# An unfitted hybridizable discontinuous Galerkin method for the Poisson interface problem and its error analysis 

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In this article, we present and analyse an unfitted mesh method for the Poisson interface problem. By constructing a novel ansatz function in the vicinity of the interface, we are able to derive an extended Poisson problem whose interface fits a given quasi-uniform triangular mesh. Then we adopt a hybridizable discontinuous Galerkin method to solve the extended problem with an appropriate choice of flux for treating the jump conditions. In contrast with existing approaches, the ansatz function is designed through a delicate piecewise quadratic Hermite polynomial interpolation with a post-processing via a standard Lagrange polynomial interpolation. Such an explicit function offers a third-order approximation to the singular part of the underlying solution for interfaces of any shape. It is also essential for both stability and convergence of the proposed method. Moreover, we provide rigorous error analysis to show that the scheme can achieve a second-order convergence rate for the approximation of the solution and its gradient. Ample numerical examples with complex interfaces demonstrate the expected convergence order and robustness of the method.

Keywords: hybridizable discontinuous Galerkin method; Poisson interface equation; quasi-uniform triangular mesh; complex interface; hermite interpolation; error analysis.

## 1. Introduction

In this article, we are concerned with the Poisson interface problem

$$
\begin{align*}
-\left.\Delta\right|_{\Omega \backslash \Gamma} u(\boldsymbol{x}) & =f(\boldsymbol{x}), & & \text { in } \quad \Omega, \\
\llbracket u \rrbracket_{\Gamma} & =g_{1}, & & \text { on } \quad \Gamma, \\
\llbracket \nabla u \cdot \boldsymbol{n} \rrbracket_{\Gamma} & =g_{2}, & & \text { on } \quad \Gamma,  \tag{1.1}\\
u & =g, & & \text { on } \quad \partial \Omega,
\end{align*}
$$

on a bounded connected polygonal domain $\Omega \subset \mathbb{R}^{2}$ with a smooth interior interface $\Gamma$, where $f, g_{1}, g_{2}$ and $g$ are given functions with regularity to be specified later. As illustrated in Fig. 1, $\Gamma$ cuts $\Omega$ into two nonoverlapping subdomains $\Omega_{1}$ and $\Omega_{2}$, i.e., $\Omega_{1} \cap \Omega_{2}=\emptyset$ and $\bar{\Omega}=\bar{\Omega}_{1} \cup \bar{\Omega}_{2}$. The Laplace operator is


Fig. 1. A sketch of the domain $\Omega$ and the interface $\Gamma$.
defined on $\Omega \backslash \Gamma$ and $\boldsymbol{n}$ denotes the unit normal to $\Gamma$ pointing from $\Omega_{1}$ to $\Omega_{2}$. As usual, the jump of a function $v$ on $\Gamma$ is defined as

$$
\begin{equation*}
\llbracket v \rrbracket(\boldsymbol{x}):=\lim _{\varepsilon \rightarrow 0}(v(\boldsymbol{x}+\varepsilon \boldsymbol{n})-v(\boldsymbol{x}-\varepsilon \boldsymbol{n})), \quad \boldsymbol{x} \in \Gamma . \tag{1.2}
\end{equation*}
$$

This problem has important applications in the simulation of immiscible multi-phase flows (Kummer \& Oberlack, 2013), the electroporation state of a biological cell under an electric field (Guyomarc'h et al., 2009; Hu et al., 2015) and many other fields.

Various numerical approaches have been proposed for this problem which particularly include the finite difference (FD), finite elements (FEs) and discontinuous Galerkin (DG) method. In general, these methods can be classified into two categories, i.e., fitted mesh method and unfitted mesh method. The former is built on body fitted mesh that does not allow the interface to cut across any of the elements in the mesh, so the jump conditions across the interface can be easily incorporated into a standard FE formulation (Bramble \& King, 1996; Chen \& Zou, 1998). However, generating a body fitted mesh of relatively high quality is challenging and computationally prohibitive, especially when complex and/or moving interfaces are involved. The latter is more desirable as it only uses a fixed quasi-uniform mesh regardless of the location of the interface. The success of the unfitted method relies on how to effectively handle the jump conditions. The well-known immersed interface method (IIM) developed in Leveque \& Li (1994) is a second-order (see Huang \& Li, 1999) FD-based unfitted method, where the stencils and FD approximation should be carefully designed near the interface. This approach has been further studied and applied to various interface problems (see, e.g., Li, 1997; Zhang \& LeVeque, 1997; Li \& Lai, 2001; Gong et al., 2008). The immersed FE methods have also been intensively investigated (see, e.g., Li et al., 2003; He et al., 2011; Adjerid et al., 2014; Ben-Romdhane et al., 2014) which use special basis functions constructed from jump conditions. Besides, other FE-based methods are also available (see, e.g., Hansbo \& Hansbo, 2002; Huang \& Zou, 2002; Hou \& Liu, 2005; Wu \& Xiao, 2010; Li et al., 2010; Hiptmair et al., 2012; Hou et al., 2013; Guzman et al., 2015 and the references therein for further information).

It is known that the DG method (see, e.g., Arnold, 1982; Arnold et al., 2002) allows the approximation to be discontinuous across the element boundaries, so it enjoys significant advantages in solving interface problems. Recently, several fitted and unfitted DG methods have been proposed (see, e.g., Guyomarc'h et al., 2009; Massjung, 2009; Huynh et al., 2013; Kummer \& Oberlack, 2013; Wang \& Chen, 2014). Typically, the proper introduction of numerical fluxes or penalty terms becomes crucial for such schemes. In the last decade, the hybridizable discontinuous Galerkin (HDG) method has emerged as an important family of various DG schemes, largely attributed to the remarkable reduction of degree of freedom (see, e.g., Cockburn et al., 2008, 2009a,b, 2010). It has been successfully applied to solve a variety of PDEs, but there has been very limited study available for the interface problem. Fitted HDG methods were proposed for elliptic interface problems in Huynh et al. (2013) and for Stokes interface problems in Wang \& Khoo (2013). It was demonstrated that the jump conditions could be naturally incorporated into the HDG scheme with a judicious choice of the numerical flux, and such treatment only resulted in some additional terms in the right-hand side of the global linear system, so the coefficient matrix remains unchanged. Recently, a new extension technique with HDG discretization was presented for solving PDEs on curved domains by using unfitted meshes (see Cockburn \& Solano, 2012, 2014). Optimal convergence rates are maintained if the distance of the unfitted mesh to the curved boundary of the computational domain is of order $h$.

The purpose of this article is to establish and analyse an unfitted HDG method for Poisson interface problems. Motivated by the idea of deriving an extended problem via introducing an ansatz function in Kummer \& Oberlack (2013), we propose a novel ansatz function, derive a new extended problem and then solve it by a HDG method. More precisely, the ingredients of the algorithm and contributions of the paper lie in the following aspects.
(i) We construct a piecewise polynomial ansatz function (denoted by $u_{a, h}$ ) in the vicinity of the interface by a quadratic Hermite interpolation with a post-processing via a standard Lagrange polynomial interpolation. Our delicate construction is accomplished by appropriate choice of the interpolation constraints according to the jump conditions. It provides a third-order approximation to the singular part of solution and leads to stable computation for interfaces of arbitrary shape. It is noteworthy that the technique to devise the ansatz function is essentially different from that in Kummer \& Oberlack (2013). A signed distance function was used in Kummer \& Oberlack (2013) to design the anastz function, so it could accurately represent simple interfaces (e.g., circular and spherical interfaces), but it became much more involved for general interfaces.
(ii) By subtracting the singular part $u_{a}$ from the solution $u$ of (1.1), we can obtain an equivalent intermediate problem involving unknown $u_{a}$. With the ansatz function at our disposal, we further derive an extended problem from the intermediate problem with $u_{a, h}$ in place of $u_{a}$. On account of the good approximability of $u_{a, h}$ to $u_{a}$, we can achieve a second-order accuracy between the solutions of these two problems. Remarkably, the interface of the extended problem aligns with the given quasi-uniform mesh, and the HDG (see Huynh et al., 2013; Wang \& Khoo, 2013) becomes the method of choice.
(iii) We rigorously show that the unfitted HDG method has a second-order $L^{2}$-convergence in both the potential $u$ and its gradient. We remark that the convergence analysis of unfitted methods for interface problems is challenging yet and very limited results are available along this line.

The outline of this article is as follows. In Section 2, we first introduce some notation to be used throughout this article, then derive an extended Poisson interface problem by using a novel ansatz function.

In Section 3, the delicated techniques for the construction of the ansatz function are discussed in detail. Rigorous theoretical analysis for the numerical solution are presented in Section 4. Then, in Section 5 we provide various numerical examples to validate the proposed method and the theoretical results. Concluding remarks and future works are given in Section 6. Proofs of the error estimates have been gathered in the appendix.

## 2. The unfitted HDG method

### 2.1 Notation and well-posedness of (1.1)

We first introduce some notation to be used throughout the article. Let $L^{2}(D)$ be the space of square integrable functions on a generic domain $D \subset \mathbb{R}^{2}$, equipped with the standard $L^{2}$-norm $\|\cdot\|_{L^{2}(D)}$. Moreover, let $\boldsymbol{L}^{2}(D)=\left[L^{2}(D)\right]^{2}$ and denote its norm by $\|\cdot\|_{L^{2}(D)}$. Besides, denote by $H^{r}(D)$ with real $r$ the standard Sobolev space equipped with the norm $\|\cdot\|_{H^{r}(D)}$, and correspondingly, denote by $\boldsymbol{H}^{r}(D):=\left[H^{r}(D)\right]^{2}$ the vector version with the norm $\|\boldsymbol{v}\|_{\boldsymbol{H}^{r}(D)}:=\sum_{i=1}^{2}\left\|v_{i}\right\|_{H^{r}(D)}$ (Adams \& Fournier, 2003). Further, let $C^{k}(D)$ be the continuous function space consisting of functions up to $k$ th derivatives being continuous with the norm

$$
\begin{equation*}
\|u\|_{C^{k}(D)}=\inf \left\{\|\tilde{u}\|_{C^{k}\left(R^{2}\right)}: \tilde{u} \in C^{k}\left(R^{2}\right) \text { and }\left.\tilde{u}\right|_{D}=u\right\} . \tag{2.1}
\end{equation*}
$$

Note that we have

$$
\begin{equation*}
\max _{|\alpha| \leq k} \max _{x \in D}\left|D^{\alpha} u(\boldsymbol{x})\right| \leq\|u\|_{C^{k}(D)} \tag{2.2}
\end{equation*}
$$

In order to characterize the regularity of piecewise functions, we define the space and its norm by

$$
\begin{equation*}
X^{r}=L^{2}(\Omega) \cap H^{r}\left(\Omega_{1}\right) \cap H^{r}\left(\Omega_{2}\right), \quad\|v\|_{X^{r}}=\|v\|_{H^{r}\left(\Omega_{1}\right)}+\|v\|_{H^{r}\left(\Omega_{2}\right)}, \quad \forall v \in X^{r} \tag{2.3}
\end{equation*}
$$

We recall the following result (see Rotberg \& Seftel, 1969; Bramble \& King, 1996), which asserts the well-posedness of the model problem (1.1).

Theorem 2.1 Assume that $f \in L^{2}(\Omega), g \in H^{r-1 / 2}(\partial \Omega), g_{1} \in H^{r-1 / 2}(\Gamma)$ and $g_{2} \in H^{r-3 / 2}(\Gamma)$. Then the problem (1.1) has a unique solution $u \in X^{r}$, and $u$ satisfies the following a priori estimate:

$$
\begin{equation*}
\|u\|_{X^{r}} \leq C\left(\|f\|_{L^{2}(\Omega)}+\|g\|_{H^{r-1 / 2}(\partial \Omega)}+\left\|g_{1}\right\|_{H^{r-1 / 2}(\Gamma)}+\left\|g_{2}\right\|_{H^{r-3 / 2}(\Gamma)}\right), \quad 0 \leq r \leq 2 \tag{2.4}
\end{equation*}
$$

where $C$ is a positive constant independent of $u$ and the given data.

### 2.2 Extended problem

Without loss of generality, let $\mathcal{T}_{h}$ be a quasi-uniform triangular mesh of the domain $\Omega$. As usual, the set of edges of the triangulation $\mathcal{T}_{h}$ is denoted by $\mathcal{E}_{h}$, and the boundary of any element $T$ is denoted by $\partial T$. Let $\mathcal{P}^{k}(D)$ be the space of polynomials of degree at most $k$ on a generic domain $D$. The corresponding


Fig. 2. Illustration of the set $\mathcal{T}_{h, c}$ of all cut triangles (left); subdomains $\Omega_{c}^{1}$ and $\Omega_{c}^{2}$, and the new mesh fitted interface $\widetilde{\Gamma}$ (right).
discontinuous FE spaces are given by

$$
\begin{aligned}
\mathcal{V}_{h}^{k} & :=\left\{\boldsymbol{v} \in \boldsymbol{L}^{2}\left(\mathcal{T}_{h}\right):\left.\boldsymbol{v}\right|_{T} \in\left[\mathcal{P}^{k}(T)\right]^{2}, \quad \forall T \in \mathcal{T}_{h}\right\}, \\
\mathcal{W}_{h}^{k} & :=\left\{w \in L^{2}\left(\mathcal{T}_{h}\right):\left.w\right|_{T} \in \mathcal{P}^{k}(T), \quad \forall T \in \mathcal{T}_{h}\right\}, \\
\mathcal{M}_{h}^{k} & :=\left\{\mu \in L^{2}\left(\mathcal{E}_{h}\right):\left.\mu\right|_{F} \in \mathcal{P}^{k}(F), \quad \forall F \in \mathcal{E}_{h}\right\} .
\end{aligned}
$$

For any vector functions $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{L}^{2}\left(\mathcal{T}_{h}\right)$, and any scalar functions $v, w \in L^{2}\left(\mathcal{T}_{h}\right)$, the corresponding inner products are given by

$$
(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{T}_{h}}:=\sum_{T \in \mathcal{T}_{h}}(\boldsymbol{v}, \boldsymbol{w})_{T}, \quad(v, w)_{\mathcal{T}_{h}}:=\sum_{T \in \mathcal{T}_{h}}(v, w)_{T}, \quad\langle v, w\rangle_{\partial \mathcal{T}_{h}}:=\sum_{T \in \mathcal{T}_{h}}\langle v, w\rangle_{\partial T},
$$

where

$$
(\boldsymbol{v}, \boldsymbol{w})_{T}:=\int_{T} \boldsymbol{v} \cdot \boldsymbol{w} \mathrm{~d} \boldsymbol{x}, \quad(v, w)_{T}:=\int_{T} v w \mathrm{~d} x, \quad(v, w)_{\partial T}:=\int_{\partial T} v w \mathrm{~d} s .
$$

Moreover, we need to introduce some notation and concepts related to the intersection of the interface with the triangular Cartesian mesh $\mathcal{T}_{h}$. In what follows, for any $T \in \mathcal{T}_{h}$, we assume the following hypothesis:
(H1) $\Gamma$ does not intersect an edge of $T$ at more than two points unless this edge is part of $\Gamma$;
(H2) If $\Gamma$ meets a triangle at two points, then these two points must be on different edges of $T$.
Define the set of all cut cells (see the shaped part in Fig. 2 (left)) and the neighborhood of $\Gamma$ (see Fig. 2 (right)), respectively, by

$$
\mathcal{T}_{h, c}:=\left\{T \in \mathcal{T}_{h}: T \cap \Gamma \neq \emptyset\right\}, \quad \Omega_{c}:=\bigcup_{T \in \mathcal{T}_{h, c}} \bar{T}, \quad \Omega_{c}^{i}=\Omega_{c} \cap \Omega_{i}, \quad i=1,2
$$

Note that the outer boundary of $\Omega_{c}^{2}$, denoted by $\widetilde{\Gamma}:=\partial \Omega_{c}^{2} \backslash \Gamma$, becomes the new interface of the extended problem.

We decompose the exact solution $u$ of the model problem (1.1) as

$$
u= \begin{cases}u_{1}, & \boldsymbol{x} \in \Omega_{1}, \\ u_{2}, & \boldsymbol{x} \in \Omega_{2} .\end{cases}
$$

Conventionally, we assume that $u$ has a good regularity in both $\Omega_{1}$ and $\Omega_{2}$, and the singularity of the solution only occurs on $\Gamma$. Define

$$
\begin{equation*}
\boldsymbol{V}=\left\{u \in L^{2}(\Omega):\left.u\right|_{\Omega_{i}} \in H^{3}\left(\Omega_{i}\right) \text { and }\left.u\right|_{\Omega_{c}^{i}} \in C^{3}\left(\Omega_{c}^{i}\right), i=1,2\right\} . \tag{2.5}
\end{equation*}
$$

For the sake of theoretical analysis, we assume that the solution $u \in \boldsymbol{V}$ and define the 'broken' norm of $u$ on $\Omega_{c}$ by

$$
\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}=\left\|u_{1}\right\|_{C^{3}\left(\Omega_{c}^{1}\right)}+\left\|u_{2}\right\|_{C^{3}\left(\Omega_{c}^{2}\right)} .
$$

We also introduce the Whitney's extension theorem (see Fefferman, 2007, Theorem 1).
Theorem 2.2 For an arbitrary $E \subset \mathbb{R}^{n}$ and $m \geq 1$, there exists a linear map $T: C^{m}(E) \rightarrow C^{m}\left(\mathbb{R}^{n}\right)$, such that $T \phi=\phi$ on $E$, for any $\phi \in C^{m}(E)$. Moreover, the norm of $T$ is bounded by a constant depending only on $m$ and $n$.

