colon X \to Y$.

A *computable topological space* is a quadruple (X, τ, β, ν), where (X, τ) is a T_0-space, β is a base of τ, $\nu\colon \omega \to \beta$ is a numbering of the base β, and there is a c.e. set W such that for any $i, j \in \omega$ it holds $\nu(i) \cap \nu(j) = \bigcup\{\nu(k) \mid \langle i, j, k\rangle \in W\}$.

Let (X, τ, β, ν) be a computable topological space. For a point $x \in X$, its *name* is the set $N^x = \{i \in \omega \mid x \in B_i\}$. For an open set $U \subseteq X$, its *open name* is a set $W \subseteq \omega$ such that $U = \cup_{i\in W} B_i$.

Let X and Y be computable topological spaces. A function $f\colon X \to Y$ is *effectively continuous* if there is an enumeration operator Φ that on input a name of an open set V in Y lists an open name of the set $f^{-1}(V)$ in X (or equivalently, there is an enumeration operator Ψ that given the name of a point $x \in X$, enumerates the name of the point $f(x)$). See, e.g., Section 2.1 in [MM18] for more details about effective continuity.

A computable Polish space $(X, d, (\alpha_i)_{i\in\omega})$ is called *effectively compact* if there is a computable enumeration of all of its finite covers by basic open balls of radius 2^{-n}, uniformly in n.

A *Stone space* (a *profinite space*) is a compact and totally disconnected Hausdorff space. Totally disconnected and compact computable Polish spaces are called *computable Stone spaces*.

A topological space X is *totally separated* if for any two points $a \neq b$ from X, there exist disjoint open sets A and B such that $a \in A$, $b \in B$, and $X = A \cup B$. For compact Hausdorff spaces X, the two notions "totally disconnected" and "totally separated" are equivalent.

3. The space $C[0,1]$

Let $\mathcal{C}$ denote the computable presentation of the space $C[0,1]$ (under the supremum metric) given by the set of piecewise linear functions with finitely many rational breakpoints. In this section we prove that $\mathcal{C}$ is computably and punctually universal. To make the exposition self-contained, we include the original Sierpiński's proof of the Banach–Mazur theorem.

3.1. **Sierpiński's proof.**

Theorem 3.1. *The space $C[0,1]$ of continuous functions $f\colon [0,1] \to \mathbb{R}$, with metric*

$$r(f,g) = \max_{0\le t\le 1} |f(t) - g(t)|$$

is universal.

Proof. Let M be a separable metric space, and let $Q = (p_0, p_1, \dots)$ be a countable dense subset of M.

Let ϱ denote the distance in M and we define

$$\gamma_n(p) = \varrho(p, p_n) - \varrho(p, p_0), \quad \text{for } p \in Q \text{ and } n \in \omega. \tag{1}$$

Applying the triangle inequality allows us to easily obtain

$$-\varrho(p_0, p_n) \le \varrho(p, p_n) - \varrho(p, p_0) \le \varrho(p_0, p_n)$$

and thus

$$|\gamma_n(p)| \le \varrho(p_0, p_n) \quad \text{for } p \in Q \text{ and } n \in \omega. \tag{2}$$

Now let $\varphi : [0,1] \to [-1,1]^{\mathbb{N}}$ be a space filling curve, that is, $\varphi(t) = \langle \varphi_0(t), \varphi_1(t), \dots \rangle$ is a surjective function. Furthermore, since φ is continuous, each projection, φ_n ($n \in \omega$) is also continuous.

Following (2), for any $k \in \omega$, there exists a real number $t_k \in [0,1]$ such that

$$\gamma_n(p_k) = \varrho\left(p_0, p_n\right)\left(\varphi_n\left(t_k\right)\right) \quad \forall n \in \omega \tag{3}$$

because φ is surjective.

Now consider the closure of $T \subseteq [0,1]$, $\overline{T}$, where $T = \{t_0, t_1, \dots\}$. And we define $f_n\colon \overline{T} \to \mathbb{R}$

$$f_n(t) = \varrho\left(p_0, p_n\right)\left(\varphi_n(t)\right). \tag{4}$$

If T is dense in $[0,1]$, then clearly, $f_n \in C[0,1]$. Otherwise, there exist disjoint open intervals, $\{I_\lambda\}_{\lambda\in\Lambda}$ such that $\bigsqcup_{\lambda\in\Lambda} I_\lambda = [0,1] - \overline{T}$. Then we extend the domain of f_n to $[0,1]$ as follows,

$$f_n(t) = \begin{cases} \varrho\left(p_0, p_n\right)\left(\varphi_n(t)\right), & \text{if } t \in \overline{T}, \\ \frac{f_n(b) - f_n(a)}{b-a}(t-a) + f_n(a), & \text{if } t \in I_\lambda \text{ and } I_\lambda = (a,b). \end{cases} \tag{5}$$

It is then clear that f_n as such defined is continuous on $[0,1]$.

We now aim to show that the map $p_n \mapsto f_n$ is isometric, i.e.

$$\varrho\left(p_i, p_k\right) = r\left(f_i, f_k\right) \quad \forall i,k \in \omega$$

By (1) and (4), we have that

$$\varrho\left(p_i, p_k\right) = \gamma_i\left(p_k\right) - \gamma_k\left(p_k\right) = f_i\left(t_k\right) - f_k\left(t_k\right).$$

Since $t_k \in [0,1]$, it thus follows that

$$\varrho\left(p_i, p_k\right) \le r\left(f_i, f_k\right).$$

On the other hand, again from (1) and (4), we can obtain for any $i,j,k \in \omega$

$$f_i\left(t_j\right) - f_k\left(t_j\right) = \gamma_i\left(p_j\right) - \gamma_k\left(p_j\right) = \varrho\left(p_j, p_i\right) - \varrho\left(p_j, p_k\right).$$

By the triangle inequality we have $|\varrho(p_j, p_i) - \varrho(p_j, p_k)| \leq \varrho(p_i, p_k)$, then we can obtain

$$|f_i(t_j) - f_k(t_j)| \leq \varrho(p_i, p_k)$$

and hence for any $t \in T$

$$|f_i(t) - f_k(t)| \leq \varrho(p_i, p_k).$$

Since f_n is continuous for all $n \in \omega$, and T is dense in $\overline{T}$ then we can obtain

$$|f_i(t) - f_k(t)| \leq \varrho(p_i, p_k) \quad \text{for } t \in \overline{T}.$$

And by the definition of f_n (5), any maximum difference must be attained in $\overline{T}$. Thus

$$r(f_i, f_k) \leq \varrho(p_i, p_k).$$

Finally, given any $p \in M$, let $n_0, n_1, \ldots$ be a sequence of natural numbers such that $(p_{n_k})_{k \in \omega} \to p$, where $p_{n_k} \in Q$ for all k. Then we define a map $p \to f^{(p)}$ by

$$f^{(p)}(t) = \lim_{k \to \infty} f_{n_k}(t).$$

Since for any $\varepsilon > 0$ there exists N such that for all $i, k > N$

$$\varrho(p_{n_i}, p_{n_k}) < \varepsilon,$$

then for the same N, we also have that for all $i, k > N$

$$r(f_{n_i}, f_{n_k}) < \varepsilon$$

and thus the sequence $(f_{n_k})_{k \in \omega}$ converges uniformly, which gives that $f^{(p)} \in C[0,1]$. Thus the map $p \mapsto f^{(p)}$ gives us an isometric map from M to a subspace of $C[0,1]$. □

3.2. **The issue with Sierpiński's proof.** The reader should take a few moments and convince themselves that the definition of $\overline{T}$ and, thus, of f_n, does not seem to be computable in the modern sense. Indeed, it is not clear whether the points t_k can be computed at all; recall that the inverse of a space-filling curve cannot be continuous and therefore, in particular, cannot be computable. The best we can hope for is to argue that perhaps $\overline{T}$ is a computable ('decidable') closed set. However, we suspect that it does not have to be a computable closed set in general. Even if $\overline{T}$ was computable (as a closed set), it would not necessarily be clear how exactly we would use its decidability to define f_n. We conjecture that, with a bit of effort, the construction of Sierpiński can likely be made $0'$-computable. We are not sure if Sierpiński's proof, or some insignificant modification of it, can be made computable.

To circumvent these difficulties we will replace the embedding of Sierpiński with a carefully designed approximation argument. We shall define our functions f_n so that we do not have to use t_k. We will, however, use that these t_k *exist* in the verification.

The good news is that at least the space-filling curve is indeed effective in the modern sense, as discussed below.

3.3. **Space-filling curve.** We use Schoenberg's construction here. First define

$$p(t) = \begin{cases} 0, & \text{if } 0 \le t \le \frac{1}{3}, \\ 3t-1, & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ 1, & \text{if } \frac{2}{3} \le t \le 1, \\ p(-t), & \text{if } t < 0, \\ p(t-2), & \text{if } t > 1, \end{cases}$$

and then define

$$\varphi(t) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{p\left(3^{2k}t\right)}{2^k} \quad \text{and} \quad \psi(t) = \frac{1}{2}\sum_{k=0}^{\infty} \frac{p\left(3^{2k+1}t\right)}{2^k}.$$

Let $(t_n)_{n\in\omega}$ be a fast converging sequence to t. Note that p is effectively uniformly continuous. Let $\varepsilon > 0$ be given, then pick $\delta = \frac{\varepsilon}{3}$. It is then easy to check that this δ works for all t. Thus it follows that p is computable; to obtain a name for $p(t)$, take $\varepsilon = 2^{-n}$ for any desired n, then compute $p(t_m)$, where $t_m \to t$ is a name for t and $|t_m - t| < \frac{\varepsilon}{3}$.

Now we want to compute $\varphi(t)$ to any arbitrary error we like, 2^{-n} for some $n \in \omega$. Since $0 \le p(t) \le 1$ for any $t \in \mathbb{R}$, we have that $\varphi(t) \le \frac{1}{2}\sum_{k=0}^{\infty} 2^{-k}$. Let N be such that $\frac{1}{2}\sum_{k=N+1}^{\infty} 2^{-k} < 2^{-2n}$. It is clear that such an N can be found effectively and uniformly in n. Then we compute $\frac{1}{2}\sum_{k=0}^{N} \frac{p\left(3^{2k}t\right)}{2^k}$, where each $p\left(3^{2k}t\right)$ is computed to an accuracy of ε where $\frac{1}{2}\sum_{k=0}^{N} \frac{\varepsilon}{2^k} < 2^{-2n}$. It follows then that $\left|\frac{1}{2}\sum_{k=0}^{N} \frac{p\left(3^{2k}t\right)}{2^k} - \varphi(t)\right| < 2^{-n}$ where each p is only computed to an accuracy of ε. Thus φ is also computable, and using a similar argument, ψ can also be shown to be computable. For the proof of surjectivity of $f\colon [0,1] \to [0,1]^2$ where $f(t) = \langle \varphi(t), \psi(t)\rangle$ refer to [Sag94].

To extend this idea to a $\aleph_0$-dimensional space filling curve, we define the following

$$\varphi_n(t) = \sum_{k=1}^{\infty} \frac{p\left(3^{2^{n-1}(2k-1)-1}t\right)}{2^k}.$$

A straightforward modification of the previous argument allows us to show that $\varphi_n(t)$ is computable uniformly in n, and thus $f = \langle \varphi_0, \varphi_1, \dots\rangle$ gives a computable space filling curve. For surjectivity of f, we refer again to [Sag94].

3.4. **Punctual universality.** Now we proceed to the proof of Theorem 1.2 restated below.

Theorem 1.2. *Given any punctual Polish space $\mathcal{M}$, there is a primitive recursive isometry from $\mathcal{M}$ onto a subset of $\mathcal{C}$.*

Proof of Theorem 1.2. Let $\mathcal{M} = (M, d, (\alpha_i)_{i\in\omega})$ be a punctual space. Then for all $i, j \in \omega$, $d(\alpha_i, \alpha_j)$ can be primitively recursively computed to any degree of accuracy ε. We denote this by $d^\varepsilon(\alpha_i, \alpha_j)$, and further note that this is a rational. The goal is then to construct a sequence $\{f_i\}_{i\in\omega}$ of primitive recursive points in $\mathcal{C}$ such that for all $i, j \in \omega$, it holds $d(\alpha_i, \alpha_j) = \max|f_i - f_j|$.

Definition 3.1. Given n, ε and a sequence $\langle z_0, z_1, \dots, z_n\rangle$, where $z_i \in [-1,1]$, we say that the sequence is *n, ε-correct* if $\forall i, j \leq n$, we have

$$|z_i d^\varepsilon(\alpha_i, \alpha_0) - z_j d^\varepsilon(\alpha_j, \alpha_0)| \leq d^\varepsilon(\alpha_i, \alpha_j) + 3\varepsilon.$$

When $\varepsilon = 0$, we simply say the sequence is n-correct.

