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A LEGENDRE-GAUSS COLLOCATION METHOD FOR NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce an efficient Legendre-Gauss collocation method for solving nonlinear delay differential equations with variable delay. We analyze the convergence of the single-step and multi-domain versions of the proposed method, and show that the scheme enjoys high order accuracy and can be implemented in a stable and efficient manner. We also make numerical comparison with other methods.

1. **Introduction.** Delay differential equations (DDEs) naturally arise in diverse science and engineering applications [25], and in recent years, a large body of literatures has been devoted to numerical solutions of DDEs (see, e.g., [3]-[6] and [9, 10, 14, 26]). Among the existing methods, numerical schemes based on Taylor's expansions or quadrature formulas have been frequently used (cf. [11, 12, 25, 26, 28] and the references therein), e.g., the implicit Runge-Kutta methods, which can be systematically designed and often provide accurate approximations. Over the years, spectral method has become increasingly popular and been widely used in spatial discretizations of PDEs owing to its high order of accuracy (cf. [7, 8, 13, 16, 17, 18]). The solution of a DDE globally depends on its history due to the delay variable, so a global spectral method could be a good candidate for numerical DDEs. Some work has been done along this line, and we particularly point out that Ito, Tran and Manitius [27] proposed and analyzed a Legendre-tau method for linear DDEs

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with one constant delay, where the solution was approximated by a truncated Legendre series, and the unknown expansion coefficients was solved from a Lanczos-tau formulation. Moreover, Guo and Wang [21], and Guo and Yan [23] developed the Legendre-Gauss collocation methods for ODEs based on Legendre polynomial expansions. Meanwhile, Guo and Wang [20], and Guo et al. [22] discussed the Laguerre-Gauss-type collocation methods for ODEs. However, it is more interesting but challenging to develop and analyze such type of high-order methods for nonlinear DDEs with variable delay of the form

$$\begin{cases} U'(t) = f(U(t), W(t), t), & 0 < t \leq T, \\ U(t) = V(t), & t \leq 0, \end{cases} \tag{1}$$

where $W(t) = U(t - \theta(t))$, f, θ and V are given functions and the delay variable: $\theta(t) \geq 0$. We start with a single-step scheme motivated by [21]. Basically, we approximate the solution by a finite Legendre series, and collocate the numerical scheme at Legendre-Gauss points to determine the coefficients. Due to the delay variable, it's inevitable to solve nonlinear systems likewise for the implicit Runge-Kutta methods, but the iterative solvers can be implemented efficiently thanks to the fast transform [2]. The scheme has an infinite order of accuracy both in time and the delay variable, and is particularly attractive for DDEs with highly oscillatory solutions and/or stiff behavior. Therefore, it enjoys some remarkable advantages over the Runge-Kutta-type methods.

For a more effective implementation, we suggest a multi-domain scheme to advance in time subinterval by subinterval due to the following twofold considerations. Firstly, the resultant system for the expansion coefficients can be solved more stably and efficiently for a small or modest number of unknowns. In particular, for large T, it is desirable to partition the solution interval $(0, T)$ and solve the subsystems successively. We also want to make a point that we merely need to store the Legendre coefficients of the solution in relevant "delay" subintervals to recover the solution in the subinterval that needs to resolve. Hence, the scheme can be implemented efficiently and economically. On the other hand, the multi-domain scheme provides us a flexibility to handle DDEs involving non-smooth initial data and/or solutions. In such cases, the partition of the interval can be adapted to the evolution process as the adaptive Runge-Kutta methods, and we are able to place more points in the subintervals that are needed.

The rest of the paper is organized as follows. In the next section, we present and analyze the single-step Legendre-Gauss collocation method, and provide some numerical results to justify our theoretical analysis. The multi-domain version is described, and the convergence result is also derived and numerically illustrated in Section 3. The final section is for some concluding discussions.

2. Single-step Legendre-Gauss collocation method.

In this section, we describe and analyze a single-step numerical integration process for the DDE (1) using the Legendre-Gauss interpolation, which serves as a base for the multi-domain scheme to be presented in the forthcoming section.

2.1. Preliminaries.

Let $L_l(\cdot)$ be the Legendre polynomial of degree l defined on $[-1, 1]$. The shifted Legendre polynomials $L_{T,l}(t)$, $t \in [0, T]$, are defined by (cf. [21])

$$L_{T,l}(t) = L_l\Big(\frac{2t}{T} - 1\Big) = \frac{(-1)^l}{l!} \frac{d^l}{dt^l} \Big\{ t^l \Big(1 - \frac{t}{T}\Big)^l \Big\}, \qquad l = 0, 1, 2, \cdots .$$

According to the properties of the standard Legendre polynomials, we have

$$\begin{aligned}&(l+1)L_{T,l+1}(t)-(2l+1)\Big(\frac{2t}{T}-1\Big)L_{T,l}(t)+lL_{T,l-1}(t)=0, \qquad l\geq 1,\\&L_{T,0}(t)=1, \quad L_{T,1}(t)=\frac{2t}{T}-1, \quad L_{T,2}(t)=\frac{6t^2}{T^2}-\frac{6t}{T}+1.\end{aligned} \tag{2}$$

Moreover, there holds the recursive relation

$$L'_{T,l+1}(t)-L'_{T,l-1}(t)=\frac{2}{T}(2l+1)L_{T,l}(t), \qquad l\geq 1. \tag{3}$$

The set of $L_{T,l}(t)$ forms a complete $L^2(0,T)-$orthogonal system, namely,

$$\int_0^T L_{T,l}(t)L_{T,m}(t)dt=\frac{T}{2l+1}\delta_{l,m}, \tag{4}$$

where $\delta_{l,m}$ is the Kronecker symbol. Thus for any $v\in L^2(0,T)$, we write

$$v(t)=\sum_{l=0}^{\infty}\widehat{v}_{T,l}L_{T,l}(t), \qquad \widehat{v}_{T,l}=\frac{2l+1}{T}\int_0^T v(t)L_{T,l}(t)dt. \tag{5}$$

We now turn to the Legendre-Gauss interpolation. Let $\{t_j^N,\omega_j^N\}_{j=0}^N$ be the Legendre-Gauss quadrature nodes (in $(-1,1)$ and arranged in ascending order) and weights. Accordingly, we define the shifted Legendre-Gauss interpolation nodes and weights as

$$t_{T,j}^N=\frac{T}{2}(t_j^N+1), \quad \omega_{T,j}^N=\frac{T}{2}\omega_j^N, \quad j=0,1,\cdots,N.$$

Let $\mathcal{P}_N(0,T)$ be the set of all real polynomials of degree at most N. We recall that for any $\phi\in\mathcal{P}_{2N+1}(0,T)$,

$$\int_0^T\phi(t)dt=\frac{T}{2}\int_{-1}^1\phi\Big(\frac{T}{2}(t+1)\Big)dt=\frac{T}{2}\sum_{j=0}^N\omega_j^N\phi\Big(\frac{T}{2}(t_j^N+1)\Big)=\sum_{j=0}^N\omega_{T,j}^N\phi(t_{T,j}^N). \tag{6}$$

Denote by $(u,v)_T$ and $\|v\|_T$ the inner product and the norm of the space $L^2(0,T)$, respectively. Define the following discrete inner product and norm associated with the Legendre-Gauss quadrature:

$$\langle u,v\rangle_{T,N}=\sum_{j=0}^N u(t_{T,j}^N)v(t_{T,j}^N)\omega_{T,j}^N, \qquad \|v\|_{T,N}=\langle v,v\rangle_{T,N}^{\frac{1}{2}}.$$

Thanks to (6), for any $\phi\psi\in\mathcal{P}_{2N+1}(0,T)$ and $\varphi\in\mathcal{P}_N(0,T)$,

$$(\phi,\psi)_T=\langle\phi,\psi\rangle_{T,N}, \qquad \|\varphi\|_T=\|\varphi\|_{T,N}. \tag{7}$$

For any $v\in C(0,T)$, the shifted Legendre-Gauss interpolant $\mathcal{I}_{T,N}v(t)\in\mathcal{P}_N(0,T)$ is determined by

$$\mathcal{I}_{T,N}v(t_{T,j}^N)=v(t_{T,j}^N), \qquad 0\leq j\leq N.$$

In view of (7), we have that for any $\phi\in\mathcal{P}_{N+1}(0,T)$,

$$(\mathcal{I}_{T,N}v,\phi)_T=\langle\mathcal{I}_{T,N}v,\phi\rangle_{T,N}=\langle v,\phi\rangle_{T,N}. \tag{8}$$

We can expand $\mathcal{I}_{T,N}v(t)$ as

$$\mathcal{I}_{T,N}v(t)=\sum_{l=0}^N\widetilde{v}_{T,l}^N L_{T,l}(t), \tag{9}$$

where by (4) and (8),

$$\widetilde{v}_{T,l}^N = \frac{2l+1}{T}(\mathcal{I}_{T,N}v, L_{T,l})_T = \frac{2l+1}{T}\langle v, L_{T,l}\rangle_{T,N}, \qquad 0 \le l \le N. \tag{10}$$

Furthermore, one verifies from (4), (7) and (10) that for any $\psi \in \mathcal{P}_{N+1}(0,T)$,

$$\psi(t) = \sum_{l=0}^{N+1} \widehat{\psi}_{T,l}^N L_{T,l}(t),\ \mathcal{I}_{T,N}\psi(t) = \sum_{l=0}^{N} \widetilde{\psi}_{T,l}^N L_{T,l}(t) \implies \widetilde{\psi}_{T,l}^N = \widehat{\psi}_{T,l}^N,\ 0 \le l \le N.$$

which implies that (cf. [21])

$$\|\psi\|_{T,N} \le \|\psi\|_T, \quad \forall \psi \in \mathcal{P}_{N+1}(0,T). \tag{11}$$

Let r be a nonnegative integer, $H^r(0,T)$ be the usual Sobolev space as defined in [1], and denote by its norm and semi-norm by $\|\cdot\|_{r,T}$ and $|\cdot|_{r,T}$, respectively. There holds the following estimates (see, e.g., Formulas (5.4.33) and (5.4.34) of [13]).

Lemma 2.1. *For any $u \in H^r(0,T)$ with integer $1 \le r \le N+1$,*

$$\|\mathcal{I}_{T,N}u - u\|_T \le cT^r N^{-r} |u|_{r,T}, \tag{12}$$

and

$$\big\|(\mathcal{I}_{T,N}u - u)'\big\|_T \le cT^{r-1} N^{\frac{3}{2}-r} |u|_{r,T}. \tag{13}$$

We notice that the Legendre-Gauss interpolation estimate in H^1-norm is optimal (cf. [13]). The semi-norm in the upper bound can be improved to the weighted semi-norm $\|t^{r/2}(T-t)^{r/2}u^{(r)}\|_T$ as in [19].