By using the above extension theorem, there exist linear bounded extensions $\tilde{u}_{i}$ of $u_{i}$ from subdomains $\Omega_{c}^{i}$ to the domain $\Omega_{c}$, such that

$$
\begin{equation*}
\tilde{u}_{i}(\boldsymbol{x})=u_{i}(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega_{c}^{i},\left.\quad \tilde{u}_{i}\right|_{\Omega_{c}} \in C^{3}\left(\Omega_{c}\right), \quad\left\|\tilde{u}_{i}\right\|_{C^{3}\left(\Omega_{c}\right)} \leq C\left\|u_{i}\right\|_{C^{3}\left(\Omega_{c}^{i}\right.}, \quad i=1,2 \tag{2.6}
\end{equation*}
$$

where $C$ is a positive constant independent of $\Omega_{c}^{i}$, the triangulation $\mathcal{T}_{h}$ and the mesh size $h$. Set

$$
d(\boldsymbol{x})=\tilde{u}_{2}(\boldsymbol{x})-\tilde{u}_{1}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \Omega_{c},
$$

and further define the ansatz function

$$
u_{a}(\boldsymbol{x})= \begin{cases}d(\boldsymbol{x}), & \boldsymbol{x} \in \Omega_{c}^{2}  \tag{2.7}\\ 0, & \boldsymbol{x} \in \Omega \backslash \Omega_{c}^{2}\end{cases}
$$

Note that $u_{a}$ is discontinuous on both $\Gamma$ and $\widetilde{\Gamma}$, and we can compute the jumps across $\Gamma$ as follows:

$$
\begin{align*}
\llbracket u_{a} \rrbracket_{\Gamma} & =d(\boldsymbol{x})=\llbracket u \rrbracket_{\Gamma}, & & \llbracket \Delta u_{a} \rrbracket_{\Gamma}=\Delta d(\boldsymbol{x})=\llbracket \Delta u \rrbracket_{\Gamma}, \\
\llbracket \partial_{x_{1}} u_{a} \rrbracket_{\Gamma} & =\partial_{x_{1}} d(\boldsymbol{x})=\llbracket \partial_{x_{1}} u \rrbracket_{\Gamma}, & & \llbracket \partial_{x_{2}} u_{a} \rrbracket_{\Gamma}=\partial_{x_{2}} d(\boldsymbol{x})=\llbracket \partial_{x_{2}} u \rrbracket_{\Gamma} . \tag{2.8}
\end{align*}
$$

Then the solution $u$ can be decomposed as $u=u_{p}+u_{a}$ and

$$
u_{p}=u-u_{a}= \begin{cases}u_{1}, & \text { in } \Omega_{1}  \tag{2.9}\\ \tilde{u}_{1}, & \text { in } \Omega_{c}^{2} \\ u_{2}, & \text { in } \Omega_{2} \backslash \Omega_{c}^{2}\end{cases}
$$



Fig. 3. An illustration of the decomposition: $u-u_{a}=u_{p}$.
which is a piecewise smooth function, see Fig. 3. Indeed, for $u \in \boldsymbol{V}$, we have

$$
\left.u_{p}\right|_{\Omega_{1} \backslash \Omega_{c}^{1}} \in H^{3}\left(\Omega_{\backslash} \backslash \Omega_{c}^{1}\right),\left.\quad u_{p}\right|_{\Omega_{2} \backslash \Omega_{c}^{2}} \in H^{3}\left(\Omega_{2} \backslash \Omega_{c}^{2}\right), \quad u_{p} \mid \Omega_{c} \in C^{3}\left(\Omega_{c}\right)
$$

and $u_{p}$ is discontinuous only on $\widetilde{\Gamma}$, which aligns with the triangular Cartesian mesh $\mathcal{T}_{h}$ (see Fig. 2).
Plugging the decomposition $u=u_{p}+u_{a}$ into the model problem (1.1) leads to

$$
\begin{align*}
-\left.\Delta\right|_{\Omega \backslash \tilde{\Gamma}} u_{p} & =f+\left.\Delta\right|_{\Omega \backslash(\Gamma \cup \tilde{\Gamma})} u_{a}, & & \text { in } \Omega, \\
\llbracket u_{p} \rrbracket_{\tilde{\Gamma}} & =-\llbracket u_{a} \rrbracket_{\tilde{\Gamma}}, & & \text { on } \widetilde{\Gamma}, \\
\llbracket \nabla u_{p} \cdot \boldsymbol{n} \rrbracket_{\tilde{\Gamma}} & =-\llbracket \nabla u_{a} \cdot \boldsymbol{n} \rrbracket_{\tilde{\Gamma}}, & & \text { on } \widetilde{\Gamma},  \tag{2.10}\\
u_{p} & =g, & & \text { on } \partial \Omega .
\end{align*}
$$

In contrast with (1.1), the jump conditions initially on the interface $\Gamma$ are now replaced by new jump conditions on $\widetilde{\Gamma}$. However, the function $u_{a}$ in the right-hand side of (2.10) is unknown. In fact, the model problem (2.10) will be only used as an intermediate problem for the theoretical analysis.

Indeed, the shifting of jump conditions to $\widetilde{\Gamma}$ in (2.10) significantly facilitates the numerical implementation. This inspires us to explicitly construct a good approximation: $u_{a, h} \approx u_{a}$, such that

$$
\begin{equation*}
\operatorname{supp}\left\{u_{a, h}\right\}=\Omega_{c}^{2},\left.\quad u_{a, h}\right|_{\Omega_{c}^{2}} \in C\left(\Omega_{c}^{2}\right), \tag{2.11}
\end{equation*}
$$

by using the jump conditions (2.8). With this, we define the following approximate problem of (2.10):

$$
\begin{align*}
-\left.\Delta\right|_{\Omega \backslash \tilde{\Gamma}} u_{p, h} & =f+\left.\Delta\right|_{\Omega \backslash(\Gamma \cup \tilde{\Gamma})} u_{a, h}, & & \text { in } \Omega, \\
\llbracket u_{p, h} \rrbracket_{\tilde{\Gamma}} & =-\llbracket u_{a, h} \rrbracket_{\tilde{\Gamma}}, & & \text { on } \widetilde{\Gamma},  \tag{2.12}\\
\llbracket \nabla u_{p, h} \cdot \boldsymbol{n} \rrbracket_{\tilde{\Gamma}} & =-\llbracket \nabla u_{a, h} \cdot \boldsymbol{n} \rrbracket_{\tilde{\Gamma}}, & & \text { on } \widetilde{\Gamma}, \\
u_{p, h} & =g, & & \text { on } \partial \Omega .
\end{align*}
$$

This approximate problem turns out to be a variant of (2.10), where the unknown $u_{a}$ is replaced by a known approximate function $u_{a, h}$. It is worthwhile to point out that (2.12) is well posed when $\llbracket u_{a, h} \rrbracket_{\tilde{\Gamma}} \in H^{1 / 2}(\widetilde{\Gamma})$. We postpone the detailed construction of $u_{a, h}$ in Section 3 and provide rigorous error analysis of this approach in Section 4.

### 2.3 HDG formulation

In what follows, we formulate the HDG scheme for the extended interface problem (2.12), which has interface $\widetilde{\Gamma}$ fitted with the triangular Cartesian mesh $\mathcal{T}_{h}$.

We rewrite (2.12) into a first-order system

$$
\begin{align*}
\boldsymbol{q}_{p, h} & =\left.\nabla\right|_{\Omega \backslash \tilde{\Gamma}} u_{p, h}, & & \text { in } \Omega, \\
-\left.\nabla\right|_{\Omega \backslash \backslash \tilde{\Gamma}} \cdot \boldsymbol{q}_{p, h} & =f+\left.\Delta\right|_{\Omega \backslash(\Gamma \cup \tilde{\Gamma})} u_{a, h}, & & \text { in } \Omega, \\
\llbracket u_{p, h} \rrbracket_{\tilde{\Gamma}} & =-\llbracket u_{a, h} \rrbracket_{\tilde{\Gamma}}, & & \text { on } \widetilde{\Gamma},  \tag{2.13}\\
\llbracket \nabla u_{p, h} \cdot \boldsymbol{n} \rrbracket_{\tilde{\Gamma}} & =-\llbracket \nabla u_{a, h} \cdot \boldsymbol{n} \rrbracket_{\tilde{\Gamma}}, & & \text { on } \widetilde{\Gamma}, \\
u_{p, h} & =g, & & \text { on } \partial \Omega .
\end{align*}
$$

Note that $u_{p, h}$ is double valued on the interface $\widetilde{\Gamma}$. However, the trace $\hat{u}_{p, h}^{h} \in \mathcal{M}_{h}^{1}$ solved by standard HDG method can only be the approximation of trace of $u_{p, h}$ from one side of $\widetilde{\Gamma}$. With $\hat{u}_{p, h}^{h}$ an approximation of $\left.u_{p, h}\right|_{\Omega_{1} \cap \tilde{\Gamma}}$, we take $\hat{u}_{p, h}^{h}-\llbracket u_{a, h} \rrbracket_{\tilde{\Gamma}}$ as an approximation of $\left.u_{p, h}\right|_{\Omega_{c}^{2} \cap \tilde{\Gamma}}$ to mimic the jump of $u_{p, h}$ on $\widetilde{\Gamma}$. Accordingly, we define

$$
\tilde{u}_{p, h}^{h}= \begin{cases}\hat{u}_{p, h}^{h}-\llbracket u_{a, h} \rrbracket \rrbracket_{\tilde{\Gamma}}, & \text { if } \partial T \cap \tilde{\Gamma} \neq \emptyset \text { and } T \in \Omega_{2} \backslash \Omega_{c}^{2},  \tag{2.14}\\ \hat{u}_{p, h}^{h}, & \text { otherwise }\end{cases}
$$

Apparently, $\tilde{u}_{p, h}^{h}$ is double valued on the interface $\widetilde{\Gamma}$ and satisfies $\llbracket \tilde{u}_{p, h}^{h} \rrbracket_{\tilde{\Gamma}}=-\llbracket u_{a, h} \rrbracket_{\tilde{\Gamma}}$. We first consider the first two equations of (2.13) on a typical element $T$ of $\mathcal{T}_{h}$. By multiplying the test functions $\boldsymbol{v}$ and $w$ on both sides of them, and integrating by parts, the classical DG scheme reads: find $u_{p, h}^{h} \in \mathcal{W}_{h}^{1}$ and $\boldsymbol{q}_{p, h}^{h} \in \mathcal{V}_{h}^{1}$, such that

$$
\begin{align*}
\left(\boldsymbol{q}_{p, h}^{h}, \boldsymbol{v}\right)_{T}+\left(u_{p, h}^{h}, \nabla \cdot \boldsymbol{v}\right)_{T}-\left\langle\tilde{u}_{p, h}^{h}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial T} & =0, \\
\left(\boldsymbol{q}_{p, h}^{h}, \nabla w\right)_{T}-\left\langle\tilde{\boldsymbol{q}}_{p, h}^{h} \cdot \boldsymbol{n}, w\right\rangle_{\partial T} & =(f, w)_{T}+\left(\left.\Delta\right|_{\Omega \backslash \Gamma} u_{a, h}, w\right)_{T}, \tag{2.15}
\end{align*}
$$

for any $\boldsymbol{v} \in \mathcal{V}_{h}^{1}$ and $w \in \mathcal{W}_{h}^{1}$. In order to capture the jumps across the interface, the numerical flux is taken as (cf. Huynh et al., 2013)

$$
\tilde{\boldsymbol{q}}_{p, h}^{h}=\boldsymbol{q}_{p, h}^{h}-\tau\left(u_{p, h}^{h}-\tilde{u}_{p, h}^{h}\right) \boldsymbol{n}= \begin{cases}\hat{\boldsymbol{q}}_{p, h}^{h}-\tau \llbracket u_{a, h} \rrbracket \tilde{\Gamma}_{\tilde{\Gamma}} \boldsymbol{n}, & \text { if } \partial T \cap \widetilde{\Gamma} \neq \emptyset \text { and } T \in \Omega_{2} \backslash \Omega_{c}^{2},  \tag{2.16}\\ \hat{\boldsymbol{q}}_{p, h}^{h}, & \text { otherwise },\end{cases}
$$

where $\hat{\boldsymbol{q}}_{p, h}^{h}=\boldsymbol{q}_{p, h}^{h}-\tau\left(u_{p, h}^{h}-\hat{u}_{p, h}^{h}\right) \boldsymbol{n}$ is the numerical flux of standard HDG method, and the local stabilization parameter $\tau$ which has an important effect on both the stability and accuracy of the numerical scheme is piecewise, non-negative constant defined on $\partial \mathcal{T}_{h}$. In order to weakly enforce the jump condition in the flux across the interior faces, we reformulate the flux equation as

$$
\begin{equation*}
\left\langle\tilde{\boldsymbol{q}}_{p, h}^{h} \cdot \boldsymbol{n}, \mu\right\rangle_{\left.\partial \mathcal{I}_{h}\right\rangle \partial \Omega}=\left\langle\llbracket \nabla u_{p, h} \cdot \boldsymbol{n} \rrbracket, \mu\right\rangle_{\tilde{\Gamma}}=-\left\langle\llbracket \nabla u_{a, h} \cdot \boldsymbol{n} \rrbracket, \mu\right\rangle_{\tilde{\Gamma}}, \quad \forall \mu \in \mathcal{M}_{h}^{1} . \tag{2.17}
\end{equation*}
$$

Then, substituting (2.14) and (2.16) into (2.15), summing up the resulting equations over all elements and re-arranging some terms, we obtain the HDG formulation: Find $\left(\boldsymbol{q}_{p, h}^{h}, u_{p, h}^{h}, \hat{u}_{p, h}^{h}\right) \in \mathcal{V}_{h}^{1} \times \mathcal{W}_{h}^{1} \times \mathcal{M}_{h}^{1}$ such that

$$
\begin{align*}
\left(\boldsymbol{q}_{p, h}^{h}, \boldsymbol{v}\right)_{\tau_{h}}+\left(u_{p, h}^{h}, \nabla \cdot \boldsymbol{v}\right)_{\mathcal{T}_{h}}-\left\langle\hat{u}_{p, h}^{h}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \tau_{h}} & =-\left\langle\llbracket u_{a, h} \rrbracket_{\tilde{\Gamma}}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\tilde{\Gamma}},  \tag{2.18a}\\
\left(\boldsymbol{q}_{p, h}^{h}, \nabla w\right)_{\mathcal{T}_{h}}-\left\langle\hat{\boldsymbol{q}}_{p, h}^{h} \cdot \boldsymbol{n}, w\right\rangle_{\partial \tau_{h}} & =(f, w)_{\tau_{h}}-\left\langle\tau \llbracket u_{a, h} \rrbracket_{\tilde{\Gamma}}, w\right\rangle_{\tilde{\Gamma}}+\left(\left.\Delta\right|_{\Omega \Omega \Gamma} u_{a, h}, w\right)_{\tau_{h, c}},  \tag{2.18b}\\
\left\langle\hat{\boldsymbol{q}}_{p, h}^{h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h} \backslash \Omega} & =-\left\langle\llbracket \nabla u_{a, h} \cdot \boldsymbol{n}_{\tilde{\Gamma}} \rrbracket-\tau \llbracket u_{a, h} \rrbracket, \mu\right\rangle_{\tilde{\Gamma}},  \tag{2.18c}\\
\left\langle\hat{u}_{p, h}^{h}, \mu\right\rangle_{\partial \Omega} & =\langle g, \mu\rangle_{\partial \Omega}, \tag{2.18d}
\end{align*}
$$

for all $(\boldsymbol{v}, w, \mu) \in \mathcal{V}_{h}^{1} \times \mathcal{W}_{h}^{1} \times \mathcal{M}_{h}^{1}$.
The HDG formulation (2.18) is a natural extension of the standard HDG method. In fact, the standard HDG scheme is a particular case of the formulation (2.18) with $u_{a, h}=0$.

## 3. Construction of the ansatz function

In the previous discussions, the approximation $u_{a, h}$ of the ansatz function $u_{a}$ was assumed known. In this section, we detail its construction and show that the approximation is of third order.