Lemma 3.2. *Suppose that $\langle z_0, z_1, \dots, z_n\rangle$ is n, ε-correct. Then $\exists x \in [-1,1]$ such that $\langle z_0, z_1, \dots, z_n, x\rangle$ is $n+1, \varepsilon$-correct.*

Proof. If $n = 0$, then we simply take $x = 1$ so we assume $n > 0$. Let N' be s.t. $z_{N'} = \min\{z_1, z_2, \dots, z_n\}$. For each $i \leq n$ we denote $\gamma_i = z_i d^\varepsilon(\alpha_i, \alpha_0) - d^\varepsilon(\alpha_i, \alpha_{n+1})$. Let N be s.t. $\gamma_N = \max\{\gamma_1, \gamma_2, \dots, \gamma_n\}$. It follows that $z_N d^\varepsilon(\alpha_{n+1}, \alpha_0) + 3\varepsilon \geq \gamma_N$. By triangle inequality we have $d^\varepsilon(\alpha_i, \alpha_0) - d^\varepsilon(\alpha_{n+1}, \alpha_0) \leq d^\varepsilon(\alpha_i, \alpha_{n+1}) + 3\varepsilon$. Then if $0 \leq z_N \leq 1$, $z_N d^\varepsilon(\alpha_i, \alpha_0) - z_N d^\varepsilon(\alpha_{n+1}, \alpha_0) \leq d^\varepsilon(\alpha_i, \alpha_{n+1}) + 3\varepsilon$. Then we obtain $\gamma_N \leq z_N d^\varepsilon(\alpha_{n+1}, \alpha_0) + 3\varepsilon$. If $-1 \leq z_N < 0$, then $z_N d^\varepsilon(\alpha_{n+1}, \alpha_0) \geq z_N d^\varepsilon(\alpha_i, \alpha_0) + z_N d^\varepsilon(\alpha_i, \alpha_{n+1}) + 3 z_N \varepsilon \geq z_N d^\varepsilon(\alpha_i, \alpha_0) - d^\varepsilon(\alpha_i, \alpha_{n+1}) + 3 z_N \varepsilon$ that is $z_N d^\varepsilon(\alpha_{n+1}, \alpha_0) + 3|z_N|\varepsilon \geq z_N d^\varepsilon(\alpha_i, \alpha_0) - d^\varepsilon(\alpha_i, \alpha_{n+1})$, which gives $z_N d^\varepsilon(\alpha_{n+1}, \alpha_0) + 3\varepsilon \geq \gamma_N$.

Now we take $x = \max\left\{\frac{\gamma_N - 3\varepsilon}{d^\varepsilon(\alpha_{n+1}, \alpha_0)}, z_{N'}\right\}$. Since $z_N d^\varepsilon(\alpha_{n+1}, \alpha_0) \geq \gamma_N - 3\varepsilon$, we obviously have that $x \in [-1,1]$. Now we verify that $\langle z_0, z_1, \dots, z_n, x\rangle$ is $n+1, \varepsilon$-correct. Fix $1 \leq i \leq n$, ($i = 0$ is trivial). Then

$$\begin{aligned}
x d^\varepsilon(\alpha_{n+1}, \alpha_0) - z_i d^\varepsilon(\alpha_i, \alpha_0) &\geq \gamma_N - 3\varepsilon - z_i d^\varepsilon(\alpha_i, \alpha_0) \\
&\geq \gamma_i - 3\varepsilon - z_i d^\varepsilon(\alpha_i, \alpha_0) \\
&\geq -d^\varepsilon(\alpha_i, \alpha_{n+1}) - 3\varepsilon.
\end{aligned}$$

Now if $x = \frac{\gamma_N - 3\varepsilon}{d^\varepsilon(\alpha_{n+1}, \alpha_0)}$, then

$$\begin{aligned}
x d^\varepsilon(\alpha_{n+1}, \alpha_0) - z_i d^\varepsilon(\alpha_i, \alpha_0) &= \gamma_N - 3\varepsilon - z_i d^\varepsilon(\alpha_i, \alpha_0) \\
&= z_N d^\varepsilon(\alpha_N, \alpha_0) - d^\varepsilon(\alpha_N, \alpha_{n+1}) - 3\varepsilon - z_i d^\varepsilon(\alpha_i, \alpha_0) \\
&\leq d^\varepsilon(\alpha_N, \alpha_i) - d^\varepsilon(\alpha_N, \alpha_{n+1}) \quad \text{(by } n, \varepsilon\text{-correctness)} \\
&\leq d^\varepsilon(\alpha_{n+1}, \alpha_i) + 3\varepsilon \quad \text{(by triangle inequality)},
\end{aligned}$$

and if $x = z_{N'}$, then

$$\begin{aligned}
x d^\varepsilon(\alpha_{n+1}, \alpha_0) - z_i d^\varepsilon(\alpha_i, \alpha_0) &= z_{N'} d^\varepsilon(\alpha_{n+1}, \alpha_0) - z_i d^\varepsilon(\alpha_i, \alpha_0) \\
&\leq z_i d^\varepsilon(\alpha_{n+1}, \alpha_0) - z_i d^\varepsilon(\alpha_i, \alpha_0) \\
&\leq d^\varepsilon(\alpha_{n+1}, \alpha_i) + 3\varepsilon \quad \text{(by triangle inequality)}.
\end{aligned}$$

□

Lemma 3.3. *Suppose that $\langle z_0, z_1, \dots, z_n, z_{n+1}\rangle$ is an $n+1, \varepsilon$-correct sequence, and $\langle z_0 + \varepsilon_0, z_1 + \varepsilon_1, \dots, z_n + \varepsilon_n\rangle$ is n, ε-correct for some $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n \in [-1,1]$. Then $\langle z_0 + \varepsilon_0, z_1 + \varepsilon_1, \dots, z_n + \varepsilon_n, z_{n+1} + \varepsilon_{n+1}\rangle$ is $n+1, \varepsilon$-correct for some ε_{n+1} s.t. $|\varepsilon_{n+1}| \leq \max_{1 \leq i \leq n}\left\{\frac{d^\varepsilon(\alpha_i, \alpha_0) - 3\varepsilon}{d^\varepsilon(\alpha_{n+1}, \alpha_0)}|\varepsilon_i|\right\}$.*

Proof. For each i, let $\hat{z}_i = z_i d^\varepsilon(\alpha_i, \alpha_0)$ and $\hat{\varepsilon}_i = \varepsilon_i d^\varepsilon(\alpha_i, \alpha_0)$. If for every $1 \leq i \leq n$ we have $|\hat{z}_i + \hat{\varepsilon}_i - \hat{z}_{n+1}| \leq d^\varepsilon(\alpha_i, \alpha_{n+1}) + 3\varepsilon$, we can just take $\varepsilon_{n+1} = 0$. Therefore, assume that this is not the case and let k be s.t. $|\hat{z}_k + \hat{\varepsilon}_k - \hat{z}_{n+1}| - d^\varepsilon(\alpha_k, \alpha_{n+1}) - 3\varepsilon$ is the largest.

First suppose that $\hat{z}_k + \hat{\varepsilon}_k > \hat{z}_{n+1}$. Let $\varepsilon_{n+1} = \frac{|\hat{z}_k+\hat{\varepsilon}_k-\hat{z}_{n+1}|-d^\varepsilon(\alpha_k,\alpha_{n+1})-3\varepsilon}{d^\varepsilon(\alpha_{n+1},\alpha_0)} > 0$. We check that $\langle z_0+\varepsilon_0, z_1+\varepsilon_1, \ldots, z_n+\varepsilon_n, z_{n+1}+\varepsilon_{n+1}\rangle$ is $n+1, \varepsilon$-correct. Fix $1 \le i \le n$. If $\hat{z}_{n+1}+\hat{\varepsilon}_{n+1} \le \hat{z}_i + \hat{\varepsilon}_i$, then

$$\begin{aligned}
|(\hat{z}_i+\hat{\varepsilon}_i)-(\hat{z}_{n+1}+\hat{\varepsilon}_{n+1})| &= \hat{z}_i+\hat{\varepsilon}_i-\hat{z}_{n+1}-\hat{\varepsilon}_{n+1}\\
&\le |\hat{z}_i+\hat{\varepsilon}_i-\hat{z}_{n+1}|-\hat{\varepsilon}_{n+1}\\
&= |\hat{z}_i+\hat{\varepsilon}_i-\hat{z}_{n+1}|-(|\hat{z}_k+\hat{\varepsilon}_k+\hat{z}_{n+1}|-d^\varepsilon(\alpha_k,\alpha_{n+1})-3\varepsilon)\\
&\text{(by maximal property of } k)\\
&\le |\hat{z}_i+\hat{\varepsilon}_i-\hat{z}_{n+1}|-(|\hat{z}_i+\hat{\varepsilon}_i+\hat{z}_{n+1}|-d^\varepsilon(\alpha_i,\alpha_{n+1})-3\varepsilon)\\
&= d^\varepsilon(\alpha_i,\alpha_{n+1})+3\varepsilon.
\end{aligned}$$

On the other hand, if $\hat{z}_i+\hat{\varepsilon}_i < \hat{z}_{n+1}+\hat{\varepsilon}_{n+1}$ we have

$$\begin{aligned}
|(\hat{z}_{n+1}+\hat{\varepsilon}_{n+1})-(\hat{z}_i+\hat{\varepsilon}_i)| &= [(\hat{z}_k+\hat{\varepsilon}_k)-(\hat{z}_i+\hat{\varepsilon}_i)]-[(\hat{z}_k+\hat{\varepsilon}_k)-(\hat{z}_{n+1}+\hat{\varepsilon}_{n+1})]\\
&\text{(by correctness)}\\
&\le d^\varepsilon(\alpha_k,\alpha_i)+3\varepsilon-((\hat{z}_k+\hat{\varepsilon}_k)-(\hat{z}_{n+1}+\hat{\varepsilon}_{n+1}))\\
&\text{by } (\hat{z}_k+\hat{\varepsilon}_k-\hat{z}_{n+1}>0)\\
&= d^\varepsilon(\alpha_k,\alpha_i)+3\varepsilon-|\hat{z}_k+\hat{\varepsilon}_k-\hat{z}_{n+1}|+\hat{\varepsilon}_{n+1}\\
&\text{(by definition of } \hat{\varepsilon}_{n+1})\\
&= d^\varepsilon(\alpha_k,\alpha_i)-d^\varepsilon(\alpha_k,\alpha_{n+1})\\
&\text{(by triangle inequality)}\\
&\le d^\varepsilon(\alpha_{n+1},\alpha_i)+3\varepsilon.
\end{aligned}$$

For the symmetric case where $\hat{z}_k+\hat{\varepsilon}_k \le \hat{z}_{n+1}$, take

$$\varepsilon_{n+1} = -\frac{|\hat{z}_k+\hat{\varepsilon}_k-\hat{z}_{n+1}|-d^\varepsilon(\alpha_k,\alpha_{n+1})-3\varepsilon}{d^\varepsilon(\alpha_{n+1},\alpha_0)} < 0.$$

A similar argument applies, by switching the arguments for the subcases $\hat{z}_{n+1}+\hat{\varepsilon}_{n+1} \le \hat{z}_i+\hat{\varepsilon}_i$ and $\hat{z}_i+\hat{\varepsilon}_i \le \hat{z}_{n+1}+\hat{\varepsilon}_{n+1}$. The bound follows from the choice of ε_{n+1}. □

Lemma 3.4. *Suppose that $\langle z_0, z_1, \ldots, z_n\rangle$ is a n, ε-correct sequence, then the set of all x such that $\langle z_0, z_1, \ldots, z_n, x\rangle$ is $n+1, \varepsilon$-correct is a non-empty closed interval.*

Proof. Let $I = \{x \in [-1,1] \mid \langle z_0, z_1, \ldots, z_n, x\rangle$ is $n+1, \varepsilon$-correct$\}$. By Lemma 3.2, $I \ne \emptyset$. I is clearly closed, by the definition of correctness. Now fix some $x, y \in I$ and let w be s.t. $x < w < y$. Fix some $1 \le i \le n$. If $z_i d^\varepsilon(\alpha_i,\alpha_0) \le w d^\varepsilon(\alpha_{n+1},\alpha_0)$, then

$$\begin{aligned}
|wd^\varepsilon(\alpha_{n+1},\alpha_0)-z_i d^\varepsilon(\alpha_i,\alpha_0)| &= wd^\varepsilon(\alpha_{n+1},\alpha_0)-z_i d^\varepsilon(\alpha_i,\alpha_0)\\
&< yd^\varepsilon(\alpha_{n+1},\alpha_0)-z_i d^\varepsilon(\alpha_i,\alpha_0)\\
&\le d^\varepsilon(\alpha_i,\alpha_{n+1})+3\varepsilon \quad \text{(by correctness of } y).
\end{aligned}$$

On the other hand, if $z_i d^\varepsilon(\alpha_i,\alpha_0) > w d^\varepsilon(\alpha_{n+1},\alpha_0)$, then we use x instead of y. □

In view of Lemma 3.4, given an n, ε-correct sequence $\langle z_0, z_1, \ldots, z_n\rangle$, let $U^\varepsilon\langle z_0, z_1, \ldots, z_n\rangle$ and $L^\varepsilon\langle z_0, z_1, \ldots, z_n\rangle$ denote the largest and smallest $x \in [-1,1]$ such that $\langle z_0, z_1, \ldots, z_n, x\rangle$ is $n+1, \varepsilon$-correct respectively.