2.2. **The single-step scheme.** We next present the single-step scheme for the delay differential equation (1). For this purpose, we denote the grid set by $\Lambda_N := \big\{t_{T,k}^N\ :\ 0 \le k \le N\big\} \subset (0,T)$. The single-step Legendre-Gauss collocation approximation to (1) is to find $u^N(t) \in \mathcal{P}_{N+1}(0,T)$, such that

$$\frac{d}{dt}u^N(t) = f(u^N(t), v^N(t), t), \quad \forall\, t \in \Lambda_N; \quad u^N(0) = U(0) = V(0), \tag{14}$$

where the delay term

$$v^N(t) = \begin{cases} u^N(t-\theta(t)), & \forall\, t \in \Lambda_N \cap \{t : t > \theta(t)\}, \\ V(t-\theta(t)), & \forall\, t \in \Lambda_N \cap \{t : t \le \theta(t)\}. \end{cases} \tag{15}$$

Here, we recall that $\theta(t) \ge 0$ and $V(\cdot)$ is a known function. Denote by

$$\Lambda_N^0 = \big\{t \in \Lambda_N\ :\ t \le \theta(t)\big\}, \quad \Lambda_N^1 = \big\{t \in \Lambda_N\ :\ t > \theta(t)\big\},$$

and set $w^N(t) = u^N(t-\theta(t))$. The collocation scheme (14)-(15) can be reformulated as: Find $u^N(t) \in \mathcal{P}_{N+1}(0,T)$ such that

$$\frac{d}{dt}u^N(t) = \begin{cases} f\big(u^N(t), w^N(t), t\big), & \forall\, t \in \Lambda_N^1, \\ f\big(u^N(t), V(t-\theta(t)), t\big), & \forall\, t \in \Lambda_N^0, \end{cases} \tag{16}$$

supplemented with the initial condition $u^N(0) = V(0)$. We notice that it is an implicit scheme.

An important problem is how to resolve (16). Indeed, we may follow Lambert [28] to design an algorithm (mainly for ODEs) to resolve the discrete system (16) based on the Lagrange interpolation. However, it is known that Lagrange interpolation is not stable for large N. Hence, we propose a stable approach by expanding $u^N(t)$ directly in terms of the shifted Legendre polynomials and determining the unknown coefficients of the collocation scheme (16). This approach is stable for large N, and much easier to implement particularly for DDEs, since one only needs to store the coefficients of the numerical solution at each step. One may refer to Remark 2.1 of [21] for some more features of this method for ODEs.

We next describe the numerical implementation of the collocation scheme (16). For this purpose, we expand the collocation solution as

$$u^N(t)=\sum_{l=0}^{N+1}\widehat{u}_{T,l}^N L_{T,l}(t)\in\mathcal{P}_{N+1}(0,T),\quad 0<t\le T. \tag{17}$$

Furthermore, let $[l]$ be the integer part of l. According to [21], we have

$$\frac{d}{dt}L_{T,l}(t)=\frac{2}{T}\sum_{m=0}^{[\frac{l-1}{2}]}(2l-4m-1)L_{T,l-2m-1}(t).$$

On the other hand, $L_{T,l}(0)=(-1)^l$. Hence, (16) is equivalent to

$$\sum_{l=1}^{N+1}a_{T,k,l}^N\widehat{u}_{T,l}^N=f_{T,k}^N,\quad 0\le k\le N;\quad \sum_{l=0}^{N+1}(-1)^l\widehat{u}_{T,l}^N=V(0), \tag{18}$$

where

$$a_{T,k,l}^N=\frac{2}{T}\sum_{m=0}^{[\frac{l-1}{2}]}(2l-4m-1)L_{T,l-2m-1}(t_{T,k}^N),\quad 0\le k\le N,\ 1\le l\le N+1,$$

and

$$\begin{aligned}
f_{T,k}^N &= \begin{cases} f\Big(\sum\limits_{l=0}^{N+1}\widehat{u}_{T,l}^N L_{T,l}(t_{T,k}^N),\sum\limits_{l=0}^{N+1}\widehat{u}_{T,l}^N L_{T,l}(t_{T,k}^N-\theta(t_{T,k}^N)),t_{T,k}^N\Big), & t_{T,k}^N\in\Lambda_N^1,\\ f\Big(\sum\limits_{l=0}^{N+1}\widehat{u}_{T,l}^N L_{T,l}(t_{T,k}^N),V(t_{T,k}^N-\theta(t_{T,k}^N)),t_{T,k}^N\Big), & t_{T,k}^N\in\Lambda_N^0,\end{cases}\\
&= \begin{cases} f\Big(V(0)+\sum\limits_{l=1}^{N+1}\widehat{u}_{T,l}^N(L_{T,l}(t_{T,k}^N)-(-1)^l),V(0) & \\ \quad +\sum\limits_{l=1}^{N+1}\widehat{u}_{T,l}^N(L_{T,l}(t_{T,k}^N-\theta(t_{T,k}^N))-(-1)^l),t_{T,k}^N\Big), & t_{T,k}^N\in\Lambda_N^1,\\ f\Big(V(0)+\sum\limits_{l=1}^{N+1}\widehat{u}_{T,l}^N(L_{T,l}(t_{T,k}^N)-(-1)^l), & \\ \quad V(t_{T,k}^N-\theta(t_{T,k}^N)),t_{T,k}^N\Big), & t_{T,k}^N\in\Lambda_N^0.\end{cases}
\end{aligned}$$

Further, let

$$\widehat{\mathbf{u}}_T^N=\big(\widehat{u}_{T,1}^N,\widehat{u}_{T,2}^N,\cdots,\widehat{u}_{T,N+1}^N\big)',\qquad \mathbf{F}_T^N(\widehat{\mathbf{u}}_T^N)=\big(f_{T,0}^N,f_{T,1}^N,\cdots,f_{T,N}^N\big)',$$

and $\mathbb{A}_T^N$ be the matrix with the entries $a_{T,k,l}^N$, $0\le k\le N,\ 1\le l\le N+1$. Then we can rewrite (18) into the following matrix form:

$$\mathbb{A}_T^N\widehat{\mathbf{u}}_T^N=\mathbf{F}_T^N(\widehat{\mathbf{u}}_T^N);\quad \widehat{u}_{T,0}^N=V(0)-\sum_{l=1}^{N+1}(-1)^l\widehat{u}_{T,l}^N. \tag{19}$$

In actual computations, we solve $\{\widehat{u}_{T,l}^N\}_{l=0}^{N+1}$ from (19), and recover the collocation solution $u^N(t),\ 0<t\le T$ from (17).

We tabulate in Table 1 the condition numbers of $\mathbb{A}_T^N$ with $T=1$ and various N, which indicates that the condition numbers grow like $N^2/4$ as with the normal collocation scheme using Lagrangian basis.

Table 1. The condition numbers with $T=1$.

N	Cond. Num.	N	Cond. Num.
5	13.07571278675941	10	35.50775936479631
20	118.6786221997361	40	430.2160290393170
60	935.2398483834137	80	1633.731574173880
100	2525.687353973011	120	3611.105891809397

2.3. **Error analysis.** In this subsection, we analyze the convergence of the scheme (16). In particular, we shall prove the spectral accuracy of numerical solution $u^N(t)$. Let $\mathcal{I}_{T,N}$ be the Legendre-Gauss interpolation operator as defined before. Denote

$$E^N(t)=u^N(t)-\mathcal{I}_{T,N}U(t).$$

Lemma 2.1. *Let U and u^N be respectively solutions of (1) and (16). If $U\in H^r(0,T)$, with integer $2\le r\le N+1$, then for any $\varepsilon>0$,*

$$\begin{aligned}&(\tfrac{1}{2}-\epsilon)\|t^{-1}(E^N-E^N(0))\|_T^2+T^{-1}\big|E^N(T)-E^N(0)\big|^2\\ \le\,& c\epsilon^{-1}T^{2r-2}N^{3-2r}|U|_{r,T}^2+2\|G_{T,1}^N\|_{T,N}^2,\end{aligned}\tag{20}$$

where

$$G_{T,1}^N(t)=\begin{cases} f\big(u^N(t),w^N(t),t\big)-f\big(\mathcal{I}_{T,N}U(t),\mathcal{I}_{T,N}W(t),t\big), & t\in\Lambda_N^1,\\ f\big(u^N(t),V(t-\theta(t)),t\big)-f\big(\mathcal{I}_{T,N}U(t),V(t-\theta(t)),t\big), & t\in\Lambda_N^0.\end{cases}\tag{21}$$

Proof. Let

$$G_{T,2}^N(t)=\mathcal{I}_{T,N}\frac{d}{dt}U(t)-\frac{d}{dt}\mathcal{I}_{T,N}U(t).$$

Then we have from (1) that

$$\begin{cases}\dfrac{d}{dt}\mathcal{I}_{T,N}U(t)=f\big(U(t),W(t),t\big)-G_{T,2}^N(t), & t\in\Lambda_N^1,\\ \dfrac{d}{dt}\mathcal{I}_{T,N}U(t)=f\big(U(t),V(t-\theta(t)),t\big)-G_{T,2}^N(t), & t\in\Lambda_N^0.\end{cases}\tag{22}$$

Subtracting (22) from (16) yields

$$\begin{cases}\dfrac{d}{dt}E^N(t)=G_{T,1}^N(t)+G_{T,2}^N(t), \qquad t\in\Lambda_N^j,\quad j=0,1,\\ E^N(0)=U(0)-\mathcal{I}_{T,N}U(0).\end{cases}\tag{23}$$

Clearly, $t^{-1}(E^N(t)-E^N(0))\in\mathcal{P}_N(0,T)$. Thereby, by (7) and integration by parts, we deduce that

$$
\begin{aligned}
&\quad 2\big\langle E^N - E^N(0), \frac{d}{dt}(t^{-1}(E^N - E^N(0)))\big\rangle_{T,N} \\
&= 2\big(E^N - E^N(0), \frac{d}{dt}(t^{-1}(E^N - E^N(0)))\big)_T \\
&= -2\big(E^N - E^N(0), t^{-2}(E^N - E^N(0))\big)_T \\
&\quad +2\big(E^N - E^N(0), t^{-1}\frac{d}{dt}(E^N - E^N(0))\big)_T \\
&= -\|t^{-1}(E^N - E^N(0))\|_T^2 + T^{-1}\big|E^N(T) - E^N(0)\big|^2.
\end{aligned} \tag{24}
$$

On the other hand, we have from (23) that

$$
\begin{aligned}
&\frac{d}{dt}\big(t^{-1}(E^N(t) - E^N(0))\big) = -t^{-2}(E^N(t) - E^N(0)) + t^{-1}\frac{d}{dt}E^N(t) \\
= &- t^{-2}\big(E^N(t) - E^N(0)\big) + t^{-1}\big(G^N_{T,1}(t) + G^N_{T,2}(t)\big), \quad t \in \Lambda^j_N,\ j = 0, 1.
\end{aligned}
$$

The above fact with (7) leads to that

$$
\begin{aligned}
&\quad 2\big\langle E^N - E^N(0), \frac{d}{dt}(t^{-1}(E^N - E^N(0)))\big\rangle_{T,N} \\
&= -2\big\langle E^N - E^N(0), t^{-2}(E^N - E^N(0))\big\rangle_{T,N} \\
&\quad +2\big\langle E^N - E^N(0), t^{-1}(G^N_{T,1} + G^N_{T,2})\big\rangle_{T,N} \\
&= -2\|t^{-1}(E^N - E^N(0))\|_T^2 + A^N_{T,1} + A^N_{T,2},
\end{aligned} \tag{25}
$$

where

$$
A^N_{T,1} = 2\big\langle t^{-1}(E^N - E^N(0)), G^N_{T,1}\big\rangle_{T,N}, \quad A^N_{T,2} = 2\big\langle t^{-1}(E^N - E^N(0)), G^N_{T,2}\big\rangle_{T,N}.
$$