For the approximation of the ansatz function $u_{a}$, more information regarding the jumps of the derivative of $u$ on $\Gamma$ are needed. For this purpose, we set up a local coordinate system in the normal and the tangential directions at a given point $\left(x_{1}^{*}, x_{2}^{*}\right) \in \Gamma$ as follows:

$$
\begin{equation*}
\xi=\left(x_{1}-x_{1}^{*}\right) \cos \theta+\left(x_{2}-x_{2}^{*}\right) \sin \theta, \quad \eta=-\left(x_{1}-x_{1}^{*}\right) \sin \theta+\left(x_{2}-x_{2}^{*}\right) \cos \theta \tag{3.1}
\end{equation*}
$$

where $\theta$ is the angle between the $x_{1}$-axis and $\xi$-axis (see Fig. 4 (left)). In the neighborhood of the point $\left(x_{1}^{*}, x_{2}^{*}\right)$, the interface can be written as $\xi=\gamma(\eta)$, where $\gamma(\eta)$ is a smooth function with respect to $\eta$ due to the smoothness of $\Gamma$ and $\gamma(0)=\gamma_{\xi}(0)=0$. Note that the coordinate transformation (3.1) is invertible. As a matter of fact, we have

$$
x_{1}=x_{1}(\xi, \eta)=\xi \cos \theta-\eta \sin \theta+x_{1}^{*}, \quad x_{2}=x_{2}(\xi, \eta)=\xi \sin \theta+\eta \cos \theta+x_{2}^{*} .
$$

Therefore, in the neighborhood of the point $\left(x_{1}^{*}, x_{2}^{*}\right)$, the jump functions $g_{i}(\boldsymbol{x})$ can be expressed as functions of $\eta$, i.e.,

$$
g_{i}(\boldsymbol{x})=g_{i}\left(x_{1}(\gamma(\eta), \eta), x_{2}(\gamma(\eta), \eta)\right), \quad i=1,2
$$

For simplicity, $g_{1}^{\prime}$ and $g_{2}^{\prime}$ are used to denote $\frac{\mathrm{d} g_{1}}{\mathrm{~d} \eta}$ and $\frac{\mathrm{d} g_{2}}{\mathrm{~d} \eta}$, respectively. Then under local coordinate system, the following jump conditions hold:

$$
\begin{equation*}
\llbracket u_{\xi} \rrbracket_{\Gamma}=g_{2}, \quad \llbracket u_{\eta} \rrbracket_{\Gamma}=g_{1}^{\prime} . \tag{3.2}
\end{equation*}
$$

A detailed derivation can be found in Li \& Ito (2006). Thus the jump conditions in the $x_{1}$ and $x_{2}$ directions are given by

$$
\begin{equation*}
\llbracket u_{x_{1}} \rrbracket_{\Gamma}=g_{2} \cos \theta-g_{1}^{\prime} \sin \theta, \quad \llbracket u_{x_{2}} \rrbracket_{\Gamma}=g_{2} \sin \theta+g_{1}^{\prime} \cos \theta \tag{3.3}
\end{equation*}
$$



FIg. 4. A diagram of the local coordinates in the normal and tangential directions at a point $\left(x_{1}^{*}, x_{2}^{*}\right) \in \Gamma$ (left). A typical triangle element $T$ with an interface cutting through, the curve between $E$ and $F$ is part of the interface $\Gamma$ (right).

Moreover, the jump condition involving the Laplacian of $u$

$$
\begin{equation*}
\llbracket \Delta u \rrbracket_{\Gamma}=-\llbracket f \rrbracket_{\Gamma}, \tag{3.4}
\end{equation*}
$$

can be derived from the governing equation (1.1) straightforwardly.
Recalling the definition of $u_{a}$ in (2.7), we actually need to establish an approximation of the function $d(\boldsymbol{x})$ on the subdomain $\Omega_{c}$. Importantly, we have the information of $d(\boldsymbol{x})$ on the interface $\Gamma$ available, i.e., the jump conditions (2.8). We first construct a multi-variable Hermit type interpolation $I_{h}^{D} d$ element by element. Let $T=\triangle A B C \in \mathcal{T}_{h, c}$ be a cut triangle with $E$ and $F$ the two intersection points of the interface $\Gamma$ and the edges of $T$ (see Fig. 4 (right)). Moreover, denote the coordinates of two intersection points $E, F$ by $\boldsymbol{a}=\left(a_{1}, a_{2}\right)$ and $\boldsymbol{b}=\left(b_{1}, b_{2}\right)$, respectively. The triangle $T$ is cut by the interface $\Gamma$ into two subsets denoted by $T_{1}=T \cap \Omega_{1}$ and $T_{2}=T \cap \Omega_{2}$, respectively. We now establish $p(\boldsymbol{x}):=\left.I_{h}^{D} d(\boldsymbol{x})\right|_{T} \in \mathcal{P}^{2}(T)$ such that

$$
\begin{array}{cll}
p=\llbracket u \rrbracket, & p_{x_{1}}=\llbracket u_{x_{1}} \rrbracket, & p_{x_{2}}=\llbracket u_{x_{2}} \rrbracket, \\
p=\llbracket u \rrbracket, & \nabla p \cdot \boldsymbol{n}=\llbracket \nabla u \rrbracket \cdot \boldsymbol{n}, & \Delta p=\llbracket \Delta u \rrbracket, \tag{3.5}
\end{array}
$$

Here, additional jump conditions are given in (3.3) and (3.4).
The following Lemma shows the existence and uniqueness of this interpolation.
Lemma 3.1 Under the mesh hypothesis (H1)-(H2) in Subsection 2.2, for a given interface element $T$ as indicated in Fig. 4, the quadratic polynomial $p(\boldsymbol{x})=I_{h}^{D} d(\boldsymbol{x})$ is uniquely determined by the interpolation conditions in (3.5).

Proof. The quadratic polynomial $p(\boldsymbol{x})$ reads

$$
p(\boldsymbol{x})=c_{1}+c_{2} x_{1}+c_{3} x_{2}+c_{4} x_{1}^{2}+c_{5} x_{1} x_{2}+c_{6} x_{2}^{2},
$$

where the coefficients $\left\{c_{i}\right\}_{i=1}^{6}$ are to be determined.

The unit normal direction of the line segment $\overline{E F}$ can be expressed as

$$
\boldsymbol{n}=\left(n_{1}, n_{2}\right)=\left(b_{2}-a_{2}, a_{1}-b_{1}\right) / \sqrt{\left(b_{2}-a_{2}\right)^{2}+\left(a_{1}-b_{1}\right)^{2}}
$$

By a straightforward calculation, the interpolation conditions (3.5) result in a linear system of $\left\{c_{i}\right\}_{i=1}^{6}$ with the coefficient matrix

$$
\mathbb{A}=\left(\begin{array}{cccccc}
1 & a_{1} & a_{2} & a_{1}^{2} & a_{1} a_{2} & a_{2}^{2} \\
1 & b_{1} & b_{2} & b_{1}^{2} & b_{1} b_{2} & b_{2}^{2} \\
0 & 1 & 0 & 2 a_{1} & a_{2} & 0 \\
0 & 0 & 1 & 0 & a_{1} & 2 a_{2} \\
0 & n_{1} & n_{2} & 2 b_{1} n_{1} & b_{2} n_{1}+b_{1} n_{2} & 2 b_{2} n_{2} \\
0 & 0 & 0 & 2 & 0 & 2
\end{array}\right)
$$

whose determinant is

$$
\operatorname{det}(\mathbb{A})=-2\left(\left(a_{1}-b_{1}\right)^{2}+\left(a_{2}-b_{2}\right)^{2}\right)^{\frac{3}{2}}
$$

As long as the two points $E, F$ are $\operatorname{distinct}, \operatorname{det}(\mathbb{A}) \neq 0$. Then the uniqueness of $p(\boldsymbol{x})$ is proved.
The following error estimate for the interpolation which is crucial for the error analysis later.

Theorem 3.2 Let $T \in \mathcal{T}_{h, c}$ be a cut triangle and $p(\boldsymbol{x})=I_{h}^{D} d(\boldsymbol{x})$ be the interpolation constructed above. If $u \in \boldsymbol{V}$ (defined in (2.5)), then

$$
\begin{equation*}
\max _{\boldsymbol{x} \in T}|d(\boldsymbol{x})-p(\boldsymbol{x})| \leq C h^{3}\|u\|_{\tilde{C}^{3}(T)}, \tag{3.6}
\end{equation*}
$$

where $C$ is a generic constant independent of $h, u$ and the ratio

$$
\alpha= \begin{cases}\left(b_{2}-a_{2}\right) /\left(b_{1}-a_{1}\right), & \text { if } a_{1} \neq b_{1}, \\ \left(b_{1}-a_{1}\right) /\left(b_{2}-a_{2}\right), & \text { otherwise }\end{cases}
$$

Proof. Define the error function

$$
e(\boldsymbol{x})=d(\boldsymbol{x})-p(\boldsymbol{x}) .
$$

We first consider the error function $e(\boldsymbol{x})$ on the line segment $\overline{E F}$, which can be parameterized by

$$
\boldsymbol{x}(t)=(1-t) \boldsymbol{a}+t \boldsymbol{b}, \quad t \in[0,1] .
$$

A direct calculation gives

$$
\begin{align*}
\frac{\mathrm{d}^{2} e}{\mathrm{~d} t^{2}}(\boldsymbol{x}(t)) & =\left(\left(b_{1}-a_{1}\right)^{2} \frac{\partial^{2}}{\partial x_{1}^{2}}+2\left(b_{1}-a_{1}\right)\left(b_{2}-a_{2}\right) \frac{\partial^{2}}{\partial x_{1} \partial x_{2}}+\left(b_{2}-a_{2}\right)^{2} \frac{\partial^{2}}{\partial x_{2}^{2}}\right) e(\boldsymbol{x}(t)) \\
& =\left(1+\alpha^{2}\right)\left(b_{1}-a_{1}\right)^{2}\left(\frac{1}{1+\alpha^{2}} \frac{\partial^{2}}{\partial x_{1}^{2}}+\frac{2 \alpha}{1+\alpha^{2}} \frac{\partial^{2}}{\partial x_{1} \partial x_{2}}+\frac{\alpha^{2}}{1+\alpha^{2}} \frac{\partial^{2}}{\partial x_{2}^{2}}\right) e(\boldsymbol{x}(t)) \\
& \triangleq\left(1+\alpha^{2}\right)\left(b_{1}-a_{1}\right)^{2} r_{1}(\boldsymbol{x}(t)) \tag{3.7}
\end{align*}
$$

with

$$
\begin{equation*}
r_{1}(\boldsymbol{x}(t))=\frac{1}{1+\alpha^{2}} \frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{1}^{2}}+\frac{2 \alpha}{1+\alpha^{2}} \frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{1} \partial x_{2}}+\frac{\alpha^{2}}{1+\alpha^{2}} \frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{2}^{2}} . \tag{3.8}
\end{equation*}
$$

Without loss of generality, we assume that $a_{1} \neq b_{1}$ and define the ratio $\alpha=\left(b_{2}-a_{2}\right) /\left(b_{1}-a_{1}\right)$, otherwise, the ratio can be defined as $\alpha=\left(b_{1}-a_{1}\right) /\left(b_{2}-a_{2}\right)$ and the analysis is similar. The unit normal of the line segment $\overline{E F}$ can be expressed in terms of $\alpha$ by

$$
\begin{equation*}
\boldsymbol{n}=(\alpha,-1) / \sqrt{\alpha^{2}+1} . \tag{3.9}
\end{equation*}
$$

The normal derivative of $e(\boldsymbol{x})$ along the segment $\overline{E F}$ is

$$
s(\boldsymbol{x}(t)) \triangleq \nabla e(\boldsymbol{x}(t)) \cdot \boldsymbol{n}=\frac{1}{\sqrt{\alpha^{2}+1}}\left(\frac{\partial e(\boldsymbol{x}(t))}{\partial x_{1}} \alpha-\frac{\partial e(\boldsymbol{x}(t))}{\partial x_{2}}\right) .
$$

Accordingly, its derivative with respect to $t$ is

$$
\begin{align*}
\frac{\mathrm{d} s}{\mathrm{~d} t}(\boldsymbol{x}(t)) & =\sqrt{\alpha^{2}+1}\left(b_{1}-a_{1}\right)\left(\frac{\alpha}{\alpha^{2}+1} \frac{\partial^{2}}{\partial x_{1}^{2}}+\frac{\alpha^{2}-1}{\alpha^{2}+1} \frac{\partial^{2}}{\partial x_{1} \partial x_{2}}-\frac{\alpha}{\alpha^{2}+1} \frac{\partial^{2}}{\partial x_{2}^{2}}\right) e(\boldsymbol{x}(t)) \\
& \triangleq \sqrt{\alpha^{2}+1}\left(b_{1}-a_{1}\right) r_{2}(\boldsymbol{x}(t)) \tag{3.10}
\end{align*}
$$

with

$$
\begin{equation*}
r_{2}(\boldsymbol{x}(t))=\frac{\alpha}{\alpha^{2}+1} \frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{1}^{2}}+\frac{\alpha^{2}-1}{\alpha^{2}+1} \frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{1} \partial x_{2}}-\frac{\alpha}{\alpha^{2}+1} \frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{2}^{2}} . \tag{3.11}
\end{equation*}
$$

Further, denote the Laplacian of $e(\boldsymbol{x})$ by

$$
\begin{equation*}
r_{3}(x(t)) \triangleq \Delta e(\boldsymbol{x}(t))=\frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{1}^{2}}+\frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{2}^{2}} \tag{3.12}
\end{equation*}
$$

Then, combining the definitions of $r_{i}(\boldsymbol{x}(t)), i=1,2,3$, in (3.8), (3.11) and (3.12), we obtain

$$
\left(\begin{array}{ccc}
\frac{1}{1+\alpha^{2}} & \frac{2 \alpha}{1+\alpha^{2}} & \frac{\alpha^{2}}{1+\alpha^{2}}  \tag{3.13}\\
\frac{\alpha}{1+\alpha^{2}} & \frac{\alpha^{2}-1}{1+\alpha^{2}} & -\frac{\alpha}{1+\alpha^{2}} \\
1 & 0 & 1
\end{array}\right)\left(\begin{array}{c}
\frac{\partial^{2} e(\boldsymbol{x}(t))}{x_{1}^{2}} \\
\frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{1} \partial x_{2}} \\
\frac{\partial^{2} e(x(t))}{\partial x_{2}^{2}}
\end{array}\right)=\left(\begin{array}{l}
r_{1}(\boldsymbol{x}(t)) \\
r_{2}(\boldsymbol{x}(t)) \\
r_{3}(\boldsymbol{x}(t))
\end{array}\right) .
$$

A direct computation shows that the determinant of the coefficient matrix is -1 . Thus $\frac{\partial^{2} e(x(t))}{\partial x_{i} \partial x_{j}}, i, j=1,2$, can be uniquely determined by $r_{i}, i=1,2,3$. Actually the solution of (3.13) is

$$
\left\{\begin{array}{l}
\frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{1}}=-\frac{\alpha^{2}-1}{\alpha^{2}+1} r_{1}(\boldsymbol{x}(t))+\frac{2 \alpha}{\alpha^{2}+1} r_{2}(\boldsymbol{x}(t))+\frac{\alpha^{2}}{\alpha^{2}+1} r_{3}(\boldsymbol{x}(t)), \\
\frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{1} \partial x_{2}}=\frac{2 \alpha}{\alpha^{2}+1} r_{1}(\boldsymbol{x}(t))+\frac{\alpha^{2}-1}{\alpha^{2}+1} r_{2}(\boldsymbol{x}(t))-\frac{\alpha}{\alpha^{2}+1} r_{3}(\boldsymbol{x}(t)), \\
\frac{\partial^{2} e(x(t))}{\partial x_{2}^{2}}=\frac{\alpha^{2}-1}{\alpha^{2}+1} r_{1}(\boldsymbol{x}(t))-\frac{2 \alpha}{\alpha^{2}+1} r_{2}(\boldsymbol{x}(t))+\frac{1}{\alpha^{2}+1} r_{3}(\boldsymbol{x}(t)) .
\end{array}\right.
$$

Thanks to the fact that the absolute value of all coefficients of $r_{i}(\boldsymbol{x}(t))$ on the right-hand side are less than 1 , we have estimates

$$
\begin{equation*}
\left|\frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{i} \partial x_{j}}\right| \leq\left|r_{1}(\boldsymbol{x}(t))\right|+\left|r_{2}(\boldsymbol{x}(t))\right|+\left|r_{3}(\boldsymbol{x}(t))\right|, \quad t \in[0,1], \quad i, j=1,2 . \tag{3.14}
\end{equation*}
$$

Invoking the interpolation conditions (3.5), we obtain

$$
e(\boldsymbol{x}(0))=e(\boldsymbol{x}(1))=0, \quad \frac{\mathrm{~d} e}{\mathrm{~d} t}(\boldsymbol{x}(0))=\left(b_{1}-a_{1}\right) \frac{\partial e}{\partial x_{1}}(\boldsymbol{a})+\left(b_{2}-a_{2}\right) \frac{\partial e}{\partial x_{2}}(\boldsymbol{a})=0 .
$$

Applying the Rolle's Theorem twice, there exists $\xi \in(0,1)$ such that $\left.\frac{\mathrm{d}^{2} e}{\mathrm{~d} t^{2}} \boldsymbol{x}(\xi)\right)=0$, which implies $r_{1}(\boldsymbol{x}(\xi))=0$ by (3.7). Thus the Taylor expansion of the function $r_{1}(\boldsymbol{x}(t))$ at $t=\xi$ leads to

$$
\begin{align*}
\left|r_{1}(\boldsymbol{x}(t))\right| & =\left|r_{1}(\boldsymbol{x}(\xi))+\frac{\mathrm{d} r_{1}}{\mathrm{~d} t}\left(\boldsymbol{x}\left(\zeta_{1}\right)\right)(t-\xi)\right| \\
& \leq\left|\left(b_{1}-a_{1}\right)\left(\frac{1}{1+\alpha^{2}} \frac{\partial^{3}}{\partial x_{1}^{3}}+\frac{2 \alpha}{1+\alpha^{2}} \frac{\partial^{3}}{\partial x_{1}^{2} \partial x_{2}}+\frac{\alpha^{2}}{1+\alpha^{2}} \frac{\partial^{3}}{\partial x_{1} \partial x_{2}^{2}}\right) e\left(\boldsymbol{x}\left(\zeta_{1}\right)\right)\right| \\
& +\left|\left(b_{2}-a_{2}\right)\left(\frac{1}{1+\alpha^{2}} \frac{\partial^{3}}{\partial x_{1}^{2} \partial x_{2}}+\frac{2 \alpha}{1+\alpha^{2}} \frac{\partial^{3}}{\partial x_{1} \partial x_{2}^{2}}+\frac{\alpha^{2}}{1+\alpha^{2}} \frac{\partial^{3}}{\partial x_{2}^{3}}\right) e\left(\boldsymbol{x}\left(\zeta_{1}\right)\right)\right| \\
& \leq h\left(\left|\frac{\partial^{3} d\left(\boldsymbol{x}\left(\zeta_{1}\right)\right)}{\partial x_{1}^{3}}\right|+2\left|\frac{\partial^{3} d\left(\boldsymbol{x}\left(\zeta_{1}\right)\right)}{\partial x_{1}^{2} \partial x_{2}}\right|+2\left|\frac{\partial^{3} d\left(\boldsymbol{x}\left(\zeta_{1}\right)\right)}{\partial x_{1} \partial x_{2}^{2}}\right|+\left|\frac{\partial^{3} d\left(\boldsymbol{x}\left(\zeta_{1}\right)\right)}{\partial x_{2}^{3}}\right|\right) \\
& \leq C h\left(\left\|\tilde{u}_{1}\right\|_{C^{3}(T)}+\left\|\tilde{u}_{2}\right\|_{C^{3}(T)}\right) \\
& \leq C h\|u\|_{\tilde{C}^{3}(T)} \tag{3.15}
\end{align*}
$$

where $\zeta_{1}$ is between $t$ and $\xi$ and then in $(0,1)$, and the fact $p(\boldsymbol{x})$ is a quadratic polynomial and the definition of the extensions in (2.6) are used.