Lemma 3.5. *Suppose that* $\langle z_0, z_1, \dots, z_n\rangle$ *and* $\langle z_0+\varepsilon_0, z_1+\varepsilon_1, \dots, z_n+\varepsilon_n\rangle$ *are both* n,ε*-correct and* $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_n \in [-1,1]$*. Then both* $|U^\varepsilon\langle z_0, z_1, \dots, z_n\rangle - U^\varepsilon\langle z_0+\varepsilon_0, z_1+\varepsilon_1, \dots, z_n+\varepsilon_n\rangle|$ *and* $|L^\varepsilon\langle z_0, z_1, \dots, z_n\rangle - L^\varepsilon\langle z_0+\varepsilon_0, z_1+\varepsilon_1, \dots, z_n+\varepsilon_n\rangle|$ *are bounded above by* $\max_{1\le i\le n}\left\{\frac{d^\varepsilon(\alpha_i,\alpha_0)-3\varepsilon}{d^\varepsilon(\alpha_{n+1},\alpha_0)}|\varepsilon_i|\right\}$.

Proof. Apply Lemma 3.3 to conclude that $\langle z_0+\varepsilon_0, z_1+\varepsilon_1, \dots, z_n+\varepsilon_n, x\rangle$ is $n+1,\varepsilon$-correct, where $x = U^\varepsilon\langle z_0, z_1, \dots, z_n\rangle + \varepsilon_{n+1}$ and

$$|\varepsilon_{n+1}| \le \max_{1\le i\le n}\left\{\frac{d^\varepsilon(\alpha_i,\alpha_0)-3\varepsilon}{d^\varepsilon(\alpha_{n+1},\alpha_0)}|\varepsilon_i|\right\}.$$

This shows that

$$|U^\varepsilon\langle z_0, z_1, \dots, z_n\rangle - U^\varepsilon\langle z_0+\varepsilon_0, z_1+\varepsilon_1, \dots, z_n+\varepsilon_n\rangle| \le \max_{1\le i\le n}\left\{\frac{d^\varepsilon(\alpha_i,\alpha_0)-3\varepsilon}{d^\varepsilon(\alpha_{n+1},\alpha_0)}|\varepsilon_i|\right\}.$$

The same argument can be repeated for L. □

Now we define the sequence $\{f_i\}_{i\in\omega}$. Let $\varphi : [0,1] \to [-1,1]^{\mathbb{N}}$ be a primitive recursive surjection. Refer to Subsection 3.3 for a primitive recursive version of φ. For each $t \in [0,1]$, let $\varphi(t) = \langle\varphi_i(t)\rangle_{i\in\mathbb{N}}$ and $\langle\varphi_i(t)\rangle$ is a uniformly computable sequence of primitive recursive functions mapping $[0,1] \to [-1,1]^{\mathbb{N}}$.

Let $f_0(t) = 0$ for all $t \in [0,1]$. For each $t \in [0,1]$ we define $f_1(t), f_2(t), \dots$ inductively. Assume that $f_0(t), f_1(t), \dots, f_n(t)$ have all been defined, with the property that $f_i(t) = z_i d^\varepsilon(\alpha_i, \alpha_0)$ for some $z_i \in [-1,1]$ and $\langle 0, z_1, \dots, z_n\rangle$ is n-correct. Then we define

$$f_{n+1}(t) = \begin{cases} L\langle 0, z_1, z_2, \dots, z_n\rangle d(\alpha_{n+1},\alpha_0), & \text{if } \varphi_{n+1}(t) < L\langle 0, z_1, z_2, \dots, z_n\rangle, \\ U\langle 0, z_1, z_2, \dots, z_n\rangle d(\alpha_{n+1},\alpha_0), & \text{if } \varphi_{n+1}(t) > U\langle 0, z_1, z_2, \dots, z_n\rangle, \\ \varphi_{n+1}(t) d(\alpha_{n+1},\alpha_0), & \text{otherwise.} \end{cases}$$

Then we will have that $\left\langle 0, z_1, z_2, \dots, z_n, \frac{f_{n+1}(t)}{d(\alpha_{n+1},\alpha_0)}\right\rangle$ is $n+1$-correct for any $t \in [0,1]$. This defines a sequence of total functions $\{f_i\}_{i\in\omega}$. Furthermore, by Lemma 3.5 it is easy to see that each f_n is continuous.

Now we analyze the effectivity of our definitions. Checking for n,ε-correctness is clearly primitive recursive since it only requires comparing rationals.

Lemma 3.6. *Suppose that* $\langle z_0, z_1, \dots, z_n\rangle$ *is* n,ε*-correct and* $z_i \in \mathbb{Q}\cap[-1,1]$ *for each* $0 \le i \le n$*, then* $U^\varepsilon\langle z_0, z_1, \dots, z_n\rangle$ *and* $L^\varepsilon\langle z_0, z_1, \dots, z_n\rangle$ *are primitively recursively computable reals uniformly in* $\langle z_0, z_1, \dots, z_n\rangle$*.*

Proof. Let $U = U^\varepsilon\langle z_0, z_1, \dots, z_n\rangle$ and $L = L^\varepsilon\langle z_0, z_1, \dots, z_n\rangle$. Suppose that we want to compute U to an accuracy of 2^{-n}, then by Lemma 3.2 we can find some x s.t. $\langle z_0, z_1, \dots, z_n, x\rangle$ is $n+1,\varepsilon$-correct. Note that such an $x \in \mathbb{Q}$ by the proof of the lemma, then we compute $x + m2^{-n}$ for $m \in \mathbb{N}$ until we find $x + m2^{-n}$ that is $n+1,\varepsilon$-correct and $x + (m+1)2^{-n}$ that is not. This search is clearly bounded since $m \le 2^n(1-x)$. If no such m is found, then it must be that $U = 1$. L can be computed in the same way by considering $x - m2^{-n}$ instead, and again m must be bounded by $2^n(x+1)$. If no such m is found, then $L = -1$. □

Lemma 3.7. $\{f_n\}_{n\in\omega}$ *is a uniformly primitive recursive sequence of elements in* $\mathcal{C}$*.*

Proof. We produce a primitive recursive function H s.t. for every $i, n \in \omega$ and every $q \in \mathbb{Q} \cap [0,1]$, we have that $|H(n,i,q) - f_n(q)| < 2^{-i}$. Fix $q \in \mathbb{Q} \cap [0,1]$. For $n = 0$ we can just take $H(n,i,q) = 0$ for all i. Suppose that $H(m,i,q)$ is defined for all i and $m \leq n$. Again let $f_m(q) = z_m d(\alpha_m, \alpha_0)$ for $1 \leq m \leq n$. Using the values of $H(m,i,q)$, we are able to estimate $z_1, \dots, z_n$ to any precision, ε we want. Let $y_i \in \mathbb{Q} \cap [-1,1]$ be the approximations of z_i as given by $H(m,i,q)$. By Lemma 3.2, we can primitively recursively find an $x \in \mathbb{Q} \cap [-1,1]$ s.t. $\langle 0, y_1, \dots, y_n, x\rangle$ is $n+1, \varepsilon$-correct. Then by Lemma 3.6, we are able to approximate $U^\varepsilon\langle 0, y_1, y_2, \dots, y_n\rangle, L^\varepsilon\langle 0, y_1, y_2, \dots, y_n\rangle$ up to any precision we desire, and since each y_i is close to z_i, by Lemma 3.5, this is also close to $U^\varepsilon\langle 0, z_1, z_2, \dots, z_n\rangle$ and $L^\varepsilon\langle 0, z_1, z_2, \dots, z_n\rangle$. From the discussion in Subsection 3.3, we are also able to primitively recursively approximate φ_{n+1}, then we define

$$H(n+1, i, q) = \begin{cases} L^\varepsilon\langle 0, y_1, y_2, \dots, y_n\rangle d^\varepsilon(\alpha_{n+1}, \alpha_0), & \text{if } \varphi^\varepsilon_{n+1}(q) < L^\varepsilon\langle 0, y_1, y_2, \dots, y_n\rangle, \\ U^\varepsilon\langle 0, y_1, y_2, \dots, y_n\rangle d^\varepsilon(\alpha_{n+1}, \alpha_0), & \text{if } \varphi^\varepsilon_{n+1}(q) > U^\varepsilon\langle 0, y_1, y_2, \dots, y_n\rangle, \\ \varphi^\varepsilon_{n+1}(q) d^\varepsilon(\alpha_{n+1}, \alpha_0), & \text{otherwise,} \end{cases}$$

where ε can be chosen appropriately to obtain $|H(n+1, i, q) - f_{n+1}(q)| < 2^{-i}$. Now we want to check that f_n is effectively (in n) uniformly continuous. We find a primitive recursive function $m(n, \varepsilon')$ s.t. for each n and ε', if $|t - t'| < m(n, \varepsilon')$ then $|f_n(t) - f_n(t')| < \varepsilon$. If $n = 0$, then simply take $m(0, \varepsilon') = \varepsilon'$. Suppose then that $m(i, \varepsilon')$ is already defined for $i \leq n$, then we can compute $m(n+1, \varepsilon')$ primitively recursively by a reasoning similar to above and using the fact that in Lemma 3.5 the difference between $|U^\varepsilon\langle z_0, z_1, \dots, z_n\rangle - U^\varepsilon\langle z_0 + \varepsilon_0, z_1 + \varepsilon_1, \dots, z_n + \varepsilon_n\rangle|$ is bounded only by the various ε_i and $d^\varepsilon(\alpha_i, \alpha_0)$. It then follows that each f_n is a primitive recursive element of $\mathcal{C}$. □

Lemma 3.8. *For each i, j, $d(\alpha_i, \alpha_j) = \max|f_i - f_j|$.*

Proof. Fix i, j and $t \in [0,1]$. By the definition of the functions $\{f_n\}_{n\in\omega}$, we have $|f_i(t) - f_j(t)| \leq d(\alpha_i, \alpha_j)$. Now, for each k, n, note that $|d(\alpha_k, \alpha_n) - d(\alpha_k, \alpha_0)| \leq d(\alpha_n, \alpha_0)$. By the surjectiveness of φ, for each k, there is some $t_k \in [0,1]$ s.t. for every n, we have $\varphi_n(t_k) d(\alpha_n, \alpha_0) = d(\alpha_k - \alpha_n) - d(\alpha_k, \alpha_0)$. ($t_k$ cannot be found effectively, of course). Now for each k and n, the initial segment $\langle \varphi_0(t_k), \varphi_1(t_k), \varphi_2(t_k), \dots\rangle$ of length n is n-correct, and therefore, $f_n(t_k) = \varphi_n(t_k) d(\alpha_n, \alpha_0)$ for every n.

Now given any i, k we see that

$$\begin{aligned} |f_i(t_k) - f_k(t_k)| &= |\varphi_i(t_k) d(\alpha_i, \alpha_0) - \varphi_k(t_k) d(\alpha_k, \alpha_0)| \\ &= |d(\alpha_k, \alpha_i) - d(\alpha_k, \alpha_0) + d(\alpha_k, \alpha_0)| \\ &= d(\alpha_k, \alpha_i). \end{aligned}$$

Thus for every i, j, $\max|f_i - f_j| = d(\alpha_i, \alpha_j)$. □

This concludes the proof of Theorem 1.2. □

4. The Urysohn space

The Urysohn space $\mathbb{U}$, constructed in [Ury27], is the unique up to isometry universal and ultrahomogeneous Polish space, where the property of ultrahomogeneity states that any isometry of two finite subspaces of $\mathbb{U}$ extends to an isometry of $\mathbb{U}$ onto itself. Urysohn's original construction defines $\mathbb{U}$ as the completion of the *rational Urysohn space,* universal for the class of all countable rational-valued metric spaces. To prove that $\mathbb{U}$ with the rational Urysohn space as the dense subset is computably and punctually universal, we recall Urysohn's original proof (which we claim is inherently highly constructive) and make some necessary adjustments so as to make the proof primitive recursive. We will mainly follow the paper [Ury51] which is the Russian version of [Ury27] published in a collection of works of Urysohn. Main definitions and notations we need can be found in the paper [Mel08]; below we recall some of them.

Definition 4.1. Let (X, d) be a metric space. A mapping $f\colon X \to \mathbb{R}^+$ is a *Katětov map* if for all $x, y \in X$ the following holds.

$$|f(x) - f(y)| \leqslant d(x, y) \leqslant f(x) + f(y).$$

Katětov maps over X correspond to one-point extensions of X. Given a Katětov map over X, we can associate it with a point z extending the metric space (X, d) by defining $d(x, z) = f(x)$ for each $x \in X$. It is evident that $d(x, z) > 0$ for any $x \in X$ and the triangle inequality still holds in $(X \cup \{z\}, d)$ because $d(x, y) \leq f(x) + f(y)$. The set of all Katětov maps over X is denoted by $E(X)$. Endowed with the sup-metric, $E(X)$ forms a complete metric space.