Since $t^{-1}(E^N(t) - E^N(0)) \in \mathcal{P}_N(0,T)$ and $G^N_{T,2}(t) \in \mathcal{P}_N(0,T)$, we use (7) to obtain that for any $\epsilon > 0$,

$$
|A^N_{T,2}| = 2|(t^{-1}(E^N - E^N(0)), G^N_{T,2})_T| \le \epsilon\|t^{-1}(E^N - E^N(0))\|_T^2 + \epsilon^{-1}\|G^N_{T,2}\|_T^2.
$$

Inserting the above estimate and (24) into (25) gives that

$$
(1-\epsilon)\|t^{-1}(E^N - E^N(0))\|_T^2 + T^{-1}\big|E^N(T) - E^N(0)\big|^2 \le \epsilon^{-1}\|G^N_{T,2}\|_T^2 + A^N_{T,1}. \tag{26}
$$

We next estimate $\|G^N_{T,2}\|_T$. Clearly, by (12) with $\frac{dU}{dt}$ and $r-1$ instead of u and r, we have that for integer $r \ge 2$,

$$
\|\mathcal{I}_{T,N}\frac{d}{dt}U - \frac{d}{dt}U\|_T \le cT^{r-1}N^{1-r}|U|_{r,T}. \tag{27}
$$

Moreover, by (13),

$$
\|\frac{d}{dt}(\mathcal{I}_{T,N}U - U)\|_T \le cT^{r-1}N^{\frac{3}{2}-r}|U|_{r,T}. \tag{28}
$$

Therefore

$$
\|G^N_{T,2}\|_T \le \|\frac{d}{dt}(\mathcal{I}_{T,N}U - U)\|_T + \|\mathcal{I}_{T,N}\frac{d}{dt}U - \frac{d}{dt}U\|_T \le cT^{r-1}N^{\frac{3}{2}-r}|U|_{r,T}. \tag{29}
$$

On the other hand, by (7),

$$
|A^N_{T,1}| \le \frac{1}{2}\|t^{-1}(E^N - E^N(0))\|_T^2 + 2\|G^N_{T,1}\|_{T,N}^2.
$$

Substituting the above and (29) into (26), we obtain (20). $\square$

We now consider several typical f and analyze the numerical errors. Hereafter, β denotes any positive number less than $\frac{1}{4}$.

Case I. Consider (16) with the linear delay:

$$\theta(t) = \lambda t, \quad 0 \le \lambda < 1. \tag{30}$$

Assume that $f(x,y,t)$ satisfies the following Lipschitz conditions in x and y. That is, there exists real numbers $\gamma_1 \ge 0$ and $\gamma_2 \ge 0$ such that

$$|f(x_1,y,t) - f(x_2,y,t)| \le \gamma_1 |x_1 - x_2|, \tag{31}$$

and

$$|f(x,y_1,t) - f(x,y_2,t)| \le \gamma_2 |y_1 - y_2|. \tag{32}$$

Theorem 2.2. *If the conditions (30)-(32) hold, $U \in H^r(0,T)$, with integer $2 \le r \le N+1$, and for certain $\delta > 0$,*

$$(1+\delta)T^2\gamma_1^2 + (1+\delta^{-1})(1-\lambda)^{-1}T^2\gamma_2^2 \le \beta < \frac{1}{4}, \tag{33}$$

then

$$\|U - u^N\|_T^2 \le c_\beta T^{2r} N^{3-2r} |U|_{r,T}^2, \tag{34}$$

$$|U(T) - u^N(T)|^2 \le c_\beta T^{2r-1} N^{3-2r} |U|_{r,T}^2. \tag{35}$$

In particular

$$\max_{t\in[0,T]} |U(t) - u^N(t)|^2 \le c_\beta T^{2r-1} N^{3-2r} |U|_{r,T}^2, \tag{36}$$

where c_β is a positive constant depending only on β.

Proof. Obviously, in this case, $\Lambda_N^0 = \emptyset$. Moreover, $w^N(t) \in \mathcal{P}_{N+1}(0,T)$. Therefore, by virtue of (11), (31) and (32), for any $\delta > 0$,

$$\begin{aligned} \|G_{T,1}^N\|_{T,N}^2 \le\ & (1+\delta)\|f(u^N, w^N, \cdot) - f(\mathcal{I}_{T,N}U, w^N, \cdot)\|_{T,N}^2 \\ & + (1+\delta^{-1})\|f(\mathcal{I}_{T,N}U, w^N, \cdot) - f(\mathcal{I}_{T,N}U, \mathcal{I}_{T,N}W, \cdot)\|_{T,N}^2 \\ \le\ & (1+\delta)\gamma_1^2\|E^N\|_T^2 + (1+\delta^{-1})\gamma_2^2\|w^N - \mathcal{I}_{T,N}W\|_T^2. \end{aligned} \tag{37}$$

On the other hand, by virtue of (12) and a direct calculation, we deduce that for any $\epsilon > 0$,

$$\begin{aligned} \|w^N - \mathcal{I}_{T,N}W\|_T^2 &\le (1+\epsilon)\|w^N - W\|_T^2 + (1+\epsilon^{-1})\|W - \mathcal{I}_{T,N}W\|_T^2 \\ &\le (1+\epsilon)(1-\lambda)^{-1}\|U - u^N\|_T^2 + (1+\epsilon^{-1})\|W - \mathcal{I}_{T,N}W\|_T^2 \\ &\le (1+\epsilon)(1-\lambda)^{-1}\|U - u^N\|_T^2 + c\epsilon^{-1}T^{2r}N^{-2r}|W|_{r,T}^2 \\ &\le (1+\epsilon)(1-\lambda)^{-1}\|U - u^N\|_T^2 + c\epsilon^{-1}T^{2r}N^{-2r}|U|_{r,T}^2. \end{aligned} \tag{38}$$

Due to (33), we have that $(1+\delta^{-1})T^2\gamma_2^2 < \frac{1}{4}$. Hence, inserting the above two inequalities into (20), we obtain from (33) that

$$\begin{aligned} & \Big(\frac{1}{2} - \epsilon\Big)\|t^{-1}(E^N - E^N(0))\|_T^2 + T^{-1}\big|E^N(T) - E^N(0)\big|^2 \\ \le\ & 2(1+\delta)\gamma_1^2\|E^N\|_T^2 + 2(1+\delta^{-1})(1+\epsilon)(1-\lambda)^{-1}\gamma_2^2\|U - u^N\|_T^2 \\ & + c\epsilon^{-1}(1+\delta^{-1})\gamma_2^2 T^{2r}N^{-2r}|U|_{r,T}^2 + c\epsilon^{-1}T^{2r-2}N^{3-2r}|U|_{r,T}^2 \\ \le\ & 2(1+\delta)\gamma_1^2\|E^N\|_T^2 + 2(1+\delta^{-1})(1+\epsilon)(1-\lambda)^{-1}\gamma_2^2\|U - u^N\|_T^2 \\ & + c\epsilon^{-1}T^{2r-2}N^{3-2r}|U|_{r,T}^2. \end{aligned} \tag{39}$$

Moreover

$$\begin{aligned}&\Big(\frac{1}{2}-\epsilon\Big)\|E^N\|_T^2 \le \Big(\frac{1}{2}-\epsilon\Big)\Big((1+\epsilon)\|E^N-E^N(0)\|_T^2+(1+\epsilon^{-1})\|E^N(0)\|_T^2\Big)\\ \le&\Big(\frac{1}{2}-\epsilon\Big)\Big((1+\epsilon)T^2\|t^{-1}(E^N-E^N(0))\|_T^2+(1+\epsilon^{-1})T(E^N(0))^2\Big).\end{aligned} \tag{40}$$

Therefore, we use (39) and (40) to derive that

$$\begin{aligned}&\Big(\frac{1}{2}-\epsilon\Big)\|E^N\|_T^2+(1+\epsilon)T\big|E^N(T)-E^N(0)\big|^2\\ &\le(1+\epsilon)T^2\Big((\frac{1}{2}-\epsilon)\|t^{-1}(E^N-E^N(0))\|_T^2+T^{-1}\big|E^N(T)-E^N(0)\big|^2\Big)\\ &+c\epsilon^{-1}T(E^N(0))^2\\ \le&(1+\epsilon)T^2\Big(2(1+\delta)\gamma_1^2\|E^N\|_T^2+2(1+\delta^{-1})(1+\epsilon)(1-\lambda)^{-1}\gamma_2^2\|U-u^N\|_T^2\Big)\\ &+c\epsilon^{-1}T(E^N(0))^2+c\epsilon^{-1}T^{2r}N^{3-2r}|U|_{r,T}^2,\end{aligned} \tag{41}$$

or equivalently,

$$\begin{aligned}&\Big(\frac{1}{2}-\epsilon-2(1+\epsilon)(1+\delta)T^2\gamma_1^2\Big)\|E^N\|_T^2+(1+\epsilon)T\big|E^N(T)-E^N(0)\big|^2\\ \le&2(1+\delta^{-1})(1+\epsilon)^2(1-\lambda)^{-1}T^2\gamma_2^2\|U-u^N\|_T^2+c\epsilon^{-1}T(E^N(0))^2\\ &+c\epsilon^{-1}T^{2r}N^{3-2r}|U|_{r,T}^2.\end{aligned} \tag{42}$$

Thanks to (12), we have that

$$\begin{aligned}\|U-u^N\|_T^2&\le(1+\epsilon)\|E^N\|_T^2+(1+\epsilon^{-1})\|U-\mathcal{I}_{T,N}U\|_T^2\\ &\le(1+\epsilon)\|E^N\|_T^2+c\epsilon^{-1}T^{2r}N^{-2r}|U|_{r,T}^2.\end{aligned} \tag{43}$$

The above with (42) yields

$$\begin{aligned}&\Big(\tfrac{1}{2}-\epsilon-2(1+\epsilon)(1+\delta)T^2\gamma_1^2\Big)\|U-u^N\|_T^2+(1+\epsilon)^2T\big|E^N(T)-E^N(0)\big|^2\\ &\le(1+\epsilon)\Big(\tfrac{1}{2}-\epsilon-2(1+\epsilon)(1+\delta)T^2\gamma_1^2\Big)\|E^N\|_T^2\\ &\quad+(1+\epsilon)^2T\big|E^N(T)-E^N(0)\big|^2+c\epsilon^{-1}T^{2r}N^{-2r}|U|_{r,T}^2\\ &\le 2(1+\delta^{-1})(1+\epsilon)^3(1-\lambda)^{-1}T^2\gamma_2^2\|U-u^N\|_T^2\\ &\quad+c\epsilon^{-1}T^{2r}N^{3-2r}|U|_{r,T}^2+c\epsilon^{-1}T(E^N(0))^2,\end{aligned} \tag{44}$$

or equivalently,

$$\begin{aligned}&\Big(\tfrac{1}{2}-\epsilon-2(1+\epsilon)(1+\delta)T^2\gamma_1^2-2(1+\delta^{-1})(1+\epsilon)^3(1-\lambda)^{-1}T^2\gamma_2^2\Big)\|U-u^N\|_T^2\\ &+(1+\epsilon)^2T\big|E^N(T)-E^N(0)\big|^2\le c\epsilon^{-1}T^{2r}N^{3-2r}|U|_{r,T}^2+c\epsilon^{-1}T(E^N(0))^2.\end{aligned} \tag{45}$$

On the other hand, for any $v\in H^1(0,T)$ (see (3.9) of [21]),

$$\max_{t\in[0,T]}|v(t)|^2\le\frac{2}{T}\|v\|_T^2+2T\|\frac{dv}{dt}\|_T^2. \tag{46}$$