Moreover, it is clear that by (3.5),

$$
s(\boldsymbol{x}(0))=\frac{1}{\sqrt{\alpha^{2}+1}}\left(\frac{\partial e(\boldsymbol{a})}{\partial x_{1}} \alpha-\frac{\partial e(\boldsymbol{a})}{\partial x_{2}}\right)=0, \quad s(\boldsymbol{x}(1))=\nabla e(\boldsymbol{b}) \cdot \boldsymbol{n}=0 .
$$

Therefore, there exists $\eta \in(0,1)$ such that $\frac{\mathrm{d} s}{\mathrm{~d} t}(\boldsymbol{x}(\eta))=0$ by Rolle's Theorem in the interval $[0,1]$. This gives $r_{2}(\boldsymbol{x}(\eta))=0$ by (3.10). In addition, $r_{3}(\boldsymbol{x}(1))=0$ due to the interpolation conditions (3.5). In
terms of the expressions of $r_{2}(\boldsymbol{x}(t))$ and $r_{3}(\boldsymbol{x}(t))$ in (3.11) and (3.12), and the facts that $r_{2}(\boldsymbol{x}(\eta))=0$ for some $\eta \in(0,1)$ and $r_{3}(\boldsymbol{x}(1))=0$, the Taylor expansion is used for $r_{2}(\boldsymbol{x}(t))$ and $r_{3}(\boldsymbol{x}(t))$ at $t=\eta$ and 1 , respectively. Then just following the same process as that in (3.15), we obtain

$$
\begin{equation*}
\left|r_{i}(\boldsymbol{x}(t))\right| \leq C h\|u\|_{\tilde{C}^{3}(T)}, \quad i=2,3 . \tag{3.16}
\end{equation*}
$$

Consequently, a combination of (3.14)-(3.16) yields

$$
\begin{equation*}
\left|\frac{\partial^{2} e(\boldsymbol{x}(t))}{\partial x_{i} \partial x_{j}}\right| \leq C h\|u\|_{\tilde{C}^{3}(T)}, \quad i, j=1,2 . \tag{3.17}
\end{equation*}
$$

Now we are ready to estimate $e(\boldsymbol{x})$ in the element $T$. The interpolation conditions (3.5) imply both $e$ and $\nabla e$ vanish at the point $E$. As a result, using the Taylor expansion of $e(\boldsymbol{x})$ at the point $E$ leads to

$$
e(\boldsymbol{x})=\frac{1}{2!}(\Delta \boldsymbol{x})^{T} D^{2} e(\boldsymbol{a}) \Delta \boldsymbol{x}+\frac{1}{3!} \sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial^{3} e(\boldsymbol{\xi})}{\partial x_{j} \partial x_{k} \partial x_{l}}\left(x_{j}-a_{j}\right)\left(x_{k}-a_{k}\right)\left(x_{l}-a_{l}\right),
$$

where $\Delta \boldsymbol{x}=\boldsymbol{x}-\boldsymbol{a}$ and $\boldsymbol{\xi}=\boldsymbol{a}+\theta(\boldsymbol{x}-\boldsymbol{a}) \in T, \theta \in(0,1)$. Consequently,

$$
\begin{align*}
|e(\boldsymbol{x})| & \leq\left|\frac{1}{2!}(\Delta \boldsymbol{x})^{T} D^{2} e(\boldsymbol{a}) \Delta \boldsymbol{x}\right|+h^{3}\left|\sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{2} \frac{\partial^{3} e(\boldsymbol{\xi})}{\partial x_{j} \partial x_{k} \partial x_{l}}\right| \\
& \leq \frac{1}{2}\left(\left|\frac{\partial^{2} e(\boldsymbol{a})}{\partial x_{1}^{2}}\right|+2\left|\frac{\partial^{2} e(\boldsymbol{a})}{\partial x_{1} \partial x_{2}}\right|+\left|\frac{\partial^{2} e(\boldsymbol{a})}{\partial x_{2}^{2}}\right|\right) h^{2}+h^{3} \sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{l=1}^{2}\left|\frac{\partial^{3} e(\boldsymbol{\xi})}{\partial x_{j} \partial x_{k} \partial x_{l}}\right| \\
& \leq C h^{3}\|u\|_{\tilde{C}^{3}(T)} \tag{3.18}
\end{align*}
$$

where (3.17) is implemented.
Remark 3.3 The fact that the constant $C$ in the estimate (3.6) is independent of the ratio $\alpha$ implies that the estimate holds for interfaces of arbitrary shape.

Although $I_{h}^{D} d(\boldsymbol{x})$ approximates $d(\boldsymbol{x})$ with the accuracy $O\left(h^{3}\right)$ in maximum norm, we shall not use it to produce $u_{a, h}$ directly due to its discontinuity across the element boundary in $\mathcal{T}_{h, c}$. Instead, it is used to construct another $C^{0}$ interpolation

$$
\begin{equation*}
I_{h}^{C} d(\boldsymbol{x}) \in\left\{d_{h} \in C\left(\Omega_{c}\right):\left.\quad d_{h}\right|_{T} \in \mathcal{P}^{2}(T), \quad \forall T \in \mathcal{T}_{h, c}\right\} \tag{3.19}
\end{equation*}
$$

which also approximates $d(\boldsymbol{x})$, but with better regularity. Let $\mathcal{N}=\left\{\boldsymbol{x}_{i}\right\}$ be the set of all conventional Lagrange interpolation nodes on the mesh $\mathcal{T}_{h, c}$, see Fig. 5 (left). Since some nodes are associated with at least two elements, $I_{h}^{D} d(\boldsymbol{x})$ is usually multi-valued at certain node $\boldsymbol{x}_{i} \in \mathcal{N}$. For example, at the node $\boldsymbol{x}_{i}$ in Fig. 5 (right), $I_{h}^{D} d(\boldsymbol{x})$ can take different values from six associated triangles. There always holds

$$
\left|I_{h}^{D} d\left(\boldsymbol{x}_{i}\right)-d\left(\boldsymbol{x}_{i}\right)\right| \leq C h^{3}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}
$$



FIG. 5. Lagrange interpolation nodes of $I_{h}^{C} d(\boldsymbol{x})$ on $\mathcal{T}_{h, c}$ (left). A diagram for the determination of the approximate node value $\tilde{d}_{i}$ (right).
no matter which associated triangle of $\boldsymbol{x}_{i}$ the value of $I_{h}^{D} d\left(\boldsymbol{x}_{i}\right)$ is taken from. Thus, for each node $\boldsymbol{x}_{i} \in \mathcal{N}$, we only need to pick up a fixed associated element $T_{i}$ of $\boldsymbol{x}_{i}$ and take $I_{h}^{D} d\left(\boldsymbol{x}_{i}\right)$ in it as an approximate value of $d\left(\boldsymbol{x}_{i}\right)$. Denoting this approximate value by $\tilde{d}_{i}$, we have

$$
\begin{equation*}
\left|d\left(\boldsymbol{x}_{i}\right)-\tilde{d}_{i}\right| \leq C h^{3}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} \tag{3.20}
\end{equation*}
$$

Define $I_{h}^{C} d(\boldsymbol{x})$ as the classical piecewise quadratical Lagrange interpolation on nodes $\left\{\boldsymbol{x}_{i}\right\}$ in $\mathcal{T}_{h, c}$, satisfying

$$
I_{h}^{C} d\left(\boldsymbol{x}_{i}\right)=\tilde{d}_{i}, \quad \forall \boldsymbol{x}_{i} \in \mathcal{N} .
$$

Then $I_{h}^{C} d(\boldsymbol{x})$ is implemented to define

$$
u_{a, h}:= \begin{cases}I_{h}^{C} d(\boldsymbol{x}), & \text { for } \boldsymbol{x} \in \Omega_{c}^{2}  \tag{3.21}\\ 0, & \text { otherwise }\end{cases}
$$

REMARK 3.4 In the above, we propose a new approach based on interpolation technique to construct the ansatz function, which only relies on the location of the interface and jump conditions. Moreover, the resulted function $u_{a, h}$ is a piecewise polynomial, its gradient and Laplacian can be calculated exactly. However, the ansatz function in Kummer \& Oberlack (2013) heavily depend on a signed-distance function. The construction of signed-distance level set fields appears not that simple for complex interfaces. Moreover, the accuracy of the HDG method relies on extra high order accuracy of the approximate signed-distance function due to the fact that a new term involving the Laplacian of the ansatz function appears in the righthand side of the extended Poisson equation.

According to Theorem 3.2, we immediately have the following error estimate for $I_{h}^{C} d(\boldsymbol{x})$.
Corollary 3.5 Assume that $u \in \boldsymbol{V}$. Then

$$
\max _{x \in T}\left|D^{k}\left(I_{h}^{C} d(\boldsymbol{x})-d(\boldsymbol{x})\right)\right| \leq C h^{3-k}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}
$$

for $k=0,1,2$ and any $T \in \mathcal{T}_{h, c}$.
Proof. The error can be decomposed into two parts by using triangle inequality as follows:

$$
\left|D^{k}\left(I_{h}^{C} d(\boldsymbol{x})-d(\boldsymbol{x})\right)\right| \leq\left|D^{k}\left(I_{h}^{C} d(\boldsymbol{x})-I_{h} d(\boldsymbol{x})\right)\right|+\left|D^{k}\left(I_{h} d(\boldsymbol{x})-d(\boldsymbol{x})\right)\right| .
$$

Here $I_{h} d(\boldsymbol{x})$ is the classical piecewise quadratical Lagrange interpolation at nodes $\mathcal{N}$ in $\mathcal{T}_{h, c}$ satisfying

$$
I_{h} d(\boldsymbol{x})=d\left(x_{i}\right), \quad \forall \boldsymbol{x}_{i} \in \mathcal{N}
$$

It is noted that the standard estimate for Lagrange interpolation can be used for estimate of the second part. We only need to deal with the first term. It is clear that

$$
\left|D^{k}\left(I_{h}^{C} d(\boldsymbol{x})-I_{h} d(\boldsymbol{x})\right)\right|=\left|\sum_{i=1}^{6}\left(\tilde{d}_{i}-d\left(\boldsymbol{x}_{i}\right)\right) D^{k} N_{i}\right| \leq C h^{3}\|u\|_{\tilde{C}\left(\Omega_{c}\right)} \sum_{i=1}^{6}\left|D^{k} N_{i}\right|
$$

with $\left\{N_{i}\right\}_{i=1}^{6}$ the quadratic Lagrange interpolation basis functions, where (3.20) is used. The implementation of the chain rule leads to

$$
\nabla_{x} N_{k}=\mathbb{J}^{-1} \nabla_{\xi} N_{k}, \quad\left(\frac{\partial^{2} N_{k}}{\partial x_{i} \partial x_{j}}\right)_{2 \times 2}=\mathbb{J}^{-2}\left(\frac{\partial^{2} N_{k}}{\partial \xi_{i} \partial \xi_{j}}\right)_{2 \times 2}, \quad k=1, \cdots, 6,
$$

where $\xi_{i}, i=1,2$, are barycentric coordinates in the reference triangle and

$$
\mathbb{J}=\left(\begin{array}{ll}
\frac{\partial x_{1}}{\partial \xi_{1}} & \frac{\partial x_{2}}{\partial \xi_{1}} \\
\frac{\partial x_{1}}{\partial \xi_{2}} & \frac{\partial x_{2}}{\partial \xi_{2}}
\end{array}\right),
$$

is the Jacobian matrix. Thanks to the uniform triangulation, $\mathbb{J}=h \mathbb{I}$ with $\mathbb{I}$ the identity matrix. This completes the proof.

Recalling the definitions of $u_{a}$ and $u_{a, h}$ in (2.7) and (3.21), we obtain the following approximation result from the above.

Lemma 3.6 Under the conditions in Theorem 3.2, we have that for any $T \in \mathcal{T}_{h, c}$,

$$
\begin{equation*}
\max _{x \in T \cap \Omega_{c}^{2}}\left|D^{k} u_{a}(\boldsymbol{x})-D^{k} u_{a, h}(\boldsymbol{x})\right| \leq C h^{3-k}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}, \quad k=0,1,2, \tag{3.22}
\end{equation*}
$$

where $C$ is a positive constant independent of $h, u$ and the shape of the interface.

In the end of this section, we outline some basic ideas on how to construct similar ansatz functions in solving three dimensional interface problems, and highlight the underlying difficulties. Jump conditions like (3.5) can also be derived for 3D problem, see Li \& Ito (2006, Chapter 4). Note that in 3D case, the interface may cut the edges of a tetrahedral element at three or four points, which must be treated differently in the construction of the ansatz function $u_{a, h}$. Although we can use similar polynomial interpolation techniques to construct $u_{a, h}$, it is challenging yet to choose appropriate interpolation conditions (from the jump conditions) so that $u_{a, h}$ always offers a stable approximation of $u_{a}$, regardless of the shape and position of the interface. Once $u_{a, h}$ and an estimate similar to Theorem 3.2 are available, the HDG method and the analysis of the whole algorithm can be extended to 3D case straightforwardly.

In principle, one can construct $u_{a, h}$ to be a higher order approximation of $u_{a}$ by using more interpolation conditions, and then increase the polynomial degree in HDG discretization to devise a high-order unfitted method. Interestingly, Guzman et al. (2015) introduced some other techniques for constructing higherorder interpolation of the singular part of the discontinuous solution.

## 4. Error analysis

In our unfitted mesh method, the HDG solver is actually used to solve the approximate interface problem (2.12). Therefore, the well-posedness of it and the estimate of the error $u_{p}-u_{p, h}$ play a crucial role in the analysis.

We introduce the standard broken Sobolev space

$$
H^{r}\left(\mathcal{T}_{h}\right):=\left\{u: u \in L^{2}(\Omega),\left.u\right|_{T} \in H^{r}(T), \forall T \in \mathcal{T}_{h}\right\}
$$

with the norm

$$
\|u\|_{H^{r}\left(\mathcal{T}_{h}\right)}=\sum_{T \in \mathcal{T}_{h}}\|u\|_{H^{r}(T)}
$$

and

$$
X_{\tilde{\Gamma}}^{r}=L^{2}(\Omega) \cap H^{r}\left(\Omega_{1} \cup \Omega_{c}\right) \cap H^{r}\left(\Omega_{2} \backslash \Omega_{c}^{2}\right), \quad X_{\Gamma \cup \widetilde{\Gamma}}^{r}=L^{2}(\Omega) \cap H^{r}\left(\Omega_{1}\right) \cap H^{r}\left(\Omega_{c}^{2}\right) \cap H^{r}\left(\Omega_{2} \backslash \Omega_{c}^{2}\right),
$$

equipped with norms

$$
\|v\|_{X_{\tilde{T}}^{r}}=\|v\|_{H^{r}\left(\Omega_{1} \cup \Omega_{c}\right)}+\|v\|_{H^{r}\left(\Omega_{2} \backslash \Omega_{c}^{2}\right)}, \quad\|v\|_{\Gamma \cup \tilde{\Gamma}}^{r}=\|v\|_{H^{r}\left(\Omega_{1}\right)}+\|v\|_{H^{r}\left(\Omega_{c}^{2}\right)}+\|v\|_{H^{r}\left(\Omega_{2} \backslash \Omega_{c}^{2}\right)} .
$$

The first result is on the regularity of the solution of (2.12), and the error estimate between the solutions of the intermediate problem (2.10) and the extended problem (2.12).