If Y is a subspace of X and $f \in E(Y)$, the *Katětov extension* of f is the mapping $\hat{f}\colon X \to \mathbb{R}^+$ defined by $\hat{f}(x) = \inf_{y \in Y}(f(y) + d(x, y))$. It is easy to see that $\hat{f} \in E(X)$.

A metric space (X, d) has the *approximate extension property* if for any finite subspace $A \subseteq X$, any $f \in E(A)$, and any $\varepsilon > 0$, there exists a point $z \in X$ satisfying

$$\forall a \in A \ |d(z, a) - f(a)| \leqslant \varepsilon.$$

When ε is changed to 0 in the above definition, we say that (X, d) has the *extension property*.

It is well-known that for Polish metric spaces both of these properties are equivalent to isometricity to the Urysohn space.

4.1. **Urysohn's construction of $\mathbb{U}$.** As mentioned above, $\mathbb{U}$ is the completion of the rational Urysohn space $\mathbb{QU} = (a_n)_{n \in \omega}$, which is constructed in [Ury51] as follows. Fix a primitive recursive numbering of the set of all nonempty finite sequences of positive rational numbers $(Q_n)_{n \in \omega}$ such that the length of the sequence Q_n does not exceed n. Define the distances $d(a_i, a_j)$ between $a_i, a_j \in \mathbb{QU}$ in stages.

Stage 0. Let $d(a_0, a_0) = 0$.

Stage $n + 1$. Suppose that $d(a_i, a_k)$ have been defined for all $i, k \leqslant n$. Set $d(a_{n+1}, a_{n+1}) = 0$ and consider the sequence Q_n. Let $Q_n = (q_0, \ldots, q_s)$. There are two possibilities:

Case 1. For the mapping $f(a_i) = q_i$, observe that $f \notin E(\{a_0, \ldots, a_s\})$, i.e., we cannot extend the space $\{a_0, \ldots, a_s\}$ by setting the distances via the sequence Q_n. Then Q_n is called *incorrect* and for all $j \leqslant n$ we let

$$d(a_{n+1}, a_j) = \max_{i,k \leqslant n} d(a_i, a_k).$$

Case 2. Mapping $f(a_i) = q_i$ is a Katětov map over $\{a_0, \ldots, a_s\}$. Then we call the sequence Q_n *correct* and let

$$d(a_{n+1}, a_j) = \hat{f}(a_j) \text{ for } j \leqslant n.$$

It is clear that d is a metric. It is also easy to see that the triple $\mathcal{U} = (\mathbb{U}, d, \mathbb{QU})$ is a primitive recursive metric space. We call this space the *standard copy* of $\mathbb{U}$.

The following property of $\mathbb{QU}$ that implies its universality for all countable rational-valued spaces is easily verified.

Theorem 4.1 (Rational extension property of $\mathbb{QU}$, [Ury51])**.** *For any finite subspace $A \subseteq \mathbb{QU}$ and any rational-valued Katětov map $f \in E(A)$, there is a point $b \in \mathbb{QU}$ realizing f, i.e., $d(a,b) = f(a)$ for all $a \in A$.*

4.2. **Punctual universality of $\mathbb{U}$.** We now prove the primitive recursive version of the universality of the Urysohn space restated below.

Theorem 1.4. *The presentation $\mathcal{U}$ of the Urysohn space constructed by Urysohn is punctually universal, in the sense that every punctual Polish space can be punctually isometrically embedded into $\mathcal{U}$.*

This fact readily follows from Theorem 1.2: we only need to punctually embed $(C[0,1], d, (\alpha_i)_{i\in\omega})$ into $\mathcal{U}$, where $(\alpha_i)_{i\in\omega}$ is the family of all piecewise linear functions with rational breakpoints. Let $k_0 = 0$.

Stage 0. Let $g(\alpha_0) = a_{k_0} \in \mathbb{QU}$.

Stage $n+1$. Suppose that the images $g(\alpha_0) = a_{k_0}, \dots, g(\alpha_n) = a_{k_n}$ have been defined. Mapping f defined by $f(a_{k_i}) = d(\alpha_i, \alpha_{n+1})$ is a rational-valued Katětov map over $\{a_{k_0}, \dots, a_{k_n}\}$. Let $M = \max\{k_0, \dots, k_n\}$. Then $\hat{f}$, the Katětov extension of f to the whole initial segment $\{a_0, a_1, \dots, a_M\}$, can be written as a rational sequence $Q = (\hat{f}(a_0), \dots, \hat{f}(a_M))$. Primitively recursively find the number $s \geqslant n$ of this sequence in $\mathbb{QU}$, then we have $d(a_s, a_i) = \hat{f}(a_i)$, in particular, $d(a_s, a_{k_i}) = f(a_{k_i}) = d(\alpha_{n+1}, \alpha_i)$, $i \leqslant n$. Let $g(\alpha_{n+1}) = a_s$.

This proof essentially repeats the proof of the rational extension property of $\mathbb{QU}$ (cf. [Ury51]). We use the fact that the distances between piecewise linear functions $\alpha_i \in C[0,1]$ are rational and can be determined *exactly.* Nonetheless we will also provide a more direct proof of Theorem 1.4 by modifying Urysohn's proof of universality of $\mathbb{U}$. In 2005, Kamo [Kam05, Kam06] used a similar modification of the mentioned proof to show that $\mathcal{U}$ is computably universal, that is, every computable metric space computably embeds into it.

The more direct proof of Theorem 1.4 is given in Appendix A.

4.3. **Punctual non-categoricity of $\mathbb{U}$.** Recall that a structure is categorical if there is only one presentation of it up to isomorphism. In the sense of primitive recursive presentations of metric spaces, we prove that up to punctual isometry (isometries that are primitive recursive with a primitive recursive inverse), the Urysohn space is not categorical.

Theorem 1.3. *Urysohn space is not punctually categorical. Moreover, for every punctual presentation $\mathcal{U}_1 = (\mathbb{U}, d, \mathbb{QU}_1)$ of $\mathbb{U}$ there exists a punctual presentation $\mathcal{U}_2 = (\mathbb{U}, d, \mathbb{QU}_2)$ of $\mathbb{U}$ such that there is no primitive recursive isometry $F\colon \mathcal{U}_1 \to \mathcal{U}_2$.*

Proof of Theorem 1.3. Fix a computable list $(p_e)_{e\in\omega}$ of all binary primitive recursive functions. We will satisfy a series of requirements

R_e : p_e does not induce a primitive recursive isometry F between $\mathcal{U}_1$ and $\mathcal{U}_2$ (to be elaborated below).

Construct $\mathbb{QU}_2 = \{b_i\}_{i\in\omega}$ exactly as the standard copy, interrupting occasionally to meet the next requirement of our list. Suppose we first start working with requirement R_e as stage s. Starting from this stage, we put the process of copying

$\mathbb{QU}$ into $\mathbb{QU}_2$ on hold and set the distances between new points $b_s, b_{s+1}, \dots$ added to $\mathbb{QU}_2$ as follows:

$$d(b_i, b_j) = M = \max_{k,l<s} d(b_k, b_l) \text{ for } i \geqslant s \text{ and } j < i,$$

as in case 1 from the construction of the standard copy. This way, from now on, the maximal distance between points of $\mathcal{U}_2$ does not grow. Continue this until the copy $\mathbb{QU}_1$ shows us points a_k, a_l such that $d(a_k, a_l) > M + 2$. Compute p_e-images of these points with precision 1, i.e. compute $p_e(a_k, 0)$, $p_e(a_l, 0)$. As soon as these computations halt, we win, because we see that p_e does not preserve distances. Resume the copying process and then turn to the next requirement. □

5. Cantor space

Cantor space is the set $\mathcal{C} = 2^\omega$ of all infinite binary strings endowed with the metric $d_C(\alpha, \beta) = 2^{-k}$, where k is the least i such that $\alpha(i) \neq \beta(i)$.

This representation of Cantor space can be viewed as a full binary tree. Topology $\tau_{\mathcal{C}}$ induced by the metric d_C has the base consisting of clopen sets $\mathcal{C}_\sigma = \{\alpha \in 2^\omega : \sigma \subset \alpha\}$ for arbitrary finite binary strings $\sigma \in 2^{<\omega}$. It is easy to see that Cantor space is a punctual Stone space.

Theorem 1.5. *Every punctual, effectively compact (in the Turing sense) Stone space X is primitively recursively homeomorphically embeddable into Cantor space.*

The Idea behind the formal proof below is as follows. Using effective compactness and known techniques developed in [HKS23, HTMN20, DM23, BHTM23], we can computably list all clopen splits of the given punctual space. We will use this enumeration to produce a *primitive recursive* binary tree T with no dead ends such that the space X is primitively recursively homeomorphic to the set of paths $[T]$ through this tree. The enumeration of clopen splits will not be punctual, however, we will produce a primitive recursive T by simply extending its branches one level further while we wait for the next split to occur. We will then use primitive recursiveness of the distances between points to punctually decide, for any point x, in which of the finitely many clopen sets built so far x is located. Thus, the natural homeomorphism from the space to $[T]$ will be primitive recursive. (We however conjecture that its inverse will not be primitive recursive, in general.) Since the tree T can be interpreted as a subset of $2^{<\omega}$, the result will follow.

As mentioned above, the technique of splitting a computable Stone space into clopen components is not new. However, the punctual (primitive recursive) analysis of this aspect of Stone duality is new, and the sufficiency of *computable* effective compactness to establish this *punctual* result seems rather unexpected. We now give the technical details.

Proof of Theorem 1.5. Let $(X, d, (\alpha_i)_{i\in\omega})$ be a punctual, effectively compact Stone space. We will build a primitive recursive binary tree $T \subseteq 2^{<\omega}$. The tree T will be pruned, i.e., for every $\sigma \in T$ there exists a path $p \in [T]$ such that $\sigma \subset p$

In order to prove the theorem, it is sufficient to establish that there exists a primitive recursive homeomorphic embedding Φ acting from our Stone space X into the Cantor space 2^ω such that range$(\Phi) \subseteq [T]$.

Here we assume that the space $[T]$ (of paths through the tree T) is endowed with the Cantor metric d_C: if $\alpha \neq \beta$, then $d_C(\alpha, \beta) = 2^{-k}$ for the least k such that $\alpha(k) \neq \beta(k)$.

Our construction proceeds as follows:

(A) firstly, we build a *computable* pruned tree $S \subseteq 2^{<\omega}$;

(B) after that, we transform the tree S into a primitive recursive tree T, and we construct the desired homeomorphic embedding Φ from X into 2^ω such that $\mathrm{range}(\Phi) \subseteq [T]$.

Intuitively speaking, the tree S satisfies almost everything that we need, except that S is not necessarily primitive recursive. More formally, we will map every point $x \in X$ to a path $p_x \in [S]$, and this mapping will be a computable homeomorphic embedding Ψ from X into 2^ω such that $\mathrm{range}(\Psi) \subseteq [S]$. We note that the first part of the proof works for any effectively compact, computable Stone space X.

In the second part of the proof, we have to do some additional work to ensure that everything works well in the primitive recursive setting.

Construction, part (A). In order to build our computable tree S, firstly, we collect the necessary definitions and ancillary results. The construction of S itself is given after the proof of Lemma 5.6 below.

For a basic open ball B, by $\mathrm{cen}(B)$ we denote the center of B, and $\mathrm{rad}(B)$ denotes the radius of B. By $B^c(\alpha, r)$ we denote the *closed ball* of radius r:

$$B^c(\alpha, r) = \{x : d(\alpha, x) \leq r\}.$$

Similarly, if B is a basic open ball, then B^c denotes the corresponding closed ball $B^c(\mathrm{cen}(B), \mathrm{rad}(B))$.

We say that a basic open ball $B(\alpha, r)$ is *formally included* into the ball $B(\beta, q)$ if $d(\beta, \alpha) + r < q$. Notice that the formal inclusion of balls implies the usual set-theoretic inclusion of these balls.

We say that basic open balls $B(\alpha, r)$ and $B(\beta, q)$ are *formally disjoint* if $d(\alpha, \beta) > r + q$. It is clear that formally disjoint balls are disjoint. Note that for our space X, we can computably enumerate all tuples $(i, j, r, q) \in \omega^2 \times \mathbb{Q}^2_+$ such the balls $B(\alpha_i, r)$ and $B(\alpha_j, q)$ are formally disjoint (in fact, here enumerability follows from the computability of the space X). Similarly, for the space X, one can computably enumerate all (indices of) pairs of formally included balls.

Let $\vec{B} = (B_0, B_1, \dots, B_k)$ and $\vec{C} = (C_0, C_1, \dots, C_\ell)$ be finite tuples of basic open balls. We say that the tuples $\vec{B}$ and $\vec{C}$ are *formally disjoint* if for all $i \leq k$ and $j \leq \ell$, the balls B_i and C_j are formally disjoint. Notice that in this case, the open sets $\bigcup_{i \leq k} B_i$ and $\bigcup_{j \leq \ell} C_j$ do not intersect.

