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# INTEGRATION PROCESSES OF ORDINARY DIFFERENTIAL EQUATIONS BASED ON LAGUERRE-RADAU INTERPOLATIONS

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Abstract. In this paper, we propose two integration processes for ordinary differential equations based on modified Laguerre-Radau interpolations, which are very efficient for long-time numerical simulations of dynamical systems. The global convergence of proposed algorithms are proved. Numerical results demonstrate the spectral accuracy of these new approaches and coincide well with theoretical analysis.

## 1. Introduction

Numerous problems in science and engineering are governed by ordinary differential equations. There have been fruitful results on their numerical solutions; see, e.g., Butcher [2, 4], Hairer, Norsett and Wanner [16], Hairer and Wanner [17], Higham [18] and Stuart and Humphries [25]. For Hamiltonian systems, we refer to the powerful symplectic difference method of Feng [5]; see also [6, 15, 22] and the references therein.

As a basic tool, the Runge-Kutta method plays an important role in numerical integrations of ordinary differential equations. We usually design these kinds of numerical schemes in two ways. The first way is based on Taylor's expansion coupled with other techniques. The next is to construct numerical schemes by using collocation approximation. For instance, Butcher [3] provided some implicit Runge-Kutta processes based on the Radau quadrature formulas; see also [4, 16, 17] and the references therein. On the other hand, Babuska and Janik [1], and TalJ-Ezer [26] used the same trick in time discretization for parabolic equations.

In the existing work, one often used the Legendre-Radau interpolation to design the Runge-Kutta processes. However, the Legendre-Radau interpolation is available for finite interval essentially. Conversely, if we use the Laguerre-Radau interpolation, we can approximate the exact solutions on the half line. Thereby, the related algorithms might be more appropriate for long-time calculations. In particular, the corresponding Runge-Kutta processes often possess the global convergence. As we

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know, some authors developed the Laguerre approximation with successful applications to spatial approximations of various partial differential equations on the half line and a large class of other related problems; see, e.g., Funaro [5], Guo and Shen [8], Guo, Shen and Xu [9], Guo and Xu [12], Iranzo and Falques [19], Mastroianni and Monegate [21], Maday, Pernaud-Thomas, and Vandeven [20], Shen [24], and Xu and Guo [27]. But so far, to our knowledge, there is no work concerning the applications of Laguerre approximation to integration processes for ordinary differential equations.

This paper discusses two new integration processes based on modified Laguerre-Radau interpolation. In the next section, we propose the first algorithm by using the modified Laguerre polynomials. This process has several advantages. First, it is easier to be implemented, especially for nonlinear systems. Next, it provides the global numerical solutions and the global convergence in certain weighted Sobolev space. Hence, it is very applicable to long-time calculations. Furthermore, by adjusting a parameter involved in the process, we may weaken the conditions on the underlying problems, and so enlarge its applications essentially. In Section 3, by taking the modified Laguerre functions as base functions, we design the second integration process. This process not only has the same merits as the first process, but also possesses the global convergence in the space $L^2(0,\infty)$. This implies that the pointwise numerical errors decay to zero rapidly as time goes to infinity. Therefore, it is more suitable for long-time calculations. We also develop a technique for refining numerical results in Section 4. In other words, we first use the above methods with moderate mode to obtain numerical solutions, and then use the shifted Laguerre approximation to refine them. This simplifies actual computations and provides more precise numerical solutions. We present numerical results in Section 5, which demonstrate the spectral accuracy of proposed methods and coincide well with analysis. The final section is for concluding remarks.

## 2. The first numerical integration process

In this section, we propose the first integration process. Let $\omega_\beta(t) = e^{-\beta t}$, $\beta > 0$, and define the weighted space $L^2_{\omega_\beta}(0,\infty)$ as usual, with the following inner product and norm,

$$(u,v)_{\omega_\beta} = \int_0^\infty u(t)v(t)\omega_\beta(t)dt, \qquad \|v\|_{\omega_\beta} = (v,v)^{\frac{1}{2}}_{\omega_\beta}.$$

The modified Laguerre polynomial of degree $l$ is defined by (cf. [13])

$$\mathcal{L}_l^{(\beta)}(t) = \frac{1}{l!}e^{\beta t}\frac{d^l}{dt^l}(t^l e^{-\beta t}), \qquad l \geq 0.$$

They satisfy the recurrence relation

$$\frac{d}{dt}\mathcal{L}_l^{(\beta)}(t) = \frac{d}{dt}\mathcal{L}_{l-1}^{(\beta)}(t) - \beta\mathcal{L}_{l-1}^{(\beta)}(t), \qquad l \geq 1. \tag{2.1}$$

The set of Laguerre polynomials is a complete $L^2_{\omega_\beta}(0,\infty)$-orthogonal system, namely,

$$(\mathcal{L}_l^{(\beta)}, \mathcal{L}_m^{(\beta)})_{\omega_\beta} = \frac{1}{\beta}\delta_{\ell,m} \tag{2.2}$$

where $\delta_{l,m}$ is the Kronecker symbol. Thus, for any $v \in L^2_{\omega_\beta}(0,\infty)$,

$$v(t) = \sum_{l=0}^{\infty} \hat{v}_l \mathcal{L}_l^{(\beta)}(t), \qquad \hat{v}_l = \beta(v, \mathcal{L}_l^{(\beta)})_{\omega_\beta}.$$

Now, let $N$ be any positive integer, and $\mathcal{P}_N(0,\infty)$ the set of all algebraic polynomials of degree at most $N$. We denote by $t^N_{\beta,j}$ the nodes of modified Laguerre-Radau interpolation. Indeed, $t^N_{\beta,0} = 0$, and $t^N_{\beta,j} (1 \le j \le N)$ are the distinct zeros of $\frac{d}{dt}\mathcal{L}^{(\beta)}_{N+1}(t)$. By using (2.1) and the formula (2.12) of [10], the corresponding Christoffel numbers are as follows:

$$\omega^N_{\beta,0} = \frac{1}{\beta(N+1)}, \quad \omega^N_{\beta,j} = \frac{1}{\beta(N+1)\mathcal{L}^{(\beta)}_N(t^N_{\beta,j})\mathcal{L}^{(\beta)}_{N+1}(t^N_{\beta,j})}, \quad 1 \le j \le N. \tag{2.3}$$

For any $\phi \in \mathcal{P}_{2N}(0,\infty)$,

$$\sum_{j=0}^{N} \phi(t^N_{\beta,j})\omega^N_{\beta,j} = \int_0^\infty \phi(t)\omega_\beta(t)dt.$$

Next, we define the following discrete inner product and norm,

$$(u,v)_{\omega_\beta,N} = \sum_{j=0}^{N} u(t^N_{\beta,j})v(t^N_{\beta,j})\omega^N_{\beta,j}, \qquad \|v\|_{\omega_\beta,N} = (v,v)^{\frac{1}{2}}_{\omega_\beta,N}.$$

For any $\phi, \psi \in \mathcal{P}_N(0,\infty)$,

$$(\phi,\psi)_{\omega_\beta} = (\phi,\psi)_{\omega_\beta,N}, \qquad \|\phi\|_{\omega_\beta} = \|\phi\|_{\omega_\beta,N}. \tag{2.4}$$

The modified Laguerre-Radau interpolant $I_{\beta,N}v \in \mathcal{P}_N(0,\infty)$, is determined by

$$I_{\beta,N}v(t^N_{\beta,j}) = v(t^N_{\beta,j}), \qquad 0 \le j \le N.$$

By (2.4), for any $\phi \in \mathcal{P}_N(0,\infty)$,

$$(I_{\beta,N}v,\phi)_{\omega_\beta} = (I_{\beta,N}v,\phi)_{\omega_\beta,N} = (v,\phi)_{\omega_\beta,N}. \tag{2.5}$$

The interpolant $I_{\beta,N}v$ can be expanded as

$$I_{\beta,N}v(t) = \sum_{l=0}^{N} \widetilde{v}^N_{\beta,l}\mathcal{L}^{(\beta)}_l(t).$$

By virtue of (2.2) and (2.4),

$$\widetilde{v}^N_{\beta,l} = \beta(I_{\beta,N}v, \mathcal{L}^{(\beta)}_l)_{\omega_\beta} = \beta(v, \mathcal{L}^{(\beta)}_l)_{\omega_\beta,N}. \tag{2.6}$$

We now consider the following model problem

$$\begin{cases} \dfrac{d}{dt}U(t) = f(U(t),t), & t > 0, \\ U(0) = U_0. \end{cases} \tag{2.7}$$

We suppose that $\dfrac{d}{dt}U(t)$ is continuous for $t \ge 0$. Let

$$G^N_{\beta,1}(t) = \frac{d}{dt}I_{\beta,N}U(t) - I_{\beta,N}\frac{d}{dt}U(t).$$

Then we obtain from (2.7) that

$$\frac{d}{dt}I_{\beta,N}U(t^N_{\beta,k}) = f(U(t^N_{\beta,k}),t^N_{\beta,k}) + G^N_{\beta,1}(t^N_{\beta,k}), \qquad 1 \le k \le N. \tag{2.8}$$