Theorem 4.1 Assume that $f \in L^{2}(\Omega), g \in H^{1}(\partial \Omega)$ and $u \in \boldsymbol{V}$. Then the approximate problem (2.12) has a unique solution $u_{p, h} \in X_{\widetilde{\Gamma}}^{3 / 2}$ satisfying the a priori estimate:

$$
\begin{equation*}
\left\|u_{p, h}\right\|_{X_{\Gamma}^{3 / 2}} \leq C\left(\|f\|_{L^{2}(\Omega)}+\|g\|_{H^{1}(\partial \Omega)}+\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}\right) . \tag{4.1}
\end{equation*}
$$

Moreover, we have the error estimate

$$
\begin{equation*}
\left\|u_{p}-u_{p, h}\right\|_{X_{\tilde{\Gamma}}^{r}} \leq C h^{3-r}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}, \quad r=1,3 / 2 \tag{4.2}
\end{equation*}
$$

where the positive constant $C$ only depends on the area of $\Omega_{c}$ and the length of $\tilde{\Gamma}$.
Proof. The construction of $u_{a}$ and $u_{a, h}$ shows that $u_{a} \in C^{3}\left(\Omega_{c}^{2}\right)$ and $u_{a, h} \in C^{0}\left(\Omega_{c}^{2}\right)$. Thus, $\llbracket u_{a}-u_{a, h} \rrbracket_{\tilde{\Gamma}} \in$ $H^{1}(\widetilde{\Gamma})$, and

$$
\begin{equation*}
\left\|\llbracket u_{a}-u_{a, h} \rrbracket\right\|_{H^{1}(\widetilde{\Gamma})} \leq C\left(\left\|\llbracket \nabla u_{a}-\nabla u_{a, h} \rrbracket\right\|_{L^{2}(\widetilde{\Gamma})}+\left\|\llbracket u_{a}-u_{a, h} \rrbracket\right\|_{L^{2}(\tilde{\Gamma})}\right) \leq C|\widetilde{\Gamma}|^{1 / 2} h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}, \tag{4.3}
\end{equation*}
$$

by using the maximum norm estimate (3.22). According to the definition of $u_{a}$, we can derive

$$
\begin{equation*}
\left\|\llbracket u_{a} \rrbracket\right\|_{H^{1}(\tilde{\Gamma})} \leq C\left(\left\|u_{2}\right\|_{H^{3 / 2}\left(\Omega_{c}^{2}\right)}+\left\|\tilde{u}_{1}\right\|_{H^{3 / 2}\left(\Omega_{c}^{2}\right)}\right) \leq C \mid \Omega_{c}^{2} 1^{1 / 2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}, \tag{4.4}
\end{equation*}
$$

by using the trace theorem and the estimate (2.6). It is worthwhile to point out that as shown in Ding (1996), the constant in the trace theorem of Sobolev spaces on Lipschitz domains only depends on the Lipschitz constant of the boundary and the dimension of the space. Note that here the Lipschitz constant of $\widetilde{\Gamma}$ is independent of $h$, so the constant $C$ in (4.4) is independent of $h$ as well.

Applying the triangular inequality to (4.3) leads to

$$
\left\|\llbracket u_{a, h} \rrbracket\right\|_{H^{1}(\widetilde{\Gamma})} \leq\left\|\llbracket u_{a} \rrbracket\right\|_{H^{1}(\tilde{\Gamma})}+C|\widetilde{\Gamma}| h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} \leq C\left(\left|\Omega_{c}^{2}\right|^{1 / 2}+|\widetilde{\Gamma}|^{1 / 2}\right)\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} .
$$

This estimate also implies that

$$
\left\|\llbracket \nabla u_{a, h} \cdot \boldsymbol{n} \rrbracket\right\|_{L^{2}(\tilde{\Gamma})} \leq C\left(\left|\Omega_{c}^{2}\right|^{1 / 2}+|\widetilde{\Gamma}|^{1 / 2}\right)\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} .
$$

Then the existence and uniqueness of the solution and the regularity result (4.1) can be derived straightforwardly from Theorem 2.1.

Now, we turn to the error estimate (4.2). Subtracting the problem (2.12) from the (2.10), we obtain the following interface problem

$$
\begin{align*}
-\left.\Delta\right|_{\Omega \backslash \tilde{\Gamma}}\left(u_{p}-u_{p, h}\right) & =\left.\Delta\right|_{\Omega \backslash(\Gamma \cup \tilde{\Gamma})}\left(u_{a}-u_{a, h}\right), & & \text { in } \Omega, \\
\llbracket u_{p}-u_{p, h} \rrbracket_{\tilde{\Gamma}} & =-\llbracket u_{a}-u_{a, h} \rrbracket_{\tilde{\Gamma}}, & & \text { on } \widetilde{\Gamma}, \\
\llbracket \nabla\left(u_{p}-u_{p, h}\right) \cdot \boldsymbol{n} \rrbracket_{\tilde{\Gamma}} & =-\llbracket \nabla\left(u_{a}-u_{a, h}\right) \cdot \boldsymbol{n} \rrbracket_{\tilde{\Gamma}}, & & \text { on } \widetilde{\Gamma},  \tag{4.5}\\
u_{p}-u_{p, h} & =0, & & \text { on } \partial \Omega .
\end{align*}
$$

Recovering the original interface $\Gamma$, the interface problem (4.5) is equivalent to

$$
\begin{align*}
-\left.\Delta\right|_{\Omega \backslash(\Gamma \cup \widetilde{\Gamma})}\left(u_{p}-u_{p, h}-u_{a, h}+u_{a}\right) & =0, & & \text { in } \Omega, \\
\llbracket u_{p}-u_{p, h}-u_{a, h}+u_{a} \rrbracket_{\Gamma} & =\llbracket u_{a}-u_{a, \hbar} \rrbracket_{\Gamma}, & & \text { on } \Gamma, \\
\llbracket \nabla\left(u_{p}-u_{p, h}-u_{a, h}+u_{a}\right) \cdot \boldsymbol{n} \rrbracket_{\Gamma} & =\llbracket \nabla\left(u_{a}-u_{a, h}\right) \cdot \boldsymbol{n} \rrbracket_{\Gamma}, & & \text { on } \Gamma, \\
\llbracket u_{p}-u_{p, h}-u_{a, h}+u_{a} \rrbracket_{\tilde{\Gamma}} & =0, & & \text { on } \widetilde{\Gamma},  \tag{4.6}\\
\llbracket \nabla\left(u_{p}-u_{p, h}-u_{a, h}+u_{a}\right) \cdot \boldsymbol{n} \rrbracket_{\tilde{\Gamma}} & =0, & & \text { on } \widetilde{\Gamma}, \\
u_{p}-u_{p, h}-u_{a, h}+u_{a} & =0, & & \text { on } \partial \Omega,
\end{align*}
$$

with two interface $\Gamma$ and $\widetilde{\Gamma}$. Since $u_{a}-u_{a, h} \in H^{1}(\Gamma)$, we use Theorem 2.1 again to derive

$$
\left\|u_{p}-u_{p, h}-u_{a, h}+u_{a}\right\|_{\tilde{\Gamma} \cup \Gamma}^{r} \leq\left\|\llbracket u_{a}-u_{a, h} \rrbracket_{\Gamma}\right\|_{H^{r-1 / 2}(\Gamma)}+\left\|\llbracket \nabla\left(u_{a}-u_{a, h}\right) \cdot \boldsymbol{n} \rrbracket_{\Gamma}\right\|_{H^{r-3 / 2}(\Gamma)}, \quad 0 \leq r \leq 3 / 2 .
$$

Especially, together with estimates (3.22), we give more specific estimations for $r=1,3 / 2$, respectively. Note that

$$
\begin{aligned}
& \|\phi\|_{H^{-1 / 2}(\widetilde{\Gamma})}=\sup _{v \in H^{1 / 2}(\widetilde{\Gamma})\{\{0\}} \frac{\langle\phi, v\rangle_{\widetilde{\Gamma}}}{\|v\|_{H^{1 / 2}(\widetilde{\Gamma})}} \leq \sup _{v \in H^{1 / 2}(\widetilde{\Gamma}) \backslash\{0\}} \frac{\|\phi\|_{L^{2}(\widetilde{\Gamma}}\|v\|_{L^{2}(\widetilde{\Gamma})}}{\|v\|_{H^{1 / 2}(\widetilde{\Gamma})}} \\
& \leq \sup _{v \in H^{1 / 2}(\widetilde{\Gamma}) \backslash\{0\}} \frac{C|\widetilde{\Gamma}|^{1 / 2}\|\phi\|_{L^{\infty}\left(\Omega_{c}^{2}\right)}\|v\|_{L^{2}(\widetilde{\Gamma})}}{\|v\|_{H^{1 / 2}(\widetilde{\Gamma})}} \leq C|\widetilde{\Gamma}|^{1 / 2}\|\phi\|_{L^{\infty}\left(\Omega_{c}^{2}\right)},
\end{aligned}
$$

holds for any $\phi \in L^{\infty}\left(\Omega_{c}^{2}\right)$, so we can derive

$$
\begin{aligned}
\left\|u_{p}-u_{p, h}\right\|_{X_{\tilde{\Gamma} \cup \Gamma}^{1}} & \leq\left\|u_{a}-u_{a, h}\right\|_{X_{\tilde{\Gamma} \cup \Gamma}^{1}}+\left\|\llbracket u_{a}-u_{a, h} \rrbracket_{\Gamma}\right\|_{H^{1 / 2}(\Gamma)}+\left\|\llbracket \nabla\left(u_{a}-u_{a, h}\right) \cdot \boldsymbol{n} \rrbracket_{\Gamma}\right\|_{H^{-1 / 2}(\Gamma)} \\
& \leq C\left(\left|\Omega_{c}^{2}\right|^{1 / 2}+|\widetilde{\Gamma}|^{1 / 2}\right) \sum_{T \subset \mathcal{T}_{h, c}}\left(\max _{x \in T \cap \Omega_{c}^{2}}\left|u_{a}-u_{a, h}\right|+\max _{x \in T \cap \Omega_{c}^{2}}\left|\nabla\left(u_{a}-u_{a, h}\right)\right|\right) \\
& \leq C\left(\left|\Omega_{c}^{2}\right|^{1 / 2}+|\widetilde{\Gamma}|^{1 / 2}\right) h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)},
\end{aligned}
$$

where the triangular inequality, error estimate (3.22) and the fact $\operatorname{supp}\left(u_{a}-u_{a, h}\right)=\Omega_{c}^{2}$ were used. Since $u_{p}, u_{p, h} \in X_{\tilde{\Gamma}}^{1}$, we obtain

$$
\begin{equation*}
\left\|u_{p}-u_{p, h}\right\|_{X_{\tilde{\Gamma}}^{1}} \leq\left\|u_{p}-u_{p, h}\right\|_{X_{\tilde{\Gamma} \cup \Gamma}^{1}} \leq C\left(\left|\Omega_{c}^{2}\right|^{1 / 2}+|\widetilde{\Gamma}|^{1 / 2}\right) h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} . \tag{4.7}
\end{equation*}
$$

Furthermore, using the interpolation inequality and error estimate (3.22) leads to

$$
\begin{aligned}
\left\|u_{p}-u_{p, h}\right\|_{X_{\Gamma \cup \Gamma}^{3 / 2}} & \leq\left\|u_{a}-u_{a, h}\right\|_{X_{\Gamma}^{3 / 2}}^{3 / 2}+\left\|\llbracket u_{a}-u_{a, h} \rrbracket_{\Gamma}\right\|_{H^{1}(\Gamma)}+\| \| \nabla\left(u_{a}-u_{a, h}\right) \cdot \boldsymbol{n} \rrbracket_{\Gamma} \|_{L^{2}(\Gamma)} \\
& \leq C\left(\left\|u_{a}-u_{a, h}\right\|_{X_{\tilde{\Gamma}}^{1} \cup \Gamma}^{1 / 2}\left\|u_{a}-u_{a, h}\right\|_{X_{\tilde{\Gamma} \cup \Gamma}^{2}}^{1 / 2}+\left\|\llbracket u_{a}-u_{a, h} \rrbracket_{\Gamma}\right\|_{H^{1}(\Gamma)}+\left\|\llbracket \nabla\left(u_{a}-u_{a, h}\right) \cdot \boldsymbol{n} \rrbracket_{\Gamma}\right\|_{L^{2}(\Gamma)}\right) \\
& \leq C\left(\left|\Omega_{c}^{2}\right|^{1 / 2}+|\widetilde{\Gamma}|^{1 / 2}\right) h^{3 / 2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} .
\end{aligned}
$$

Since $u_{p} \in X_{\widetilde{\Gamma}}^{3}, u_{p, h} \in X_{\tilde{\Gamma}}^{3 / 2}$, then $u_{p}-u_{p, h} \in X_{\tilde{\Gamma}}^{3 / 2}$ and

$$
\begin{equation*}
\left\|u_{p}-u_{p, h}\right\|_{X_{\tilde{\Gamma}}^{3 / 2}} \leq\left\|u_{p}-u_{p, h}\right\|_{X_{\tilde{\Gamma} \cup \Gamma}^{3 / 2}} \leq C\left(\left|\Omega_{c}^{2}\right|^{1 / 2}+|\widetilde{\Gamma}|^{1 / 2}\right) h^{3 / 2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} . \tag{4.8}
\end{equation*}
$$

This ends the proof.
Another important part of the theoretical analysis is the error analysis of the HDG formulation (2.18). We follow the technique in Cockburn et al. (2010). The analysis relies on the projection $\Pi_{h}$ on $u_{p, h}$ and its gradient $\boldsymbol{q}_{p, h}$, defined by

$$
\Pi_{h}: \boldsymbol{L}^{2}(\Omega) \times L^{2}(\Omega) \rightarrow \mathcal{V}_{h}^{1} \times \mathcal{M}_{h}^{1}, \quad \Pi_{h}\left(\boldsymbol{q}_{p, h}, u_{p, h}\right):=\left(\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}, \Pi_{\mathrm{w}} u_{p, h}\right),
$$

where for any $T \in \mathcal{T}_{h}$ and all edges/faces $F$ of $T$, the functions $\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}$ and $\Pi_{\mathrm{w}} u_{p, h}$ satisfy

$$
\begin{array}{rlrl}
\left(\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}, \boldsymbol{v}\right)_{T} & =\left(\boldsymbol{q}_{p, h}, \boldsymbol{v}\right)_{T}, & & \forall \boldsymbol{v} \in\left[\mathcal{P}_{0}(T)\right]^{2}, \\
\left(\Pi_{\mathrm{w}} u_{p, h}, w\right)_{T} & =\left(u_{p, h}, w\right)_{T}, & & \forall w \in \mathcal{P}_{0}(T), \\
\left\langle\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h} \cdot \boldsymbol{n}-\tau \Pi_{\mathrm{w}} u_{p, h}, \mu\right\rangle_{F} & =\left\langle\boldsymbol{q}_{p, h} \cdot \boldsymbol{n}-\tau u_{p, h}, \mu\right\rangle_{F}, & \forall \mu \in \mathcal{P}_{1}(F) . \tag{4.9c}
\end{array}
$$

According to the regularity result in Theorem 4.1, we know that $u_{p, h}, \boldsymbol{q}_{p, h} \in \operatorname{dom}\left(\Pi_{h}\right)$. However, the regularity conclusion $u_{p, h} \in X_{\tilde{\Gamma}}^{3 / 2}$ is not enough to ensure a second-order convergence rate for this projection. We need to use another important fact that $u_{p, h}$ is an $O\left(h^{2}\right)$ approximation of $u_{p} \in H^{3}\left(\mathcal{T}_{h}\right)$. By introducing the intermediate function $u_{p}$ in the error analysis, we prove a second-order convergence rate for the projection $\Pi_{h}\left(\boldsymbol{q}_{p, h}, u_{p, h}\right)$ as stated in the following theorem.