This, along with (12) and (13), leads to that

$$\begin{aligned}(E^N(0))^2=|\mathcal{I}_{T,N}U(0)-U(0)|^2&\le\frac{2}{T}\|\mathcal{I}_{T,N}U-U\|_T^2+2T\|\frac{d}{dt}(\mathcal{I}_{T,N}U-U)\|_T^2\\ &\le cT^{2r-1}N^{3-2r}|U|_{r,T}^2.\end{aligned} \tag{47}$$

Next let $\epsilon = (\frac{3}{4\beta+2})^{\frac{1}{3}} - 1 > 0$. Then $(1+\epsilon)^3(1+2\beta) = \frac{3}{2}$. Hence, by (33),

$$\begin{aligned}&\epsilon + 2(1+\epsilon)(1+\delta)T^2\gamma_1^2 + 2(1+\delta^{-1})(1+\epsilon)^3(1-\lambda)^{-1}T^2\gamma_2^2\\ &< (1+\epsilon)^3(1+2(1+\delta)T^2\gamma_1^2 + 2(1+\delta^{-1})(1-\lambda)^{-1}T^2\gamma_2^2) - 1\\ &\le (1+\epsilon)^3(1+2\beta) - 1 = \tfrac{1}{2}.\end{aligned}$$

A combination of the above estimate, (45) and (47) leads to (34). Further, by (45) and (47), we deduce that

$$(E^N(T))^2 \le 2\big|E^N(T) - E^N(0)\big|^2 + 2\big|E^N(0)\big|^2 \le c_\beta T^{2r-1}N^{3-2r}|U|^2_{r,T}.$$

Moreover, an argument similar to (47) yields

$$|\mathcal{I}_{T,N}U(T) - U(T)|^2 \le cT^{2r-1}N^{3-2r}|U|^2_{r,T}. \tag{48}$$

Consequently,

$$|U(T) - u^N(T)|^2 \le 2|\mathcal{I}_{T,N}U(T) - U(T)|^2 + 2\big|E^N(T)\big|^2 \le c_\beta T^{2r-1}N^{3-2r}|U|^2_{r,T}.$$

This leads to (35). Next, by using (37), (38), (34) and (12), we deduce that

$$\begin{aligned}\|G^N_{T,1}\|^2_{T,N} &\le (1+\delta)\gamma_1^2\|E^N\|^2_T + (1+\delta^{-1})(1+\epsilon)(1-\lambda)^{-1}\gamma_2^2\|U-u^N\|^2_T\\ &\quad + c\epsilon^{-1}(1+\delta^{-1})\gamma_2^2T^{2r}N^{-2r}|U|^2_{r,T}\\ &\le 2(1+\delta)\gamma_1^2(\|U-\mathcal{I}_{T,N}U\|^2_T + \|U-u^N\|^2_T) + c_\beta\gamma_2^2T^{2r}N^{3-2r}|U|^2_{r,T}\\ &\le c_\beta(\gamma_1^2+\gamma_2^2)T^{2r}N^{3-2r}|U|^2_{r,T}.\end{aligned} \tag{49}$$

Hence, by (7), (23), (28), (29), (49) and (33), we deduce that

$$\begin{aligned}\|\frac{d}{dt}(U-u^N)\|^2_T &\le 2\|\frac{d}{dt}E^N\|^2_T + 2\|\frac{d}{dt}(U-\mathcal{I}_{T,N}U)\|^2_T\\ &= 2\|\frac{d}{dt}E^N\|^2_{T,N} + 2\|\frac{d}{dt}(U-\mathcal{I}_{T,N}U)\|^2_T\\ &\le 4\|G^N_{T,1}\|^2_{T,N} + 4\|G^N_{T,2}\|^2_T + cT^{2r-2}N^{3-2r}|U|^2_{r,T}\\ &\le c_\beta T^{2r-2}N^{3-2r}|U|^2_{r,T}.\end{aligned} \tag{50}$$

Finally, by virtue of (46), (34) and (50), we obtain (36). □

Remark 2.1. The condition (33) is necessary for the proof, but it should not be essential. In fact, some numerical examples do not meet this condition, but the numerical scheme still converges. This remark also applies to multiple-domain cases.

Case II. Assume that the delay function satisfies:

$$t - \theta(t) \le 0, \quad t \in [0,T]. \tag{51}$$

Moveover, $f(x,y,t)$ satisfies the Lipschitz condition (31).

Theorem 2.3. *If the conditions* (51) *and* (31) *hold,* $U \in H^r(0,T)$, *with integer* $2 \le r \le N+1$, *and*

$$T^2\gamma_1^2 \le \beta < \frac{1}{4}, \tag{52}$$

then

$$\|U-u^N\|^2_T \le c_\beta T^{2r}N^{3-2r}|U|^2_{r,T}, \tag{53}$$

$$|U(T)-u^N(T)|^2 \le c_\beta T^{2r-1}N^{3-2r}|U|^2_{r,T}. \tag{54}$$

In particular

$$\max_{t\in[0,T]}|U(t)-u^N(t)|^2 \le c_\beta T^{2r-1}N^{3-2r}|U|^2_{r,T}. \tag{55}$$

Proof. Obviously, in this case, $\Lambda_N^1 = \emptyset$. Hence, by virtue of (11) and (31),

$$\|G_{T,1}^N\|_{T,N}^2 \le \gamma_1^2 \|E^N\|_T^2. \tag{56}$$

Inserting the above inequality into (20), we obtain

$$\begin{aligned}&(\tfrac{1}{2}-\epsilon)\|t^{-1}(E^N - E^N(0))\|_T^2 + T^{-1}\big|E^N(T)-E^N(0)\big|^2\\ &\le 2\gamma_1^2\|E^N\|_T^2 + c\epsilon^{-1}T^{2r-2}N^{3-2r}|U|_{r,T}^2.\end{aligned} \tag{57}$$

Therefore, we use (40), (57) and (47) to derive that

$$\begin{aligned}&(\tfrac{1}{2}-\epsilon)\|E^N\|_T^2 + (1+\epsilon)T\big|E^N(T)-E^N(0)\big|^2\\ &\le (1+\epsilon)T^2\Big((\tfrac{1}{2}-\epsilon)\|t^{-1}(E^N-E^N(0))\|_T^2 + T^{-1}\big|E^N(T)-E^N(0)\big|^2\Big)\\ &\quad + c\epsilon^{-1}T(E^N(0))^2 \le 2(1+\epsilon)T^2\gamma_1^2\|E^N\|_T^2 + c\epsilon^{-1}T^{2r}N^{3-2r}|U|_{r,T}^2,\end{aligned} \tag{58}$$

or equivalently,

$$\Big(\frac{1}{2}-\epsilon-2(1+\epsilon)T^2\gamma_1^2\Big)\|E^N\|_T^2 + (1+\epsilon)T\big|E^N(T)-E^N(0)\big|^2 \le c\epsilon^{-1}T^{2r}N^{3-2r}|U|_{r,T}^2. \tag{59}$$

Let $\epsilon = \frac{3}{8\beta+4} - \frac{1}{2} > 0$. Then by (52),

$$\epsilon + 2(1+\epsilon)T^2\gamma_1^2 = (1+\epsilon)(1+2T^2\gamma_1^2) - 1 \le (1+\epsilon)(1+2\beta) - 1 = \beta + \frac{1}{4} < \frac{1}{2}.$$

A combination of the above estimate, (43) and (59) leads to (53). Further, by (59) and (47), we deduce that

$$(E^N(T))^2 \le 2\big|E^N(T)-E^N(0)\big|^2 + 2\big|E^N(0)\big|^2 \le c_\beta T^{2r-1}N^{3-2r}|U|_{r,T}^2.$$

This, along with (48) yields (54). We can derive the result (55) easily. ☐

Remark 2.2. We see from (34), (35), (53) and (54) that the errors $|U(T)-u^N(T)|$ and $\|U-u^N\|_T$ decay rapidly as N and r increase. The convergence rate is $\mathcal{O}(N^{\frac{3}{2}-r})$. Thus, the smoother the exact solution, the smaller the numerical errors. In other words, the scheme (16) possesses the spectral accuracy.

Remark 2.3. If $\dfrac{d^rU}{dt^r} \in L^\infty(0,T)$, $N \ge r-1$ and $T \le 1$, then we use Theorems 2.1 and 2.2 to deduce that

$$\|U-u^N\|_T \le c_\beta^{\frac{1}{2}} T^{r+\frac{1}{2}} N^{\frac{3}{2}-r}\Big\|\frac{d^rU}{dt^r}\Big\|_{L^\infty(0,T)}, \tag{60}$$

$$|U(T)-u^N(T)| \le c_\beta^{\frac{1}{2}} T^{r} N^{\frac{3}{2}-r}\Big\|\frac{d^rU}{dt^r}\Big\|_{L^\infty(0,T)}. \tag{61}$$

In particular, if $\frac{1}{N} \le T \le 1$, then we may take $r = N+1$ in (60) and (61), to reach that

$$\|U-u^N\|_T = \mathcal{O}(T^{2N+1}), \qquad |U(T)-u^N(T)| = \mathcal{O}(T^{2N+\frac{1}{2}}). \tag{62}$$

2.4. **Numerical results.** In this subsection, we present some numerical results to illustrate the efficiency of our single-step algorithm.

❶ Linear variable delay (Case I):

$$\begin{cases} \dfrac{d}{dt}u(t) = \dfrac{1}{2}e^{\frac{t}{2}}u\Big(\dfrac{t}{2}\Big) + \dfrac{1}{2}u(t), \quad \text{for } 0 \le t \le T,\\ u(0) = 1. \end{cases} \tag{63}$$

As pointed out in [15], the exact solution is $u(t) = e^t$. Obviously, the conditions (31) and (32) hold with $\gamma_1 = \dfrac{1}{2}$ and $\gamma_2 = \dfrac{1}{2}e^{\frac{T}{2}}$. Moreover, the inequality (33) is satisfied for $T = 0.3$. But for $T = 1$, the inequality (33) is no longer valid. In Fig. 1, we plot the numerical errors at $t = T$ with $T = 0.3$, 1 and various values of N. They indicate that the numerical errors decay exponentially as N increases. In particular, we can observe from Fig. 1 that even if the condition (33) is not satisfied (for instance, T=1), our algorithm is still valid.

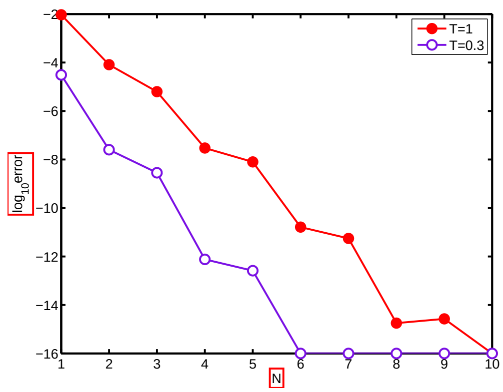

FIGURE 1. The numerical errors of Legendre-Gauss collocation method at $t = T$.

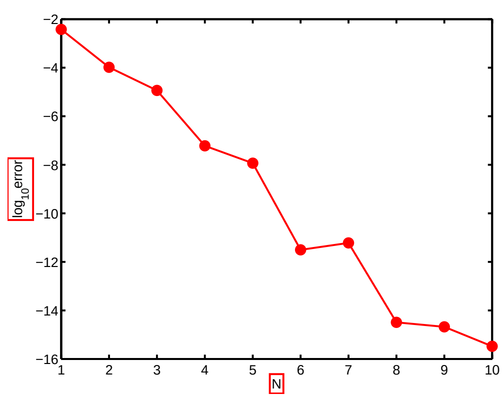

FIGURE 2. The numerical errors of Legendre-Gauss collocation method at $t = 1$.