Next, we derive an explicit expression for the left side of (2.8). Let $\tilde{U}^N_{\beta,l}$ be the coefficients of $I_{\beta,N}U$ in terms of $\mathcal{L}^{(\beta)}_l(t)$. By virtue of (2.1) and (2.6), we deduce that

$$\begin{aligned}\frac{d}{dt}I_{\beta,N}U(t) &= \sum_{l=1}^{N}\tilde{U}^N_{\beta,l}\frac{d}{dt}\mathcal{L}^{(\beta)}_l(t) = -\beta\sum_{l=1}^{N}\tilde{U}^N_{\beta,l}(\sum_{m=0}^{l-1}\mathcal{L}^{(\beta)}_m(t))\\ &= -\beta^2\sum_{l=1}^{N}(\sum_{j=0}^{N}U(t^N_{\beta,j})\mathcal{L}^{(\beta)}_l(t^N_{\beta,j})\omega^N_{\beta,j})(\sum_{m=0}^{l-1}\mathcal{L}^{(\beta)}_m(t))\\ &= -\beta^2\sum_{j=0}^{N}(\sum_{l=1}^{N}\mathcal{L}^{(\beta)}_l(t^N_{\beta,j})(\sum_{m=0}^{l-1}\mathcal{L}^{(\beta)}_m(t)))U(t^N_{\beta,j})\omega^N_{\beta,j}.\end{aligned} \tag{2.9}$$

For simplicity, we set

$$a^N_{\beta,k,j} = -\beta^2\omega^N_{\beta,j}\sum_{l=1}^{N}\mathcal{L}^{(\beta)}_l(t^N_{\beta,j})(\sum_{m=0}^{l-1}\mathcal{L}^{(\beta)}_m(t^N_{\beta,k})), \quad 1 \le k \le N, \quad 0 \le j \le N. \tag{2.10}$$

Then, (2.9) reads

$$\frac{d}{dt}I_{\beta,N}U(t^N_{\beta,k}) = \sum_{j=0}^{N}a^N_{\beta,k,j}U(t^N_{\beta,j}), \qquad 1 \le k \le N. \tag{2.11}$$

Furthermore, let

$$\mathbb{U}^N = (U(0), U(t^N_{\beta,1}), \cdots, U(t^N_{\beta,N}))^T,$$
$$\mathbb{F}^N_\beta(\mathbb{U}^N) = (f(U(t^N_{\beta,1}), t^N_{\beta,1}), f(U(t^N_{\beta,2}), t^N_{\beta,2}), \ldots, f(U(t^N_{\beta,N}), t^N_{\beta,N}))^T,$$
$$\mathbb{G}^N_{\beta,1} = (G^N_{\beta,1}(t^N_{\beta,1}), G^N_{\beta,1}(t^N_{\beta,2}), \cdots, G^N_{\beta,1}(t^N_{\beta,N}))^T,$$

and

$$\mathbb{A}^N_\beta = \begin{pmatrix} a^N_{\beta,1,0} & a^N_{\beta,1,1} & \cdots & a^N_{\beta,1,N}\\ a^N_{\beta,2,0} & a^N_{\beta,2,1} & \cdots & a^N_{\beta,2,N}\\ \cdots & & & \\ a^N_{\beta,N,0} & a^N_{\beta,N,1} & \cdots & a^N_{\beta,N,N}\end{pmatrix}.$$

Accordingly, we can rewrite (2.8) as

$$\mathbb{A}^N_\beta\mathbb{U}^N = \mathbb{F}^N_\beta(\mathbb{U}^N) + \mathbb{G}^N_{\beta,1}. \tag{2.12}$$

We are now in a position to construct the numerical scheme for (2.7). To do this, we approximate $U(t)$ by $u^N(t) \in \mathcal{P}_N(0,\infty)$. Clearly $I_{\beta,N}u^N(t) = u^N(t)$. Furthermore, we set

$$\mathbf{u}^N = (u^N(0), u^N(t^N_{\beta,1}), \cdots, u^N(t^N_{\beta,N}))^T,$$
$$\mathbb{F}^N_\beta(\mathbf{u}^N) = \Big(f(u^N(t^N_{\beta,1}), t^N_{\beta,1}), f(u^N(t^N_{\beta,2}), t^N_{\beta,2}), \cdots, f(u^N(t^N_{\beta,N}), t^N_{\beta,N})\Big)^T.$$

By replacing $\mathbb{U}^N$ by $\mathbf{u}^N$ and neglecting $\mathbb{G}^N_{\beta,1}$ in (2.12), we derive the following scheme

$$\begin{cases} \mathbb{A}^N_\beta\mathbf{u}^N = \mathbb{F}^N_\beta(\mathbf{u}^N),\\ u^N(0) = U_0.\end{cases} \tag{2.13}$$

This is an implicit scheme. If $f(z,t)$ is a nonlinear function for $z$, then we need a nonlinear iteration to solve this system. In this work, we shall use the Newton-Raphson iteration. Finally, the global numerical solution

$$u^N(t) = \sum_{l=0}^{N} \tilde{u}_{\beta,l}^N \mathcal{L}_l^{(\beta)}(t), \qquad t \ge 0, \tag{2.14}$$

where by (2.6),

$$\tilde{u}_{\beta,l}^N = \beta(u^N, \mathcal{L}_l^{(\beta)})_{\omega_\beta,N} = \beta \sum_{j=0}^{N} u^N(t_{\beta,j}^N)\mathcal{L}_l^{(\beta)}(t_{\beta,j}^N)\omega_{\beta,j}^N.$$

The system (2.13) is equivalent to the system

$$\begin{cases} \dfrac{d}{dt}u^N(t_{\beta,k}^N) = f(u^N(t_{\beta,k}^N), t_{\beta,k}^N), & 1 \le k \le N, \\ u^N(0) = U_0. \end{cases} \tag{2.15}$$

It has a unique solution, if $f(z,t)$ fulfills some reasonable conditions; see Appendix of this paper.

We next analyze the numerical error of (2.13). In particular, we shall prove its spectral accuracy. It means that for any fixed $N$, the smoother the exact solution $U$, the higher the order of convergence of the numerical solution. To do this, let $E^N(t) = u^N(t) - I_{\beta,N}U(t)$. Subtracting (2.15) from (2.8) yields that

$$\begin{cases} \dfrac{d}{dt}E^N(t_{\beta,k}^N) = G_{\beta,2}^N(t_{\beta,k}^N) - G_{\beta,1}^N(t_{\beta,k}^N), & 1 \le k \le N, \\ E^N(0) = 0 \end{cases} \tag{2.16}$$

where

$$G_{\beta,2}^N(t_{\beta,k}^N) = f(u^N(t_{\beta,k}^N), t_{\beta,k}^N) - f(I_{\beta,N}U(t_{\beta,k}^N), t_{\beta,k}^N).$$

We next multiply the first formula of (2.16) by $2E^N(t_{\beta,k}^N)\omega_{\beta,k}^N$ and sum the resulting equality for $1 \le k \le N$. Since $E^N(0) = 0$, we deduce that

$$2(E^N, \frac{d}{dt}E^N)_{\omega_\beta,N} = A_{\beta,1}^N + A_{\beta,2}^N \tag{2.17}$$

where

$$A_{\beta,1}^N = -2(G_{\beta,1}^N, E^N)_{\omega_\beta,N}, \qquad A_{\beta,2}^N = 2(G_{\beta,2}^N, E^N)_{\omega_\beta,N}.$$

By (2.4) and integration by parts, we get

$$2(E^N, \frac{d}{dt}E^N)_{\omega_\beta,N} = 2(E^N, \frac{d}{dt}E^N)_{\omega_\beta} = \beta\|E^N\|_{\omega_\beta}^2.$$

Due to $G_{\beta,1}^N(t) \in \mathcal{P}_N(0,\infty)$, we use (2.4) and the Cauchy inequality to obtain that

$$|A_{\beta,1}^N| \le 2\|G_{\beta,1}^N\|_{\omega_\beta,N}\|E^N\|_{\omega_\beta,N} = 2\|G_{\beta,1}^N\|_{\omega_\beta}\|E^N\|_{\omega_\beta}.$$

Substituting the above two estimates into (2.17), we assert that

$$\beta\|E^N\|_{\omega_\beta}^2 \le A_{\beta,2}^N + 2\|G_{\beta,1}^N\|_{\omega_\beta}\|E^N\|_{\omega_\beta}. \tag{2.18}$$

We now assume that there exists a real number $\gamma$ such that

$$(f(z_1,t) - f(z_2,t))(z_1 - z_2) \le \gamma|z_1 - z_2|^2, \quad \forall z_1, z_2 \in \mathbb{R}. \tag{2.19}$$

Then by (2.4),

$$A_{\beta,2}^N \le 2\gamma\|E^N\|_{\omega_\beta,N}^2 = 2\gamma\|E^N\|_{\omega_\beta}^2.$$

The above fact, along with (2.18), shows that for any $\beta > 2\gamma$,

$$\|E^N\|_{\omega_\beta} \le \frac{2}{\beta - 2\gamma}\|G^N_{\beta,1}\|_{\omega_\beta}. \tag{2.20}$$

Thus it remains to estimate $\|G^N_{\beta,1}\|_{\omega_\beta}$.