Theorem 4.2 If $\tau$ in (2.16) is non-negative and $u \in \boldsymbol{V}$, then the system (4.9) is uniquely solvable for $\Pi_{\mathrm{v}} \boldsymbol{q}_{p, h}$ and $\Pi_{\mathrm{w}} u_{p, h}$. Furthermore, there is a positive constant $C$ only depending on the maximum of $\tau$, the area of $\Omega_{c}$ and the length of $\tilde{\Gamma}$, such that

$$
\begin{align*}
& \left\|\Pi_{\mathrm{w}} u_{p, h}-u_{p, h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)} \leq C \max \left\{1, \frac{1}{\tau^{\max }}\right\} h^{2}\left(\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}+\|u\|_{H^{3}\left(\mathcal{T}_{h} \mid \mathcal{T}_{h, c}\right)}\right),  \tag{4.10a}\\
& \left\|\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)} \leq C \max \left\{1, \tau^{\max }\right\} h^{2}\left(\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}+\|u\|_{H^{3}\left(\mathcal{T}_{h} \mid \mathcal{I}_{h, c}\right)}\right), \tag{4.10b}
\end{align*}
$$

where $\tau^{\text {max }}:=\left.\max \tau\right|_{\partial \tau_{h}}$.
With the aid of the projection $\Pi_{h}$, we can derive the error estimate for the HDG formulation (2.18).
Theorem 4.3 Let $\left(\boldsymbol{q}_{p, h}^{h}, u_{p, h}^{h}, \hat{u}_{p, h}^{h}\right)$ solve the system (2.18), and let the solution $\boldsymbol{q}_{p, h}, u_{p, h}$ be in the domain of $\Pi_{h}$. Then

$$
\begin{align*}
& \left\|\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}^{h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)} \leq\left\|\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)}, \\
& \left\|\Pi_{\mathrm{w}} u_{p, h}-u_{p, h}^{h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)} \leq C h\left\|\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}\right\|_{L^{2}\left(\mathcal{I}_{h}\right)} . \tag{4.11}
\end{align*}
$$

To avoid distraction from the main result, we postpone the proofs of the above two theorems to the Appendix A and B, respectively.

Let us introduce $u_{h}=u_{p, h}^{h}+u_{a, h}$ and $\boldsymbol{q}_{h}=\boldsymbol{q}_{p, h}^{h}+\nabla u_{a, h}$. Then the main result is stated as follows.
Theorem 4.4 Let $u \in \boldsymbol{V}$ such that $(\boldsymbol{q}, u)$ be the solution of (1.1), and let $\left(\boldsymbol{q}_{p, h}^{h}, u_{p, h}^{h}\right)$ be the solution of (2.18). Then the following error estimate holds:

$$
\begin{align*}
& \left\|u-u_{h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)} \leq C h^{2}\left(\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}+\|u\|_{H^{3}\left(\mathcal{T}_{h} \backslash \tau_{h, c}\right)}\right),  \tag{4.12a}\\
& \left\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)} \leq C h^{2}\left(\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}+\|u\|_{H^{3}\left(\mathcal{T}_{h} \backslash \mathcal{T}_{h, c}\right)}\right) \tag{4.12b}
\end{align*}
$$

where the positive constant $C$ only depends on $\tau$, the area of $\Omega_{c}$ and the length of $\widetilde{\Gamma}$.

Proof. Firstly, the triangle inequality gives

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)} \leq\left\|u_{p}-u_{p, h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)}+\left\|u_{p, h}-u_{p, h}^{h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)}+\left\|u_{a}-u_{a, h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)}, \tag{4.13}
\end{equation*}
$$

where the facts $u=u_{p}+u_{a}$ and $u_{h}=u_{p, h}^{h}+u_{a, h}$ were used. Then, it follows from Theorems 4.2 and 4.3 that

$$
\begin{aligned}
\left\|u_{p, h}-u_{p, h}^{h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)} & \leq C\left(\left\|u_{p, h}-\Pi_{\mathrm{w}} u_{p, h}\right\|_{L^{2}\left(\mathcal{I}_{h}\right)}+\left\|\Pi_{\mathrm{w}} u_{p, h}-u_{p, h}^{h}\right\|_{L^{2}\left(\mathcal{T}_{h}\right)}\right) \\
& \leq C h^{2}\left(\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)}+\|u\|_{H^{3}\left(\mathcal{T}_{h} \backslash \mathcal{T}_{h, c}\right)}\right) .
\end{aligned}
$$

Moreover, by using the following fact

$$
\left\|u_{a}-u_{a, h}\right\|_{L^{2}\left(\mathcal{I}_{h}\right)} \leq C\left|\Omega_{c}\right|^{\frac{1}{2}} \max _{x \in \Omega_{c}}\left|u_{a}-u_{a, h}\right| \leq C\left|\Omega_{c}\right|^{\frac{1}{2}} h^{3}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)},
$$

and Theorem 4.1, we can obtain the estimate (4.12a). The result (4.12b) can be proved in a similar fashion.

Remark 4.5 The algorithm and analysis are for the Poisson interface problem (1.1). To solve a general elliptic interface problem with discontinuous coefficients, one can use the iterative procedure in Kummer \& Oberlack (2013), and resort to the above proposed algorithm to solve a similar type of problem at each iteration. Another more effective approach is to construct the ansatz function $u_{a, h}$ in a similar fashion, but the resulted $u_{a, h}$ will depend on the unknown function $u_{p, h}$ (see Hou et al. (2013) for some basic ideas). Accordingly, we need to come up with a stable algorithm to solve them simultaneously. In a nutshell, the main techniques introduced in this article pave the way for solving and analyzing more general interface problems.

## 5. Numerical results

In this section, we present some numerical examples with general curved interfaces to demonstrate the accuracy and robustness of our approach. In all examples, the computational domain is $\Omega=[-1,1] \times$ $[-1,1]$ and divided uniformly by triangular mesh $\mathcal{T}_{h}$ with $h_{x}=h_{y}=\frac{1}{2^{n}}$.

### 5.1 Circular interface

We first give two examples with the same circular interface $\Gamma$ given by $\sqrt{x_{1}^{2}+x_{2}^{2}}=0.5$, but with different jump conditions.

Example 5.1 Choose $f$ such that the exact solution is

$$
u= \begin{cases}\sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right)+1, & \left(x_{1}, x_{2}\right) \in \Omega_{1} \\ \sin \left(\pi x_{1}\right) \sin \left(\pi x_{2}\right), & \left(x_{1}, x_{2}\right) \in \Omega_{2}\end{cases}
$$

Accordingly, the jump conditions across the interface are

$$
\llbracket u \rrbracket=1, \quad \llbracket \nabla u \cdot \boldsymbol{n} \rrbracket=0, \quad \text { on } \Gamma .
$$

TABLE $1 L^{2}$-error and order of convergence for $u_{h}, \boldsymbol{q}_{h}$ and $u_{h}^{*}$ (Example 5.1)

| $k$ | Mesh | $\left\\|u-u_{h}\right\\|_{\mathcal{T}_{h}}$ | Order | $\left\\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\\|_{\mathcal{T}_{h}}$ | Order | $\left\\|u-u_{h}^{*}\right\\|_{\mathcal{T}_{h}}$ | Order |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $8 \times 8$ | $9.2055 \mathrm{e}-2$ | - | $3.1281 \mathrm{e}-1$ | - | $2.3124 \mathrm{e}-2$ | - |
|  | $16 \times 16$ | $2.8092 \mathrm{e}-2$ | 1.774 | $9.2055 \mathrm{e}-2$ | 1.765 | $3.4159 \mathrm{e}-3$ | 2.759 |
|  | $32 \times 32$ | $7.6896 \mathrm{e}-3$ | 1.870 | $2.5108 \mathrm{e}-2$ | 1.874 | $4.6538 \mathrm{e}-4$ | 2.876 |
|  | $64 \times 64$ | $2.0177 \mathrm{e}-3$ | 1.930 | $6.5693 \mathrm{e}-3$ | 1.934 | $6.0799 \mathrm{e}-5$ | 2.936 |
|  | $128 \times 128$ | $5.1716 \mathrm{e}-4$ | 1.964 | $1.6811 \mathrm{e}-3$ | 1.966 | $7.7723 \mathrm{e}-6$ | 2.968 |
| 2 | $8 \times 8$ | $6.5609 \mathrm{e}-3$ | - | $2.1953 \mathrm{e}-2$ | - | $9.3507 \mathrm{e}-4$ | - |
|  | $16 \times 16$ | $9.2047 \mathrm{e}-4$ | 2.833 | $3.0354 \mathrm{e}-3$ | 2.855 | $5.9630 \mathrm{e}-5$ | 3.971 |
|  | $32 \times 32$ | $1.2212 \mathrm{e}-4$ | 2.914 | $4.0025 \mathrm{e}-4$ | 2.923 | $3.7516 \mathrm{e}-6$ | 3.990 |
|  | $64 \times 64$ | $1.5735 \mathrm{e}-5$ | 2.956 | $5.1429 \mathrm{e}-5$ | 2.960 | $2.3509 \mathrm{e}-7$ | 3.996 |
|  | $128 \times 128$ | $1.9971 \mathrm{e}-6$ | 2.978 | $6.5193 \mathrm{e}-6$ | 2.980 | $1.4711 \mathrm{e}-8$ | 3.998 |
| 3 | $8 \times 8$ | $3.3237 \mathrm{e}-4$ | - | $1.1061 \mathrm{e}-3$ | - | $6.7615 \mathrm{e}-5$ | - |
|  | $16 \times 16$ | $2.2629 \mathrm{e}-5$ | 3.877 | $7.4572 \mathrm{e}-5$ | 3.891 | $2.1463 \mathrm{e}-6$ | 4.977 |
|  | $32 \times 32$ | $1.4770 \mathrm{e}-6$ | 3.938 | $4.8464 \mathrm{e}-6$ | 3.944 | $6.7372 \mathrm{e}-8$ | 4.994 |
|  | $64 \times 64$ | $9.4349 \mathrm{e}-8$ | 3.969 | $3.0898 \mathrm{e}-7$ | 3.971 | $2.4222 \mathrm{e}-9$ | 4.798 |

Note that $u_{a}(\boldsymbol{x})=u_{2}(\boldsymbol{x})-u_{1}(\boldsymbol{x})=-1$. Therefore, $u_{a, h}$ is exactly equal to $u_{a}$. Under this circumstance, we have $u_{p, h}=u_{p}$. Thus, optimal convergence rate can be obtained for both $u_{h}$ and $\boldsymbol{q}_{h}$ in $L^{2}$-norm, see Table 1 for $k=1,2,3$.

According to Stenberg (1989) and Huynh et al. (2013), if the function $u_{p, h}$ is sufficiently smooth in any element $T \in \mathcal{T}_{h}$, the accuracy of the numerical solution will be improved by using a local postprocessing. On every element $T \in \mathcal{T}_{h}$, we define a new approximate solution $u_{p, h}^{h *} \in \mathcal{P}^{k+1}(T)$ such that it satisfies

$$
\begin{align*}
\left(\nabla u_{p, h}^{h *}, \nabla w\right)_{T} & =\left(\boldsymbol{q}_{p, h}^{h}, \nabla w\right)_{T}, \quad \forall w \in \mathcal{P}^{k+1}(T), \\
\left(u_{p, h}^{h *}, 1\right)_{T} & =\left(u_{p, h}^{h}, 1\right)_{T} . \tag{5.1}
\end{align*}
$$

Since the construction of $u_{p, h}^{h *}$ is done elementwise, therefore it is very efficient. The new approximation $u_{h}^{*}:=u_{p, h}^{h *}+u_{a, h}$ has superconvergence of order $k+2$, see Table 1 . The numerical solution $u_{h}$ on $32 \times 32$ mesh with $k=1$ is depicted in Fig. 6 (left). We need to point out that finer mesh is used in all plots presented in this section for a better resolution of the interface.

Example 5.2 In this example, the jumps $\llbracket u \rrbracket$ in the solution, $\llbracket \nabla u \cdot \boldsymbol{n} \rrbracket$ in the flux and $\llbracket f \rrbracket$ in the source term are computed from the following exact solution:

$$
u= \begin{cases}\sin \left(x_{1}+x_{2}\right)+\cos \left(x_{1}+x_{2}\right), & \left(x_{1}, x_{2}\right) \in \Omega_{1}, \\ -\left(x_{1}^{2}+x_{2}^{2}\right), & \left(x_{1}, x_{2}\right) \in \Omega_{2}\end{cases}
$$



Fig. 6. The numerical solution $u_{h}$ of the circular interface.

TABLE $2 L^{2}$-error and order of convergence for $u_{h}$ and $\boldsymbol{q}_{h}$ (Examples 5.2-5.6)

| Example | Mesh | $\left\\|u-u_{h}\right\\|_{0}$ | Order | $\left\\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\\|_{0}$ | Order |
| :--- | :---: | :---: | :---: | :---: | :---: |
|  | $8 \times 8$ | $1.0674 \mathrm{e}-2$ | - | $2.4986 \mathrm{e}-2$ | - |
|  | $16 \times 16$ | $2.6670 \mathrm{e}-3$ | 2.001 | $6.4708 \mathrm{e}-3$ | 1.949 |
| 5.2 | $32 \times 32$ | $6.7042 \mathrm{e}-4$ | 1.992 | $1.6548 \mathrm{e}-3$ | 1.967 |
|  | $64 \times 64$ | $1.6832 \mathrm{e}-4$ | 1.994 | $4.1888 \mathrm{e}-4$ | 1.982 |
|  | $128 \times 128$ | $4.2180 \mathrm{e}-5$ | 1.997 | $1.0541 \mathrm{e}-4$ | 1.991 |
|  | $16 \times 16$ | $4.8309 \mathrm{e}-3$ | - | $2.4484 \mathrm{e}-2$ | - |
| 5.3 | $32 \times 32$ | $1.1520 \mathrm{e}-3$ | 2.068 | $6.0670 \mathrm{e}-3$ | 2.013 |
|  | $64 \times 64$ | $2.8090 \mathrm{e}-4$ | 2.036 | $1.5074 \mathrm{e}-3$ | 2.009 |
|  | $128 \times 128$ | $6.9345 \mathrm{e}-5$ | 2.018 | $3.7566 \mathrm{e}-4$ | 2.005 |
|  | $32 \times 32$ | $2.6760 \mathrm{e}-3$ | - | $2.1770 \mathrm{e}-2$ | - |
| 5.4 | $64 \times 64$ | $6.6576 \mathrm{e}-3$ | 2.007 | $5.6110 \mathrm{e}-3$ | 1.956 |
|  | $128 \times 128$ | $1.6642 \mathrm{e}-4$ | 2.000 | $1.4253 \mathrm{e}-3$ | 1.977 |
|  | $8 \times 8$ | $1.3909 \mathrm{e}-2$ | - | $1.0934 \mathrm{e}-1$ | - |
|  |  | $16 \times 16$ | $2.7914 \mathrm{e}-3$ | 2.317 | $2.6939 \mathrm{e}-2$ |
| 5.5 | $32 \times 32$ | $6.1992 \mathrm{e}-4$ | 2.171 | $6.6936 \mathrm{e}-3$ | 2.021 |
|  | $64 \times 64$ | $1.4619 \mathrm{e}-4$ | 2.084 | $1.6663 \mathrm{e}-3$ | 2.006 |
|  |  | $128 \times 128$ | $3.6205 \mathrm{e}-5$ | 2.014 | $4.1538 \mathrm{e}-4$ |
|  |  | 2.004 |  |  |  |
|  | $32 \times 32$ | $6.1768 \mathrm{e}-4$ | - | $6.9771 \mathrm{e}-3$ | - |
| 5.6 | $64 \times 64$ | $1.5144 \mathrm{e}-4$ | 2.028 | $1.7877 \mathrm{e}-3$ | 1.965 |
|  |  | $128 \times 128$ | $3.7671 \mathrm{e}-5$ | 2.007 | $4.5215 \mathrm{e}-4$ |
|  |  | 1.983 |  |  |  |

This problem has jumps on both $u$ and flux $\nabla u \cdot \boldsymbol{n}$. We only use HDG solver with linear polynomial basis because $u_{p, h}$ has a low regularity. The $L^{2}$-errors and their corresponding convergence rate are tabulated in Table 2. A second convergence rate is achieved for both $u_{h}$ and $\boldsymbol{q}_{h}$. The numerical solution $u_{h}$ on $32 \times 32$ mesh is depicted in Fig. 6 (right).


FIG. 7. The numerical solution $u_{h}$ of the elliptic interface (left) and the star shaped interface (right), respectively.

### 5.2 Interface of complicated shape

In this section, we present some examples with complicated interfaces. The jump in the solution as well as the jump in the flux is nonzero, which can be obtained from the expression of the exact solution by a simple calculation.

Example 5.3 The interface $\Gamma$ is now defined by $\frac{x_{1}^{2}}{0.8^{2}}+\frac{x_{2}^{2}}{0.64^{2}}=1$. Choose $f$ such that the exact solution

$$
u= \begin{cases}-3 x_{1}^{2}+3 x_{2}^{2}+2, & \left(x_{1}, x_{2}\right) \in \Omega_{1} \\ e^{x_{1}} \cos x_{2}, & \left(x_{1}, x_{2}\right) \in \Omega_{2}\end{cases}
$$

The $L^{2}$-errors and their corresponding convergence rate are listed in Table 2. Figure 7 (left) shows the numerical solution $u_{h}$ on $32 \times 32$ mesh.

Example 5.4 The interface is the star-shaped line which is defined by

$$
r=0.7+0.2 \sin (5 \theta), \quad \theta \in[0,2 \pi),
$$

where $r=\sqrt{x_{1}^{2}+x_{2}^{2}}$. Suppose that the exact solution is

$$
u= \begin{cases}e^{x_{1}^{2}+x_{2}^{2}}, & \left(x_{1}, x_{2}\right) \in \Omega_{1} \\ \left(x_{1}^{2}+x_{2}^{2}\right)^{2}-0.1 \ln \left(2 \sqrt{x_{1}^{2}+x_{2}^{2}}\right), & \left(x_{1}, x_{2}\right) \in \Omega_{2}\end{cases}
$$

The $L^{2}$-errors and their corresponding convergence rates are presented in Table 2. We depict the numerical solution $u_{h}$ on $32 \times 32$ mesh in Fig. 7 (right).