❷ Nonlinear variable delay (Case I):

$$\begin{cases} \dfrac{d}{dt}u(t) = 1 - 2u^2\Big(\dfrac{t}{2}\Big), & \text{for } 0 \le t \le 1, \\ u(0) = 0. \end{cases} \tag{64}$$

This is a nonlinear DDE with the exact solution $u(t) = \sin(t)$, see [15]. In Fig. 2, we plot the numerical errors at $t = 1$ using the single-step scheme (16) and Newton-Raphson iteration method with various values of N. They indicate that the numerical errors decay exponentially as N increases. We also note that the condition (32) is not satisfied for problem (64), but our algorithm is still valid.

❸ Linear constant delay (Case II):

$$\begin{cases} \dfrac{d}{dt}u(t) = -u\Big(t-\dfrac{\pi}{2}\Big), & 0 \le t \le \frac{\pi}{2}, \\ u(t) = \sin(t), & -\frac{\pi}{2} \le t \le 0. \end{cases} \tag{65}$$

The exact solution is $u(t) = \sin(t)$. Obviously, the condition (31) hold with $\gamma_1 = 0$. Moreover, the inequality (51) is satisfied. In Fig. 3, we plot the numerical errors at $t = \frac{\pi}{2}$ with various values of N. They indicate that the numerical errors decay exponentially as N increases.

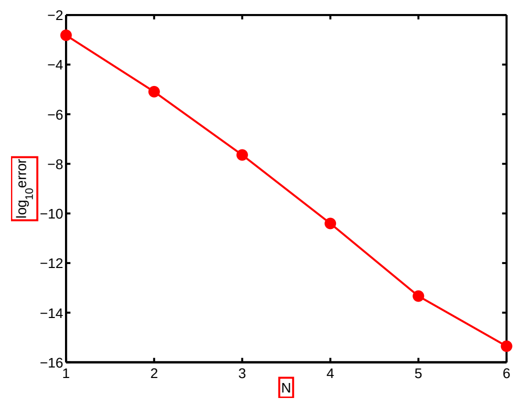

FIGURE 3. The numerical errors of Legendre-Gauss collocation method at $t = \frac{\pi}{2}$.

3. **Multiple-domain Legendre-Gauss collocation method.** In the last section, we investigated the single-step Legendre-Gauss collocation method. The numerical errors decay very rapidly as N and r increase. While the foregoing single-step collocation method provide accurate results, in actual computation, it is not convenient to resolve the discrete system (19) with very large mode N. As commented in the introduction, it is advisable to partition the interval $(0, T)$ into a finite number of subintervals and solve the equations subsequently on each subinterval.

3.1. **The multiple-domain scheme.** We now describe the multiple-domain scheme. Let M and N_m, $1 \le m \le M$ be any positive integers. We first decompose the interval $(0, T]$ into M subintervals $(T_{m-1}, T_m]$, $1 \le m \le M$, such that the set of T_m includes all breaking points, where $T_0 = 0$ and $T_M = T$. Denote by $\tau_m = T_m - T_{m-1}$, $1 \le m \le M$. We shall use $u_m^{N_m}(t) \in \mathcal{P}_{N_m+1}(0, \tau_m)$ to approximate the solution U in the subinterval $(T_{m-1}, T_m]$.

Firstly, replacing T and N by τ_1 and N_1 in (16) and all other formulas in Subsection 2.2, we can derive an alternative algorithm, with which we obtain the numerical solution $u_1^{N_1}(t) \in \mathcal{P}_{N_1+1}(0, \tau_1)$. Then we evaluate the numerical solutions $u_m^{N_m}(t) \in \mathcal{P}_{N_m+1}(0, \tau_m)$, $2 \le m \le M$, step by step. Finally, the global numerical solution of (1) is given by

$$u^N(T_{m-1} + t) = u_m^{N_m}(t), \qquad 0 \le t \le \tau_m, \qquad 1 \le m \le M. \tag{66}$$

We now present the numerical scheme for $u_m^{N_m}(t)$. Denote by $t_{\tau_m,k}^{N_m}$ and $\omega_{\tau_m,k}^{N_m}$, $0 \le k \le N_m$ the nodes and the corresponding Christoffel numbers of the shifted Legendre-Gauss interpolation on the interval $(0, \tau_m)$. Let

$$\Lambda_{N,m}^0 = \{t_{\tau_m,k}^{N_m} \mid T_{m-1} + t_{\tau_m,k}^{N_m} - \theta(T_{m-1} + t_{\tau_m,k}^{N_m}) \le 0,\ 0 \le k \le N_m\},$$

and

$$\begin{aligned}&\Lambda_{N,m}^j\\ =&\{t_{\tau_m,k}^{N_m} \mid T_{m-1} + t_{\tau_m,k}^{N_m} - \theta(T_{m-1} + t_{\tau_m,k}^{N_m}) \in (T_{j-1}, T_j],\ 0 \le k \le N_m\},\ 1 \le j \le m.\end{aligned}$$

The multiple-domain collocation method for (1) is to seek $u_m^{N_m}(t) \in \mathcal{P}_{N_m+1}(0, \tau_m)$, such that

$$\begin{cases} \dfrac{d}{dt}u_m^{N_m}(t) = f(u_m^{N_m}(t), w_m^j(t), T_{m-1} + t), t \in \Lambda_{N,m}^j,\ j > 0,\\ \dfrac{d}{dt}u_m^{N_m}(t) = f(u_m^{N_m}(t), V(T_{m-1} + t - \theta(T_{m-1} + t)), T_{m-1} + t), t \in \Lambda_{N,m}^0,\\ u_m^{N_m}(0) = u_{m-1}^{N_{m-1}}(\tau_{m-1}), 2 \le m \le M, \end{cases} \tag{67}$$

where

$$w_m^j(t) = u^N(T_{m-1} + t - \theta(T_{m-1} + t)) = u_j^{N_j}(T_{m-1} - T_{j-1} + t - \theta(T_{m-1} + t)).$$

Denote by $U_m(t) = U(T_{m-1} + t)$ for $0 \le t \le \tau_m$. Then by (1),

$$\begin{cases} \dfrac{d}{dt}U_m(t) = f(U_m(t), W_m^j(t), T_{m-1} + t), t \in \Lambda_{N,m}^j,\ j > 0,\\ \dfrac{d}{dt}U_m(t) = f(U_m(t), V(T_{m-1} + t - \theta(T_{m-1} + t)), T_{m-1} + t), t \in \Lambda_{N,m}^0,\\ U_m(0) = U_{m-1}(\tau_{m-1}), 2 \le m \le M,\\ U_1(0) = U(0) = V(0), \end{cases} \tag{68}$$

where

$$W_m^j(t) = U(T_{m-1} + t - \theta(T_{m-1} + t)) = U_j(T_{m-1} - T_{j-1} + t - \theta(T_{m-1} + t)).$$

We see from (67) and (68) that the local numerical solution $u_m^{N_m}(t)$ is actually an approximation to the local exact solution $U_m(t)$, with the approximate initial data $u_m^{N_m}(0) = u_{m-1}^{N_{m-1}}(\tau_{m-1})$.

3.2. **Error analysis.** We next analyze the numerical errors. Denote

$$E_m^{N_m}(t) = u_m^{N_m}(t) - \mathcal{I}_{\tau_m,N_m}U_m(t).$$

Lemma 3.1. *Let U_m and $u_m^{N_m}$ be respectively solutions of (68) and (67). If $U_m \in H^r(0, \tau_m)$, with integer $2 \le r \le N_m + 1$, then for any $\varepsilon > 0$,*

$$\begin{aligned}&(\frac{1}{2} - \epsilon)\|t^{-1}(E_m^{N_m} - E_m^{N_m}(0))\|_{\tau_m}^2 + \tau_m^{-1}\big|E_m^{N_m}(\tau_m) - E_m^{N_m}(0)\big|^2\\ \le& c\epsilon^{-1}\tau_m^{2r-2}N_m^{3-2r}|U_m|_{r,\tau_m}^2 + 2\|G_{\tau_m,1}^{N_m}\|_{\tau_m,N_m}^2,\end{aligned} \tag{69}$$

where

$$G_{\tau_m,1}^{N_m}(t) = \begin{cases} f(u_m^{N_m}(t), w_m^j(t), T_{m-1} + t) - f(\mathcal{I}_{\tau_m,N_m}U_m(t), W_m^j(t), T_{m-1} + t),\\ \qquad\qquad\qquad\qquad t \in \Lambda_{N,m}^j,\ j > 0,\\ f(u_m^{N_m}(t), V(T_{m-1} + t - \theta(T_{m-1} + t)), T_{m-1} + t)\\ -f(\mathcal{I}_{\tau_m,N_m}U_m(t), V(T_{m-1} + t - \theta(T_{m-1} + t)), T_{m-1} + t),\ t \in \Lambda_{N,m}^0. \end{cases}$$

Proof. Let

$$G^{N_m}_{\tau_m,2}(t) = \mathcal{I}_{\tau_m,N_m}\frac{d}{dt}U_m(t) - \frac{d}{dt}\mathcal{I}_{\tau_m,N_m}U_m(t).$$

Then, following the same line as in the derivation of (23), we obtain from (67) and (68) that

$$\begin{cases} \frac{d}{dt}E^{N_m}_m(t) = G^{N_m}_{\tau_m,1}(t) + G^{N_m}_{\tau_m,2}(t), & t \in \Lambda^j_{N,m},\ j \geq 0, \\ E^{N_m}_m(0) = u^{N_m}_m(0) - \mathcal{I}_{\tau_m,N_m}U_m(0), & 2 \leq m \leq M, \\ E^{N_1}_1(0) = U(0) - \mathcal{I}_{\tau_1,N_1}U(0). \end{cases} \tag{70}$$

Like (24), we have

$$\begin{aligned} &2\big\langle E^{N_m}_m - E^{N_m}_m(0), \frac{d}{dt}(t^{-1}(E^{N_m}_m - E^{N_m}_m(0)))\big\rangle_{\tau_m,N_m} \\ &= -\,\|t^{-1}(E^{N_m}_m - E^{N_m}_m(0))\|^2_{\tau_m} + \tau_m^{-1}\big|E^{N_m}_m(\tau_m) - E^{N_m}_m(0)\big|^2. \end{aligned} \tag{71}$$

Next, we use (70) to derive that

$$\begin{aligned} &\frac{d}{dt}(t^{-1}(E^{N_m}_m(t) - E^{N_m}_m(0))) = -t^{-2}(E^{N_m}_m(t) - E^{N_m}_m(0)) + t^{-1}\frac{d}{dt}E^{N_m}_m(t) \\ &= -t^{-2}(E^{N_m}_m(t) - E^{N_m}_m(0)) + t^{-1}(G^{N_m}_{\tau_m,1}(t) + G^{N_m}_{\tau_m,2}(t)), \qquad t \in \Lambda^j_{N,m},\ j \geq 0. \end{aligned}$$