In order to estimate $\|G^N_{\beta,1}\|_{\omega_\beta}$, we need some approximation results on the modified Laguerre-Radau interpolation. For this purpose, we use the following notations:

$$\mathcal{R}^{(1)}_{N,r,\beta}(v) = \beta^{-1}||t^{\frac{r-1}{2}}\frac{d^r v}{dt^r}||_{\omega_\beta} + (1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}||t^{\frac{r}{2}}\frac{d^r v}{dt^r}||_{\omega_\beta},$$

$$\begin{aligned}\mathcal{R}^{(2)}_{N,r,\beta}(v) = \beta^{-1}||t^{\frac{r+1}{2}}\tfrac{d^{r+2}v}{dt^{r+2}}||_{\omega_\beta} + N^{-\frac{1}{2}}||t^{\frac{r+1}{2}}\tfrac{d^{r+2}v}{dt^{r+2}}||_{\omega_\beta}\\ +(1+\beta^{-\frac{1}{2}})(\ln N)^{\frac{1}{2}}||t^{\frac{r+2}{2}}\tfrac{d^{r+2}v}{dt^{r+2}}||_{\omega_\beta}.\end{aligned}$$

According to Theorems 3.7 and 3.8 of [10], we know that for the integer $r \ge 1$,

$$\begin{aligned}\|I_{\beta,N}v - v\|_{\omega_\beta} \le c(\beta N)^{\frac{1}{2}-\frac{r}{2}}\mathcal{R}^{(1)}_{N,r,\beta}(v),\\ \|\frac{d}{dt}(I_{\beta,N}v - v)\|_{\omega_\beta} \le c(\beta N)^{\frac{1}{2}-\frac{r}{2}}\mathcal{R}^{(2)}_{N,r,\beta}(v).\end{aligned} \tag{2.21}$$

We now go back to (2.20). By (2.21), we get

$$\begin{aligned}\|G^N_{\beta,1}\|_{\omega_\beta} &\le \|\tfrac{d}{dt}(I_{\beta,N}U - U)\|_{\omega_\beta} + \|\tfrac{dU}{dt} - I_{\beta,N}\tfrac{dU}{dt}\|_{\omega_\beta}\\ &\le c(\beta N)^{\frac{1}{2}-\frac{r}{2}}(\mathcal{R}^{(2)}_{N,r,\beta}(U) + \mathcal{R}^{(1)}_{N,r,\beta}(\tfrac{dU}{dt})).\end{aligned}$$

Therefore, (2.20) reads

$$\|E^N\|_{\omega_\beta} \le \frac{c}{\beta - 2\gamma}(\beta N)^{\frac{1}{2}-\frac{r}{2}}(\mathcal{R}^{(2)}_{N,r,\beta}(U) + \mathcal{R}^{(1)}_{N,r,\beta}(\frac{dU}{dt})).$$

Finally, we use (2.21) again to reach the following result.

**Theorem 2.1.** *Let* (2.19) *hold and* $\beta > 2\gamma$. *If* $\mathcal{R}^{(1)}_{N,r,\beta}(U), \mathcal{R}^{(2)}_{N,r,\beta}(U)$ *and* $\mathcal{R}^{(1)}_{N,r,\beta}(\frac{dU}{dt})$ *are finite, then*

$$\begin{aligned}&\|U - u^N\|_{\omega_\beta} \le \|I_{\beta,N}U - U\|_{\omega_\beta} + \|E^N\|_{\omega_\beta}\\ &\le \tfrac{c}{\beta-2\gamma}(\beta N)^{\frac{1}{2}-\frac{r}{2}}((\beta-2\gamma)\mathcal{R}^{(1)}_{N,r,\beta}(U) + \mathcal{R}^{(2)}_{N,r,\beta}(U) + \mathcal{R}^{(1)}_{N,r,\beta}(\tfrac{dU}{dt})).\end{aligned} \tag{2.22}$$

*Remark* 2.1. According to (2.22),

$$\|U - u^N\|_{\omega_\beta} = \frac{\beta - 2\gamma + 1}{\beta - 2\gamma}\mathcal{O}((1 + \frac{1}{\beta})(\beta N)^{\frac{1}{2}-\frac{r}{2}}(\ln N)^{\frac{1}{2}}).$$

Moreover, a suitable choice of $\beta$ may improve the numerical accuracy.

*Remark* 2.2. The norms involved on the right side of (2.22) are finite as long as $f(z,t)$ satisfies certain conditions and $\beta > 2\gamma$. For instance, by (2.7) and (2.10), for any $\delta > 0$,

$$\begin{aligned}2U(t)\tfrac{d}{dt}U(t) &= 2(f(U(t),t) - f(0,t))U(t) + 2f(0,t)U(t)\\ &\le 2\gamma U^2(t) + 2f(0,t)U(t) \le (2\gamma+\delta)U^2(t) + \tfrac{f^2(0,t)}{\delta}.\end{aligned}$$

Thus integrating the above leads to

$$U^2(t) \le U_0^2 + (2\gamma+\delta)\int_0^t U^2(s)ds + \frac{1}{\delta}\int_0^t f^2(0,s)ds.$$

Then by the Gronwell inequality,

$$U^2(t) \le e^{(2\gamma+\delta)t}(U_0^2 + \frac{1}{\delta}\int_0^t e^{-(2\gamma+\delta)s} f^2(0,s)ds).$$

This with $\beta > 2\gamma$ ensures the finiteness of the norm $||u||_{\omega_\beta}$. We can check the finiteness of other norms on the right side of (2.22), provided that $f(z,t)$ satisfy certain conditions.

*Remark* 2.3. If (2.19) holds and integer $r > 1$, then scheme (2.13) with $\beta > 2\gamma$ has the global convergence and the spectral accuracy in the weighted space $L^2_{\omega_\beta}(0,\infty)$.

*Remark* 2.4. For the validity of convergence of usual integration processes, we impose certain conditions on the constant $\gamma$ in (2.19). This limits its applications seriously. However, for any $\gamma$, we could use the scheme (2.13) with the suitable parameter $\beta > 2\gamma$ to solve (2.7) efficiently. Therefore, our new process is available for a large class of dynamical systems.

*Remark* 2.5. The algorithm (2.13) with fixed parameter $\beta$ is still applicable, even if $\beta \le 2\gamma$. For example, we assume that for a certain real number $\alpha_1 \ge \frac{1}{2}\beta$,

$$(f(z_1(t)),t) - f(z_2(t),t))(z_1 - z_2) \le \alpha_1 |z_1 - z_2|^2. \tag{2.23}$$

In this case, we take $\alpha > \alpha_1 - \frac{1}{2}\beta$ and make the variable transformation

$$U(t) = e^{\alpha t}V(t), \qquad F(V(t),t) = e^{-\alpha t} f(e^{\alpha t}V(t),t) - \alpha V(t). \tag{2.24}$$

Then (2.7) becomes

$$\begin{cases} \dfrac{dV(t)}{dt} = F(V(t),t), \quad t > 0, \\ V(0) = U_0. \end{cases} \tag{2.25}$$

We may use (2.13) to resolve (2.25), and obtain the numerical solution $v^N$. Moreover, the condition (2.23) ensures the global spectral accuracy of $v^N$. The numerical solution of (2.7) is given by $u^N(t) = e^{\alpha t}v^N(t)$.

*Remark* 2.6. Suppose that $f(z,t)$ fulfills the following Lipschitz condition:

$$|f(z_1,t) - f(z_2,t)| \le L|z_1 - z_2|, \quad L \ge 0. \tag{2.26}$$

Then we have an error estimate similar to (2.22) for any $\beta > 2L$.

*Remark* 2.7. It is easy to generalize the method (2.13) to a system of ordinary differential equations. If it fulfills certain conditions like (2.19) or (2.26), then the same result as in Theorem 2.1 holds.

*Remark* 2.8. The proposed method is also applicable to Hamilton systems, which do not satisfy (2.26). For example, we consider the system

$$\begin{cases} \dfrac{d}{dt}P(t) = -4Q(t), \qquad \dfrac{d}{dt}Q(t) = P(t), \qquad t > 0, \\ P(0) = P_0, \qquad\qquad Q(0) = Q_0. \end{cases} \tag{2.27}$$

The corresponding Hamiltonian function is $H(P,Q) = \frac{1}{2}P^2 + 2Q^2$. We approximate $P(t)$ and $Q(t)$ by $p^N(t)$ and $q^N(t)$, respectively. The numerical algorithm for (2.27) is as follows:

$$
(2.28)\quad \begin{cases} \dfrac{d}{dt}p^N(t^N_{\beta,k}) = -4q^N(t^N_{\beta,k}), & \dfrac{d}{dt}q^N(t^N_{\beta,k}) = p^N(t^N_{\beta,k}), \qquad 1 \le k \le N, \\ p^N(0) = P_0, & q^N(0) = Q_0. \end{cases}
$$

By using (2.21), we can prove that for any $\beta > 0$,

$$
\begin{aligned} \|P - p^N\|_{\omega_\beta} + 2\|Q - q^N\|_{\omega_\beta} \le c(1 + \tfrac{1}{\beta})(\beta N)^{\frac{1}{2}-\frac{r}{2}}(&\mathcal{R}^{(1)}_{N,r,\beta}(P) + \mathcal{R}^{(1)}_{N,r,\beta}(Q) \\ &+ \mathcal{R}^{(2)}_{N,r,\beta}(P) + \mathcal{R}^{(2)}_{N,r,\beta}(Q) + \mathcal{R}^{(1)}_{N,r,\beta}(\frac{dP}{dt}) + \mathcal{R}^{(1)}_{N,r,\beta}(\frac{dQ}{dt})). \end{aligned}
$$

This implies the global convergence and the spectral accuracy of numerical solution.