We say that a tuple $\vec{B}$ is *formally included* into $\vec{C}$ if for each $i \leq k$, the ball B_i is formally included into some ball C_j, $j \leq \ell$. In this case, we have $\bigcup_{i \leq k} B_i \subseteq \bigcup_{j \leq \ell} C_j$.

For a tuple $\vec{B} = (B_0, B_1, \dots, B_k)$ of basic open balls, we introduce the following notation:

$$\bigcup \vec{B} := \bigcup_{i \leq k} B_i.$$

In what follows, we identify tuples $\vec{B}$ and their Gödel codes.

We will use the following known results:

Proposition 5.1 (see, e.g., Theorems 1.1 and 3.3 in [DM23]). *A computable Polish space M is effectively compact if and only if there exists a computable enumeration of* all *finite covers of M by basic open balls.*

Lemma 5.2 (folklore, see the proof in [DM23, p. 190]). *Let M be a compact Polish space. Suppose that $(B_0, B_1, \ldots, B_k)$ is a finite cover of M by basic open balls. Then there exists $\varepsilon > 0$ such that for any $y \in M$, the ball $B(y, \varepsilon)$ is formally included into some B_i, $i \leq k$.*

The following definition (and its properties given below) will be crucial in the construction of our computable tree S.

Definition 5.1. Let $\vec{B} = (B_0, B_1, \ldots, B_k)$ and $\vec{C} = (C_0, C_1, \ldots, C_\ell)$ be tuples of basic open balls. We say that the pair $(\vec{B}, \vec{C})$ is a *clopen split* of the space X if $B_i \cap C_j = \emptyset$ for all i and j, and $(\bigcup \vec{B}) \cup (\bigcup \vec{C}) = X$.

Remark 5.3. Since in Theorem 1.5 the space X is compact and totally separated, for any points $b \neq c$ from X, there exists a clopen split $(\vec{B}, \vec{C})$ such that $b \in \bigcup \vec{B}$ and $c \in \bigcup \vec{C}$.

Our first lemma below allows us to refine a given clopen split $(\vec{B}, \vec{C})$ "up to the precision 2^{-m}". Intuitively speaking, we get "kind of the same" clopen split, but the resulting new split is defined via some balls of radius $< 2^{-m}$, and every pair $(B \in \vec{B}, C \in \vec{C})$ "becomes" formally disjoint.

Lemma 5.4. *Let $(X, d, (\alpha_i)_{i\in\omega})$ be an effectively compact, computable Polish space. Suppose that $(\vec{B}, \vec{C})$ is a clopen split of the space X. Suppose also that $m \in \omega$. Then one can computably find (uniformly in $\vec{B}, \vec{C}, m$) a number $t_m > m$ and tuples $\vec{D}, \vec{E}$ of basic open balls such that:*

1. $\bigcup \vec{D} = \bigcup \vec{B}$ *and* $\bigcup \vec{E} = \bigcup \vec{C}$*;*
2. *for each ball $V \in \vec{D}, \vec{E}$, we have* $\mathrm{rad}(V) < 2^{-m}$*;*
3. *for all $D \in \vec{D}$ and $E \in \vec{E}$, we have*

$$d(\mathrm{cen}(D), \mathrm{cen}(E)) > \mathrm{rad}(D) + \mathrm{rad}(E) + 2^{-t_m}. \tag{6}$$

Consequently, the tuples $\vec{D}$ and $\vec{E}$ are formally disjoint, and $(\vec{D}, \vec{E})$ is a clopen split of the space X.

Lemma 5.4 allows us to introduce the following notions:

Definition 5.2. If a clopen split $(\vec{D}, \vec{E})$ satisfies Eq. (6) for some $t_m \in \omega$, then we say that $(\vec{D}, \vec{E})$ is a *fd-clopen split*. Here *fd* stands for 'formally disjoint'.

It is clear that for an effectively compact, computable Polish space X, one can computably enumerate all its fd-clopen splits.

Definition 5.3. We say that the triple $(t_m, \vec{D}, \vec{E})$ obtained in Lemma 5.4 is a 2^{-m}*-formal refinement* of the clopen split $(\vec{B}, \vec{C})$.

Proof of Lemma 5.4. Here it is sufficient to (non-constructively) prove the existence of tuples $\vec{D}$ and $\vec{E}$ of basic open balls such that:

(a) the tuple $\vec{D}, \vec{E}$ forms a cover of the space X;
(b) the tuple $\vec{D}$ is formally included into $\vec{B}$, and $\vec{E}$ is formally included into $\vec{C}$;
(c) for every ball $V \in \vec{D}, \vec{E}$, we have $\mathrm{rad}(V) < 2^{-m}$;
(d) the tuples $\vec{D}$ and $\vec{E}$ are formally disjoint.

Note that items (a)–(b) imply that $\bigcup \vec{D} = \bigcup \vec{B}$ and $\bigcup \vec{E} = \bigcup \vec{C}$. In addition, if, say, balls $D_0 \in \vec{D}$ and $E_0 \in \vec{E}$ are formally disjoint, then one can find (computably in D_0, E_0, m) an integer $t' > m$ such that

$$d(\mathrm{cen}(D_0), \mathrm{cen}(E_0)) > \mathrm{rad}(D_0) + \mathrm{rad}(E_0) + 2^{-t'}.$$

Notice that in the space X, the properties of 'being formally included' and 'being formally disjoint' are Σ^0_1. Therefore, if such a nice pair $(\vec{D}, \vec{E})$ exists, then we can find such a pair by computably enumerating all finite covers of X by basic open balls (recall Proposition 5.1). When this $(\vec{D}, \vec{E})$ is found, we compute the additional parameter $t_m > m$.

Now we establish the existence of $\vec{D}, \vec{E}$ satisfying items (a)–(d). Note that the tuple $\vec{B}, \vec{C}$ forms a cover of the whole space X. Thus, by Lemma 5.2, there exists a rational $\varepsilon > 0$ such that every ball $B(\alpha_i, \varepsilon)$ is formally included into some ball taken from $\vec{B}, \vec{C}$. Therefore, by choosing sufficiently small radii $r < \min(2^{-m}, \varepsilon)$, we will automatically satisfy items (b) and (c).

Let $U := \bigcup \vec{B}$. We claim that

$$U = \bigcup\{B(\alpha_j, r) : r > 0 \text{ is a sufficiently small rational}, \quad B^c(\alpha_j, 2r) \subseteq B_i \text{ for some } i \leq k\}. \tag{7}$$

Indeed, suppose that $x \in U$. Without loss of generality, we may assume that $x \in B_0$. Choose a sufficiently small rational δ such that $B(x, \delta) \subseteq B_0$. Find a special point α_j such that $d(\alpha_j, x) < \delta/3$. If $d(\alpha_j, y) \leq 2\delta/3$, then

$$d(x, y) \leq d(x, \alpha_j) + d(\alpha_j, y) < \delta/3 + 2\delta/3 = \delta$$

and $y \in B(x, \delta) \subseteq B_0$. Thus, for $r := \delta/3$, we have $x \in B(\alpha_j, r)$ and $B^c(\alpha_j, 2r) \subseteq B_0$.

Since U is a closed subset of X, the set U is compact. From Eq. (7), we choose a finite open subcover $\vec{D}$ which covers the set U. Similarly, we can choose a tuple $\vec{E}$ which forms a cover of $\bigcup \vec{C}$ such that for every $E \in \vec{E}$, we have

$$B^c(\mathrm{cen}(E), 2 \cdot \mathrm{rad}(E)) \subseteq C_j \text{ for some } C_j \in \vec{C}.$$

Since $\vec{B}, \vec{C}$ is a cover of X, it is clear that $\vec{D}, \vec{E}$ also covers X. In addition, for all $D \in \vec{D}$ and $E \in \vec{E}$, we have

$$d(\mathrm{cen}(D), \mathrm{cen}(E)) > 2\max(\mathrm{rad}(D), \mathrm{rad}(E)) \geq \mathrm{rad}(D) + \mathrm{rad}(E).$$

Hence, the tuples $\vec{D}$ and $\vec{E}$ are formally disjoint. We conclude that the tuple $\vec{D}, \vec{E}$ satisfies the items (a)–(d). Lemma 5.4 is proved. □

The next lemma is essentially our main tool in constructing the desired tree S.

Lemma 5.5. *Suppose that $(\vec{B}^0, \vec{C}^0), (\vec{B}^1, \vec{C}^1), \ldots, (\vec{B}^s, \vec{C}^s)$ are clopen splits of the space X. Given clopen sets $V_j \in \{\bigcup \vec{B}^j, \bigcup \vec{C}^j\}$ for $j \leq s$, one can do the following:*

(i) *We can computably decide whether the intersection* $V = V_0 \cap V_1 \cap \cdots \cap V_s$ *is empty or not.*
(ii) *Given a Cauchy name* f *of a point* $x \in X$*, we can computably in* f *decide whether* $x \in \bigcup \vec{B}^j$ *or* $x \in \bigcup \vec{C}^j$*.*

In addition, these procedures are uniform in the parameters $\vec{B}^j, \vec{C}^j, V_j$*, and* f*.*

Lemma 5.5 is a consequence of the following more technical result:

Lemma 5.6. *Suppose that* $(\vec{B}^0, \vec{C}^0), (\vec{B}^1, \vec{C}^1), \ldots, (\vec{B}^s, \vec{C}^s)$ *are clopen splits of the space* X*. One can computably find (uniformly in the parameters* $\vec{B}^j, \vec{C}^j$*) the following:*

- *a number* $m \in \omega$*,*
- *a sequence of basic open balls* $\vec{U} = (U_0, U_1, \ldots, U_\ell)$*,*
- *a finite map* ξ *which maps each pair* (U_i, j)*, for* $i \le \ell$ *and* $j \le s$*, to some tuple* $\vec{D}^{i,j} \in \{\vec{B}^j, \vec{C}^j\}$*,*

satisfying the following conditions:

(1) $\vec{U}$ *is a cover of the space* X*;*
(2) *for every* i *and* j*,* U_i *is a subset of the set* $\bigcup \xi(U_i, j)$*;*
(3) *for every* $i \le \ell$*,* $j \le s$*, and* $k \in \omega$*, the condition* $d(\mathrm{cen}(U_i), \alpha_k) < \mathrm{rad}(U_i) + 2^{-m}$ *implies* $B(\alpha_k, 2^{-m}) \subseteq \bigcup \xi(U_i, j)$*.*

First, we give a proof of Lemma 5.5 (assuming that Lemma 5.6 is true). Secondly, we prove Lemma 5.6.

Proof of Lemma 5.5. For the clopen splits $(\vec{B}^j, \vec{C}^j)$ we compute the values m, $\vec{U}$, and ξ from Lemma 5.6.

(i) Without loss of generality, assume that $V_j = \bigcup \vec{B}^j$ for all $j \le s$. Then the set $(\bigcup \vec{B}^0) \cap (\bigcup \vec{B}^1) \cap \cdots \cap (\bigcup \vec{B}^s)$ is non-empty if and only if some $U_i \in \vec{U}$ satisfies $\xi(U_i, j) = \vec{B}^j$ for all $j \le s$.

(ii) Firstly, by using f we find a special point α_k such that $x \in B(\alpha_k, 2^{-m})$. Secondly, since the space X is computable Polish, for each $i \le \ell$ we computably find a rational number q_i such that $|d(\mathrm{cen}(U_i), \alpha_k) - q_i| < 2^{-m-1}$. Since the sequence $\vec{U}$ forms a cover of X, there exists some i such that $d(\mathrm{cen}(U_i), \alpha_k) < \mathrm{rad}(U_i)$, and hence, $q_i < \mathrm{rad}(U_i) + 2^{-m-1}$.

We find (the least) $i \le \ell$ such that $q_i < \mathrm{rad}(U_i) + 2^{-m-1}$. Then we have $d(\mathrm{cen}(U_i), \alpha_k) < q_i + 2^{-m-1} < \mathrm{rad}(U_i) + 2^{-m}$, and thus, by Property (3) of Lemma 5.6, $x \in B(\alpha_k, 2^{-m}) \subseteq \bigcup \xi(U_i, j)$.

Therefore, we conclude that $x \in \bigcup \vec{B}^j$ if and only if $\xi(U_i, j) = \vec{B}^j$. Lemma 5.5 is proved. □

Proof of Lemma 5.6. We give a proof for the case $s = 2$: i.e., we consider clopen splits $(\vec{B}^0, \vec{C}^0)$, $(\vec{B}^1, \vec{C}^1)$, and $(\vec{B}^2, \vec{C}^2)$. A proof for the general case could be arranged similarly.

Recall Definition 5.3. By applying Lemma 5.4, we compute $(t_0, \vec{D}^0, \vec{E}^0)$ as a 2^{-1}-formal refinement of the split $(\vec{B}^0, \vec{C}^0)$. Let $(t_1, \vec{D}^1, \vec{E}^1)$ be a 2^{-t_0}-formal refinement of the clopen split $(\vec{B}^1, \vec{C}^1)$. Finally, choose $(t_2, \vec{D}^2, \vec{E}^2)$ as a 2^{-t_1}-formal refinement of the split $(\vec{B}^2, \vec{C}^2)$. Notice that $t_0 < t_1 < t_2$.