Hence, by (7) with τ_m and N_m instead of T and N, we deduce from the above inequality that

$$\begin{aligned} &2\big\langle E^{N_m}_m - E^{N_m}_m(0), \frac{d}{dt}(t^{-1}(E^{N_m}_m - E^{N_m}_m(0)))\big\rangle_{\tau_m,N_m} \\ &= -\,2(E^{N_m}_m - E^{N_m}_m(0), t^{-2}(E^{N_m}_m - E^{N_m}_m(0)))_{\tau_m} \\ &\quad + 2\big\langle t^{-1}(E^{N_m}_m - E^{N_m}_m(0)), G^{N_m}_{\tau_m,1}\big\rangle_{\tau_m,N_m} + 2(t^{-1}(E^{N_m}_m - E^{N_m}_m(0)), G^{N_m}_{\tau_m,2})_{\tau_m} \\ &\leq -\,2\|t^{-1}(E^{N_m}_m - E^{N_m}_m(0))\|^2_{\tau_m} + (\frac{1}{2} + \epsilon)\|t^{-1}(E^{N_m}_m - E^{N_m}_m(0))\|^2_{\tau_m} \\ &\quad + 2\|G^{N_m}_{\tau_m,1}\|^2_{\tau_m,N_m} + \epsilon^{-1}\|G^{N_m}_{\tau_m,2}\|^2_{\tau_m}. \end{aligned} \tag{72}$$

According to (29) with the parameters N_m and τ_m, we have

$$\|G^{N_m}_{\tau_m,2}\|_{\tau_m} \leq c\tau_m^{r-1}N_m^{\frac{3}{2}-r}|U_m|_{r,\tau_m}. \tag{73}$$

A combination of (71)-(73) leads to (69). ☐

We now consider several typical f, and analyze the numerical errors. Hereafter, let $\tau = \max_{1\leq j\leq M}\tau_j$ and $N = \min_{1\leq j\leq M} N_j$. Moreover, we always assume that for any $1 \leq i \leq j \leq M$, $\frac{\tau_i}{\tau_j}$ is bounded above.

Case I. Consider (67) with the linear delay:

$$\theta(t) = \lambda t, \quad 0 < \lambda < 1. \tag{74}$$

Assume that $f(x, y, t)$ satisfies the Lipschitz conditions (31) and (32).

Theorem 3.2. *If*

❶ *the conditions* (74), (31) *and* (32) *hold,*
❷ $(1 - \lambda)T_m \leq T_{m-1}$ *for all* $2 \leq m \leq M$,
❸ $U \in H^r(0, T)$ *with integer* $2 \leq r \leq N + 1$,

❹ *for certain* $\delta > 0$ *and* $c_0 > 0$,

$$\begin{cases} (1+\delta)\tau_1^2\gamma_1^2 + (1+\delta^{-1})(1-\lambda)^{-1}\tau_1^2\gamma_2^2 \le \beta < \dfrac{1}{4}, \\ \tau_m\gamma_1 < \dfrac{1}{2}, \quad \tau_m\gamma_2 \le c_0, \quad \forall\ m > 1, \end{cases} \tag{75}$$

then for any $1 \le m \le M$,

$$\|U - u^N\|^2_{L^2(T_{m-1},T_m)} \le c_\beta \tau^{2r} N^{3-2r} |U|^2_{r,T_m}, \tag{76}$$

$$|U(T_m) - u^N(T_m)|^2 \le c_\beta \tau^{2r-1} N^{3-2r} |U|^2_{r,T_m}. \tag{77}$$

In particular,

$$\max_{t\in[T_{m-1},T_m]} |U(t) - u^N(t)|^2 \le c_\beta \tau^{2r-1} N^{3-2r} |U|^2_{r,T_m}. \tag{78}$$

Proof. Clearly, in this case, $\Lambda^0_{N,m} = \emptyset$ for $m \ge 1$ and $\Lambda^m_{N,m} = \emptyset$ for $m > 1$. Therefore, by (31), (32), (11), (46) and the fact $\sum_{k=0}^{N_m} \omega^{N_m}_{\tau_m,k} = \tau_m$, we deduce that for any $\epsilon > 0$ and $m > 1$,

$$\begin{aligned} \|G^{N_m}_{\tau_m,1}\|^2_{\tau_m,N_m} &\le (1+\epsilon)\sum_{j=1}^{m-1}\sum_{t^{N_m}_{\tau_m,k}\in\Lambda^j_{N,m}} \Big(f(u^{N_m}_m(t^{N_m}_{\tau_m,k}), w^j_m(t^{N_m}_{\tau_m,k}), T_{m-1}+t^{N_m}_{\tau_m,k}) \\ &\quad - f(\mathcal{I}_{\tau_m,N_m}U_m(t^{N_m}_{\tau_m,k}), w^j_m(t^{N_m}_{\tau_m,k}), T_{m-1}+t^{N_m}_{\tau_m,k})\Big)^2 \omega^{N_m}_{\tau_m,k} \\ &\quad + (1+\epsilon^{-1})\sum_{j=1}^{m-1}\sum_{t^{N_m}_{\tau_m,k}\in\Lambda^j_{N,m}} \Big(f(\mathcal{I}_{\tau_m,N_m}U_m(t^{N_m}_{\tau_m,k}), w^j_m(t^{N_m}_{\tau_m,k}), T_{m-1}+t^{N_m}_{\tau_m,k}) \\ &\quad - f(\mathcal{I}_{\tau_m,N_m}U_m(t^{N_m}_{\tau_m,k}), W^j_m(t^{N_m}_{\tau_m,k}), T_{m-1}+t^{N_m}_{\tau_m,k})\Big)^2 \omega^{N_m}_{\tau_m,k} \\ &\le (1+\epsilon^{-1})\gamma_2^2 \sum_{j=1}^{m-1}\sum_{t^{N_m}_{\tau_m,k}\in\Lambda^j_{N,m}} (w^j_m(t^{N_m}_{\tau_m,k}) - W^j_m(t^{N_m}_{\tau_m,k}))^2 \omega^{N_m}_{\tau_m,k} \\ &\quad + (1+\epsilon)\gamma_1^2 \|E^{N_m}_m\|^2_{\tau_m,N_m} \\ &\le (1+\epsilon^{-1})\tau_m\gamma_2^2 \max_{1\le j\le m-1} \|U_j - u^{N_j}_j\|^2_{L^\infty(0,\tau_j)} + (1+\epsilon)\gamma_1^2\|E^{N_m}_m\|^2_{\tau_m} \\ &\le (1+\epsilon^{-1})\tau_m\gamma_2^2 \max_{1\le j\le m-1} (\frac{2}{\tau_j}\|U_j - u^{N_j}_j\|^2_{\tau_j} + 2\tau_j\|\frac{d}{dt}(U_j - u^{N_j}_j)\|^2_{\tau_j}) \\ &\quad + (1+\epsilon)\gamma_1^2\|E^{N_m}_m\|^2_{\tau_m}. \end{aligned} \tag{79}$$

Inserting the above inequality into (69), we obtain that for $m > 1$,

$$\begin{aligned} &(\tfrac{1}{2} - \epsilon)\|t^{-1}(E^{N_m}_m - E^{N_m}_m(0))\|^2_{\tau_m} + \tau_m^{-1}\big|E^{N_m}_m(\tau_m) - E^{N_m}_m(0)\big|^2 \\ &\le 2(1+\epsilon)\gamma_1^2\|E^{N_m}_m\|^2_{\tau_m} + c\epsilon^{-1}\tau_m^{2r-2}N_m^{3-2r}|U_m|^2_{r,\tau_m} \\ &\quad + 2(1+\epsilon^{-1})\tau_m\gamma_2^2 \max_{1\le j\le m-1}(\frac{2}{\tau_j}\|U_j - u^{N_j}_j\|^2_{\tau_j} + 2\tau_j\|\frac{d}{dt}(U_j - u^{N_j}_j)\|^2_{\tau_j}). \end{aligned} \tag{80}$$

On the other hand, like (40),

$$\begin{aligned} &(\tfrac{1}{2} - \epsilon)\|E^{N_m}_m\|^2_{\tau_m} \\ &\le (\tfrac{1}{2} - \epsilon)\Big((1+\epsilon)\tau_m^2\|t^{-1}(E^{N_m}_m - E^{N_m}_m(0))\|^2_{\tau_m} + (1+\epsilon^{-1})\tau_m|E^{N_m}_m(0)|^2\Big). \end{aligned} \tag{81}$$

Therefore, we use (80) and (81) to derive that for $m > 1$,

$$\begin{aligned}
&(\frac{1}{2}-\epsilon)\|E_m^{N_m}\|_{\tau_m}^2 + (1+\epsilon)\tau_m\big|E_m^{N_m}(\tau_m) - E_m^{N_m}(0)\big|^2 \le c\epsilon^{-1}\tau_m|E_m^{N_m}(0)|^2 \\
&\quad + (1+\epsilon)\tau_m^2\Big((\frac{1}{2}-\epsilon)\|t^{-1}(E_m^{N_m} - E_m^{N_m}(0))\|_{\tau_m}^2 + \tau_m^{-1}\big|E_m^{N_m}(\tau_m) - E_m^{N_m}(0)\big|^2\Big) \\
&\le (1+\epsilon)\tau_m^2\Big(2(1+\epsilon)\gamma_1^2\|E_m^{N_m}\|_{\tau_m}^2 + 2(1+\epsilon^{-1})\tau_m\gamma_2^2 \max_{1\le j\le m-1}(\frac{2}{\tau_j}\|U_j - u_j^{N_j}\|_{\tau_j}^2 \\
&\quad + 2\tau_j\|\frac{d}{dt}(U_j - u_j^{N_j})\|_{\tau_j}^2)\Big) + c\epsilon^{-1}\tau_m^{2r}N_m^{3-2r}|U_m|_{r,\tau_m}^2 + c\epsilon^{-1}\tau_m|E_m^{N_m}(0)|^2,
\end{aligned} \tag{82}$$

or equivalently,

$$\begin{aligned}
&(\tfrac{1}{2}-\epsilon-2(1+\epsilon)^2\tau_m^2\gamma_1^2)\|E_m^{N_m}\|_{\tau_m}^2 + (1+\epsilon)\tau_m\big|E_m^{N_m}(\tau_m) - E_m^{N_m}(0)\big|^2 \\
&\le 2(1+\epsilon^{-1})(1+\epsilon)\tau_m^3\gamma_2^2 \max_{1\le j\le m-1}(\frac{2}{\tau_j}\|U_j - u_j^{N_j}\|_{\tau_j}^2 + 2\tau_j\|\frac{d}{dt}(U_j - u_j^{N_j})\|_{\tau_j}^2) \\
&\quad + c\epsilon^{-1}\tau_m^{2r}N_m^{3-2r}|U_m|_{r,\tau_m}^2 + c\epsilon^{-1}\tau_m|E_m^{N_m}(0)|^2.
\end{aligned} \tag{83}$$

Furthermore, like (43), we have

$$\|U_m - u_m^{N_m}\|_{\tau_m}^2 \le (1+\epsilon)\|E_m^{N_m}\|_{\tau_m}^2 + c\epsilon^{-1}\tau_m^{2r}N_m^{-2r}|U_m|_{r,\tau_m}^2. \tag{84}$$