## 3. The second integration process

In the last section, we provided an integration process with the spectral accuracy in the weighted space $L^2_{\omega_\beta}(0, \infty)$. However, the small error in the weighted space does not imply the small error in the maximum norm. On the other hand, such a measurement is not the most appropriate, if the exact solution decays fast enough as $t \to \infty$. In this section, we develop another integration process for the model problem (2.7), with the global spectral accuracy in the space $L^2(0, \infty)$. The main idea is to take the modified Laguerre functions $\widetilde{\mathcal{L}}^{(\beta)}_l(t) = e^{-\frac{1}{2}\beta t}\mathcal{L}^{(\beta)}_l(t)$ as the base functions, instead of $\mathcal{L}^{(\beta)}_l(t)$.

According to (2.1), the functions $\widetilde{\mathcal{L}}^{(\beta)}_l(t)$ satisfy the recurrence relation (cf. [14])

$$
(3.1)\qquad \frac{d}{dt}\widetilde{\mathcal{L}}^{(\beta)}_l(t) = \frac{d}{dt}\widetilde{\mathcal{L}}^{(\beta)}_{l-1}(t) - \frac{1}{2}\beta\widetilde{\mathcal{L}}^{(\beta)}_l(t) - \frac{1}{2}\beta\widetilde{\mathcal{L}}^{(\beta)}_{l-1}(t), \qquad l \ge 1.
$$

Denote by $(u, v)$ and $\|v\|$ the inner product and the norm of the space $L^2(0, \infty)$, respectively. The set of $\widetilde{\mathcal{L}}^{(\beta)}_l(t)$ is a complete $L^2(0, \infty)$-orthogonal system, i.e.,

$$
(3.2)\qquad (\widetilde{\mathcal{L}}^{(\beta)}_l, \widetilde{\mathcal{L}}^{(\beta)}_m) = \frac{1}{\beta}\delta_{l,m}.
$$

We now introduce the new Laguerre-Radau interpolation. Set $Q_N(0, \infty) = \mathrm{span}\{\widetilde{\mathcal{L}}^{(\beta)}_0, \widetilde{\mathcal{L}}^{(\beta)}_1, \cdots, \widetilde{\mathcal{L}}^{(\beta)}_N\}$. Let $t^N_{\beta,j}$ and $\omega^N_{\beta,j}$ be the same as in (2.3), and take the nodes and weights of the new Laguerre-Radau interpolation as

$$
(3.3)\qquad \widetilde{t}^N_{\beta,j} = t^N_{\beta,j}, \qquad \widetilde{\omega}^N_{\beta,j} = \frac{1}{\widetilde{\mathcal{L}}^{(\beta)}_N(t^N_{\beta,j})\widetilde{\mathcal{L}}^{(\beta)}_{N+1}(t^N_{\beta,j})} = e^{\beta t^N_{\beta,j}}\omega^N_{\beta,j}.
$$

We also define the following discrete inner product and norm,

$$
(u, v)_{\beta,N} = \sum_{j=0}^{N} u(t^N_{\beta,j})v(t^N_{\beta,j})\widetilde{\omega}^N_{\beta,j}, \qquad \|v\|_{\beta,N} = (v, v)^{\frac{1}{2}}_{\beta,N}.
$$

For any $\phi_1, \phi_2 \in Q_N(0, \infty)$, we have $\phi_1 = e^{-\frac{1}{2}\beta t}\psi_1$, $\phi_2 = e^{-\frac{1}{2}\beta t}\psi_2$ and $\psi_1, \psi_2 \in \mathcal{P}_N(0, \infty)$. Thus by (2.4),

$$
(3.4)\qquad (\phi_1, \phi_2)_{\beta,N} = (\psi_1, \psi_2)_{\omega_\beta,N} = (\psi_1, \psi_2)_{\omega_\beta} = (\phi_1, \phi_2).
$$

The new Laguerre-Radau interpolant $\widetilde{I}_{\beta,N}v \in Q_N(0, \infty)$ is determined by

$$
\widetilde{I}_{\beta,N}v(t^N_{\beta,j}) = v(t^N_{\beta,j}), \qquad 0 \le j \le N.
$$

Thanks to (3.4), for any $\phi \in Q_N(0,\infty)$,

$$(\widetilde{I}_{\beta,N}v,\phi) = (\widetilde{I}_{\beta,N}v,\phi)_{\beta,N} = (v,\phi)_{\beta,N}. \tag{3.5}$$

Let

$$\widetilde{I}_{\beta,N}v(t) = \sum_{l=0}^{N} \widetilde{v}_{\beta,l}^{N} \widetilde{\mathcal{L}}_l^{(\beta)}(t).$$

Then, with the aid of (3.2) and (3.5), we derive that

$$\widetilde{v}_{\beta,l}^{N} = \beta(\widetilde{I}_{\beta,N}v,\widetilde{\mathcal{L}}_l^{(\beta)}) = \beta(\widetilde{I}_{\beta,N}v,\widetilde{\mathcal{L}}_l^{(\beta)})_{\beta,N} = \beta(v,\widetilde{\mathcal{L}}_l^{(\beta)})_{\beta,N}. \tag{3.6}$$

There is a close relation between $I_{\beta,N}$ and $\widetilde{I}_{\beta,N}$. Indeed by the previous two equalities,

$$\begin{aligned} e^{\frac{1}{2}\beta t}\widetilde{I}_{\beta,N}v(t) &= \sum_{l=0}^{N} \widetilde{v}_{\beta,l}^{N}\mathcal{L}_l^{(\beta)}(t) = \beta\sum_{l=0}^{N}(v,\widetilde{\mathcal{L}}_l^{(\beta)})_{\beta,N}\mathcal{L}_l^{(\beta)}(t) \\ &= \beta\sum_{l=0}^{N}(e^{\frac{1}{2}\beta t}v,\mathcal{L}_l^{(\beta)})_{\omega_\beta,N}\mathcal{L}_l^{(\beta)}(t). \end{aligned}$$

This with (2.6) implies

$$\widetilde{I}_{\beta,N}v(t) = e^{-\frac{1}{2}\beta t}I_{\beta,N}(e^{\frac{1}{2}\beta t}v(t)). \tag{3.7}$$

Now, we turn to the model problem (2.7). Let

$$G_{\beta,1}^{N}(t) = \frac{d}{dt}\widetilde{I}_{\beta,N}U(t) - \widetilde{I}_{\beta,N}\frac{d}{dt}U(t).$$

Then we obtain from (2.7) that

$$\frac{d}{dt}\widetilde{I}_{\beta,N}U(t_{\beta,k}^{N}) = f(U(t_{\beta,k}^{N}),t_{\beta,k}^{N}) + G_{\beta,1}^{N}(t_{\beta,k}^{N}), \qquad 1\le k\le N. \tag{3.8}$$

Next, we derive an explicit expression of the left side of (3.8). To this end, let $\widetilde{U}_{\beta,l}^{N}$ be the coefficients of $\widetilde{I}_{\beta,N}U(t)$ in terms of $\widetilde{\mathcal{L}}_l^{(\beta)}(t)$. Due to (3.1), we verify that

$$\begin{aligned} \frac{d}{dt}\widetilde{I}_{\beta,N}U(t) &= \sum_{l=0}^{N}\tilde{U}_{\beta,l}^{N}\frac{d}{dt}\widetilde{\mathcal{L}}_l^{(\beta)}(t) \\ &= -\tfrac{1}{2}\beta\sum_{l=1}^{N}\tilde{U}_{\beta,l}^{N}(2\sum_{m=0}^{l-1}\widetilde{\mathcal{L}}_m^{(\beta)}(t)+\widetilde{\mathcal{L}}_l^{(\beta)}(t)) - \frac{1}{2}\beta\tilde{U}_{\beta,0}^{N}\widetilde{\mathcal{L}}_0^{(\beta)}(t). \end{aligned}$$

The above with (3.6) implies that

$$\begin{aligned} \frac{d}{dt}\widetilde{I}_{\beta,N}U(t_{\beta,k}^{N}) = &-\frac{1}{2}\beta^2\sum_{l=1}^{N}(\sum_{j=0}^{N}U(t_{\beta,j}^{N})\widetilde{\mathcal{L}}_l^{(\beta)}(t_{\beta,j}^{N})\widetilde{\omega}_{\beta,j}^{N})(2\sum_{m=0}^{l-1}\widetilde{\mathcal{L}}_m^{(\beta)}(t_{\beta,k}^{N})+\widetilde{\mathcal{L}}_l^{(\beta)}(t_{\beta,k}^{N})) \\ &-\frac{1}{2}\beta^2(\sum_{j=0}^{N}U(t_{\beta,j}^{N})\widetilde{\mathcal{L}}_0^{(\beta)}(t_{\beta,j}^{N})\widetilde{\omega}_{\beta,j}^{N})\widetilde{\mathcal{L}}_0^{(\beta)}(t_{\beta,k}^{N}). \end{aligned}$$

Furthermore, we put

$$\begin{aligned} a_{\beta,k,j}^{N} = &-\tfrac{1}{2}\beta^2\widetilde{\omega}_{\beta,j}^{N}(\sum_{l=1}^{N}\widetilde{\mathcal{L}}_l^{(\beta)}(t_{\beta,j}^{N})(2\sum_{m=0}^{l-1}\widetilde{\mathcal{L}}_m^{(\beta)}(t_{\beta,k}^{N})+\widetilde{\mathcal{L}}_l^{(\beta)}(t_{\beta,k}^{N})) \\ &+\widetilde{\mathcal{L}}_0^{(\beta)}(t_{\beta,j}^{N})\widetilde{\mathcal{L}}_0^{(\beta)}(t_{\beta,k}^{N})), \quad 0\le j\le N, 1\le k\le N. \end{aligned} \tag{3.9}$$