Example 5.5 In this example, we consider the Kidney-line interface which is determined by

$$
\left(1.5\left(\left(x_{1}+0.5\right)^{2}+x_{2}^{2}\right)-0.5\left(x_{1}+0.5\right)\right)^{2}-1.2\left(\left(x_{1}+0.5\right)^{2}+x_{2}^{2}\right)+0.14=0
$$



FIG. 8. The numerical solution $u_{h}$ of the kidney-line interface (left) and butterfly-line interface (right), respectively.

The exact solution is

$$
u= \begin{cases}\sin \left(2 x_{1}^{2}+x_{2}^{2}+2\right)+x_{1}, & \left(x_{1}, x_{2}\right) \in \Omega_{1} \\ 0.1 \cos \left(1-x_{1}^{2}-x_{2}^{2}\right), & \left(x_{1}, x_{2}\right) \in \Omega_{2}\end{cases}
$$

The corresponding $L^{2}$-errors and convergence rate in Table 2 shows that a second-order convergence rate is obtained for both $u_{h}$ and $\boldsymbol{q}_{h}$. The numerical solution $u_{h}$ on $32 \times 32$ mesh is depicted in Fig. 8 (left).

Example 5.6 In the last example, we consider a butterfly-line interface governed by $6.25 x_{2}^{2}-12.25 x_{1}^{2}+$ $18.7578 x_{1}^{4}-0.5147=0$. The exact solution is

$$
u= \begin{cases}e^{-x_{1}^{2}-x_{2}^{2}}, & \left(x_{1}, x_{2}\right) \in \Omega_{1} \\ 0, & \left(x_{1}, x_{2}\right) \in \Omega_{2}\end{cases}
$$

The $L^{2}$-errors and their corresponding convergence rates are presented in Table 2. In Fig. 8 (right), we depict the numerical solution $u_{h}$ on $32 \times 32$ mesh.

In Table 2, the beginning meshes with different mesh size are just to meet the mesh assumptions $\left(\mathbf{H}_{\mathbf{1}}\right),\left(\mathbf{H}_{\mathbf{2}}\right)$ given in Subsection 2.2. All the numerical results validate that our unfitted mesh method possesses a second-order accuracy for both the solution and its gradient for Poisson interface problems of complicate shape.

## 6. Conclusions

In this article, we presented an unfitted mesh method for solving Poisson interface problems. A new piecewise polynomial ansatz function was constructed to approximate the singular part of the solution. By using this ansatz function, we derived an extended interface problem with interface fitted with a quasi-uniform triangular mesh. Then, an HDG method was used to solve it. At the end, we proved that this method had second-order convergence rate not only for the potential $u$, but also for flux $\boldsymbol{q}$. To our best knowledge, the second-order estimate for gradient is the first work concerning the unfitted mesh methods. At present, our method can only handle the jump condition of the form $\llbracket \frac{\partial u}{\partial n} \rrbracket$. However, in many
applications what we have is the jump condition of the form $\llbracket \beta \frac{\partial u}{\partial n} \rrbracket$. Fortunately, the main techniques proposed in this article can also be used to develop unfitted mesh method for that general case. We leave this as a future work.

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## Appendix A. Proof of Theorem 4.2

Some techniques for the proof below are similar to those in Cockburn et al. (2010), but we feel it is necessary to provide some details with an emphasis on some different aspects. The proof relies on a projection $\boldsymbol{B}_{\mathrm{v}}^{k}$ introduced and studied in Cockburn \& Dong (2007). Let $F$ be a face of $T$ and $F^{*}$ be a face at which $\left.\tau\right|_{\partial T}$ attains its maximum. For any function $\sigma \in L^{2}(T)$ and $\left.\sigma\right|_{F} \in L^{2}(F)$ in the domain of $\Pi_{\mathbf{v}}$, the restriction of $\boldsymbol{B}_{\mathbf{v}}^{k}$ to $T$ is defined to be the unique element of $\mathcal{P}^{k}(T)$ satisfying

$$
\begin{align*}
\left(\boldsymbol{B}_{\mathbf{v}}^{k} \boldsymbol{\sigma}, \boldsymbol{v}\right)_{T} & =(\boldsymbol{\sigma}, \boldsymbol{v})_{T}, & & \forall \boldsymbol{v} \in\left[\mathcal{P}^{k-1}(T)\right]^{2}  \tag{A.1a}\\
\left\langle\boldsymbol{B}_{\mathbf{v}}^{k} \boldsymbol{\sigma} \cdot \boldsymbol{n}, \mu\right\rangle_{F} & =\langle\boldsymbol{\sigma} \cdot \boldsymbol{n}, \mu\rangle_{F}, & & \forall \mu \in \mathcal{P}^{k}(F) \tag{A.1b}
\end{align*}
$$

for all faces $F$ of $T$ different from $F^{*}$.

Proof. Here, we mainly focus on the interface triangle $T \in \mathcal{T}_{h, c}$, and the estimates on noninterface elements can be derived in a similar fashion. Note that

$$
\left\|\Pi_{\mathrm{w}} u_{p, h}-u_{p, h}\right\|_{L^{2}(T)} \leq\left\|\Pi_{\mathrm{w}} u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h}\right\|_{L^{2}(T)}+\left\|\mathbb{P}_{h}^{1} u_{p, h}-u_{p, h}\right\|_{L^{2}(T)}
$$

where $\mathbb{P}_{h}^{1} u_{p, h}$ is the $L^{2}$-projection of $u_{p, h}$ into $\mathcal{P}_{1}(T)$. For the second term, we have

$$
\begin{aligned}
\left\|\mathbb{P}_{h}^{1} u_{p, h}-u_{p, h}\right\|_{L^{2}(T)} & \leq\left\|\mathbb{P}_{h}^{1}\left(u_{p, h}-u_{p}\right)\right\|_{L^{2}(T)}+\left\|\mathbb{P}_{h}^{1} u_{p}-u_{p}\right\|_{L^{2}(T)}+\left\|u_{p}-u_{p, h}\right\|_{L^{2}(T)} \\
& \leq 2\left\|u_{p, h}-u_{p}\right\|_{L^{2}(T)}+C h^{2}\left\|u_{p}\right\|_{H^{2}(T)}
\end{aligned}
$$

Summing up over all interface elements, we obtain

$$
\begin{equation*}
\left\|\mathbb{P}_{h}^{1} u_{p, h}-u_{p, h}\right\|_{L^{2}\left(\mathcal{T}_{h, c}\right)} \leq 2\left\|u_{p, h}-u_{p}\right\|_{L^{2}\left(\mathcal{T}_{h, c}\right)}+C h^{2}\left\|u_{p}\right\|_{H^{2}\left(\mathcal{T}_{h, c}\right)} \leq C h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} \tag{A.2}
\end{equation*}
$$

by using the error estimate (4.2).
Next, we come to deal with the first term. The definition of $\Pi_{\mathrm{w}}$ implies that

$$
\Pi_{\mathrm{w}} u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h} \in \mathcal{P}_{\perp}^{1}(T):=\left\{w \in \mathcal{P}^{1}(T):(w, \zeta)_{T}=0, \forall \zeta \in \mathcal{P}^{0}(T)\right\}
$$

Define

$$
b_{u_{p, h}}(w)=\left\langle\tau\left(\Pi_{\mathrm{w}} u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h}\right), w\right\rangle_{\partial T}, \quad \forall w \in \mathcal{P}_{\perp}^{1}(T)
$$

Moreover, we have the following estimate (see Lemma A. 1 in Cockburn et al. (2010))

$$
\begin{equation*}
\left\|\Pi_{\mathrm{w}} u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h}\right\|_{L^{2}(T)} \leq C \frac{h}{\tau_{T}^{\max }}\left\|b_{u_{p, h}}\right\|_{L^{2}(T)} \tag{A.3}
\end{equation*}
$$

with $\tau_{T}^{\max }=\left.\max \tau\right|_{\partial T}$. Now we come to estimate $\left\|b_{u_{p, h}}\right\|_{L^{2}(T)}$. By (4.9c), we obtain

$$
b_{u_{p, h}}(w)=\left\langle\nabla u_{p, h}-\Pi_{\mathrm{v}} \nabla u_{p, h}, w\right\rangle_{\partial T}+\left\langle\tau\left(u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h}\right), w\right\rangle_{\partial T} .
$$

Since $w \in \mathcal{P}_{\perp}^{1}(T)$, we have

$$
\begin{aligned}
\left\langle\left(\nabla u_{p}-\Pi_{\mathrm{v}} \nabla u_{p}\right) \cdot \boldsymbol{n}, w\right\rangle_{\partial T} & =\left(\nabla \cdot\left(\nabla u_{p}-\Pi_{\mathrm{v}} \nabla u_{p}\right), w\right)_{T}+\left(\nabla u_{p}-\Pi_{\mathrm{v}} \nabla u_{p}, \nabla w\right)_{T} \\
& =\left(\nabla \cdot\left(\nabla u_{p}-\Pi_{\mathrm{v}} \nabla u_{p}\right), w\right)_{T}=\left(\nabla \cdot \nabla u_{p}, w\right)_{T} \\
& =\left(\Delta u_{p}-\mathbb{P}_{h}^{0} \Delta u_{p}, w\right)_{T},
\end{aligned}
$$

and

$$
\begin{aligned}
\left\langle\left(\Pi_{\mathrm{v}} \nabla u_{p}-\Pi_{\mathrm{v}} \nabla u_{p, h}\right) \cdot \boldsymbol{n}, w\right\rangle_{\partial T} & =\left(\nabla \cdot\left(\Pi_{\mathrm{v}} \nabla u_{p}-\Pi_{\mathrm{v}} \nabla u_{p, h}\right), w\right)_{T}+\left(\Pi_{\mathbf{v}}\left(\nabla u_{p}-\nabla u_{p, h}\right), \nabla w\right)_{T} \\
& =\left(\Pi_{\mathbf{v}}\left(\nabla u_{p}-\nabla u_{p, h}\right), \nabla w\right)_{T}=\left(\nabla u_{p}-\nabla u_{p, h}, \nabla w\right)_{T} .
\end{aligned}
$$

Consequently,

$$
\begin{align*}
\left\langle\left(\nabla u_{p, h}-\Pi_{\mathrm{v}} \nabla u_{p, h}\right) \cdot \boldsymbol{n}, w\right\rangle_{\partial T}= & \left\langle\left(\nabla u_{p, h}-\nabla u_{p}\right) \cdot \boldsymbol{n}, w\right\rangle_{\partial T}+\left\langle\left(\nabla u_{p}-\Pi_{\mathrm{v}} \nabla u_{p}\right) \cdot \boldsymbol{n}, w\right\rangle_{\partial T} \\
& +\left\langle\left(\Pi_{\mathrm{v}} \nabla u_{p}-\Pi_{\mathrm{v}} \nabla u_{p, h}\right) \cdot \boldsymbol{n}, w\right\rangle_{\partial T} \\
= & \left\langle\left(\nabla u_{p, h}-\nabla u_{p}\right) \cdot \boldsymbol{n}, w\right\rangle_{\partial T}+\left(\Delta u_{p}-\mathbb{P}_{h}^{0} \Delta u_{p}, w\right)_{T} \\
& +\left(\nabla u_{p}-\nabla u_{p, h}, \nabla w\right)_{T} \\
\leq & C h^{-1 / 2}\left\|\nabla u_{p, h}-\nabla u_{p}\right\|_{L^{2}(\partial T)}\|w\|_{L^{2}(T)}+C h\left\|u_{p}\right\|_{H^{3}(T)}\|w\|_{L^{2}(T)} \\
& +C h^{-1}\left\|\nabla\left(u_{p}-u_{p, h}\right)\right\|_{L^{2}(T)}\|w\|_{L^{2}(T)} . \tag{A.4}
\end{align*}
$$

Further, by virtue of a trace inequality and the approximation of $L^{2}$-projection, we have

$$
\begin{align*}
\left\|\mathbb{P}_{h}^{1} u_{p, h}-u_{p, h}\right\|_{L^{2}(\partial T)} & \leq\left\|\mathbb{P}_{h}^{1} u_{p, h}-\mathbb{P}_{h}^{1} u\right\|_{L^{2}(\partial T)}+\left\|\mathbb{P}_{h}^{1} u_{p}-u_{p}\right\|_{L^{2}(\partial T)}+\left\|u_{p}-u_{p, h}\right\|_{L^{2}(\partial T)} \\
& \leq 2\left\|u_{p, h}-u_{p}\right\|_{L^{2}(\partial T)}+\left\|\mathbb{P}_{h}^{1} u_{p}-u_{p}\right\|_{L^{2}(\partial T)}, \\
& \leq 2\left\|u_{p, h}-u_{p}\right\|_{L^{2}(\partial T)}+h^{-1 / 2}\left(\left\|\mathbb{P}_{h}^{1} u_{p}-u_{p}\right\|_{L^{2}(T)}+h\left|\mathbb{P}_{h}^{1} u_{p}-u_{p}\right|_{H^{1}(T)}\right) \\
& \leq 2\left\|u_{p, h}-u_{p}\right\|_{L^{2}(\partial T)}+C h^{3 / 2}\left\|u_{p}\right\|_{H^{2}(T)} . \tag{A.5}
\end{align*}
$$

Thus

$$
\begin{align*}
\left\langle\tau\left(u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h}\right), w\right\rangle_{\partial T} & \leq C \tau_{T}^{\max }\left\|\mathbb{P}_{h}^{1} u_{p, h}-u_{p, h}\right\|_{L^{2}(\partial T)}\|w\|_{L^{2}(\partial T)} \\
& \leq C \tau_{T}^{\max }\left(h^{-1 / 2}\left\|u_{p, h}-u_{p}\right\|_{L^{2}(\partial T)}+h\left\|u_{p}\right\|_{H^{2}(T)}\right)\|w\|_{L^{2}(T)} \tag{A.6}
\end{align*}
$$

We derive from (A.4) and (A.6) that

$$
\begin{align*}
\left\|b_{u, p}\right\|_{L^{2}(T)} \leq & C \max \left(1, \tau_{T}^{\max }\right)\left(h^{-1 / 2}\left\|\nabla u_{p, h}-\nabla u_{p}\right\|_{L^{2}(\partial T)}+h\left\|u_{p}\right\|_{H^{3}(T)}\right. \\
& \left.+h^{-1}\left\|\nabla\left(u_{p}-u_{p, h}\right)\right\|_{L^{2}(T)}+h^{-1 / 2}\left\|u_{p, h}-u_{p}\right\|_{L^{2}(T)}+h\left\|u_{p}\right\|_{H^{2}(T)}\right) \tag{A.7}
\end{align*}
$$

Summing up (A.3) over all interface elements leads to

$$
\begin{align*}
\left\|\Pi_{\mathrm{w}} u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h}\right\|_{L^{2}\left(\mathcal{I}_{h, c}\right)} \leq & C \max \left(1, \frac{1}{\tau^{\max }}\right) h\left(h^{-1 / 2}\left\|\nabla u_{p, h}-\nabla u_{p}\right\|_{X_{\widetilde{\Gamma}}^{1 / 2}}+h\left\|u_{p}\right\|_{H^{3}\left(\mathcal{T}_{h, c}\right)}\right. \\
& \left.+h^{-1}\left\|\nabla\left(u_{p}-u_{p, h}\right)\right\|_{L^{2}\left(\mathcal{I}_{h, c}\right)}+h^{-1 / 2}\left\|u_{p, h}-u_{p}\right\|_{X_{\Gamma}^{1 / 2}}+h\left\|u_{p}\right\|_{H^{2}\left(\mathcal{T}_{h, c}\right)}\right) \\
\leq & C \max \left(1, \frac{1}{\tau^{\max }}\right) h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} \tag{A.8}
\end{align*}
$$

where $\tau^{\max }=\left.\max \tau\right|_{\partial \tau_{h, c}}$. It follows from (A.2) and (A.8) that

$$
\begin{equation*}
\left\|\Pi_{\mathrm{w}} u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h}\right\|_{L^{2}\left(\mathcal{I}_{h, c}\right)} \leq C \max \left(1, \frac{1}{\tau^{\max }}\right) h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} . \tag{A.9}
\end{equation*}
$$

Now we begin to deal with the error estimate $\left\|\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}\right\|$. A basis for $\mathbb{R}^{n}$ is furnished by the set of unit normals $\boldsymbol{n}_{F}$ for the $n$ faces $F \neq F^{*}$ of $K$. Letting $\left\{\tilde{\boldsymbol{n}}_{F}: F \neq F^{*}\right\}$ denote its dual basis, i.e., $\tilde{\boldsymbol{n}}_{F} \cdot \boldsymbol{n}_{F^{\prime}}=\delta_{F F^{\prime}}$, we can write $\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}=\sum_{F \neq F^{*}}\left(\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h} \cdot \boldsymbol{n}_{F}\right) \tilde{\boldsymbol{n}}_{F}$. Hence, it is enough to estimate all components $\left(\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}\right) \cdot \boldsymbol{n}_{F}$. Since we have

$$
\left\|\left(\boldsymbol{\Pi}_{\mathbf{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}\right) \cdot \boldsymbol{n}_{F}\right\|_{L^{2}(T)} \leq\left\|\left(\boldsymbol{\Pi}_{\mathbf{v}} \boldsymbol{q}_{p, h}-\boldsymbol{B}_{\mathbf{v}}^{1} \boldsymbol{q}_{p, h}\right)\right\|_{L^{2}(T)}+\left\|\boldsymbol{B}_{\mathbf{v}}^{1} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}\right\|_{L^{2}(T)}
$$