The above two inequalities with (75) lead to that for $m > 1$,

$$\begin{aligned}
&(\tfrac{1}{2}-\epsilon-2(1+\epsilon)^2\tau_m^2\gamma_1^2)\|U_m - u_m^{N_m}\|_{\tau_m}^2 + (1+\epsilon)^2\tau_m\big|E_m^{N_m}(\tau_m) - E_m^{N_m}(0)\big|^2 \\
&\le (1+\epsilon)(\tfrac{1}{2}-\epsilon-2(1+\epsilon)^2\tau_m^2\gamma_1^2)\|E_m^{N_m}\|_{\tau_m}^2 + (1+\epsilon)^2\tau_m\big|E_m^{N_m}(\tau_m) - E_m^{N_m}(0)\big|^2 \\
&\quad + c\epsilon^{-1}\tau_m^{2r}N_m^{-2r}|U_m|_{r,\tau_m}^2 \\
&\le 2(1+\epsilon^{-1})(1+\epsilon)^2\tau_m^3\gamma_2^2 \max_{1\le j\le m-1}(\frac{2}{\tau_j}\|U_j - u_j^{N_j}\|_{\tau_j}^2 + 2\tau_j\|\frac{d}{dt}(U_j - u_j^{N_j})\|_{\tau_j}^2) \\
&\quad + c\epsilon^{-1}\tau_m^{2r}N_m^{3-2r}|U_m|_{r,\tau_m}^2 + c\epsilon^{-1}\tau_m\big|E_m^{N_m}(0)\big|^2.
\end{aligned} \tag{85}$$

Thanks to (46), an argument similar to (47) yields

$$\begin{aligned}
&\big|E_m^{N_m}(0)\big|^2 = \big|u_m^{N_m}(0) - \mathcal{I}_{\tau_m,N_m}U_m(0)\big|^2 \\
&\le 2\big|u_{m-1}^{N_{m-1}}(\tau_{m-1}) - U_{m-1}(\tau_{m-1})\big|^2 + 2\big|U_m(0) - \mathcal{I}_{\tau_m,N_m}U_m(0)\big|^2 \\
&\le 2\big|u_{m-1}^{N_{m-1}}(\tau_{m-1}) - U_{m-1}(\tau_{m-1})\big|^2 + c\tau_m^{2r-1}N_m^{3-2r}|U_m|_{r,\tau_m}^2.
\end{aligned}$$

Similarly

$$\begin{aligned}
(U_m(\tau_m) - u_m^{N_m}(\tau_m))^2 &\le 2(E_m^{N_m}(\tau_m))^2 + 2(U_m(\tau_m) - \mathcal{I}_{\tau_m,N_m}U_m(\tau_m))^2 \\
&\le 2(E_m^{N_m}(\tau_m))^2 + c\tau_m^{2r-1}N_m^{3-2r}|U_m|_{r,\tau_m}^2.
\end{aligned}$$

A combination of the above three inequalities gives that for $m > 1$,

$$
\begin{aligned}
&(\frac{1}{2} - \epsilon - 2(1+\epsilon)^2\tau_m^2\gamma_1^2)\|U_m - u_m^{N_m}\|_{\tau_m}^2 + \frac{1}{4}(1+\epsilon)^2\tau_m\big|U_m(\tau_m) - u_m^{N_m}(\tau_m)\big|^2 \\
&\le (\frac{1}{2} - \epsilon - 2(1+\epsilon)^2\tau_m^2\gamma_1^2)\|U_m - u_m^{N_m}\|_{\tau_m}^2 + \frac{1}{2}(1+\epsilon)^2\tau_m\big|E_m^{N_m}(\tau_m)\big|^2 \\
&\quad + c\tau_m^{2r}N_m^{3-2r}|U_m|_{r,\tau_m}^2 \le (\frac{1}{2} - \epsilon - 2(1+\epsilon)^2\tau_m^2\gamma_1^2)\|U_m - u_m^{N_m}\|_{\tau_m}^2 \\
&\quad + (1+\epsilon)^2\tau_m(\big|E_m^{N_m}(\tau_m) - E_m^{N_m}(0)\big|^2 + \big|E_m^{N_m}(0)\big|^2) + c\tau_m^{2r}N_m^{3-2r}|U_m|_{r,\tau_m}^2 \\
&\le 2(1+\epsilon^{-1})(1+\epsilon)^2\tau_m^3\gamma_2^2 \max_{1\le j\le m-1}(\frac{2}{\tau_j}\|U_j - u_j^{N_j}\|_{\tau_j}^2 + 2\tau_j\|\frac{d}{dt}(U_j - u_j^{N_j})\|_{\tau_j}^2) \\
&\quad + c\epsilon^{-1}\tau_m\big|u_{m-1}^{N_{m-1}}(\tau_{m-1}) - U_{m-1}(\tau_{m-1})\big|^2 + c\epsilon^{-1}\tau_m^{2r}N_m^{3-2r}|U_m|_{r,\tau_m}^2.
\end{aligned}
\tag{86}
$$

Thus by (50), (34), (35) and (75), we obtain from (86) that for a suitable value of ϵ,

$$\|U - u^N\|_{L^2(T_1,T_2)}^2 = \|U_2 - u_2^{N_2}\|_{\tau_2}^2 \le c_\beta\tau^{2r}N^{3-2r}(|U_1|_{r,\tau_1}^2 + |U_2|_{r,\tau_2}^2).$$

Similarly

$$|U(T_2) - u^N(T_2)|^2 = |U_2(\tau_2) - u_2^{N_2}(\tau_2)|^2 \le c_\beta\tau^{2r-1}N^{3-2r}(|U_1|_{r,\tau_1}^2 + |U_2|_{r,\tau_2}^2).$$

Since $\frac{\tau_i}{\tau_j}$ is bounded above for $1 \le i \le j \le M$, we use (86), (79), (75) and repeat the above process, to obtain (76) and (77). In particular, by virtue of (46), (76) and a result similar to (50), we get (78). □

Remark 3.1. The estimates (76)-(78) indicate the spectral accuracy of scheme (67).

Remark 3.2. If $\frac{d^rU}{dt^r} \in L^\infty(0,T_m)$, $N \ge r-1$ and $\tau \le 1$, then for $1 \le m \le M$,

$$\|U - u^N\|_{L^2(0,T_m)} \le c_\beta^{\frac{1}{2}} m\tau^{r+\frac{1}{2}}N^{\frac{3}{2}-r}\|\frac{d^rU}{dt^r}\|_{L^\infty(0,T_m)}, \tag{87}$$

$$|U(T_m) - u^N(T_m)| \le c_\beta^{\frac{1}{2}} m^{\frac{1}{2}}\tau^r N^{\frac{3}{2}-r}\|\frac{d^rU}{dt^r}\|_{L^\infty(0,T_m)}, \tag{88}$$

$$\max_{t\in[T_{m-1},T_m]}|U(t) - u^N(t)| \le c_\beta^{\frac{1}{2}} m^{\frac{1}{2}}\tau^r N^{\frac{3}{2}-r}\|\frac{d^rU}{dt^r}\|_{L^\infty(0,T_m)}. \tag{89}$$

If, in addition, $\frac{1}{N} \le \tau \le 1$, then we may take $r = N+1$ in (87) and (88), to reach that for $1 \le m \le M$,

$$\|U - u^N\|_{L^2(0,T)} = \mathcal{O}(\tau^{2N}), \qquad |U(T_m) - u^N(T_m)| = \mathcal{O}(\tau^{2N}). \tag{90}$$

Case II. Assume that the delay function satisfies:

$$T_{m-1} + t - \theta(T_{m-1} + t) \le T_{m-1}, \qquad \text{for } 1 \le m \le M \text{ and } t \in (0,\tau_m). \tag{91}$$

Moveover, $f(x,y,t)$ satisfies the Lipschitz condition (31) and (32).

Theorem 3.3. *If*

❶ *the conditions* (31), (32) *and* (91) *hold,*

❷ $U \in H^r(0,T)$ *with integer* $2 \le r \le N+1$,

❸ $\tau_m\gamma_1 < \frac{1}{2}$, $\forall\ m \ge 1$,

❹ *for certain* $c_0 > 0$ *and* $m > 1$, $\tau_m\gamma_2 \le c_0$,

then we have the same results (76)-(78).

Proof. Obviously, in this case, $\Lambda^m_{N,m} = \emptyset$. Therefore, by (31), (32) and a similar argument as in the derivation of (79), we deduce that for any $\epsilon > 0$,

$$\begin{aligned}
\|G^{N_m}_{\tau_m,1}\|^2_{\tau_m,N_m} \leq & \sum_{j=1}^{m-1} \sum_{t^{N_m}_{\tau_m,k}\in\Lambda^j_{N,m}} \Big(f(u^{N_m}_m(t^{N_m}_{\tau_m,k}), w^j_m(t^{N_m}_{\tau_m,k}), T_{m-1}+t^{N_m}_{\tau_m,k}) \\
& - f(\mathcal{I}_{\tau_m,N_m}U_m(t^{N_m}_{\tau_m,k}), W^j_m(t^{N_m}_{\tau_m,k}), T_{m-1}+t^{N_m}_{\tau_m,k})\Big)^2 \omega^{N_m}_{\tau_m,k} \\
& + \sum_{t^{N_m}_{\tau_m,k}\in\Lambda^0_{N,m}} \Big(f(u^{N_m}_m(t^{N_m}_{\tau_m,k}), V(T_{m-1}+t^{N_m}_{\tau_m,k}-\theta(T_{m-1}+t^{N_m}_{\tau_m,k})), T_{m-1}+t^{N_m}_{\tau_m,k}) \\
& - f(\mathcal{I}_{\tau_m,N_m}U_m(t^{N_m}_{\tau_m,k}), V(T_{m-1}+t^{N_m}_{\tau_m,k}-\theta(T_{m-1}+t^{N_m}_{\tau_m,k})), T_{m-1}+t^{N_m}_{\tau_m,k})\Big)^2 \omega^{N_m}_{\tau_m,k} \\
\leq & (1+\epsilon^{-1})\tau_m\gamma_2^2 \max_{1\leq j\leq m-1}\Big(\frac{2}{\tau_j}\|U_j-u^{N_j}_j\|^2_{\tau_j} + 2\tau_j\|\frac{d}{dt}(U_j-u^{N_j}_j)\|^2_{\tau_j}\Big) \\
& + (1+\epsilon)\gamma_1^2\|E^{N_m}_m\|^2_{\tau_m}.
\end{aligned} \tag{92}$$

Moreover, (80)-(86) still hold. Furthermore, by (56), (53) and (12), we obtain

$$\begin{aligned}
\|G^{N_1}_{\tau_1,1}\|^2_{\tau_1,N_1} &\leq \gamma_1^2\|E^{N_1}_1\|^2_{\tau_1} \leq 2\gamma_1^2(\|U_1-\mathcal{I}_{\tau_1,N_1}U_1\|^2_{\tau_1} + \|U_1-u^{N_1}_1\|^2_{\tau_1}) \\
&\leq c_\beta\gamma_1^2\tau_1^{2r}N_1^{3-2r}|U_1|^2_{r,\tau_1}.
\end{aligned}$$

Therefore, we can derive the same results (76)-(78). □

3.3. **Numerical results.** In this subsection, we present some numerical results to illustrate the efficiency of our multiple-domain algorithm.

❶ Linear variable delay (Case I).