Then

$$\frac{d}{dt}\widetilde{I}_{\beta,N}U(t^N_{\beta,k}) = \sum_{j=0}^{N} a^N_{\beta,k,j}U(t^N_{\beta,j}). \tag{3.10}$$

We define the vectors $\mathbb{U}^N$, $\mathbb{F}^N_\beta$, $\mathbb{G}^N_{\beta,1}$, and the matrix $\mathbb{A}^N_\beta$ in the same manner as in the last section. But the entries of $\mathbb{G}^N_{\beta,1}$ and $\mathbb{A}^N_\beta$ are now given as in (3.8) and (3.9). Finally, we obtain

$$\begin{cases} \mathbb{A}^N_\beta \mathbb{U}^N = \mathbb{F}^N_\beta(\mathbb{U}^{\mathbb{N}}) + \mathbb{G}^N_{\beta,1}, \\ U(0) = U_0. \end{cases} \tag{3.11}$$

We now approximate $U(t)$ by $u^N(t) \in Q_N(0,\infty)$. Clearly, $\widetilde{I}_{\beta,N}u^N(t) = u^N(t)$. We also use the notations $\mathbf{u}^N$ and $\mathbb{F}^N_\beta(\mathbf{u}^N)$ as in the last section. By replacing $\mathbb{U}^N$ by $\mathbf{u}^N$ and neglecting $\mathbb{G}^N_{\beta,1}$ in (3.11), we derive the new integration process, which is to find $\mathbf{u}^N$ such that

$$\begin{cases} \mathbb{A}^N_\beta \mathbf{u}^N = \mathbb{F}^N_\beta(\mathbf{u}^N), \\ u^N(0) = U_0. \end{cases} \tag{3.12}$$

This is also an implicit scheme. The global numerical solution is

$$u^N(t) = \sum_{l=0}^{N} \widetilde{u}^N_{\beta,l}\widetilde{\mathcal{L}}^{(\beta)}_l(t), \qquad t \geq 0,$$

with

$$\widetilde{u}^N_{\beta,l} = \beta(u^N, \widetilde{\mathcal{L}}^{(\beta)}_l)_{\beta,N} = \beta\sum_{j=0}^{N} u^N(t^N_{\beta,j})\widetilde{\mathcal{L}}^{(\beta)}_l(t^N_{\beta,j})\widetilde{\omega}^N_{\beta,j}.$$

Indeed, scheme (3.12) is equivalent to the system

$$\begin{cases} \dfrac{d}{dt}u^N(t^N_{\beta,k}) = f(u^N(t^N_{\beta,k}), t^N_{\beta,k}), & 1 \leq k \leq N, \\ u^N(0) = U_0. \end{cases} \tag{3.13}$$

Next, we estimate the error of numerical solution. Let $E^N(t) = u^N(t) - \widetilde{I}_{\beta,N}U(t)$. Subtracting (3.8) from (3.13) gives that

$$\begin{cases} \dfrac{d}{dt}E^N(t^N_{\beta,k}) = G^N_{\beta,2}(t^N_{\beta,k}) - G^N_{\beta,1}(t^N_{\beta,k}), & 1 \leq k \leq N, \\ E^N(0) = 0 \end{cases} \tag{3.14}$$

where

$$G^N_{\beta,2}(t^N_{\beta,k}) = f(u^N(t^N_{\beta,k}), t^N_{\beta,k}) - f(\widetilde{I}_{\beta,N}U(t^N_{\beta,k}), t^N_{\beta,k}).$$

We now multiply (3.14) by $2E^N(t^N_{\beta,k})\widetilde{\omega}^N_{\beta,k}$ and sum the result for $1 \leq k \leq N$. Due to $E^N(0) = 0$, we obtain that

$$2(E^N, \frac{d}{dt}E^N)_{\beta,N} = A^N_{\beta,1} + A^N_{\beta,2} \tag{3.15}$$

where

$$A^N_{\beta,1} = -2(G^N_{\beta,1}, E^N)_{\beta,N}, \qquad A^N_{\beta,2} = 2(G^N_{\beta,2}, E^N)_{\beta,N}.$$

Thanks to (3.4) and the Cauchy inequality, we deduce that

$$2(E^N, \frac{d}{dt}E^N)_{\beta,N} = 2(E^N, \frac{d}{dt}E^N) = |E^N(\infty)|^2,$$
$$|A^N_{\beta,1}| \le 2\|G^N_{\beta,1}\|_{\beta,N}\|E^N\|_{\beta,N} = 2\|G^N_{\beta,1}\|\|E^N\|.$$

Thus (3.15) reads

$$|E^N(\infty)|^2 \le A^N_{\beta,2} + 2\|G^N_{\beta,1}\|\|E^N\|. \tag{3.16}$$

We assume that

$$(f(z_1,t) - f(z_2,t))(z_1 - z_2) \le -\gamma_0|z_1 - z_2|^2, \quad \gamma_0 > 0, \quad \forall z_1, z_2 \in \mathbb{R}. \tag{3.17}$$

Then $A^N_{\beta,2} \le -2\gamma_0\|E^N\|^2$ and so by (3.17),

$$|E^N(\infty)|^2 + \gamma_0\|E^N\|^2 \le \frac{1}{\gamma_0}\|G^N_{\beta,1}\|^2. \tag{3.18}$$

Hence it suffices to estimate $\|G^N_{\beta,1}\|^2$.

With the aid of (2.21) and (3.7), we deduce that for $r \ge 1$,

$$\|\widetilde{I}_{\beta,N}v - v\| = \|I_{\beta,N}(e^{\frac{1}{2}\beta t}v) - e^{\frac{1}{2}\beta t}v\|_{\omega_\beta} \le c(\beta N)^{\frac{1}{2}-\frac{r}{2}}\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}v). \tag{3.19}$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}(\widetilde{I}_{\beta,N}v - v) = &-\frac{1}{2}\beta e^{-\frac{1}{2}\beta t}(I_{\beta,N}(e^{\frac{1}{2}\beta t}v) - e^{\frac{1}{2}\beta t}v))\\ &+ e^{-\frac{1}{2}\beta t}\frac{d}{dt}(I_{\beta,N}(e^{\frac{1}{2}\beta t}v) - e^{\frac{1}{2}\beta t}v).\end{aligned} \tag{3.20}$$

Using the above result, along with (2.21), we assert that for $r \ge 1$,

$$\|\frac{d}{dt}(\widetilde{I}_{\beta,N}v - v)\| \le c(\beta N)^{\frac{1}{2}-\frac{r}{2}}(\beta\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}v) + \mathcal{R}^{(2)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}v)).$$

Consequently,

$$\begin{aligned}\|G^N_{\beta,1}\| &\le \|\tfrac{d}{dt}(\widetilde{I}_{\beta,N}U - U)\| + \|\tfrac{d}{dt}U - \widetilde{I}_{\beta,N}\tfrac{d}{dt}U\|\\ &\le c(\beta N)^{\frac{1}{2}-\frac{r}{2}}(\beta\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U) + \mathcal{R}^{(2)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U) + \mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}\tfrac{dU}{dt})).\end{aligned}$$

Thus, (3.18) implies that

$$\begin{aligned}|E^N(\infty)| + \gamma_0^{\frac{1}{2}}\|E^N\| \le \tfrac{c}{\gamma_0^{\frac{1}{2}}}(\beta N)^{\frac{1}{2}-\frac{r}{2}}(\beta\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U) + \mathcal{R}^{(2)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U)\\ +\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}\tfrac{dU}{dt})).\end{aligned}$$

Finally, the following conclusion follows from the previous statements, (3.19) and the fact that

$$\begin{aligned}|U(\infty) - u^N(\infty)| &\le |\widetilde{I}_{\beta,N}U(\infty) - U(\infty)| + |E^N(\infty)|\\ &\le 2\|\widetilde{I}_{\beta,N}U - U\|^{\frac{1}{2}}\|\widetilde{I}_{\beta,N}U - U\|_1^{\frac{1}{2}} + |E^N(\infty)|.\end{aligned}$$

**Theorem 3.1.** *Let* (3.17) *hold and* $\beta > 0$. *If* $\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U), \mathcal{R}^{(2)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U)$ *and* $\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}\frac{dU}{dt})$ *are finite, then*

$$\begin{aligned}\|U - u^N\| \le \frac{c}{\gamma_0}(\beta N)^{\frac{1}{2}-\frac{r}{2}}((\gamma_0 + \beta)\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U)\\ + \mathcal{R}^{(2)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U) + \mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}\frac{dU}{dt}))\end{aligned} \tag{3.21}$$

*and*

$$\begin{aligned}|U(\infty)-u^N(\infty)| \le\ & c(\beta N)^{\frac{1}{2}-\frac{r}{2}}((\beta\gamma_0^{-\frac{1}{2}}+\beta+1)\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U) \\ & +(1+\gamma_0^{-\frac{1}{2}})\mathcal{R}^{(2)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}U)+\gamma_0^{-\frac{1}{2}}\mathcal{R}^{(1)}_{N,r,\beta}(e^{\frac{1}{2}\beta t}\tfrac{dU}{dt})).\end{aligned} \tag{3.22}$$