We show that the tuple $\vec{U} := \vec{D}^2, \vec{E}^2$ and the number $m := t_2 + 1$ satisfy the conditions of the lemma. Notice that $\vec{U}$ is a cover of the whole space X.

Firstly, we compute $\xi(U_i, j)$ such that Property (2) of the lemma is satisfied. Note that we have $\bigcup \vec{D}^2 = \bigcup \vec{B}^2$ and $\bigcup \vec{E}^2 = \bigcup \vec{C}^2$, and thus, it is straightforward to appropriately define the value $\xi(U_i, 2)$ for each $U_i \in \vec{U}$. In addition, notice that $\mathrm{rad}(U_i) < 2^{-t_1}$.

We describe how to compute the value $\xi(U_i, 1)$ (the value $\xi(U_i, 0)$ is defined in a similar manner). Since $\vec{D}^1, \vec{E}^1$ is a cover of the space X, we can computably find some ball $F \in \vec{D}^1, \vec{E}^1$ such that $\mathrm{cen}(U_i) \in F$. If $F \in \vec{D}^1$, then we declare that $\xi(U_i, 1) = \vec{B}^1$. If $F \in \vec{E}^1$, then put $\xi(U_i, 1) = \vec{C}^1$. Since the sets $\bigcup \vec{B}^1 = \bigcup \vec{D}^1$ and $\bigcup \vec{C}^1 = \bigcup \vec{E}^1$ are disjoint, it is clear that the value $\xi(U_i, 1)$ is well-defined.

Now we show that $\xi(U_i, 1)$ satisfies Property (2) of the lemma. It is sufficient to establish the following:

$$\mathrm{cen}(U_i) \in \bigcup \vec{B}^1 \;\Rightarrow\; U_i \subseteq \bigcup \vec{B}^1. \tag{8}$$

Suppose that $\mathrm{cen}(U_i) \in D$ for some $D \in \vec{D}^1$. Then every $x \in U_i$ satisfies

$$d(\mathrm{cen}(D), x) \le d(\mathrm{cen}(D), \mathrm{cen}(U_i)) + d(\mathrm{cen}(U_i), x) < \mathrm{rad}(D) + 2^{-t_1}.$$

Towards a contradiction, assume that $x \in E$ for some $E \in \vec{E}^1$. Then we have

$$d(\mathrm{cen}(D), \mathrm{cen}(E)) \le d(\mathrm{cen}(D), x) + d(x, \mathrm{cen}(E)) < \mathrm{rad}(D) + 2^{-t_1} + \mathrm{rad}(E),$$

which contradicts Eq. (6) from the definition of the 2^{-t_0}-formal refinement $(t_1, \vec{D}^1, \vec{E}^1)$. Thus, we deduce that $U_i \cap (\bigcup \vec{E}^1) = \emptyset$ and $U_i \subseteq \bigcup \vec{B}^1$.

We prove that Property (3) of the lemma is satisfied. It is sufficient to consider Property (3) for the case $j = 2$, the other cases are treated similarly.

Without loss of generality, assume that $\xi(U_i, 2) = \vec{B}^2$. This implies that $U_i \subseteq \bigcup \vec{B}^2 = \bigcup \vec{D}^2$. Suppose that α_k satisfies $d(\mathrm{cen}(U_i), \alpha_k) < \mathrm{rad}(U_i) + 2^{-t_2-1}$. Choose an arbitrary point $x \in B(\alpha_k, 2^{-t_2-1})$. Towards a contradiction, assume that $x \in E$ for some $E \in \vec{E}^2$. Then we have

$$\begin{aligned} d(\mathrm{cen}(U_i), \mathrm{cen}(E)) \le d(\mathrm{cen}(U_i), \alpha_k) + d(\alpha_k, x) + d(x, \mathrm{cen}(E)) < \mathrm{rad}(U_i) + 2^{-t_2-1} + \\ + 2^{-t_2-1} + \mathrm{rad}(E) = \mathrm{rad}(U_i) + \mathrm{rad}(E) + 2^{-t_2}. \end{aligned}$$

This contradicts Eq. (6) from the definition of our 2^{-t_1}-formal refinement $(t_2, \vec{D}^2, \vec{E}^2)$. Therefore, the ball $B(\alpha_k, 2^{-t_2-1})$ and the set $\bigcup \vec{E}^2 = \bigcup \vec{C}^2$ are disjoint, and hence, we have $B(\alpha_k, 2^{-t_2-1}) \subseteq \bigcup \vec{B}^2$. Lemma 5.6 is proved. □

Building the tree S. We fix a computable enumeration $\{(\vec{B}^t, \vec{C}^t) : t \in \omega\}$ of all fd-clopen splits in our Stone space X (recall Definition 5.2). We define clopen sets:

$$U_t := \bigcup \vec{B}^t, \quad V_t := \bigcup \vec{C}^t.$$

The empty string Λ belongs to the tree S. If σ is a finite binary string such that $|\sigma| \ge 1$, then we add σ into S if and only if the clopen set

$$W_\sigma := \big(\bigcap \{U_t : t < |\sigma|,\ \sigma(t) = 0\}\big) \cap \big(\bigcap \{V_t : t < |\sigma|,\ \sigma(t) = 1\}\big)$$

is non-empty. By Lemma 5.5, one can computably check (uniformly in σ) whether W_σ is non-empty. Therefore, the tree S is computable.

In addition, if W_σ is non-empty, then at least one of the sets $W_{\sigma^\frown 0}$ or $W_{\sigma^\frown 1}$ is also non-empty (recall that $(\vec{B}^{|\sigma|}, \vec{C}^{|\sigma|})$ splits X into two disjoint parts). This implies that the tree S is pruned.

We additionally define a computable function G as follows. Put $G(\Lambda) = 0$. For a given non-empty string $\sigma \in S$, $G(\sigma)$ is equal to $(\vec{D}, \vec{E}, w)$, where $\vec{D}$, $\vec{E}$, and w are computed by the following rules:

- Let $t := |\sigma| - 1$. By applying Lemma 5.6 to the clopen splits $(\vec{B}^0, \vec{C}^0), \dots, (\vec{B}^t, \vec{C}^t)$, we compute the corresponding values m, $\vec{U}$, and ξ.
- We put $\vec{D} := \vec{U}$ and $w := m$.
- By using the map ξ, we collect into the tuple $\vec{E}$ precisely those balls U_i from $\vec{U}$ such that $U_i \subseteq W_\sigma$. Or more formally, U_i is added into $\vec{E}$ if and only if for all $k < |\sigma|$, we have
$$\xi(U_i, k) = \vec{B}^k \;\Leftrightarrow\; \sigma(k) = 0.$$
Since the set W_σ is non-empty, the tuple $\vec{E}$ is also non-empty.

For a point $x \in X$, we define the path $\Psi(x) \in [S]$ as follows. For $\sigma \neq \Lambda$, we have $\sigma \subset \Psi(x)$ if and only if $x \in W_\sigma$.

The following fact will be useful for us in the second part of the proof of the theorem.

Lemma 5.7. *(a) If $x \neq y$, then the paths $\Psi(x)$ and $\Psi(y)$ are different.*

(b) Suppose that $G(\sigma) = (\vec{D}, \vec{E}, w)$ and $d(\alpha_k, x) < 2^{-w}$ for some α_k. Then we have:

$$x \in W_\sigma \;\Leftrightarrow\; \alpha_k \in W_\sigma. \tag{9}$$

Proof. **(a)** By Remark 5.3 and Lemma 5.4, there exists an fd-clopen split $(\vec{B}^t, \vec{C}^t)$ such that $x \in \bigcup \vec{B}^t$ and $y \in \bigcup \vec{C}^t$. This implies that $\Psi(x)(t) = 0$ and $\Psi(y)(t) = 1$, hence, $\Psi(x) \neq \Psi(y)$.

(b) First, suppose that $\alpha_k \in W_\sigma$. Then we have $d(\mathrm{cen}(U_i), \alpha_k) < \mathrm{rad}(U_i)$ for some $U_i \in \vec{E}$. Then Property (3) of Lemma 5.6 implies that $x \in B(\alpha_k, 2^{-w}) \subseteq W_\sigma$.

Now suppose that $x \in W_\sigma$. Then $d(\mathrm{cen}(U_i), x) < \mathrm{rad}(U_i)$ for some $U_i \in \vec{E}$. We have
$$d(\mathrm{cen}(U_i), \alpha_k) \leq d(\mathrm{cen}(U_i), x) + d(x, \alpha_k) < \mathrm{rad}(U_i) + 2^{-w}.$$
Property (3) of Lemma 5.6 implies that $\alpha_k \in B(\alpha_k, 2^{-w}) \subseteq W_\sigma$. Lemma 5.7 is proved. □

In fact, here one could additionally prove that the map Ψ is a computable homeomorphic embedding from X into 2^ω such that $\mathrm{range}(\Psi) \subseteq [S]$, but we will not use this fact in the subsequent proof.

Construction, part (B). Now we construct a primitive recursive tree T with additional primitive recursive function $H\colon T \to S$, where S is the computable tree built above.

The tree T is constructed by stages. At a stage s, for each finite binary string σ of length s, we say whether σ belongs to T or not. This ensures that the resulting T will be primitive recursive.

Stage 0. We add the empty string Λ into the tree T, and we put $H(\Lambda) := \Lambda$.

Stage $s+1$. At the beginning of a given stage $s+1$, we have a finite tree T_s such that:

- for each $\sigma \in T_s$, there exists $\tau \in T_s$ such that $|\tau| = s$ and $\tau \supseteq \sigma$.

This condition is needed to ensure that the resulting tree $T = \bigcup_{s\in\omega} T_s$ will be pruned.

A *leaf* of T_s is a node $\sigma \in T_s$ such that $|\sigma| = s$.

Suppose that for each leaf σ of T_s, the following could be computed in $\leq s+1$ computational steps:

- for each $u \in \{0,1\}$, whether $H(\sigma)^\frown u$ belongs to the tree S,
- the value $G(H(\sigma)^\frown u)$ for each $H(\sigma)^\frown u$ belonging to S.

Then we set:

$$\sigma^\frown u \in T \;\Leftrightarrow\; H(\sigma)^\frown u \in S,$$

and $H(\sigma^\frown u) := H(\sigma)^\frown u$ for $\sigma^\frown u \in T$.

If the values above could not be computed in $\leq s+1$ steps, then for each leaf σ, we put $\sigma^\frown 0 \in T_{s+1}$, $\sigma^\frown 1 \notin T$, and $H(\sigma^\frown 0) := H(\sigma)$.

This concludes the description of the construction. It is clear that the constructed binary tree $T = \bigcup_{s\in\omega} T_s$ is pruned and primitive recursive. In addition, the function $H\colon T \to S$ is primitive recursive.

Verification. We need to show that there exists a primitive recursive homeomorphic embedding Φ from X into 2^ω with range$(\Phi) \subseteq [T]$.

The embedding Φ is defined as follows. For $x \in X$ and $\tau \in T$, we put $\tau \subset \Phi(x)$ if and only if $H(\tau) \subset \Psi(x)$.

Note that the map Φ is injective. Indeed, if $x \neq y$, then by Lemma 5.7.(a), there exists $t \in \omega$ such that $\Psi(x)(t) \neq \Psi(y)(t)$. Then there exists a sufficiently large $t' \geq t$ such that for each leaf τ of the finite tree $T_{t'}$, we have $|H(\tau)| > t$. Thus, there are incomparable leafs τ_0, τ_1 in the tree $T_{t'}$ such that $\tau_0 \subset \Phi(x)$ and $\tau_1 \subset \Phi(y)$. Therefore, $\Phi(x) \neq \Phi(y)$.

Now we show that the constructed map Φ is primitive recursive. Suppose that f is a Cauchy name for a point $x \in X$. We sketch how to compute a Cauchy name g for $\Phi(x)$, primitively recursively in f.

For each $t \in \omega$, we define a string $\tau_t \in T$ such that:

- $|\tau_t| = t$ and $\tau_t \subset \tau_{t+1}$,
- $\Phi(x) = \bigcup_{t\in\omega} \tau_t$,
- $d_C(\Phi(x), \tau_t^\frown 0^\omega) < 2^{-t+1}$.

The Cauchy name g is easily recovered from the sequence $(\tau_t)_{t\in\omega}$.

We set $\tau_0 := \Lambda$. Suppose that τ_t is already defined. If $\tau_t^\frown u \notin T$ for some $u \in \{0,1\}$, then put $\tau_{t+1} := \tau_t^\frown v$ for $v = 1-u$.

Otherwise, we have $\tau_t^\frown 0, \tau_t^\frown 1 \in T$. The construction then implies that we can compute the values $G(H(\tau_t)^\frown 0), G(H(\tau_t)^\frown 1)$ in $\leq t+1$ computational steps. Without loss of generality, we assume that $G(H(\tau_t)^\frown 0) = (\vec{D}, \vec{E}, w)$ and $G(H(\tau_t)^\frown 1) = (\vec{D}, \vec{F}, w)$.