We consider the example (63) with $T = 1$ and uniform step-size grid. As pointed out before, the conditions (31) and (32) hold with $\gamma_1 = \frac{1}{2}$ and $\gamma_2 = \frac{1}{2}e^{\frac{1}{2}}$. Furthermore, the inequality (75) is satisfied for $\tau_m \equiv 0.25$, but for $\tau_m \equiv 0.5$, the inequality (75) is no longer valid. Moreover, $(1-\lambda)T_m \leq T_{m-1}$ with $\lambda = \frac{1}{2}$. In Fig. 4, we plot the numerical errors at $t = 1$ with uniform τ_m and N_m. They indicate that the numerical errors decay exponentially as N_m increases and τ_m decreases. In particular, we can observe from Fig. 4 that even if the condition (75) is not satisfied, our multiple-domain algorithm is still valid (see the case $\tau_m \equiv 0.5$).

❷ Nonlinear variable delay (Case I).

We consider the example (64). In Fig. 5, we plot the numerical errors at $t = 1$ using the multiple-domain scheme (67) and Newton-Raphson iteration method with uniform τ_m and N_m. They indicate that the numerical errors decay exponentially as N_m increases and τ_m decreases. Again, we observe from Fig. 5 that our algorithm is still valid even if the condition (32) is not satisfied.

❸ Linear variable delay (Case II).

$$\begin{cases}
\frac{d}{dt}u(t) = u(t-1-\frac{1}{t+1}), & \text{for } t > 0, \\
u(t) = 1, & \text{for } -0.5 \leq t \leq 0, \\
u(t) = \frac{2}{3}(t+2), & \text{for } -2 \leq t < -0.5.
\end{cases} \tag{93}$$

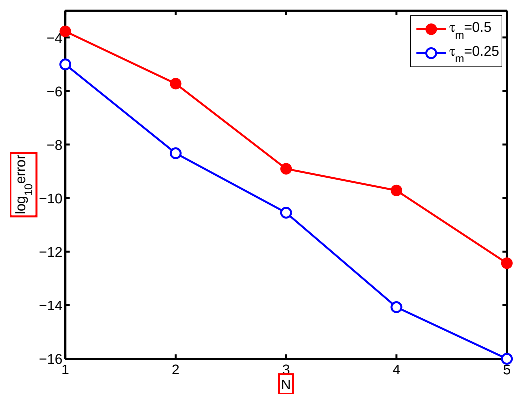

FIGURE 4. The numerical errors of Legendre-Gauss collocation method at $t = 1$.

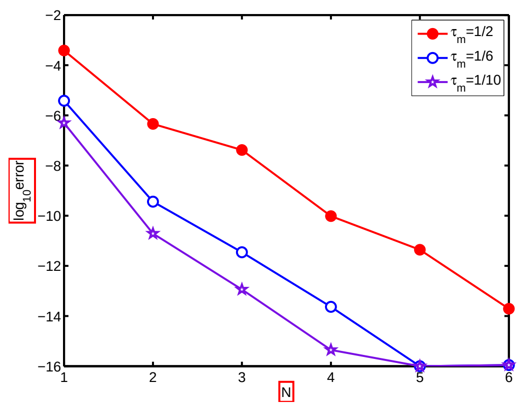

FIGURE 5. The numerical errors of Legendre-Gauss collocation method at $t = 1$.

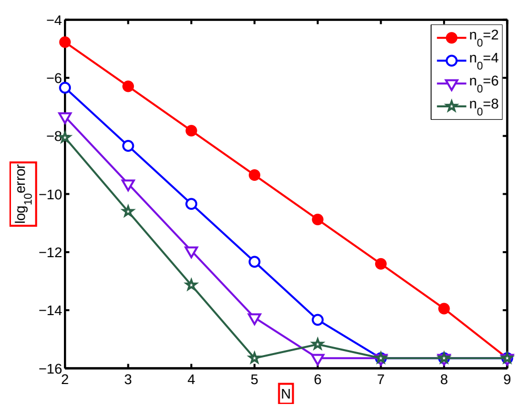

FIGURE 6. The numerical errors of Legendre-Gauss collocation method at $t = \xi_2$.

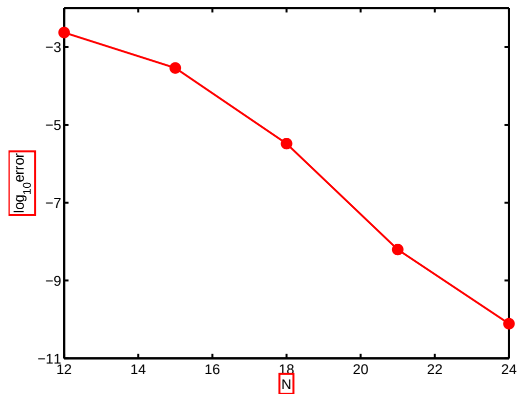

FIGURE 7. The numerical errors of Legendre-Gauss collocation method at $t = 20$.

As pointed out in [24], the exact solution is

$$u(t) = \begin{cases} 1 + \frac{2}{3}t + \frac{t^3}{3} - \frac{2}{3}\log(t+1), & \text{on } [0,1], \\ 1 - \frac{2}{3}\log 2 + t, & \text{on } [1, \sqrt{2}]. \end{cases}$$

The first derivative of the solution is not continuous at $t = -0.5$ and $t = 0$, therefore the second derivative also has a jump at the points where the time lag is equal to -0.5 and 0, for instance, $t = \xi_1 = 1$ and $t = \xi_2 = \sqrt{2}$.

Denote by $n_0 = 2k$ the total number of steps. Obviously, the conditions (31) and (32) hold with $\gamma_1 = 0$ and $\gamma_2 = 1$. Furthermore, by choosing suitably T_m, we can ensure that the condition (91) holds. In Fig. 6, we plot the numerical errors at $t = \xi_2$ with $\tau_m = \dfrac{\xi_1}{k}$, $1 \le m \le k$, $\tau_m = \dfrac{\xi_2 - \xi_1}{k}$, $k+1 \le m \le n$ and uniform N_m. They indicate that the numerical errors decay exponentially as N_m increases and τ_m decreases.

As pointed out, in actual computation, even if the conditions for Theorems 3.1 and 3.2 are not satisfied, the numerical solutions of our method match the exact solutions very well. We next present some other numerical examples to illustrate the efficiency of our method, which can be found in [5].

❹ Nonlinear constant delay (Case II):

$$\begin{cases} \frac{d}{dt}u(t) = -3u(t-1)(1+u(t)), & \text{for } t > 0, \\ u(t) = t, & \text{for } -1 \le t \le 0. \end{cases} \tag{94}$$

This is a well-known equation from biology, solution of which has the breaking points at the integers $0,\ 1,\ \cdots$.

In Table 1, we compare the errors of our method LGC3(5) ($N = 3, 5$ and the step-size is constant) with the method UCIRK4(6) (the uniformly corrected implicit Runge-Kutta method of order 4(6) presented in [5]) and the method IRKSR4(6) (the implicit Runge-Kutta superconvergence rate 4(6) presented in [4]), at the point $t = 20$ with respect to the reference solution $u(20) = 4.671437497500$. Since the three methods have the same mesh and the same number of interpolation nodes, we can observe that our method provides more accurate numerical results.

TABLE 2. Numerical results for Eq. (94).

Number of steps	**LGC3**	UCIRK4	IRKSR4	**LGC5**	UCIRK6	IRKSR6
200	**1.7e-04**	1.0e+00	1.0e+00	**1.9e-9**	2.0e-02	2.0e-02
500	**1.1e-07**	2.5e-02	2.5e-02	**4.6e-10**	7.6e-05	7.6e-05
1000	**1.2e-09**	1.5e-03	1.5e-03	**3.5e-10**	1.2e-06	1.0e-06

In particular, our method is still valid even for large time step-size. In Fig. 7, we plot the numerical errors at $t = 20$ with $\tau_m = 1,\ 1 \le m \le 20$ and uniform N_m. They indicate that the numerical errors decay exponentially as N_m increases.

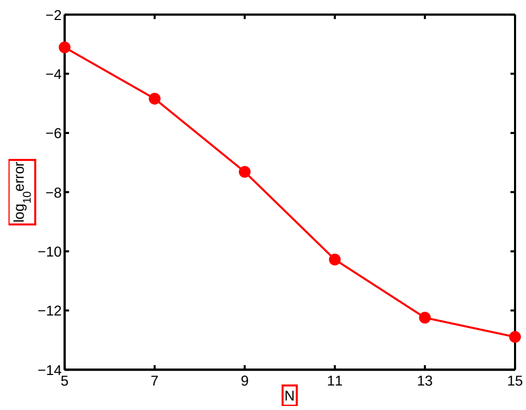

FIGURE 8. The numerical errors of Legendre-Gauss collocation method at $t = \xi_2$.

❺ Nonlinear variable delay (Case II):

$$\begin{cases} \dfrac{d}{dt}u(t) = \dfrac{t}{t+1}u(t - \log(t+1) - 1)u(t), & \text{for } t > 0, \\ u(t) = 1, & \text{for } -1 \le t \le 0. \end{cases} \tag{95}$$

The solution has the breaking points:

$$\xi_0 = 0, \quad \xi_1 = 2.1461932206205825852, \quad \xi_2 = 4.9254498245082464926, \text{ etc..}$$

In Table 2, we compare the numerical errors of various methods at the point $t = \xi_2$ with respect to the reference solution:

$$u(\xi_2) = 76.3734726693768056269.$$

Since the three methods have the same mesh and the same number of interpolation nodes, we can observe that our method provides again much better numerical results.

In particular, our method is still valid even for large time step-size. In Fig. 8, we plot the numerical errors at $t = \xi_2$ with $\tau_1 = \xi_1,\ \tau_2 = \xi_2 - \xi_1$ and uniform N_m. They indicate that the numerical errors decay exponentially as N_m increases.

4. **Concluding discussions.** In this paper, we proposed the single-step and multiple-domain Legendre-Gauss collocation integration processes for nonlinear DDEs. These approaches have several attractive features.

TABLE 3. Numerical results for Eq. (95).

Number of steps	**LGC3**	UCIRK4	IRKSR4	**LGC5**	UCIRK6	IRKSR6
8	**1.8e-05**	1.8e-01	1.3e-01	**5.4e-08**	1.6e-03	1.0e-03
84	**1.5e-11**	1.7e-05	1.2e-05	**2.1e-13**	9.6e-10	5.7e-10

- The single-step Legendre-Gauss collocation method is easy to be implemented for nonlinear DDEs, and possesses the spectral accuracy. It enjoys computational efficiency and accuracy over some popular existing methods.
- By using the multiple-domain Legendre-Gauss collocation method, we can use moderate mode N to evaluate the numerical solutions more stably and effectively. Moreover, it can be adapted to solutions with jump-discontinuities.For any fixed mode N, the numerical errors decay as $\tau \to 0$. For any fixed step size in time, the numerical error decays very rapidly as N increases. The reason of this phenomena might be that there exists the factor N^{-r} in the error estimations of our new method, whereas there is no such factor in the error estimations of the corresponding implicit Runge-Kutta method. In fact, in the derivation of algorithm of our method, we benefit from the orthogonality of Legendre polynomials, while in the derivation of algorithm of the implicit Runge-Kutta method, one used the Lagrange interpolation on the Legendre-Gauss interpolation nodes, which is not stable for large N. In particular, our methods are much easier to be implemented than the implicit Runge-Kutta method for DDEs, since we only need to save the coefficients of numerical solutions in each step.
- The numerical errors of our methods are characterized by the semi-norms of exact solutions in certain Sobolev spaces. These sharp norms are in particular necessary for problems with degenerated initial data.

The numerical results demonstrated the spectral accuracy of proposed algorithms and coincided with the theoretical analysis very well.

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