*Remark* 3.1. Because of (3.17), the norm $||U||$ is finite, as long as for any $\epsilon>0$,

$$\int_0^\infty e^{-(2\gamma-\epsilon)t}(\int_0^t e^{(2\gamma-\epsilon)\xi}f^2(0,\xi)d\xi)dt<\infty.$$

Furthermore, if $f(z,t)$ fulfills some additional conditions, then the norms appearing in the right sides of (3.21) and (3.22) are finite. Therefore, for a certain positive constant $c_*$ depending only on $\beta$,

$$\|U-u^N\|+|U(\infty)-u^N(\infty)|=c_*(1+\frac{1}{\gamma_0})(\ln N)^{\frac{1}{2}}N^{\frac{1}{2}-\frac{r}{2}}.$$

Consequently, for $r>1$, the scheme (3.12) has the global convergence and the spectral accuracy in $L^2(0,\infty)$. Moreover, at the infinity, the numerical solution has the same accuracy. This also indicates that the pointwise numerical error decays rapidly as the mode $N$ increases, with the convergence rate as $c_*(\ln N)^{\frac{1}{2}}N^{\frac{1}{2}-\frac{r}{2}}$. On the other hand, for any fixed $N$, the norm $||U-u^N||$ is bounded, and so $U(t)-u^N(t)\to 0$, a.e., as $t\to\infty$. In particular, for the smooth solution, the error decays like $o(t^{-\frac{1}{2}})$. Hence, it is very efficient for long-time numerical simulations of dynamical systems.

*Remark* 3.2. A modification of algorithm (3.12) also works well, if (2.23) holds. In this case, we make the transformation (2.24) and use (3.12) to resolve (2.25). The numerical solution of the original problem is given by $u^N(t)=e^{\alpha t}v^N(t)$. If $\alpha>\alpha_1$, then the global spectral accuracy of $v^N$ is ensured.

*Remark* 3.3. There is a close relation between schemes (2.13) and (3.12). To show this, we make the variable transformation

$$V(t)=e^{\frac{1}{2}\beta t}U(t),\qquad F(V(t),t)=e^{\frac{1}{2}\beta t}f(e^{-\frac{1}{2}\beta t}V(t),t)+\frac{1}{2}\beta V(t). \tag{3.23}$$

Then (2.7) is changed to

$$\begin{cases}\dfrac{dV(t)}{dt}=F(V(t),t), & t>0,\\ V(0)=U_0.\end{cases} \tag{3.24}$$

We may use scheme (2.13) to resolve (3.24) and obtain the numerical solution $v^N(t)$. Finally we have the numerical solution of (2.7) as $u^N(t)=e^{-\frac{1}{2}\beta t}v^N(t)$. If (3.17) holds, then the error estimates similar to those of Theorem 3.1 also hold. Conversely, if $\beta>2\gamma$, then we can use (3.12) to solve the related reformed problem with the unknown function $V(t)=e^{-\frac{1}{2}\beta t}U(t)$, with the spectral accuracy. Then $u^N(t)=e^{\frac{1}{2}\beta t}v^N(t)$.

*Remark* 3.4. It is easy to generalize the method (3.12) to a system of ordinary differential equations. If it fulfills a certain condition like in (3.17), then the same result as in Theorem 3.1 holds. The proposed method is also applicable to Hamilton systems.

## 4. Refinement of numerical results

In the previous sections, we introduced two integration processes for ordinary differential equations. Theoretically, their numerical errors with bigger mode $N$ are smaller. But in actual computation, it is not convenient to use very big mode. On the other hand, the distance between the adjacent interpolation nodes $t^N_{\beta,j}$ and $t^N_{\beta,j-1}$ increases fast as $N$ and $j$ increase, especially for the nodes which are located far from the original point $t=0$. This feature is one of advantages of the Laguerre interpolation, since we can use moderate mode $N$ to evaluate the unknown function at large $t$. But it is also its shortcoming. In fact, if the exact solution oscillates or changes very rapidly between two large adjacent interpolation nodes, then we may lose information about the structure of exact solution between those nodes. To remedy this deficiency, we may refine the numerical results. For example, let $N_0$ be a moderate positive integer, $\beta_0>0$, and the set of nodes $\{t^{N_0}_{0,\beta_0,j}\}^{N_0}_{j=0}=\{t^{N_0}_{\beta_0,j}\}^{N_0}_{j=0}$. We use (2.13) or (3.12) with the interpolation nodes $\{t^{N_0}_{0,\beta_0,j}\}^{N_0}_{j=0}$ to obtain the original numerical solution $u^{(0,N_0)}(t)=u^{N_0}(t), 0\le t<\infty$. Then we take $t^{N_1}_{1,\beta_1,0}=t^{N_0}_{0,\beta_0,N_0}$ and consider the problem

$$\begin{cases} \dfrac{d}{dt}U^{(1)}(t)=f(U^{(1)}(t),t), & t>t^{N_1}_{1,\beta_1,0}, \\ U^{(1)}(t^{N_1}_{1,\beta_1,0})=u^{(0,N_0)}(t^{N_1}_{1,\beta_1,0}). \end{cases} \tag{4.1}$$

By a shifting argument and using (2.13) or (3.12) with the parameter $\beta_1$ and $N_1$ interpolation nodes $\{t^{N_1}_{1,\beta_1,j}\}^{N_1}_{j=0}$, we get the refined numerical solution $u^{(1,N_1)}(t)$ for $t^{N_1}_{1,\beta_1,0}\le t<\infty$, especially the values of $u^{(1,N_1)}(t)$ at the interpolation points $t^{N_1}_{1,\beta_1,j}$, $0\le j\le N_1$. Repeating the above procedure, we obtain the refined numerical solution $u^{(m,N_m)}(t)$ for $t^{N_m}_{m,\beta_m,0}\le t<\infty$. This algorithm saves work and provides more accurate numerical results; see Section 5.

*Remark* 4.1. In actual computation, we may take $t^{N_m}_{m,\beta_m,0}=t^{N_{m-1}}_{m-1,\beta_{m-1},N_{m-1}-k_{m-1}}$, $k_{m-1}=0,1$ or $2$.

## 5. Numerical results

In this section, we present some numerical results. The algorithms are implemented by using MATLAB, and all calculations are carried out with a computer of CPU P4 3.0G, Mother Board I865PE/FSB 800/Dual Channel DDR400.

### 5.1. **The first interpolation process.**

We first use scheme (2.13) to solve problem (2.7) with the test function $U(t)=(t+10)^{\frac{11}{2}}+\frac{1}{2}\sin t$, which oscillates and grows to infinity as $t$ increases. The corresponding right term at (2.7) is

$$f(U(t),t)=\frac{1}{4}\exp(\sin(U(t)))+\frac{11}{2}(t+10)^{\frac{9}{2}}+\frac{1}{2}\cos t-\frac{1}{4}\exp(\sin((t+10)^{\frac{11}{2}}+\frac{1}{2}\sin t)),$$

which fulfills the condition (2.19) with $\gamma=\frac{1}{4}e$. Therefore, as predicted by (2.22), for any $\beta>2\gamma=\frac{1}{2}e\simeq 1.36$, the global numerical error $\|u^N-U\|_{\omega_\beta}$ decays exponentially as $N\to\infty$.

For a description of numerical errors, we introduce the global absolute error $E^N_{ga}=\|u^N-U\|_{\omega_\beta,N}$ and the global relative error $E^N_{gr}=\|\frac{u^N-U}{U}\|_{\omega_\beta,N}$. We are also interested in the pointwise numerical error $E^N_{pr}(t)=|\frac{u^N(t)-U(t)}{U(t)}|$.

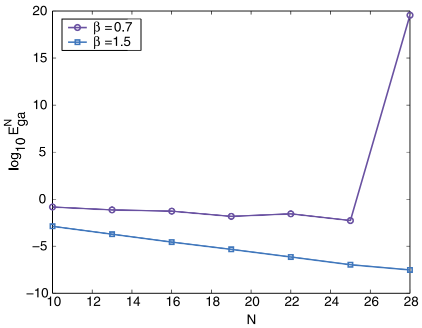


FIGURE 1. Global absolute errors of (2.13).

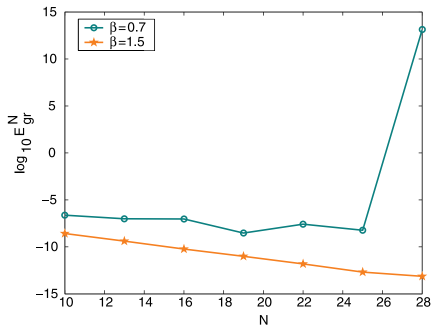


FIGURE 2. Global relative errors of (2.13).

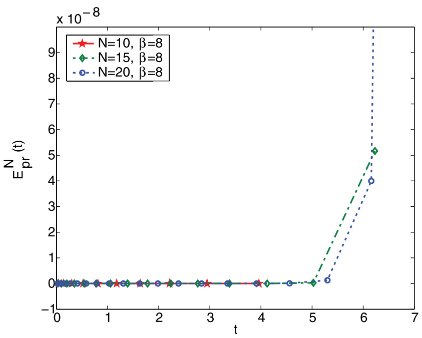


FIGURE 3. Pointwise relative errors of (2.13).