With the help of the Cauchy name f, we find (primitively recursively in f) a special point α_k such that $d(\alpha_k, x) < 2^{-w}$. For each basic ball $B \in \vec{E}, \vec{F}$, we compute a rational q_B such that

$$|d(\mathrm{cen}(B), \alpha_k) - q_B| < 2^{-w-1}.$$

We find a ball B such that $q_B < \mathrm{rad}(B) + 2^{-w-1}$. Then we have $d(\mathrm{cen}(B), \alpha_k) < \mathrm{rad}(B) + 2^{-w}$. Property (3) of Lemma 5.6 (plus Lemma 5.7.(b)) implies that:

- If $B \in \vec{E}$, then both α_k and x belong to $\bigcup \vec{E}$. Then we put $\tau_{t+1} := \tau_t \widehat{\ } 0$.
- If $B \in \vec{F}$, then $\alpha_k, x \in \bigcup \vec{F}$. We set $\tau_{t+1} := \tau_t \widehat{\ } 1$.

The described construction of the sequence $(\tau_t)_{t\in\omega}$ is clearly primitive recursive, thus the map Φ is primitive recursive.

It is known that for every compact metric space Y and every metric space Z, any injective continuous mapping $\Theta\colon Y \to Z$ is a homeomorphic embedding. Since Φ is primitive recursive, Φ is continuous. Therefore, Φ is a primitive recursive homeomorphic embedding. Theorem 1.5 is proved. $\square$

Finally, we prove the following:

Theorem 1.6. *There is a compact punctual Stone space which is not computably homeomorphically embeddable into Cantor space.*

Proof of Theorem 1.6. In the proof, we will construct a c.e. subset W of the set $\{q \in \mathbb{Q} : 0 \leq q \leq 1\}$. We will ensure (see below) that there exists an injective, primitive recursive function $\xi\colon \omega \to \mathbb{Q}$ such that $\mathrm{range}(\xi) = W$.

Then the desired Polish space $\mathcal{M} = (X, d_X, (\alpha_i)_{i\in\omega})$ is defined as follows:

- Its metric is the usual Euclidean distance on the real line, i.e., $d_X = d_{\mathbb{R}}$.
- The set X is the closure of the constructed W w.r.t. the metric $d_{\mathbb{R}}$.
- For $i \in \omega$, the special point α_i is equal to $\xi(i)$.

Such choice of $\mathcal{M}$ automatically ensures the following properties. Since X is a closed subset of the real unit interval $[0, 1]$, the space $\mathcal{M}$ is compact. In addition, notice that $\alpha_i \neq \alpha_j$ for $i \neq j$, and the distances $d(\alpha_i, \alpha_j)$ are primitive recursive, uniformly in i, j. Hence, the space $\mathcal{M}$ is punctual.

In the verification, we will additionally prove that the constructed $\mathcal{M}$ is totally separated. This implies that $\mathcal{M}$ is a punctual Stone space.

Our construction of the set W diagonalizes against all possible computable homeomorphic embeddings Θ from the closure of W (in the space $([0, 1], d_{\mathbb{R}})$) into Cantor space 2^ω. Recall that any such Θ is induced by some Turing operator Φ: i.e., for a given Cauchy name f of a point $x \in \mathcal{M}$, the function Φ^f is a Cauchy name for the point $\Theta(x) \in 2^\omega$.

We satisfy the following series of requirements:

$\mathcal{R}_e$: The Turing operator Φ_e does not induce a homeomorphic embedding Θ from our space $\mathcal{M}$ into Cantor space.

While working with a requirement $\mathcal{R}_e$, we view the functions Φ_e^f, for $f \in \omega^\omega$, as (potential) Cauchy names in Cantor space. In particular, for a finite string $\sigma \in \omega^{<\omega}$, we assume that Φ_e^σ 'encodes' a finite fragment of the Cauchy name for some point $\Theta(x)$.

We fix a family of pairwise disjoint closed intervals $I_e \subset [0,1]$:

$$I_e := \left[\frac{1}{2^{4e+1}}, \frac{1}{2^{4e}}\right].$$

The interval I_e is intended to satisfy the requirement $\mathcal{R}_e$. Our set W will be a subset of $\mathbb{Q} \cap \bigcup_{e\in\omega} I_e$.

For the sake of convenience, we use the following notations: $\ell_e := 2^{-4e-1}$ and $r_e := 2^{-4e}$.

Beforehand, we declare that $\{\ell_e, r_e : e \in \omega\} \subseteq W$. This declaration *ensures* that the constructed c.e. set W will be equal to the range of some *injective*, primitive recursive function ξ. Indeed, every non-empty c.e. W is the range of some (not necessarily injective) primitive recursive function $p(x)$. Then for the case of our W, one can define $\xi(0) := p(0)$ and

$$\xi(s+1) := \begin{cases} p(s+1), & \text{if } p(s+1) \notin \{\xi(t) : t \leq s\}, \\ \text{the greatest } w \in \{\ell_e, r_e : e \in \omega\} & \\ \qquad \text{such that } w \notin \{\xi(t) : t \leq s\}, & \text{otherwise.} \end{cases}$$

The primitive recursive function ξ enumerates our W without repetitions.

In our further discussion, we will slightly abuse the notations. We identify a Cauchy sequence $(q_s)_{s\in\omega}$, where $q_s \in W$, with the 'natural' Cauchy name f of the point $x = \lim_{s\to\infty} q_s$ in the space $\mathcal{M}$. Formally, this Cauchy name f could be defined as follows: for $s \in \omega$, find the index $k_s \in \omega$ such that $\xi(k_s) = q_s$, and then put $f(s) := k_s$. Informally, the name f just 'says' that $|x - q_s| < 2^{-s}$ for $s \in \omega$.

For example, one can view the infinite sequence $(\ell_e, \ell_e, \ell_e, \dots)$ as a Cauchy name of the point ℓ_e in $\mathcal{M}$.

$\mathcal{R}_e$-strategy. Here we assume that Θ is a computable map induced by the Turing operator Φ_e, and our goal is to ensure that this Θ is not a homeomorphic embedding from $\mathcal{M}$ into 2^ω.

The 0*-th diagonalization attempt.* Consider the following Cauchy names: $h_\ell = (\ell_e, \ell_e, \ell_e, \dots)$ for the point ℓ_e, and $h_r = (r_e, r_e, r_e, \dots)$ for the point r_e.

Search for $m \in \omega$ and finite strings $\sigma, \tau \in 2^{<\omega}$ such that:

(a) $\sigma \neq \tau$ and $|\sigma| = |\tau| = t + 1$.

(b) $\Phi_e^{h_\ell \upharpoonright (m+1)}$ says the following: for any $x \in \mathcal{M}$ which is 2^{-m}-close to ℓ_e, the point $\Theta(x)$ is 2^{-t}-close to σ.
(More formally, here we mean the following: if a Cauchy name f of a point x satisfies $f \supset h_\ell \upharpoonright (m+1)$, then the Cauchy name Φ_e^f of the point $\Theta(x)$ says that $d_C(\Theta(x), \sigma^\frown 0^\omega) < 2^{-t}$.)

(c) $\Phi_e^{h_r \upharpoonright (m+1)}$ says the following: for any x which is 2^{-m}-close to r_e, the point $\Theta(x)$ is 2^{-t}-close to τ.

If there are no such m, σ, τ, then we have:

- either one of the functions $\Phi_e^{h_\ell}, \Phi_e^{h_r}$ is not a Cauchy name, or
- the functions $\Phi_e^{h_\ell}, \Phi_e^{h_r}$ are Cauchy names of the same point $y = \Theta(\ell_e) = \Theta(r_e)$ (and hence, the map Θ is not injective).

In each of these cases, the requirement $\mathcal{R}_e$ is satisfied automatically.

So, without loss of generality, we assume that we have found the desired m, σ, τ. We define $L^* := |\sigma|$.

Case 1. Suppose that the open balls $B(\ell_e, 2^{-m})$ and $B(r_e, 2^{-m})$ intersect. (Notice that here the ball $B(\ell_e, 2^{-m})$ is just equal to the interval $(\ell_e - 2^{-m}, \ell_e + 2^{-m}) \cap [0,1]$. Thus, one can computably check whether these open balls intersect.)

Then we choose a rational number $q \in I_e$ such that $|\ell_e - q| < 2^{-m}$ and $|r_e - q| < 2^{-m}$. We add this q into the set W.

We claim that in this case, the requirement $\mathcal{R}_e$ is satisfied. Indeed, since q is 2^{-m}-close to ℓ_e, the sequence $f := (h_\ell \upharpoonright m+1)\widehat{\ }(q, q, q, \dots)$ is a Cauchy name of the point q in $\mathcal{M}$. Hence, by item (b) above, we have $d_C(\Theta(q), \sigma\widehat{\ }0^\omega) < 2^{-|\sigma|+1}$ in Cantor space. This implies that σ is an initial segment of the infinite binary string $\Theta(q)$. On the other hand, a similar argument shows that $\tau \subset \Theta(q)$. Since the strings σ and τ are incomparable, we deduce that here the Turing operator Φ_e cannot induce a well-defined map Θ.

Case 2. Suppose that the balls $B(\ell_e, 2^{-m})$ and $B(r_e, 2^{-m})$ do not intersect.

We define two clopen sets U and V in Cantor space:

- $U := B(\sigma\widehat{\ }0^\omega, 2^{-L^*+1}) = \{\alpha \in 2^\omega : \sigma \subset \alpha\}$.
- $V := 2^\omega \setminus U = \{\beta \in 2^\omega : \sigma \not\subset \beta\}$.

Notice that by the choice of m, σ, τ, we have

$$\Theta(B(\ell_e, 2^{-m})) \subseteq U \text{ and } \Theta(B(r_e, 2^{-m})) \subseteq V.$$

We put $a_0 := \ell_e$, $b_0 := r_e$, and $w_0 := m$. Then we proceed to the first diagonalization attempt.

<u>*The $(s+1)$-th diagonalization attempt.*</u> At the beginning of the $(s+1)$-th attempt, we have $w_s \in \omega$ and two rationals $a_s, b_s \in W \cap I_e$ such that:

- $a_s < b_s$ and $|a_s - b_s| = 2^{-4e-1-s}$,
- $\Theta(B(a_s, 2^{-w_s})) \subseteq U$ and $\Theta(B(b_s, 2^{-w_s})) \subseteq V$.

Firstly, we define $c := (a_s + b_s)/2$, and we add c into the set W. Put $g_c := (c, c, c, \dots)$.

We find $m' \geq w_s$ and a finite binary string ρ such that:

- $|\rho| = L^*$,
- $\Phi_e^{g_c \upharpoonright (m'+1)}$ says the following: for any $x \in \mathcal{M}$ which is $2^{-m'}$-close to c, the point $\Theta(x)$ is 2^{-L^*+1}-close to ρ.

If there are no such m' and ρ, then $\Phi_e^{g_c}$ is not a Cauchy name, and $\mathcal{R}_e$ is automatically satisfied. Hence, we assume that these m' and ρ have been found.

If $\rho = \sigma$, then it is clear that $\Theta(B(c, 2^{-m'})) \subseteq U$. Here we define $a_{s+1} := c$, $b_{s+1} := b_s$, and $w_{s+1} := m'$.

If $\rho \neq \sigma$, then we have $\Theta(B(c, 2^{-m'})) \subseteq V$. Then we define $a_{s+1} := a_s$, $b_{s+1} := c$, and $w_{s+1} := m'$.

Case A. Suppose that the open balls $B(a_{s+1}, 2^{-m'})$ and $B(b_{s+1}, 2^{-m'})$ intersect. Then we choose a rational $q \in B(a_{s+1}, 2^{-m'}) \cap B(b_{s+1}, 2^{-m'}) \cap I_e$, and we add this q into W.

Similarly to Case 1 of the 0-th diagonalization attempt, here we observe that $\Theta(q)$ must simultaneously belong to both U and V. This is impossible, and hence, $\mathcal{R}_e$ is automatically satisfied (thus, we do not need to open the $(s+2)$-th diagonalization attempt).

Case B. If the balls $B(a_{s+1}, 2^{-m'})$ and $B(b_{s+1}, 2^{-m'})$ do not intersect, we proceed to the $(s+2)$-th diagonalization attempt.

This concludes the description of the $\mathcal{R}_e$-strategy.

Construction. Notice that different strategies do not interfere with each other's actions. Hence, we can arrange the construction of our c.e. set W in a straightforward manner.

Verification. We prove that every requirement $\mathcal{R}_e$ is satisfied.

First, assume that the $\mathcal{R}_e$-strategy opens only finitely many diagonalization attempts. Then observe that $\mathcal{R}_e$ is satisfied:

- either $\mathcal{R}_e$ is satisfied automatically due to a 'bad behavior' of the Turing functional Φ_e (i.e., Φ_e does not induce a well-defined map Θ, or this Θ is not injective), or
- we have succeeded in diagonalizing by adding finitely many points $q \in I_e$ into the set W.