For the second term, the triangular equality gives

$$
\left\|\boldsymbol{B}_{\mathbf{v}}^{1} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}\right\|_{L^{2}(T)} \leq\left\|\boldsymbol{B}_{\mathbf{v}}^{1}\left(\nabla u_{p, h}-\nabla u_{p}\right)\right\|_{L^{2}(T)}+\left\|\boldsymbol{B}_{\mathbf{v}}^{1} \nabla u_{p}-\nabla u_{p}\right\|_{L^{2}(T)}+\left\|\nabla u_{p}-\nabla u_{p, h}\right\|_{L^{2}(T)} .
$$

Further, by the approximation properties of $\boldsymbol{B}_{\mathbf{v}}^{k}$ established in Cockburn \& Dong (2007), we obtain

$$
\left\|\boldsymbol{B}_{\mathbf{v}}^{1} \nabla u_{p}-\nabla u_{p}\right\|_{L^{2}(T)} \leq C h^{2}\left|\nabla u_{p}\right|_{H^{2}(T)},
$$

and

$$
\left\|\boldsymbol{B}_{\mathbf{v}}^{1}\left(\nabla u_{p}-\nabla u_{p, h}\right)\right\|_{L^{2}(T)} \leq C\left(\left\|\nabla u_{p}-\nabla u_{p, h}\right\|_{L^{2}(T)}+h^{1 / 2}\left\|\left(\nabla u_{p}-\nabla u_{p, h}\right)\right\|_{L^{2}(\partial T)}\right) .
$$

Therefore, the summation over all interface element leads to

$$
\begin{align*}
\left\|\boldsymbol{B}_{v}^{1} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}\right\|_{L^{2}\left(\mathcal{T}_{h, c}\right)} \leq & C\left\|\nabla u_{p, h}-\nabla u_{p}\right\|_{L^{2}\left(\mathcal{T}_{h, c}\right)}+C h^{2}\left|\nabla u_{p}\right|_{H^{2}\left(\mathcal{T}_{h, c}\right)} \\
& +C h^{1 / 2}\left\|\nabla u_{p}-\nabla u_{p, h}\right\|_{L^{2}\left(\partial \tau_{h, c}\right)} \\
\leq & C h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} . \tag{A.10}
\end{align*}
$$

Let $\Theta_{F}:=\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{B}_{\mathbf{v}}^{1} \boldsymbol{q}_{p, h}$. We conclude that $\Theta_{F}$ is in $\mathcal{P}_{\perp}^{1}(T)$ from (4.9a) and (A.1a). Subtracting (A.1b) from (4.9c), we obtain

$$
\left\langle\Theta_{F}, \mu\right\rangle_{\partial T}=\left\langle\tau\left(\Pi_{\mathrm{w}} u_{p, h}-u_{p, h}\right), \mu\right\rangle_{\partial T}, \quad \forall \mu \in \mathcal{P}^{1}(F), \quad \forall F \neq F^{*} .
$$

From Lemmas A. 1 and A. 2 in Cockburn et al. (2010), we derive

$$
\begin{equation*}
\left\|\Theta_{F}\right\|_{L^{2}(T)} \leq h\|b\|_{L^{2}(T)}, \tag{A.11}
\end{equation*}
$$

where $b(w):=\left\langle\tau\left(\Pi_{\mathrm{w}} u_{p, h}-u_{p, h}\right), w\right\rangle_{\partial T}$ with $w=\gamma_{F}^{-1}(\mu)$ and the trace map

$$
\gamma_{F}: \mathcal{P}_{\perp}^{1}(T) \longmapsto \mathcal{P}^{1}(F), \quad \text { defined by } \quad \gamma_{F}(w)=\mu .
$$

It only remains to estimate $\|b\|_{L^{2}(T)}$. We have

$$
\begin{aligned}
|b(w)| \leq & C \tau_{T}^{\max } h^{-1 / 2}\left\|\Pi_{\mathrm{w}} u_{p, h}-u_{p, h}\right\|_{L^{2}(\partial T)}\|w\|_{L^{2}(T)} \\
\leq & C \tau_{T}^{\max } h^{-1}\left\|\Pi_{\mathrm{w}} u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h}\right\|_{L^{2}(T)}\|w\|_{L^{2}(T)}+C \tau_{T}^{\max } h^{-1 / 2}\left\|\mathbb{P}_{h}^{1} u_{p, h}-u_{p, h}\right\|_{L^{2}(\partial T)}\|w\|_{L^{2}(T)} \\
\leq & C \tau_{T}^{\max } h^{-1}\left\|\Pi_{\mathrm{w}} u_{p, h}-\mathbb{P}_{h}^{1} u_{p, h}\right\|_{L^{2}(T)}\|w\|_{L^{2}(T)} \\
& +C \tau_{T}^{\max } h^{-1 / 2}\left(\left\|u_{p, h}-u_{p}\right\|_{L^{2}(\partial T)}+h^{3 / 2}\left\|u_{p}\right\|_{H^{2}(T)}\right)\|w\|_{L^{2}(T)} .
\end{aligned}
$$

Summing up (A.11) over all interface interment, yields

$$
\begin{align*}
\left\|\Theta_{F}\right\|_{L^{2}\left(\mathcal{I}_{h, c}\right)} & \leq C \tau^{\max } h\left(h^{-1 / 2}\left\|u_{p, h}-u_{p}\right\|_{L^{2}\left(\partial \tau_{h, c}\right)}+h\left\|u_{p}\right\|_{H^{2}\left(\mathcal{I}_{h, c}\right)}\right. \\
& \left.+h^{-1}\left\|\mathbb{P}_{h}^{1} u_{p, h}-\Pi_{\mathrm{w}} u_{p, h}\right\|_{L^{2}\left(\tau_{h, c}\right)}\right) \leq C \max \left(1, \tau^{\max }\right) h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)} \tag{A.12}
\end{align*}
$$

Therefore, a combination of (A.10) and (A.12) leads to

$$
\left\|\left(\boldsymbol{B}_{\mathbf{v}}^{1} \boldsymbol{q}_{p, h}-\boldsymbol{\Pi}_{\mathbf{v}} \boldsymbol{q}_{p, h}\right) \cdot \boldsymbol{n}_{F}\right\|_{L^{2}\left(\mathcal{I}_{h, c}\right)} \leq C \max \left(1, \tau^{\max }\right) h^{2}\|u\|_{\tilde{C}^{3}\left(\Omega_{c}\right)},
$$

which completes the proof.

## Appendix B. Proof of Theorem 4.3

Its proof is very similar to that of Theorems 3.1 and 4.1 in Cockburn et al. (2010) with the aid of a consistency result stated in Proposition B. 1 below. Note that $u_{p, h}$ is double valued on $\widetilde{\Gamma}$, the value of $u_{p, h}$ restricted on the mesh skeleton $\mathcal{E}_{h}$ taken as

$$
\breve{u}_{p, h}= \begin{cases}u_{p, h}(\boldsymbol{x}), & F \in \mathcal{E}_{h} \backslash \widetilde{\Gamma} \\ \lim _{\epsilon \rightarrow 0} u_{p, h}(\boldsymbol{x}-\epsilon \boldsymbol{n}), & F \in \widetilde{\Gamma}\end{cases}
$$

with $\boldsymbol{n}$ the unit normal on $\widetilde{\Gamma}$ pointing from $\Omega_{c}^{2}$ to $\Omega_{c} \backslash \Omega_{c}^{2}$. From the jump conditions in (2.12), we obtain

$$
u_{p, h}= \begin{cases}\breve{u}_{p, h}-\llbracket u_{a, h} \|_{\tilde{\Gamma}}, & \text { if } \partial T \cap \widetilde{\Gamma} \neq \emptyset \text { and } T \in \Omega_{2} \backslash \Omega_{c}^{2}, \\ \breve{u}_{p, h}, & \text { otherwise } .\end{cases}
$$

Also, we need the $L^{2}$-orthogonal projection

$$
\begin{equation*}
\left\langle\tau\left(P_{M} \breve{u}_{p, h}-\breve{u}_{p, h}\right), \mu\right\rangle_{\partial \tau_{h}}=0, \quad \forall \mu \in \mathcal{M}_{h}^{1}, \tag{B.1}
\end{equation*}
$$

i.e.,

$$
\begin{equation*}
\left\langle\tau\left(P_{M} \breve{u}_{p, h}-u_{p, h}\right), \mu\right\rangle_{\partial \tau_{h}}=\left\langle\tau \llbracket u_{a, h} \rrbracket_{\tilde{\Gamma}}, \mu\right\rangle_{\tilde{\Gamma}}, \quad \forall \mu \in \mathcal{M}_{h}^{1} . \tag{B.2}
\end{equation*}
$$

Proposition B. 1 Let

$$
\boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}_{p, h}}=\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}^{h}, \quad \varepsilon_{h}^{u_{p, h}}=\Pi_{\mathrm{w}} u_{p, h}-u_{p, h}^{h}, \quad \varepsilon_{h}^{\hat{u}_{p, h}}=P_{M} \breve{u}_{p, h}-\hat{u}_{p, h}^{h} .
$$

Then we have

$$
\begin{align*}
\left(\boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}_{p, h}}, \boldsymbol{v}\right)_{\tau_{h}}+\left(\varepsilon_{h}^{u_{p, h}}, \nabla \cdot \boldsymbol{v}\right)_{\tau_{h}}-\left\langle\varepsilon_{h}^{\hat{u}_{p, h}}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} & =\left(\boldsymbol{\Pi}_{\mathbf{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}, \boldsymbol{v}\right)_{\tau_{h}},  \tag{B.3a}\\
\left(\boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}_{p, h}}, \nabla w\right)_{\tau_{h}}-\left\langle\hat{\boldsymbol{\varepsilon}}_{h} \cdot \boldsymbol{n}, w\right\rangle_{\partial \tau_{h}} & =0,  \tag{B.3b}\\
\left\langle\varepsilon_{h}^{\hat{u}_{p, h}}, \mu\right\rangle_{\partial \Omega} & =0,  \tag{B.3c}\\
\left\langle\hat{\boldsymbol{\varepsilon}}_{h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \tau_{h} \partial \Omega} & =0, \tag{B.3d}
\end{align*}
$$

for all $\boldsymbol{v} \in \mathcal{V}_{h}^{1}, w \in \mathcal{W}_{h}^{1}$, and $\mu \in \mathcal{M}_{h}^{1}$, where

$$
\hat{\boldsymbol{\varepsilon}}_{h} \cdot \boldsymbol{n}:=\boldsymbol{\varepsilon}_{h}^{\boldsymbol{q}_{p, h}} \cdot \boldsymbol{n}-\tau\left(\varepsilon_{h}^{u_{p, h}}-\varepsilon_{h}^{\hat{u}_{p, h}}\right), \quad \text { on } \partial \mathcal{T}_{h} \partial \Omega
$$

Proof. Note that the exact solution $\boldsymbol{q}_{p, h}$ and $u_{p, h}$ satisfies

$$
\begin{align*}
\left(\boldsymbol{q}_{p, h}, \boldsymbol{v}\right)_{\mathcal{T}_{h}}+ & \left(u_{p, h}, \nabla \cdot \boldsymbol{v}\right)_{\mathcal{T}_{h}}-\left\langle u_{p, h}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} \tag{B.4a}
\end{align*}=0, \quad\left(\boldsymbol{q}_{p, h}, \nabla w\right)_{\mathcal{T}_{h}}-\left\langle\boldsymbol{q}_{p, h} \cdot \boldsymbol{n}, w\right\rangle_{\partial \mathcal{T}_{h}}=(f, w)_{\mathcal{T}_{h}}+\left(\left.\Delta\right|_{T \backslash \Gamma} u_{a, h}, w\right)_{\mathcal{T}_{h, c}}, ~ l
$$

for all $\boldsymbol{v} \in \mathcal{V}_{h}^{1}$ and $w \in \mathcal{W}_{h}^{1}$. By the definition of $\Pi_{h}$ and $P_{M}$, we derive from (B.4) that

$$
\begin{align*}
&\left(\boldsymbol{\Pi}_{\mathbf{v}} \boldsymbol{q}_{p, h}, \boldsymbol{v}\right)_{\tau_{h}}+\left(\Pi_{\mathrm{w}} u_{p, h}, \nabla \cdot \boldsymbol{v}\right)_{\mathcal{T}_{h}}-\left\langle P_{M} \breve{u}_{p, h}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} \\
&=\left(\boldsymbol{\Pi}_{\mathbf{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}, \boldsymbol{v}\right)_{\mathcal{T}_{h}}-\left\langle\llbracket u_{a, h} \rrbracket, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\widetilde{\Gamma}}  \tag{B.5a}\\
&\left(\boldsymbol{\Pi}_{\mathbf{v}} \boldsymbol{q}_{p, h}, \nabla w\right)_{\mathcal{T}_{h}}-\left\langle\boldsymbol{\Pi}_{\mathrm{v}} \boldsymbol{q}_{p, h} \cdot \boldsymbol{n}-\tau\left(\Pi_{\mathrm{w}} u_{p, h}\right.\right.\left.\left.-P_{M} \breve{u}_{p, h}\right), w\right\rangle_{\partial \mathcal{T}_{h}} \\
&=(f, w)_{\mathcal{T}_{h}}+\left(\left.\Delta\right|_{T \backslash \Gamma} u_{a, h}, w\right)_{\tau_{h, c}}-\left\langle\tau \llbracket u_{a, h} \rrbracket, w\right\rangle_{\widetilde{\Gamma}} \tag{B.5b}
\end{align*}
$$

for all $\boldsymbol{v} \in \mathcal{V}_{h}^{1}$ and $w \in \mathcal{W}_{h}^{1}$. Subtracting (B.5a) and (B.5b) from the two equations (2.18a) and (2.18b), respectively, we obtain (B.3a) and (B.3b).

The equation (B.3c) follows directly from the boundary condition. To prove equation (B.3d), we proceed as follows:

$$
\begin{align*}
\left\langle\mu, \hat{\boldsymbol{\varepsilon}}_{h} \cdot \boldsymbol{n}\right\rangle_{\partial \tau_{h} \backslash \partial \Omega} & =\left\langle\left(\boldsymbol{\Pi}_{\mathbf{v}} \boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}^{h}\right) \cdot \boldsymbol{n}-\tau\left(\Pi_{\mathrm{w}} u_{p, h}-u_{p, h}^{h}-P_{M} \breve{u}_{p, h}+\hat{u}_{p, h}^{h}\right), \mu\right\rangle_{\partial \tau_{h} \partial \Omega} \\
& =\left\langle\left(\boldsymbol{q}_{p, h}-\boldsymbol{q}_{p, h}^{h}\right) \cdot \boldsymbol{n}-\tau\left(u_{p, h}-u_{p, h}^{h}-P_{M} \breve{u}_{p, h}+\hat{u}_{p, h}^{h}\right), \mu\right\rangle_{\partial \tau_{h} \backslash \partial \Omega} \\
& =\left\langle\boldsymbol{q}_{p, h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \tau_{h} \backslash \partial \Omega}-\left\langle\boldsymbol{q}_{p, h}^{h} \cdot \boldsymbol{n}-\tau\left(u_{p, h}^{h}-\hat{u}_{p, h}^{h}\right), \mu\right\rangle_{\partial \tau_{h} \partial \Omega}+\left\langle\tau \llbracket u_{a, h} \rrbracket, \mu\right\rangle_{\widetilde{\Gamma}} \\
& =\left\langle\boldsymbol{q}_{p, h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \tau_{h} \backslash \partial \Omega}-\left\langle\hat{\boldsymbol{q}}_{p}^{h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \tau_{h} \partial \Omega \Omega}+\left\langle\tau \llbracket u_{a, h} \rrbracket, \mu\right\rangle_{\widetilde{\Gamma}}, \tag{B.6}
\end{align*}
$$

where we have used the definition of $\Pi_{h}$ and (B.2). Substituting equations

$$
\left\langle\boldsymbol{q}_{p, h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h} \backslash \partial \Omega}=-\left\langle\llbracket \nabla u_{a, h} \cdot \boldsymbol{n} \rrbracket, \mu\right\rangle_{\tilde{\Gamma}}
$$

and

$$
\left\langle\hat{\boldsymbol{q}}_{p}^{h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h} \backslash \Omega}=-\left\langle\llbracket \nabla u_{a, h} \cdot \boldsymbol{n} \rrbracket, \mu\right\rangle_{\tilde{\Gamma}}+\left\langle\tau \llbracket u_{a, h} \rrbracket, \mu\right\rangle_{\tilde{\Gamma}}
$$

into (B.6), we obtain

$$
\left\langle\hat{\boldsymbol{\varepsilon}}_{h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \tau_{h} \backslash \Omega}=0
$$

It ends the proof.
With this proposition, the remaining proof of Theorem 4.3 is exactly the same as in Cockburn et al. (2010).