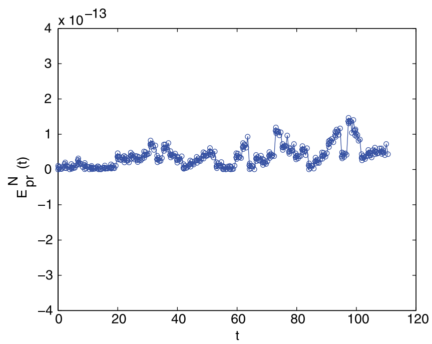


FIGURE 4. Pointwise errors of refined method.

In Figures 1 and 2, we plot the global absolute errors $\log_{10}$ of $E^N_{ga}$ and the global relative errors $\log_{10}$ of $E^N_{gr}$ with various values of $\beta$ and $N$. They indicate that for $\beta = 1.5 > 2\gamma$, the global errors decay exponentially as $N$ increases, while the scheme (2.13) is divergent for $\beta = 0.7 < 2\gamma$. They coincide very well with theoretical analysis.

As pointed out in Section 4, due to the appearance of the weight function $e^{-\beta t}$, the pointwise numerical errors for large $t$ might be bigger than the global weighted errors. In Figure 3, we plot the pointwise relative errors $E^N_{pr}(t)$ with $\beta = 8$ and $N = 10,\ 15,\ 20$, respectively. We see that the pointwise relative errors for large $j \simeq N$ are really bigger than those with small $j$. To remedy this deficiency, we use the refinement given in Section 4, with $t^{N_m}_{m,\beta_m,0} = t^{N_{m-1}}_{m-1,\beta_{m-1},N_{m-1}-k_{m-1}}$, $k_m$ being a small positive integer. In Figure 4, we plot the pointwise relative errors $E^N_{pr}(t)$ with uniform $N_m = N = 10$, $\beta_m = \beta = 8$ and $k_m = k = 2$ at all steps. Clearly, this refined approach provides more stable and accurate numerical results than the single step method (without refinement), especially for long-time calculations.

5.2. **The second interpolation process.** Next, we use (3.12) to solve (2.7) with the test function $U(t) = (2 + \sin t)e^{-\frac{1}{5}t}$, which oscillates and decays exponentially as $t$ increases. The corresponding right term at (2.7) is

$$f(U(t),t) = -U^3(t) - U(t) - \tfrac{1}{5}(2+\sin t)e^{-\frac{1}{5}t} + \cos t\ e^{-\frac{1}{5}t} \\ +(2+\sin t)^3 e^{-\frac{3}{5}t} + (2+\sin t)e^{-\frac{1}{5}t}$$

which fulfills the condition (3.17) with $\gamma_0 = 1$. According to the estimates (3.21) and (3.22), we predict that for any $\beta > 0$, both the global absolute numerical error $E^N_{ga} = \|u^N - U\|_N$ and the pointwise absolute numerical error $E^N_{pa}(t) = |u^N(t) - U(t)|$, decay exponentially as $N \to \infty$. In Figure 5, we plot the $\log_{10}$ of $E^N_{ga}$ with $\beta = 0.5,\ 1,\ 2$ and 3, respectively, which illustrates that the errors $E^N_{ga}$ decay exponentially. Moreover, we find that a suitable choice of parameter $\beta$ can raise the numerical accuracy. These facts coincide very well with theoretical analysis. In Figure 6 we plot the pointwise absolute errors $E^N_{pa}(t)$ with fixed $N = 100$ and $\beta = 1,\ 2,\ 3$, respectively. It is shown that the pointwise absolute errors also decay fast as $t \to \infty$, as mentioned in Remark 3.1, and that suitable parameter $\beta$ raises the numerical accuracy. They coincide again very well with theoretical analysis.

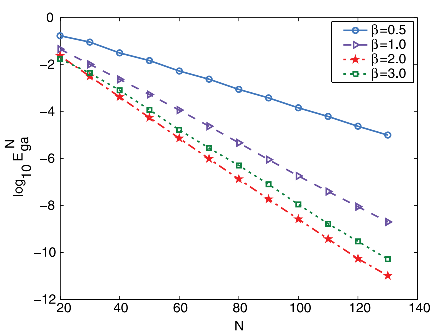


FIGURE 5. Global absolute errors.

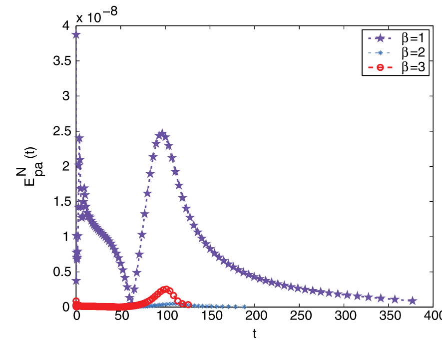


FIGURE 6. Pointwise absolute errors.

5.3. **Comparison with other numerical methods.** We now compare our new integration processes with other numerical methods. We solve the Hamiltonian system (2.27) by using algorithm (2.28) coupled with the refinement proposed in Section 4. For simplicity, we denote this method by LR method. In actual computation, we use $2.3\times10^5$ steps of refinement. At each step we take uniform $N_m = 10, \beta_m = 40$ and $k_m = 2$. Thus, the final interpolation node is $t = 1.0175 \times 10^5$. In Figure 7, we plot the numerical orbit $(p^N(t), q^N(t))$ for $t \le 1.0175\times10^5$, which is virtually indistinguishable with the exact orbit of movement governed by the Hamiltonian system (2.27). But the accurate numerical orbit does not imply the accurate numerical solution automatically, since the numerical point $(p^N(t), q^N(t))$ may be very far from the exact point $(P(t), Q(t))$ even if both of them lie on the same orbit. In other words, the numerical phase error might be big. Therefore, we are more interested in the pointwise absolute errors $E^N(t) = ((p^N(t) - P(t))^2 + (q^N(t) - Q(t))^2)^{\frac{1}{2}}$. In Figure 8, we plot the pointwise absolute errors $E^N(t)$, which grow slowly.

Next, we solve the same problem by using the Runge-Kutta scheme based on the Legendre-Gauss interpolation of order 6, and the Leap-frog scheme. For simplicity, they are denoted by RK method and LF method, respectively. Both of them are symplectic (cf. [5, 6, 17]). We take $\tau$ as the mesh size for time-discretization.

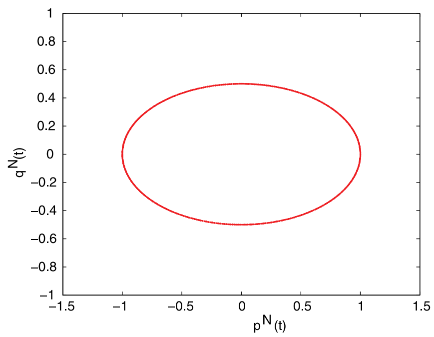

FIGURE 7. Numerical orbit $(p^N(t), q^N(t))$.

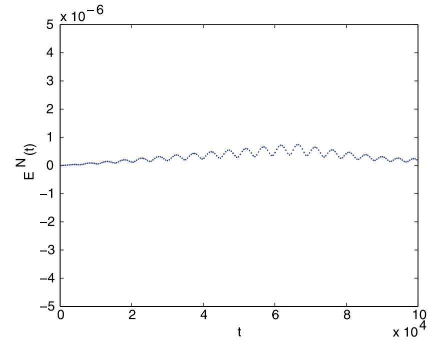

FIGURE 8. Pointwise absolute errors.

In Table 1, we list the numerical errors at $t = 1.0175 \times 10^5$ of LR, Rk and LF methods, and the corresponding CPU elapsed time. Clearly, our methods cost less computational time for obtaining the nearly same numerical accuracy.

In Table 2, we list the numerical errors at $t = 1.0175 \times 10^5$ of LR, Rk and LF methods, and the corresponding CPU elapsed time. Obviously, our methods provide much more accurate numerical results than the compared methods with the same computational time.

TABLE 1. Error $E^N(t)$ and CPU elapsed time, $t = 1.0175 \times 10^5$.

| Method | Error $E^N(t)$ | CPU elapsed time (second) |
|---|---|---|
| LR | $1.7169 \times 10^{-7}$ | $3.245 \times 10$ |
| RK, $\tau = 3.7 \times 10^{-2}$ | $3.2678 \times 10^{-7}$ | $1.333 \times 10^2$ |
| LF, $\tau = 1.6 \times 10^{-6}$ | $3.4241 \times 10^{-7}$ | $3.235 \times 10^4$ |

TABLE 2. Error $E^N(t)$ and CPU elapsed time, $t = 1.0175 \times 10^5$.

| Method | Error $E^N(t)$ | CPU elapsed time (second) |
|---|---|---|
| LR | $1.7169 \times 10^{-7}$ | $3.245 \times 104$ |
| RK,$\tau = 1.5 \times 10^{-1}$ | $1.1447 \times 10^{-3}$ | $3.334 \times 10$ |
| LF,$\tau = 1.4 \times 10^{-3}$ | $1.4899 \times 10^{-1}$ | $3.355 \times 10$ |

## 6. CONCLUDING REMARKS

In this paper, we proposed two new integration processes of ordinary differential equations, which have fascinating advantages.

- The suggested integration processes are based on the modified Laguerre-Radau interpolations on the half line. They provide the global numerical solution and the global convergence naturally, and thus are available for long-time numerical simulations of dynamical systems.