Now suppose that $\mathcal{R}_e$ opens infinitely many diagonalization attempts. Towards a contradiction, assume that the map Θ induced by Φ_e is a homeomorphic embedding.

Consider the sequences $(a_s)_{s\in\omega}$ and $(b_s)_{s\in\omega}$ constructed by the strategy $\mathcal{R}_e$. The closed intervals $J_s := [a_s, b_s]$ are nested, and we have $|a_s - b_s| = 2^{-4e-1-s}$. We deduce that there exists a point $c \in \mathcal{M}$ such that

$$\lim_{s\to\infty} a_s = c = \lim_{s\to\infty} b_s.$$

Notice that we have $\Theta(a_s) \in U$ and $\Theta(b_s) \in V$ for all $s \in \omega$. Since the sets U and V are both closed, we obtain that $\Theta(c) \in U \cap V$. But the sets U and V are disjoint, hence we get a contradiction.

Therefore, every requirement $\mathcal{R}_e$ is satisfied, and the constructed space $\mathcal{M}$ is not computably homeomorphically embeddable into Cantor space.

In order to finish the proof of the theorem, now it is sufficient to show that our space $\mathcal{M}$ is totally separated. Recall that by X we denote the closure of the set W (in the space $([0,1], d_{\mathbb{R}})$). Since $\{\ell_e : e \in \omega\} \subseteq W$, we have $0 \in X$.

Notice that for every $e \in \omega$, the set $A_e := X \cap I_e$ is clopen in $\mathcal{M}$. Indeed, we have

$$A_e = X \cap (B(\ell_e, 2^{-4e-2} + 2^{-4e-3}) \cup B(r_e, 2^{-4e-2} + 2^{-4e-3})),$$
$$X \setminus A_e = X \cap \left(B(0, 2^{-4e-3}) \cup \bigcup_{i<e} A_i \right).$$

In addition, observe the following:

- if $\mathcal{R}_e$ opens only finitely many diagonalization attempts, then the set A_e is finite;
- if $\mathcal{R}_e$ opens infinitely many diagonalization attempts, then $A_e = \{a_s, b_s : s \in \omega\} \cup \{u\}$, where the sequences $(a_s)_{s\in\omega}$ and $(b_s)_{s\in\omega}$ are built by the $\mathcal{R}_e$-strategy, and $u = \lim_s a_s = \lim_s b_s$.

Let $c < d$ be arbitrary points from X.

Case 1. If $c \in I_e$ and $d \in I_{e'}$ for some $e \neq e'$, then we split X into clopen parts A_e and $X \setminus A_e$. We have $c \in A_e$ and $d \in X \setminus A_e$.

Case 2. Suppose that $c, d \in I_e$. Then at least one of the points c, d belongs to the set W. Without loss of generality, we may assume that $c = a_s$ for some $s \in \omega$ and $a_s < a_{s+1}$ (the only other nontrivial case when $c = b_s$ and $b_{s+1} < b_s$ is treated similarly). We define $\varepsilon := (a_{s+1} - a_s)/2$, and we split X into the following two clopen parts:

$$U := \{a_s\} = X \cap B(a_s, \varepsilon),$$
$$V := X \setminus \{a_s\} = (X \setminus I_e) \cup (X \cap (B(\ell_e, a_s - \ell_e) \cup B(r_e, r_e - a_{s+1} + \varepsilon))).$$

Clearly, $c \in U$ and $d \in V$.

Case 3. Otherwise, we have $c = 0$ and $d \in I_e$ for some e. This case is treated similarly to Case 1.

We conclude that the space $\mathcal{M}$ is totally separated, and hence, $\mathcal{M}$ is a punctual Stone space (as discussed in the very beginning of the proof). Theorem 1.6 is proved. □

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Appendix A. Proving Theorem 1.4 more directly

Proof of Theorem 1.4. Let $(X, \rho, (\alpha_i)_{i\in\omega})$ be a punctual space. We construct an isometric embedding $g\colon X \to \mathcal{U}$, following the proof of Theorem 2.I of [Ury51].

Stage 0. Let $g(\alpha_0) = a_0$.

Stage $n+1$. Suppose we have defined $g(\alpha_0) = y_0, \ldots, g(\alpha_n) = y_n$. Now we want to construct a Cauchy name for the element $g(\alpha_{n+1}) = y_{n+1}$. In other words, for each ε we want to primitively recursively find a rational point realizing the Katětov mapping $f(y_i) = d(y, y_i) = \rho(\alpha_{n+1}, \alpha_i)$, $i \leqslant n$, with precision ε, where the values $f(y_i)$ are primitive recursive real numbers. To this end, we should carefully select a rational ε-approximation f^ε of the values of f and approximations y_i^ε of the points y_i and try to solve the rational system $f^\varepsilon(y_i^\varepsilon) = f^\varepsilon$. Rational extension property permits us to solve this system primitively recursively, the only problem being the choice of the good approximations of the input parameters: it may well happen that the current rational approximation f^ε of f is not a Katětov map over the approximations y_i^ε. However, the original proof can give us the tools to obtain the needed approximations, and we demonstrate that they can be found in a primitive recursive way.

We show that it is possible to primitively recursively compute approximate realizations of arbitrary primitive recursive Katětov maps f over elements of $\mathcal{U}$. To find such an ε-realization, compute approximations of the distances $d(y_i, y_j)$, $i, j \leqslant n$, with precision $\frac{\varepsilon}{6}$. Note that for some pairs $i \neq j$, it may hold $d^{\frac{\varepsilon}{6}}(y_i, y_j) \leqslant \frac{\varepsilon}{6}$, so at this moment we don't see that $d(y_i, y_j) > 0$ (though we know it is nonzero).

Inequality $d^{\frac{\varepsilon}{6}}(y_i, y_j) \leqslant \frac{\varepsilon}{6}$ means that $d(y_i, y_j) < \frac{\varepsilon}{3}$. But by the definition of a Katětov map we then have $|f(y_i) - f(y_j)| \leqslant d(y_i, y_j) < \frac{\varepsilon}{3}$. Now, we discard all excessive nearby points, or more precisely, refine the collection $Y = \{y_i^\varepsilon\}_{i\leqslant n}$ to only include such points y_i, y_j that $d^{\frac{\varepsilon}{6}}(y_i, y_j) > \frac{\varepsilon}{6}$. Clearly, such a refinement $Y' \subseteq Y$ can be defined in multiple ways, but we can always select some uniform primitive recursive way to perform it. Then, if we find an approximate realization of f for the refined system with precision $\frac{\varepsilon}{3}$, it will give us an approximate realization for the whole system with precision ε. Indeed, let y^ε be an $\frac{\varepsilon}{3}$-realization of $f \upharpoonright Y'$, i.e. $|d(y_i, y^\varepsilon) - f(y_i)| \leqslant \frac{\varepsilon}{3}$ for all $y_i \in Y'$. Then for all $y_i \in Y'$ and all $y_j \in Y$ such that $d^{\frac{\varepsilon}{6}}(y_i, y_j) \leqslant \frac{\varepsilon}{6}$ we have

$$\begin{aligned}|d(y^\varepsilon, y_j) - f(y_j)| &\leqslant |d(y^\varepsilon, y_j) - d(y^\varepsilon, y_i)| + |d(y^\varepsilon, y_i) - f(y_i)| + |f(y_i) - f(y_j)| \\ &\leqslant d(y_i, y_j) + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} \leqslant \varepsilon.\end{aligned}$$

So, henceforth we will assume that with the current precision ε we already see that $d(y_i, y_j) > 0$ for all $i \neq j \leqslant n$.

Like in [Ury51], choose a nonzero $\varepsilon_0 < \frac{\varepsilon}{6(n+1)}$ such that

$$(3n+4)\varepsilon_0 < \min_{i \neq j \leqslant n} d(y_i, y_j)$$

(we see that this minimum is nonzero thus such an ε_0 exists). Let f_i^ε be rational approximations of $f(y_i)$ with precision $\frac{\varepsilon_0}{2}$, then we have

$$f(y_i) - \tfrac{\varepsilon_0}{2} < f_i^\varepsilon < f(y_i) + \tfrac{\varepsilon_0}{2}.$$

We will assume, perhaps after a re-enumeration of Y, that $f_i^\varepsilon \geqslant f_j^\varepsilon$ for $i \leqslant j$. Using f_i^ε, find rational numbers β_i^ε such that

$$f(y_i) + (3(i+1) - 1)\varepsilon_0 < \beta_i^\varepsilon < f(y_i) + 3(i+1)\varepsilon_0.$$

Lemma A.1. *The mapping $f^\varepsilon(z_i) = \beta_i^\varepsilon$ satisfies $f^\varepsilon \in E(\{z_0, \ldots, z_n\})$ for all $z_0, \ldots, z_n$ such that $d(z_i, y_i) < \varepsilon_0$ for all i.*

Proof. We need to check that $|\beta_i^\varepsilon - \beta_j^\varepsilon| \leqslant d(z_i, z_j) \leqslant \beta_i^\varepsilon + \beta_j^\varepsilon$ for all $i, j \leqslant n$. Indeed, let $i > j$. Then $f_i^\varepsilon \leqslant f_j^\varepsilon$, which implies $f(y_i) - f(y_j) \leqslant \varepsilon_0$, and then

$$\begin{aligned}\beta_i^\varepsilon - \beta_j^\varepsilon &< f(y_i) - f(y_j) + \big(3(i+1) - (3(j+1) - 1)\big)\varepsilon_0 \\ &\leqslant \varepsilon_0 + (3(n+1) - 2)\varepsilon_0 = (3n+4)\varepsilon_0 - 2\varepsilon_0 \\ &< d(y_i, y_j) - d(y_i, z_i) - d(y_j, z_j) \leqslant d(z_i, z_j).\end{aligned}$$

On the other hand,

$$\begin{aligned}\beta_j^\varepsilon - \beta_i^\varepsilon &< f(y_j) - f(y_i) + \big(3(j+1) - (3(i+1) - 1)\big)\varepsilon_0 \\ &= f(y_j) - f(y_i) - (3(i-j))\varepsilon_0 + \varepsilon_0 \leqslant f(y_j) - f(y_i) - 2\varepsilon_0 \\ &< d(y_i, y_j) - d(y_i, z_i) - d(y_j, z_j) \leqslant d(z_i, z_j).\end{aligned}$$

Finally,

$$d(z_i, z_j) \leqslant d(y_i, y_j) + 2\varepsilon_0 \leqslant f(y_i) + f(y_j) + 2\varepsilon_0 < \beta_i^\varepsilon + \beta_j^\varepsilon. \qquad \square$$

We have obtained a rational-valued Katětov map $f^\varepsilon(z_i) = \beta_i^\varepsilon$. If the points z_i approximating y_i belong to $\mathbb{QU}$, the rational extension property allows us to primitively recursively find a realization $y^\varepsilon \in \mathbb{QU}$ of this map. Inequalities $d(z_i, y_i) < \varepsilon_0 < \frac{\varepsilon}{2}$ and $|\beta_i^\varepsilon - f(y_i)| < 3(n+1)\varepsilon_0 < \frac{\varepsilon}{2}$ readily imply $|d(y_i, y^\varepsilon) - f(y_i)| < \varepsilon$ for $i \leqslant n$. Thus, the point y^ε is the desired approximate realization of f.

Returning to the primitive recursive embedding, following the proof of Theorem 3.4 of [Mel08], we construct a sequence of points $(t_p)_p$ of $\mathbb{QU}$ that will be a Cauchy name for the point $y_{n+1} = g(\alpha_{n+1})$. More precisely, the sequence $(t_p)_p$ will satisfy $|d(t_p, y_i) - \rho(\alpha_{n+1}, \alpha_i)| \leqslant 2^{-p}$ for all i and $d(t_p, t_{p+1}) \leqslant 2^{1-p}$. By the above, we can find a 1-approximation t_0 of y_{n+1}. If a 2^{-k}-approximation t_k has already been found, let f_k be the image of t_k in $E(y_0, \dots, y_n)$ under the Kuratowski embedding: this embedding maps t_k to f_k satisfying $f_k(y_i) = d(t_k, y_i)$. In $E(y_0, \dots, y_n)$ it then holds $d(f_k, f) = \sup_{i \leqslant n} |f_k(y_i) - f(y_i)| \leqslant 2^{-k}$, where $f(y_i) = d(\alpha_{n+1}, \alpha_i)$.

The real number $d(f_k, f)$ is primitive recursive, thus the map g_k given by $g_k(y_i) = f(y_i)$, $g_k(t_k) = d(f_k, f)$ is primitive recursive Katětov over $\{y_0, \dots, y_n, t_k\}$. Find an approximate realization of this map with precision $2^{-(k+1)}$, this will be the next approximation t_{k+1} of y (see the proof of Theorem 3.4 of [Mel08]). Note that our proof provides an explicit (primitive recursive) required precision ε_0 for the input y_i, g_k that guarantees a suitable precision ε for the output, thus t_{k+1} is found primitively recursively, and g is a primitive recursive embedding of X into $\mathbb{U}$. $\square$

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