- Benefiting from the rapid convergence of the modified Laguerre-Radau interpolations, these processes possess the spectral accuracy. In particular, the numerical results fit the exact solutions well at the interpolation nodes.
- Since the distances between the adjacent nodes increase fast as $N$ and $j$ increase, we can obtain accurate numerical results even for moderate mode $N$. It in turn saves a lot of work. By taking suitable parameter $\beta$ in the first proposed process, we weaken the condition on the underlying problems, and so enlarge its applications.
- The pointwise numerical error of the second process decays with the convergence rate $\mathcal{O}(N^{\frac{1}{2}-\frac{r}{2}}(\ln N)^{\frac{1}{2}})$ as $N$ increases. It also decays with the rate $o(t^{-\frac{1}{2}})$, as $t \to \infty$. Thus, this process is very efficient.

The numerical experiments showed the efficiency of these two integration processes, and coincided with theoretical analysis very well. In particular, our methods cost less computational time and provide more accurate numerical results than other methods.

We also developed a technique for refinement of Laguerre approximation. This trick not only simplifies calculation, but also simulates the long-time behaviors of dynamical systems more properly.

Although we only considered a model problem, the suggested methods and techniques are also applicable to many other problems, such as various evolutionary partial differential equations and infinite-dimensional nonlinear dynamical systems. On the other hand, we may design other integration processes of ordinary differential equations with high accuracy based on the idea proposed in this paper; see [11].

## Appendix

We prove the existence and uniqueness of numerical solution. For simplicity, we only focus on the system (2.13) with the condition (2.26). We consider the following iteration process:

$$\begin{cases} \dfrac{d}{dt}u_m^N(t_{\beta,k}^N) = f(u_{m-1}^N(t_{\beta,k}^N), t_{\beta,k}^N), & 1 \le k \le N, \quad m \ge 1, \\ u_m^N(0) = U_0. \end{cases}$$

Furthermore, we set $\tilde{u}_m^N(t) = u_m^N(t) - u_{m-1}^N(t)$. Then by (2.15),

$$\frac{d}{dt}\tilde{u}_m^N(t_{\beta,k}^N) = f(u_{m-1}^N(t_{\beta,k}^N), t_{\beta,k}^N) - f(u_{m-2}^N(t_{\beta,k}^N), t_{\beta,k}^N), \qquad 1 \le k \le N, \quad m \ge 1.$$

We multiply the above inequality by $\tilde{u}_m^N(t_{\beta,k}^N)\omega_{\beta,k}^N$ and sum the result for $1 \le k \le N$. Due to (2.26) and $\tilde{u}_m^N(0) = 0$, we deduce that

$$(\tilde{u}_m^N, \frac{d}{dt}\tilde{u}_m^N)_{\omega_\beta,N} \le L\|\tilde{u}_m^N\|_{\omega_\beta,N}\|\tilde{u}_{m-1}^N\|_{\omega_\beta,N}.$$

On the other hand, using (2.4) and integrating by parts yield that

$$2(\tilde{u}_m^N, \frac{d}{dt}\tilde{u}_m^N)_{\omega_\beta,N} = 2(\tilde{u}_m^N, \frac{d}{dt}\tilde{u}_m^N)_{\omega_\beta} = \beta\|\tilde{u}_m^N\|_{\omega_\beta}^2, \qquad \|\tilde{u}_m^N\|_{\omega_\beta,N} = \|\tilde{u}_m^N\|_{\omega_\beta}.$$

A combination of the previous statements leads to $\|\tilde{u}_m^N\|_{\omega_\beta}^2 \le \frac{2L}{\beta}\|\tilde{u}_{m-1}^N\|_{\omega_\beta}^2$. Thus, for $\beta > 2L$, the above iteration process is convergent. This fact implies the existence of the solution of (2.13). We can prove the uniqueness of the solution of (2.13) easily.

## References


1. I. Babuska and T. Janik, The *h-p* version of the finite element method for parabolic equations, Part 1, The *p*-version in time, *Numerical Method for Partial Differential Equations*, **5**(1989), 363–399. MR1107894 (92d:65160)
2. J. C. Butcher, Implicit Runge-Kutta processes, *Math. Comput.,* **18**(1964), 50–64. MR0159424 (28:2641)
3. J. C. Butcher, Integration processes based on Radau quadrature formulas, *Math. Comput.,* **18**(1964), 233–244. MR0165693 (29:2973)
4. J. C. Butcher, *The Numerical Analysis of Ordinary Differential Equations, Runge-Kutta and General Linear Methods*, John Wiley & Sons, Chichester, 1987. MR878564 (88d:65002)
5. K. Feng, Difference schemes for Hamiltonian formulism and symplectic geometry, *J. Comput. Math,* **4**(1986), 279–289. MR860157 (88a:65094)
6. K. Feng and M. Z. Qin, *Sympletic geometric algorithms for Hamiltonian systems,* Zhejiang Science and Technology Press, Hangzhou, 2003.
7. D. Funaro, *Polynomial Approximations of Differential Equations,* Springer-Verlag, Berlin, 1992. MR1176949 (94c:65078)
8. Ben-yu Guo and Jie Shen, Laguerre-Galerkin method for nonlinear partial differential equations on a semi-infinite interval, *Numer. Math.*, **86**(2000), 635–654. MR1794346 (2001h:65152)
9. Ben-yu Guo, Jie Shen and Cheng-long Xu, Generalized Laguerre approximation and its applications to exterior problems, *J. Comp. Math.,* **23**(2005), 113–130. MR2118049 (2005m:65289)
10. Guo Ben-yu, Wang Li-lian and Wang Zhong-qing, Generalized Laguerre interpolation and pseudospectral method for unbounded domains, *SIAM J. Numer. Anal.*, **43**(2006), 2567–2589. MR2206448 (2007e:65129)
11. Guo Ben-yu and Wang Zhong-qing, Numerical Integration based on Laguerre-Gauss interpolation, *Comp. Meth. in Appl. Math. Engi.*, DOI 10.1016/j.cma, 2006, 10.10.035.
12. Guo Ben-yu and Xu Cheng-long, Mixed Laguerre-Legendre pseudospectral method for incompressible flow in an infinite strip, *Math. Comp.*, **73**( 2003), 95–125. MR2034112 (2004m:65157)
13. Guo Ben-yu and Zhang Xiao-yong, A new generalized Laguerre approximation and its applications, *J. Comp. Appl. Math.,* **181**(2005), 342–363. MR2146844 (2006e:65180)
14. Guo Ben-yu and Zhang Xiao-yong, Spectral method for differential equations of degenerate type on unbounded domains by using generalized Laguerre functions, *Appl. Numer. Math.*, **57** (2007), 455-471. MR2310760
15. E. Hairer, C. Lubich, and G. Wanner, *Geometric Numerical Integration: Structure-Preserving Algorithms for Ordinary Differential Equations, Springer Series in Comput. Mathematics*, Vol. 31, Springer-Verlag, Berlin, 2002. MR1904823 (2003f:65203)
16. E. Hairer, S. P. Norsett, and G. Wanner, *Solving Ordinary Differential Equation I: Nonstiff Problems*, Springer-Verlag, Berlin, 1987. MR868663 (87m:65005)
17. E. Hairer and G. Wanner, *Solving Ordinary Differential Equation II: Stiff and Differential—Algebraic Problems*, Springer-Verlag, Berlin, 1991. MR1111480 (92a:65016)
18. D. J. Higham, Analysis of the Enright-Kamel partitioning method for stiff ordinary differential equations, *IMA J. Numer. Anal.,* **9**(1989), 1-14. MR988786 (90m:65140)
19. V. Iranzo and A. Falquès, Some spectral approximations for differential equations in unbounded domains, *Comput. Methods Appl. Mech. Engrg.,* **98**(1992), 105-126. MR1172676 (93d:65103)
20. Y. Maday, B. Pernaud-Thomas and H. Vandeven, Reappraisal of Laguerre type spectral methods, *La Recherche Aerospatiale,* **6**(1985), 13–35. MR850680 (88b:65135)
21. G. Mastroianni and G. Monegato, Nyström interpolants based on zeros of Laguerre polynomials for some Weiner-Hopf equations, *IMA J. of Numer. Anal.,* **17**(1997), 621–642. MR1476342 (98j:45011)
22. J. M. Sanz-Serna and M. P. Calvo, *Numerical Hamiltonian Problems,* AMMC7, Chapman and Hall, London, 1994. MR1270017 (95f:65006)
23. L. I. Schiff, Nonlinear meson theory of nuclear forces, I, *Phys. Rev.,* **84**(1981), 1–9.
24. Jie Shen, Stable and efficient spectral methods in unbounded domains using Laguerre functions, *SIAM J. Numer. Anal.,* **38**(2000), 1113–1133. MR1786133 (2001g:65165)
25. A. M. Stuart and A. R. Humphries, *Dynamical systems and Numerical Analysis*, Cambridge University Press, Cambridge, 1996. MR1402909 (97g:65009)

26. H. TalJ-Ezer, Spectral methods in time for parabolic equations, *SIAM J. Numer. Anal.,* **23**(1989), 1–11.
27. Xu Cheng-long and Guo Ben-yu, Laguerre pseudospectral method for nonlinear partial differential equations, *J. Comp. Math.,* **20**(2002), 413-428. MR1914675 (2003e:65184)

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